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An Improved Criterion for Fixed Points of Contraction Mappings

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INTRODUCTION

The aim of the paper is to give a criterion of the usual functional type for the convergence of iterations generated by a contraction to a fixed point of the contraction. The criterion given is quite general and has several of the other published criteria as corollaries, but it has the drawback, common to most others, of not providing any general error estimate.

1. MAIN THEOREM

Let (X, d) be a complete metric space and let $f: X \to X$ be a contractive mapping or contraction, i.e.,

$$d(f(x), f(y)) < d(x, y) \qquad \text{for all } x \neq y \text{ in } X.$$

Let x_0 be an initial choice of point in X and let $x_n = f(x_{n-1})$ for $n > 0$. Then it is well-known [1] that if $\{x_n\}$ is a Cauchy sequence, then $x_n \to x_\infty$ for some $x_\infty \in X$ and that x_∞ is then the unique fixed point of f in X.

Following Rakotch [2], we define a class of functions T as follows:

DEFINITION. T is the class of functions $\alpha: \mathbf{R}^+ \to [0, 1)$, where $\mathbf{R}^+ = \{t \in \mathbf{R} \mid t > 0\}$, such that if t_n is monotone decreasing in $\mathbf{R}^+$ and $\alpha(t_n) \to 1$, then $t_n \to 0$.

The reader is asked to note that no continuity is assumed. The only assumption is that α does not possess a certain type of discontinuity.

THEOREM 1.1. *Let $f: X \to X$ be a contraction on a complete metric space (X, d). Assume there exists an $\alpha \in T$ such that*

$$d(f(x), f(y)) < \alpha(d(x, y))\, d(x, y) \qquad \textit{for all } x \neq y \textit{ in } X.$$

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Then for any choice of $x_0 \in X$, *letting* $x_n = f(x_{n-1})$ *for* $n > 0$, *we have that* $x_n \to x_\infty$, *the unique fixed point of* f *in* X.

The proof of this theorem will follow from two lemmas given in the next section, one of them an interesting necessary and sufficient condition for a particular iteration to converge. The lemmas are a development of results alluded to in [3], and are given here because they appear to encompass most of the criteria so far published. The results appear to be readily extendable to uniform spaces, as in [4].

2. Two Lemmas

We first provide an explicit proof of the result alluded to in [3].

Lemma 2.1. *Let* $f: X \to X$ *be a contraction of a complete metric space. Let* x_0 *be fixed in* X *and set* $x_n = f(x_{n-1})$ *for* $n > 0$. *Then the particular iteration* $x_n \to x_\infty$ *in* X, *with* x_∞ *the unique fixed point of* f, *iff for any two subsequences* x_{h_n} *and* x_{k_n}, *where* $x_{h_n} \neq x_{k_n}$, *we have* $d(x_{h_n}, x_{k_n}) = d_n$ *monotone decreasing and*

$$\Delta_n = d(f(x_{h_n}), f(x_{k_n}))/d_n \to 1$$

only if $d_n \to 0$.

Proof. First let $x_n \to x_\infty$ in X. Then any subsequences also approach x_∞ and so $d_n \to 0$ for all such pairs of subsequences. So if we have convergence, the condition is satisfied.

Now assume the condition is satisfied for the given initial point x_0 in X. Let $x_{h_n} = x_n$ and $x_{k_n} = x_{n+1}$. Then since f is a contraction, $d_n = d(x_n, x_{n+1})$ is a monotonically decreasing sequence of nonnegative numbers. So there is some $\epsilon \geqslant 0$ such that $d_n \to \epsilon$. Assume that $\epsilon > 0$. Then d_n is monotonically decreasing to ϵ, while

$$\Delta_n = d(x_{n+2}, x_{n+1})/d(x_{n+1}, x_n) \to \epsilon/\epsilon = 1.$$

So the condition is violated. Thus we have that $d_n = d(x_n, x_{n+1}) \to 0$.

As was pointed out above, we really need only show that the sequence $\{x_n\}$ is Cauchy. Assume it is not Cauchy. Then there is some $\epsilon > 0$ such that for all natural numbers N, one can find $n, m \geqslant N$ with

$$d(x_n, x_m) > \epsilon.$$

Given this ϵ, we shall construct a pair of subsequences violating the condition of the lemma, i.e., such that $d(x_{h_n}, x_{k_n})$ decreases monotonically to this $\epsilon > 0$ while $\Delta_n \to 1$.

Let n be any natural number. Choose N_n so large that for all $m \geqslant N_n$, we have

$$d(x_m, x_{m+1}) < 1/n.$$

Such an N_n exists since $d(x_m, x_{m+1}) \to 0$.

Now if $n = 1$, let h_1 be the least integer greater than N_1 such that for some $k > h_1$, we have

$$d(x_{h_1}, x_k) > \epsilon.$$

Such a pair exists by the above assumption that $\{x_n\}$ is not Cauchy. Next choose k_1 to be the least such integer above h_1.

If $n > 1$, in addition to the above, choose N_n so large that for all $m \geqslant N_n$, we have

$$d(x_m, x_{m+1}) < d(x_{h_{n-1}}, x_{k_{n-1}}) - \epsilon.$$

Such an N_n still exists since $d(x_m, x_{m+1}) \to 0$, and by the inductive assumption,

$$d(x_{h_{n-1}}, x_{k_{n-1}}) > \epsilon.$$

Next let h_n be the least integer greater than N_n such that for some $k > h_n$,

$$d(x_{h_n}, x_k) > \epsilon.$$

Such an h_n exists since $\{x_n\}$ is not Cauchy. Finally set k_n to be the least such integer k above h_n. Then either $k_n - 1 = h_n$ or else $d(x_{h_n}, x_{k_n - 1}) \leqslant \epsilon$. In either case we have

$$\epsilon < d_n = d(x_{h_n}, x_{k_n}) < \epsilon + 1/n.$$

Moreover, in either case we have

$$\epsilon < d_n < d_{n-1},$$

since $d(x_m, x_{m+1}) < d_{n-1} - \epsilon$ for $m \geqslant N_n$.

Applying the triangular inequality, we have

$$d_n = d(x_{h_n}, x_{k_n}) \leqslant d(x_{h_n}, f(x_{h_n})) + d(f(x_{h_n}), f(x_{k_n})) + d(f(x_{k_n}), x_{k_n})$$

$$\leqslant d(f(x_{h_n}), f(x_{k_n})) + 2/n.$$

Since f is a contraction, we then have

$$1 \geqslant \Delta_n = \frac{d(f(x_{h_n}), f(x_{k_n}))}{d_n} \geqslant \frac{d_n - 2/n}{d_n}.$$

Thus d_n decreases monotonically to $\epsilon > 0$, while $\Delta_n \to 1$, violating the condition. This completes the proof of Lemma 2.1.

It should be observed that the above proof is an extension of that found in [4]. The remark made in [5] concerning [4] appears to be in error.

We now wish to convert the above sequential condition to the more customary functional form.

LEMMA 2.2. *Let $f: X \to X$ be a contraction on a complete metric space. Let x_0 be chosen in X and set $x_n = f(x_{n-1})$ for $n > 0$. Then $x_n \to x_\infty$, where x_∞ is the unique fixed point of f in X iff there exists an $\alpha \in T$ such that for all $n, m \geqslant 0$ with $x_n \neq x_m$, we have*

$$d(f(x_n), f(x_m)) \leqslant \alpha(d(x_m, x_n)) \cdot d(x_m, x_n).$$

Proof. We must show that the existence of such a function $\alpha \in T$ is equivalent to the sequential condition of Lemma 2.1. First assume such an α exists. Let x_{h_n} and x_{k_n} be two subsequences of $\{x_n\}$ with $x_{h_n} \neq x_{k_n}$, and d_n *monotonically decreasing*. Assume that $\Delta_n \to 1$. Then it follows from the above inequality that $\alpha(d(x_{h_n}, x_{k_n})) \to 1$.

Then since $\alpha \in T$, we have $d_n = d(x_{h_n}, x_{k_n}) \to 0$, as desired in the condition of Lemma 2.1.

Next assume that the sequential condition of Lemma 2.1 holds. We define $\alpha: \mathbf{R}^+ \to \mathbf{R}$ as follows:

$$\alpha(t) = \begin{cases} \sup \left\{ \dfrac{d(f(x_m), f(x_n))}{d(x_m, x_n)} \,\middle|\, d(x_m, x_n) \geqslant t \right\}, \\ 0, \qquad \text{if there are no such } x_m, x_n. \end{cases}$$

Since f is a contraction, the above quotients Δ_n are all below 1 and so α is well defined for $t > 0$ and $0 \leqslant \alpha(t) \leqslant 1$ for all t. Now assume that $\alpha(t_n) \to 1$ for some monotonically decreasing sequence $t_n \in \mathbf{R}^+$. We may then assume without loss of generality that $1 - 1/n < \alpha(t_n) \leqslant 1$. We must show that $t_n \to 0$, in order to show that $\alpha \in T$.

Now $\alpha(t_n)$ is the above least upper bound (since $\alpha(t_n) > 1 - 1/n \geqslant 0$). So there is for each $n > 0$ a pair x_{h_n}, x_{k_n} in $\{x_n\}$ with

$$d(x_{h_n}, x_{k_n}) \geqslant t_n$$

and

$$1 - \frac{1}{n} < \Delta_n = \frac{d(f(x_{h_n}), f(x_{k_n}))}{d(x_{h_n}, x_{k_n})} \leqslant \alpha(t_n).$$

So the sequence $\Delta_n \to 1$. But $\Delta_n < 1$, always. Now we must indicate how the sequences x_{h_n} and x_{k_n} may be corrected to be subsequences, i.e., that we may choose h_n and k_n strictly increasing. Clearly at least one sequence is unbounded, say k_n. So we may choose a subsequence which is strictly increasing and again call it k_n. Let k_n and t_n be redefined to be the corresponding subsequences. Now assume that h_n is bounded, while k_n is strictly increasing. Let h_n' repeat infinitely often. Then

$$1 - \frac{1}{n} < \Delta_n' = \frac{d(f(x_{h_n'}), f(x_{k_n}))}{d(x_{h_n'}, x_{k_n})} < 1.$$

But $x_{h_n'}$ is fixed, while $x_{k_n} \to x_\infty$ in X. So

$$\Delta_n' \to \Delta_\infty = \frac{d(f(x_{h_n'}), f(x_\infty))}{d(x_{h_n'}, x_\infty)} < 1,$$

since $x_{h_n'} \neq x_\infty$. From this contradiction we see that h_n is unbounded, and so we may select h_n strictly increasing also. Let k_n and t_n again be redefined to be the corresponding subsequences. Now since $\{x_n\}$ is Cauchy, we then have $d_n \to 0$ trivially. But $d_n \geqslant t_n$. So $t_n \to 0$ also. But this is a subsequence of the original sequence t_n, which was monotonically decreasing. So the original sequence t_n also approaches zero, and the proof of Lemma 2.2 is complete.

The above definition of α may be also used to correct a small error in the corresponding definition in [3]. It should be clear that Theorem 1.1 follows immediately from Lemma 2.2. Unfortunately, in the extension to an arbitrary initial point, the necessary and sufficient character of the condition is lost. It appears to be only sufficient and not necessary that there be such an $\alpha \in T$ for the convergence to follow.

3. Some Applications

The more precise criterion provided above enables us to recognize many of the published results as corollaries. Among them are the following.

Corollary 3.1 (Rakotch [2, p. 463]). *If $f\colon X \to X$ is a contraction of a complete metric space satisfying*

$$d(f(x), f(y)) \leqslant \alpha(d(x, y)) \cdot d(x, y) \qquad \textit{for all } x \neq y \textit{ in } X$$

where $\alpha: \mathbf{R}^+ \to [0, 1)$ *and is monotone decreasing, then for any choice of* x_0 *in* X, *the iteration* $x_n = f(x_{n-1})$, $n > 0$, *converges to a unique fixed point* x_∞ *of* f *in* X.

Proof. Such an α is clearly in the class T.

COROLLARY 3.2. *If* $f: X \to X$ *is a contraction of a complete metric space satisfying*

$$d(f(x), f(y)) \leqslant \alpha(d(x, y)) \cdot d(x, y) \qquad \text{for all } x \neq y \text{ in } X,$$

where $\alpha: \mathbf{R}^+ \to [0, 1)$ *and is monotone increasing, then for any choice of* x_0 *in* X, *the iteration* $x_n = f(x_{n-1})$, $n > 0$, *converges to a unique fixed point* x_∞ *of* f *in* X.

Proof. The proof of this corollary indicated in [3] appears to be wrong, but it is clear that such an α is in the class T, and so convergence follows.

COROLLARY 3.3 (Boyd–Wong [6, p. 331]). *If* $f: X \to X$ *is a contraction of a complete metric space satisfying*

$$d(f(x), f(y)) \leqslant \alpha(d(x, y)) \cdot d(x, y) \qquad \text{for all } x \neq y \text{ in } X$$

where $\alpha: \mathbf{R}^+ \to [0, 1)$ *and is continuous, then for any choice of* x_0 *in* X, *the iteration* $x_n = f(x_{n-1})$, $n > 0$ *converges to a unique fixed point* x_∞ *of* f *in* X.

Proof. As in Corollary 3.1.

COROLLARY 3.4 (F. Browder [7, p. 27]). *Let* X *be a complete metric space,* M *a bounded subset of* X, f *a mapping of* M *into* M. *Suppose there exists a monotone nondecreasing function* $\psi(r)$ *for* $r \geqslant 0$, *with* ψ *continuous on the right, such that* $\psi(r) < r$ *for all* $r > 0$, *while for* x *and* y *in* M,

$$d(f(x), f(y)) \leqslant \psi(d(x, y)) \qquad \text{for all } x \neq y \text{ in } X.$$

Then for each x_0 *in* M, $f^n(x_0)$ *converges to an element* ξ *of* X, *independent of* x_0, *and*

$$d(f^n(x_0), \xi) \leqslant \psi^n(d_0),$$

where d_0 *is the diameter of* M, ψ^n *is the* nth *iterate of* ψ *and* $d_n = \psi^n(d_0) \to 0$ *as* $n \to +\infty$.

Proof. We simply define $\alpha(t) = \psi(t)/t$, for $t > 0$. Then $0 \leqslant \alpha(t) < 1$ and since it is continuous from the right we have $\alpha \in T$. Thus $f^n(x_0) = x_n$ converges to $x_\infty = \xi$. The error estimate which accompanies in this case follows from the additional assumption that ψ is monotone nondecreasing.

4. Remarks

(i) It seems clear from the above that the originally somewhat surprising result of Rakotch (Corollary 3.1) actually brought out the main point that continuity plays no role in the definition of the class T.

(ii) It seems that the functional condition given here is not necessary and sufficient for the convergence of an arbitrary iteration, but it is not yet clear that the provision of such a necessary and sufficient condition is impossible.

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