Journal of Algebra 332 (2011) 244–284

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A Clifford algebra is a weak Hopf algebra in a suitable symmetric monoidal category ☆

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A R T I C L E   I N F O

Article history:
Received 30 March 2010
Communicated by Nicolás Andruskiewitsch

MSC:
16W30

Keywords:
Clifford process
Weak braided Hopf algebra
Monoidal (co)Frobenius (co)algebra
Periodicity property

A B S T R A C T

It is well known that Clifford algebras are group algebras deformed by a 2-cocycle. Furthermore, these algebras, which are not commutative in the usual sense, can be viewed as commutative algebras in certain symmetric monoidal categories of graded vector spaces. In this note we invent a Clifford process for coalgebras that will allow us to show that Clifford algebras have also cocommutative coalgebra structures, and consequently commutative and cocommutative weak braided Hopf algebras structures, within the same symmetric monoidal categories where they lie as commutative algebras. Also, we will show that they are selfdual weak braided Hopf algebras, monoidal Frobenius algebras and monoidal coFrobenius coalgebras.

1. Introduction

Using the Cayley–Dickson processes for algebras and coalgebras we have constructed in [5] an infinite chain of weak braided Hopf algebras in some suitable symmetric monoidal categories of graded vector spaces. The procedure was initiated in [1] where the authors proved that any Cayley–Dickson algebra obtained from (k, Id_k) is an algebra deformation of a group algebra by a 2-cochain, and continued in [5] where it has been proved that any Cayley–Dickson coalgebra obtained from (k, Id_k) is, let say, a kind of coalgebra deformation of the same group algebra by the same 2-cochain. For short, they can be identified as algebras with objects of the form $k_F[G]$, as coalgebras with objects of the form $k^F[G]$, and all together with $k_F^F[G]$, where G is a finite abelian group and F is a 2-cochain

☆ Research financially supported by the contract No. 479/2009 of CNCSIS, project ID 1904, "Acţiuni, coacţiuni şi graduări pe algebre".

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.02.020

on it. By [5] we have that $k_F^F[G]$ is a weak braided Hopf algebra within the category of G-graded vector spaces endowed with the symmetric monoidal structure produced by F^{-1}, and so we got that all the Cayley–Dickson algebras and coalgebras obtained from (k, Id_k) are weak braided Hopf algebras.

The Clifford algebras associated to regular quadratic spaces have properties similar to those of Cayley–Dickson algebras. More precisely, it is well known that Clifford algebras of type $C(q_1, \dots, q_n)$, with all q_i's non-zero scalars, are involutive algebras isomorphic with deformations of group algebras by 2-cocycles (see for instance [7]). Furthermore, using the Clifford process for algebras invented by Wene [20], Albuquerque and Majid have shown in [2] that each algebra $C(q_1, \dots, q_n)$ can be obtained through this process from the involutive algebra $C(q_1, \dots, q_{n-1})$ and q_n; the Clifford algebra $C(q)$ corresponds to the involutive algebra (k, Id_k) and q. More details will be presented in Section 2.2, for the moment recall only that the Clifford algebra process associates to an involutive algebra, that is to a pair (A, σ) consisting in a finite-dimensional algebra A with unit 1 and an algebra automorphism $\sigma : A \to A$ such that $\sigma \circ \sigma = \mathrm{Id}_A$, and to a fixed element $q \in k^*$ a new involutive algebra $(\overline{A}, \overline{\sigma})$ as follows. $\overline{A} = A \times A$ with componentwise addition and multiplication given by

$$(a, b)(c, d) = \big(ac + qb\sigma(d), ad + b\sigma(c)\big),$$

for all $a, b, c, d \in A$. The involutive automorphism of $\overline{A}$ is given by

$$\overline{\sigma}(a, b) = \big(\sigma(a), -\sigma(b)\big), \quad \forall a, b \in A.$$

It is clear that $A \ni a \mapsto (a, 0) \in \overline{A}$ is an injective algebra morphism. If we identify $a \equiv (a, 0)$ and define $\mathbf{v} := (0, 1)$ we then have $b\mathbf{v} \equiv (b, 0)(0, 1) = (0, b)$, and so (a, b) identifies with $a + b\mathbf{v}$. Thus the multiplication of $\overline{A}$ transforms to

$$(a + b\mathbf{v})(c + d\mathbf{v}) = ac + qb\sigma(d) + \big(ad + b\sigma(c)\big)\mathbf{v},$$

for all $a, b, c, d \in A$, and is completely determined by the following computation rules:

$$a(b\mathbf{v}) = (ab)\mathbf{v}, \qquad (a\mathbf{v})b = \big(a\sigma(b)\big)\mathbf{v} \quad \text{and} \quad (a\mathbf{v})(b\mathbf{v}) = qa\sigma(b). \tag{1.1}$$

Consequently, $\mathbf{v}^2 = q$ and $\mathbf{v}b = \sigma(b)\mathbf{v}$, for all $b \in A$. Also, the involutive automorphism of $\overline{A}$ takes the form $\overline{\sigma} : \overline{A} \ni a + b\mathbf{v} \mapsto \sigma(a) - \sigma(b)\mathbf{v} \in \overline{A}$, for all $a, b \in A$. Clearly, a basis of $\overline{A}$ is $\{a, b\mathbf{v} \mid a, b \in \mathbb{B}\}$, where $\mathbb{B}$ is a basis of A. So $\overline{A}$ has twice the dimension of A. We will refer to $(\overline{A}, \overline{\sigma})$ as being the algebra obtained from (A, σ) and q by Clifford process for algebras.

Thus the Clifford algebras corresponding to regular quadratic spaces fit into the characterization of Cayley–Dickson algebras obtained in [1], in the sense that they are of the form $k_F[G]$, with G a finite direct sum of copies of the cyclic group $\mathbb{Z}_2$ and F a 2-cochain on it (once more all these details can be found in [2]). Now, keeping in mind the results obtained in [5] it is natural to try to see if these Clifford algebras can be also identified, this time as coalgebras, with deformations of the form $k^F[G]$ with G and F having the same definitions as in the algebra case. To this end we have adapted the techniques used in [2,5]. Firstly, we have invented a Clifford process for coalgebras and then we have shown that all the coalgebras produced by it from the trivial coalgebra (k, Id_k) are standard involutive coalgebras of the form $k^F[G]$ with G and F exactly as in the algebra case, as desired. Since $k_F[G] = k^F[G]$ as k-vector spaces we obtain that any $C(q_1, \dots, q_n)$ has a coalgebra structure that can be deduced inductively from that of k by applying to it, for n-times, the Clifford process for coalgebras. Notice that $C(q_1, \dots, q_n)$ are $(\mathbb{Z}_2)^n$-graded cocommutative coalgebras, and that any such Clifford coalgebra has a $\mathbb{Z}_2$-graded coalgebra structure as well. (All the details are presented in Section 3.) Remark also that this part of the paper in not a rather straightforward generalization of the earlier work of the author from Cayley–Dickson algebras to Clifford algebras, based on ideas of

Albuquerque and Majid. And this is because, on one hand, the Clifford process for coalgebras in not the formal dual of the Clifford process for algebras and, on the other hand, there is no guarantee that the two Clifford processes produce the same couple (G, F) when we start with the same input data (k, Id_k). Hence, from our point of view, it is particularly exciting that (G, F) is built up by two Clifford processes and that, moreover, the structures obtained from these two processes are compatible in the sense that they produce a commutative and cocommutative weak braided Hopf algebra structure on Clifford algebras.

In Section 4 we specialize the theory developed in Section 3 for $C(q_1, q_2)$ and $C(q_1, q_2, q_3)$, respectively. Our motivation is two fold. Firstly, we want to present concretely the coalgebra structure of some Clifford algebras. Secondly, these concrete descriptions will allow us to prove that the $\mathbb{Z}_2$-graded coalgebra structure of $C(q_1, q_2, q_3)$ is the graded tensor coalgebra product between $C(-q_1q_3, -q_2q_3)$, equipped with the trivial $\mathbb{Z}_2$-grading, and $C(-q_1q_2q_3)$; as we will see this fact plays a crucial role in the proof of the structure theorem stated in Theorem 5.2.

In Section 5 we show that all the Clifford algebras associated to regular quadratic spaces are commutative and cocommutative weak braided Hopf algebras. This follows from the identification of a $C(q_1, \dots, q_n)$ with a $k_F^F[G]$, and from the fact that in the inductive process for coalgebras the comultiplication behaves well with respect to the computation rules imposed by the inductive process for algebras. Since the weak braided Hopf algebra is a selfdual notion it is natural to check if the dual structure of $C(q_1, \dots, q_n)$ gives rise to a new weak braided Hopf algebra. In this direction we will prove that the categorical left and right duals of an arbitrary weak braided Hopf algebra of the form $k_F^F[G]$ are isomorphic to $k_F^F[G]$ itself. Consequently, all the Cayley–Dickson and Clifford weak braided Hopf algebras obtained from k are selfdual, and so their categorical left and right duals do not provide new classes of such objects. Nevertheless, the weak braided Hopf algebra isomorphism that will be constructed will permit us in Section 6 to show that, in general, any $k_F^F[G]$ is a Frobenius algebra and a coFrobenius coalgebra in a monoidal sense that will be explained below. As a consequence we obtain that all the Cayley–Dickson and Clifford weak braided Hopf algebras obtained from k are monoidal Frobenius algebras and monoidal coFrobenius coalgebras within the symmetric monoidal categories of graded vector spaces where they reside.

When p of q_i's are equal to 1 and the remaining ones, say in number of q, are equal to -1 the Clifford algebra $C(q_1, \dots, q_n)$ is denoted by $C^{p,q}$. In this particular case we have complete structure theorems for $C^{p,q}$'s, both at the $\mathbb{Z}_2$-graded, and respectively ungraded, algebra level (see [9,13]). In Section 7, using the tools provided by [2], we show that the same results are valid from the point of view of $\mathbb{Z}_2$-graded, and respectively ungraded, coalgebra structures of $C^{p,q}$. We will see that the "Periodicity 8" result extends to coalgebras and this will reduce the problem to calculating of $C^{p,0}$ and $C^{0,q}$ when $0 \leqslant p, q \leqslant 7$. Then some coalgebraic constructions and identifications will allow us to show that the descriptions known for $C^{p,0}$ and $C^{0,q}$ in the algebra case are valid in the coalgebra case, too.

2. Preliminaries

2.1. Categories of graded vector spaces

We assume familiarity with the basic theory of braided monoidal categories or Hopf algebra theory, see [10–12,16] for example. In what follows we are interested in the braided monoidal structures of some categories of graded vector spaces. Recall that for G a group with neutral element e a G-graded vector space over k is a k-vector space V which decomposes into a direct sum of the form $V = \bigoplus_{g \in G} V_g$, where each V_g is a vector space. For a given $g \in G$ the elements of V_g are called homogeneous elements of degree g. If $v \in V$ is a homogeneous element then by $|v| \in G$ we denote the degree of v. If $W = \bigoplus_{g \in G} W_g$ is another G-graded vector space then a k-linear map $f : V \to W$ preserves the gradings if $f(V_g) \subseteq W_g$, for all $g \in G$. Vect^G denotes the category of G-graded vector spaces and k-linear maps that preserve the gradings.

It is well known that Vect^G identifies with the category of right comodules over $k[G]$, the Hopf group algebra associated to G, for instance see [14, Example 1.6.7]. Furthermore, the "canonical" monoidal structures of $\mathcal{M}^{k[G]}$ are given by the dual quasi-bialgebra structures of $k[G]$, cf. [17]. (At this point we mention only that a dual quasi-bialgebra is a usual k-coalgebra together with a unital algebra structure such that the multiplication is associative up to conjugation by an invertible element $\varphi \in (H \otimes H \otimes H)^*$; it can be viewed as the formal dualization of the quasi-bialgebra notion that we will present below.) Since $k[G]$ is a cocommutative coalgebra we get that the monoidal structures of Vect^G are in a one-to-one correspondence with the normalized 3-cocycles ϕ on G with coefficients in k^*, this means, with k-linear maps $\phi : G \times G \times G \to k^*$ satisfying

$$\phi(y,z,t)\phi(x,yz,t)\phi(x,y,z) = \phi(x,y,zt)\phi(xy,z,t) \quad \text{and} \quad \phi(x,e,y) = 1,$$

for all $x, y, z, t \in G$. Explicitly, if $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$ are G-graded vector spaces then the tensor product of V and W in Vect^G is the same as in the category of k-vector spaces ${}_k\mathcal{M}$, and $V \otimes W$ is considered as a G-graded vector space via the grading defined by $(V \otimes W)_g := \bigoplus_{\sigma\tau=g} V_\sigma \otimes W_\tau$, for all $g \in G$. If $f : V \to V'$ and $g : W \to W'$ are morphisms in Vect^G then $f \otimes g$ in Vect^G is the tensor product morphism of f and g in the category of k-vector spaces. The unit object is k, viewed as a G-graded vector space with $k_e = k$ and $k_g = 0$, for all $G \ni g \neq e$. Now the associativity constraint a of Vect^G is completely determined by a ϕ as above, in the sense that for each a there exists a unique ϕ such that, for any $V, W, Z \in \mathrm{Vect}^G$, $a_{V,W,Z}$ is defined, on homogeneous elements $v \in V$, $w \in W$ and $z \in Z$, by

$$a_{V,W,Z}\big((v \otimes w) \otimes z\big) = \phi\big(|v|, |w|, |z|\big) v \otimes (w \otimes z).$$

The left and right unit constraints of Vect^G are always defined by identity morphisms of ${}_k\mathcal{M}$. The category Vect^G endowed with the monoidal structure given by ϕ will be denoted by Vect^G_ϕ. We then have that Vect^G_ϕ identifies, as monoidal category, with $\mathcal{M}^{k_\phi[G]}$, the category of right corepresentations of the dual quasi-Hopf algebra $k_\phi[G]$. Here $k_\phi[G]$ coincides with $k[G]$ as a Hopf algebra but since is cocommutative it can be also viewed as a dual quasi-Hopf algebra with reassociator given by $\varphi = \phi$, extended by linearity, antipode defined by $S(g) = g^{-1}$, for all $g \in G$, and distinguished elements $\alpha, \beta \in k_\phi[G]^*$ given by $\alpha(g) = 1$, and respectively by $\beta(g) = \phi(g, g^{-1}, g)^{-1}$, for all $g \in G$.

The monoidal category Vect^G admits a braided structure if and only if G is abelian. Then its braidings are into a bijective correspondence with the abelian 3-cocycles on G with coefficients in k^*, cf. [10,11]. This means, with pairs $(\phi, \mathcal{R})$ consisting of a normalized 3-cocycle ϕ on G with coefficients in k^* and a map $\mathcal{R} : G \times G \to k^*$ satisfying

$$\mathcal{R}(xy,z)\phi(x,z,y) = \phi(x,y,z)\mathcal{R}(x,z)\phi(z,x,y)\mathcal{R}(y,z)$$

and

$$\phi(x,y,z)\mathcal{R}(x,yz)\phi(y,z,x) = \mathcal{R}(x,y)\phi(y,x,z)\mathcal{R}(x,z),$$

for all $x, y, z \in G$. Actually, the braided monoidal structures on Vect^G are in a bijective correspondence with the coquasitriangular structures of $k_\phi[G]$, and so $\mathrm{Vect}^G_{\phi,\mathcal{R}}$ identifies with $\mathcal{M}^{k_\phi[G]}$ as a braided monoidal category (see [17] for more details). Here and everywhere else in this paper $\mathrm{Vect}^G_{\phi,\mathcal{R}}$ denotes the category Vect^G equipped with the braided monoidal structure given by the abelian 3-cocycle $(\phi, \mathcal{R})$ on G with coefficients in k^*.

Now, particular examples of (braided) monoidal structures on Vect^G are produced by the so-called coboundary (abelian) 3-cocycles on G. More exactly, for any 2-cochain $F \in (G \times G)^*$ on G, i.e., for any pointwise invertible map $F : G \times G \to k^*$ obeying $F(e,x) = F(y,e) = 1$, for all $x, y \in G$, the element $\phi_{F^{-1}} \in (G \times G \times G)^*$ defined by

$$\phi_{F^{-1}}(x,y,z) = \Delta_2\big(F^{-1}\big)(x,y,z) := F(y,z)^{-1}F(xy,z)F(x,yz)^{-1}F(x,y),$$

for all $x, y, z \in G$, is a normalized 3-cocycle on G. In the case that G is abelian the pair determined by $\phi_{F^{-1}}$ and the map $\mathcal{R}_{F^{-1}} : G \times G \to k^*$ given by $\mathcal{R}_{F^{-1}}(x,y) = F(x,y)F(y,x)^{-1}$, for all $x, y \in G$, is an abelian 3-cocycle on G, called a coboundary abelian 3-cocycle. (Usually, F is considered in the definition of a coboundary (abelian) 3-cocycle but we replaced it with F^{-1} in order to keep the presentation from [2].)

2.2. The Clifford process for algebras

From now on k is a field of characteristic not 2. A quadratic vector space over k is a pair (V, B) consisting in a k-vector space and a symmetric bilinear form $B : V \times V \to k$. A morphism between two quadratic spaces (V, B_V) and (W, B_W) is a k-linear morphism $f : V \to W$ obeying $B_W(f(x), f(y)) = B_V(x,y)$, for all $x, y \in V$. If, moreover, f is an isomorphism we then say that (V, B_V) and (W, B_W) are isometric quadratic spaces.

Equivalently, a quadratic vector space is a pair (V, q) with V a k-vector space and $q : V \to k$ a quadratic map, i.e., a map satisfying $q(\alpha v) = \alpha^2 q(v)$, for all $\alpha \in k$ and $v \in V$, and such that

$$Q : V \times V \ni (x,y) \mapsto \frac{1}{2}\big(q(x+y) - q(x) - q(y)\big) \in k$$

is a (symmetric) bilinear form on V. Note that the scalar $\frac{1}{2}$ is needed in order to have the above correspondence the bijective inverse of the map that associates to a bilinear form B on V the quadratic map $V \ni x \mapsto q(x) := B(x,x) \in k$ on V. When V is finite-dimensional it is well known that there exists a basis $\{e_1, \dots, e_n\}$ in V such that $B(e_i, e_j) = 0$, for any $1 \leqslant i \neq j \leqslant n$, i.e., an orthogonal basis in V with respect to the bilinear form B.

Roughly speaking a Clifford algebra associated to a quadratic space (V, q) is a unital associative algebra that contains V as a k-subspace such that $x^2 = q(x)1$, for all $x \in V$, and has to satisfy the following universal property: for any other algebra A that contains V such that $x^2 = q(x)1$, for all $x \in V$, there exists a unique algebra map $\varphi : C(V) \to A$ such that φ restricted to V is the identity morphism of V. The Clifford algebra of (V, q) always exists and is unique up to an algebra isomorphism. It will be denoted by $C(V)$.

A realization of $C(V)$ can be obtained either by deforming the algebra structure of the exterior algebra $\Lambda(V)$ of V or by considering the quotient of $T(V)$, the tensor algebra of V, by its ideal $I(q)$ generated by all the elements of the form $x \otimes x - q(x)1$, $x \in V$. Consequently, when V is finite-dimensional with an orthogonal basis $\{e_1, \dots, e_n\}$ we get that $C(V)$ is the unital algebra generated by $e_1, \dots, e_n$ with relations

$$e_i^2 = q(e_i)1, \quad \forall 1 \leqslant i \leqslant n, \quad \text{and} \quad e_ie_j = -e_je_i, \quad \forall 1 \leqslant i \neq j \leqslant n.$$

Furthermore, $\{e_x := e_1^{x_1} \cdots e_n^{x_n} \mid x_i \in \{0,1\},\ \forall 1 \leqslant i \leqslant n\}$ is a k-basis for $C(V)$, and so $\dim_k C(V) = 2^{\dim_k V}$. When $q(e_i) := q_i$ are non-zero scalars we also denote $C(V)$ by $C(q_1, \dots, q_n)$.

Let $\mathbb{Z}_2$ be the cyclic group of order two written additively. The Clifford algebra $C(V)$ is a $\mathbb{Z}_2$-graded algebra. The even part of $C(V)$ is the image of $\bigoplus_{n \in 2\mathbb{N}} T^n(V)$ under the quotient map $T(V) \to C(V,q) = T(V)/I(q)$, while the odd part of $C(V,q)$ is the image of $\bigoplus_{n \in 2\mathbb{N}+1} T^n(V)$ under the same quotient map. So under this grading any element of k has degree zero, and any element of V has degree one. By $C_0(V)$, respectively by $C_1(V)$, we denote the even, respectively odd, part of $C(V)$. In the finite-dimensional case $\{e_1^{x_1} \cdots e_n^{x_n} \mid x_i \in \{0,1\}$ and $\sum_{i=1}^n x_i \in 2\mathbb{N}\}$ is a k-basis for $C_0(V)$, while $\{e_1^{x_1} \cdots e_n^{x_n} \mid x_i \in \{0,1\}$ and $\sum_{i=1}^n x_i \in 2\mathbb{N}+1\}$ is a k-basis of $C_1(V)$, providing that $\{e_1, \dots, e_n\}$ is an orthogonal basis for (V, q).

We next see that $C(q_1, \dots, q_n)$ are $(\mathbb{Z}_2)^n$-graded algebras as well. This can be obtained by successive applications of the Clifford process for algebras that was described in introduction. For recall first that if (A, σ_A), (B, σ_B) are involutive algebras then a morphism between them is an algebra morphism $f : A \to B$ satisfying $\sigma_B f = f\sigma_A$. We say that (A, σ_A), (B, σ_B) are isomorphic involutive algebras if there exists $f : A \to B$ as above which is, moreover, bijective.

Now, when we apply the Clifford algebra process to (k, Id_k) and $q \in k^*$ we get the Clifford algebra $C(\langle q\rangle)$. More precisely, for $q \in k$ denote by $\langle q\rangle$ the isometry class of the one-dimensional quadratic space $(V = k\mathbf{e}, \mathbf{q})$ with $\mathbf{q}(\alpha\mathbf{e}) = q\alpha^2$, for all $\alpha \in k$. Thus $C(\langle q\rangle)$ is the k-algebra generated by $\mathbf{e}$ with relation $\mathbf{e}^2 = q1$, and therefore $C(\langle q\rangle) = \frac{k[X]}{(X^2-q)}$. In particular, for $q = -1$ we get $C(\langle -1\rangle) \cong k[\mathbf{i}]$, the complex number algebra over k, and for $q = 1$ we obtain $C(\langle 1\rangle) \cong k[\mathbb{Z}_2]$, the group algebra associated to the cyclic group $\mathbb{Z}_2$. Moreover, the involutive morphism corresponding through this process to $C(\langle q\rangle)$ is given by $\overline{\sigma}(\alpha 1 + \beta\mathbf{e}) = \alpha - \beta\mathbf{e}$, for all $\alpha, \beta \in k$.

For G a group and $F \in (G \times G)^*$ a 2-cochain, $k_F[G]$ is the G-graded algebra built on the group algebra $k[G]$ of G but viewed now with the new multiplication $x \bullet y = F(x, y)xy$, for all $x, y \in G$. According to [1], $k_F[G]$ is a G-graded quasialgebra with associator $\Delta_2(F^{-1})$, i.e., $k_F[G]$ is an algebra within the monoidal category $\mathrm{Vect}^G_{\Delta_2(F^{-1})}$. Furthermore, if G is abelian then $k_F[G]$ is a braided commutative algebra in $\mathrm{Vect}^G_{\Delta_2(F^{-1}), \mathcal{R}_{F^{-1}}}$.

Proposition 2.1. *(See [2].) $(C(\langle q\rangle), \overline{\sigma})$ is isomorphic to $(k_F[\mathbb{Z}_2], \sigma)$, as involutive algebra, where $F(x, y) = q^{xy}$ and $\sigma(x) = (-1)^x x$, for all $x, y \in \mathbb{Z}_2$.*

Proof. Record only that the isomorphism is produced by $\mu : C(\langle q\rangle) \to k_F[\mathbb{Z}_2]$ given by $\mu(\alpha 1 + \beta\mathbf{e}) = \alpha\bar{0} + \beta\bar{1}$, for all $\alpha, \beta \in k$, where, in order to avoid confusions, we wrote $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. □

In general, we call an algebra A a standard involutive algebra if it admits an involutive automorphism of the form $\sigma_s : A \ni x \mapsto s(x)x \in A$, for a certain map $s : A \to k^*$. It can be easily seen that in this situation s has to be multiplicative and has to satisfy the conditions $s(1) = 1$ and $s(x)^2 = 1$, for all $x \in A$. Such an example is the involutive algebra $(k_F[\mathbb{Z}_2], \sigma)$ considered in the above proposition because in this case σ can be rewritten as $\sigma(x) = s(x)x$ with $s(x) = (-1)^x$, for all $x \in \mathbb{Z}_2$. We clearly have $s(x + y) = s(x)s(y)$, $s(\bar{0}) = 1$ and $s(x)^2 = 1$, for all $x, y \in \mathbb{Z}_2$. This together with the result below implies that Clifford algebras of the type $C(q_1, \ldots, q_n)$ are all isomorphic to certain standard involutive algebras of the form $(k_F[G], \sigma_s)$.

Proposition 2.2. *(See [2, Proposition 3.1].) Let G be an abelian group and F a 2-cochain on G such that $k_F[G]$ is a standard involutive algebra via $s : G \to k^*$. Take $\overline{G} = G \times \mathbb{Z}_2$ and for $q \in k^*$ define the 2-cochain $\overline{F}$ on $\overline{G}$ by*

$$\overline{F}\big((x, \bar{0}), (y, \bar{0})\big) = \overline{F}\big((x, \bar{0}), (y, \bar{1})\big) = F(x, y),$$
$$\overline{F}\big((x, \bar{1}), (y, \bar{0})\big) = s(y)F(x, y) \quad \text{and} \quad \overline{F}\big((x, \bar{1}), (y, \bar{1})\big) = qs(y)F(x, y),$$

for all $x, y \in G$. If $\bar{s} : \overline{G} \to k^*$ is given by

$$\bar{s}\big((x, \bar{0})\big) = s(x) \quad \text{and} \quad \bar{s}\big((x, \bar{1})\big) = -s(x), \quad \forall x \in G,$$

then $(k_{\overline{F}}[\overline{G}], \sigma_{\bar{s}})$ is isomorphic to the involutive algebra obtained from $(k_F[G], \sigma_s)$ and q by Clifford process for algebras.

Proof. The isomorphism is determined by $\nu : \overline{(k_F[G], \sigma_s)} \to (k_{\overline{F}}[\overline{G}], \overline{\sigma}_{\bar{s}})$ that maps x to $(x, \bar{0})$ and $x\mathbf{v}$ to $(x, \bar{1})$, for all $x \in G$, extended by linearity. □

If the input data for the Clifford process for algebras is the involutive algebra from Proposition 2.1 (with q replaced by q_1) then at the $(n-1)$-step of this process (applied for $q_2, \ldots, q_n \in k^*$) we get the standard involutive algebra $(k_F[(\mathbb{Z}_2)^n], \sigma_s)$, where

$$F(x, y) = (-1)^{\sum_{1 \leqslant j < i \leqslant n} x_i y_j} \prod_{i=1}^{n} q_i^{x_i y_i}, \tag{2.1}$$

for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in (\mathbb{Z}_2)^n$, and σ_s is defined by $s(x) = (-1)^{\rho(x)}$ with $\rho(x) := \sum_{i=1}^{n} x_i$, for all $x = (x_1, \ldots, x_n) \in (\mathbb{Z}_2)^n$. This is because, for an arbitrary $m \in \mathbb{N}$, when we take $G = (\mathbb{Z}_2)^m$ and F a 2-cochain on it then the 2-cochain $\overline{F}$ on $\overline{G} = (\mathbb{Z}_2)^{m+1}$ defined in Proposition 2.2 can be obtained from F as

$$\overline{F}(\bar{x}, \bar{y}) = F(x, y)(-1)^{x_{m+1}\rho(y)} q_{m+1}^{x_{m+1} y_{m+1}}, \tag{2.2}$$

for all $\bar{x} = (x, x_{m+1})$ and $\bar{y} = (y, y_{m+1})$ in $(\mathbb{Z}_2)^{m+1}$. Then everything follows easily by mathematical induction (see also [2]).

In addition, $C(q_1, \ldots, q_n)$ can be identified as an involutive k-algebra with $k_F[(\mathbb{Z}_2)^n]$, where F is as in (2.1). Indeed, if $\{e_x = e_1^{x_1} \cdots e_n^{x_n} \mid x_i \in \{0, 1\},\ \forall 1 \leqslant i \leqslant n\}$ is the basis of $C(q_1, \ldots, q_n)$ determined by the orthogonal basis $\{e_1, \ldots, e_n\}$ of V then

$$e_x e_y = F(x, y) e_{x \oplus y}, \quad \text{for all } x, y \in (\mathbb{Z}_2)^n, \tag{2.3}$$

where we consider $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ as vectors in $(\mathbb{Z}_2)^n$, and where $\oplus$ is the addition modulo 2. Then $C(q_1, \ldots, q_n) \ni e_x \mapsto x \in (\mathbb{Z}_2)^n$, extended by linearity, is an algebra isomorphism, cf. [2, Proposition 2.2]. The involutive automorphism σ of $C(q_1, \ldots, q_n)$ is the so-called the main (or degree) automorphism of it, that is $\sigma(e_x) = (-1)^{\rho(x)} e_x$, for all $x \in (\mathbb{Z}_2)^n$. As a consequence we obtain that $C(q_1, \ldots, q_n)$ is a commutative algebra in $\mathrm{Vect}^{(\mathbb{Z}_2)^n}_{\mathfrak{e}, \mathcal{R}_{F^{-1}}}$, where by $\mathfrak{e}$ we have denoted the trivial 3-cocycle on $(\mathbb{Z}_2)^n$. This happens because F in (2.1) is a 2-cocycle (in other words $\Delta_2(F^{-1}) = \mathfrak{e}$), and this explains why all the Clifford algebras are associative in the usual sense. Note also that, for all $x, y \in (\mathbb{Z}_2)^n$,

$$\mathcal{R}_{F^{-1}}(x, y) = \frac{F(x, y)}{F(y, x)} = (-1)^{\sum_{1 \leqslant j < i \leqslant n}(x_i y_j - x_j y_i)} = (-1)^{\rho(x)\rho(y) + x \cdot y}, \tag{2.4}$$

where $x \cdot y = \sum_{i=1}^{n} x_i y_i$ is the dot product of the vectors x and y. Once more all these results can be found in [1,2].

Finally, if $q_1, \ldots, q_n, q_{n+1} \in k^*$ then the algebra obtained from the standard involutive algebra $(C(q_1, \ldots, q_n), \sigma_s)$ and q_{n+1} by Clifford process for algebras is isomorphic to $C(q_1, \ldots, q_n, q_{n+1})$, as standard involutive $\mathbb{Z}_2$-graded algebras. For short, if F and $\overline{F}$ are the 2-cocycles on $(\mathbb{Z}_2)^n$, and respectively on $(\mathbb{Z}_2)^{n+1}$, corresponding to $C(q_1, \ldots, q_n)$, and respectively to $C(q_1, \ldots, q_{n+1})$, as in (2.1) then $\overline{F}$ can be obtained from F as in (2.2), too. Thus $C(q_1, \ldots, q_{n+1}) \cong k_{\overline{F}}[(\mathbb{Z}_2)^{n+1}] \cong \overline{k_F[(\mathbb{Z}_2)^n]} \cong \overline{C(q_1, \ldots, q_n)}$, as standard involutive algebras, as claimed.

3. The Clifford process for coalgebras

We show that the Clifford algebras $C(q_1, \ldots, q_n)$ admit coalgebra structures as well. We also present a Clifford process for coalgebras that allows us to describe Clifford algebras of this form as cocommutative coalgebras within the same symmetric monoidal categories where they are commutative algebras. Toward this end let us start by defining the Clifford process for coalgebras.

Let k be a field of characteristic not 2, C a k-vector space and $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ two k-linear maps such that ε is a counit for Δ. So we do not assume that (C, Δ, ε) is a coassociative coalgebra. Assume in turn that C comes equipped with an involutive coalgebra map, that is, C comes with a k-linear map $\sigma : C \to C$ such that

$$\sigma(c)_1 \otimes \sigma(c)_2 = \sigma(c_1) \otimes \sigma(c_2) \quad \text{and} \quad \sigma^2(c) = c,$$

for all $c \in C$, where, $\Delta(c) = c_1 \otimes c_2$, summation understood, is the Sweedler type notation for coalgebras from [19]. We call the pair (C, σ) an involutive coalgebra.

If (C, σ) and (C', σ') are involutive coalgebras then a k-linear map $f : C \to C'$ is called a morphism of involutive coalgebras if f is a coalgebra morphism such that $\sigma' f = f\sigma$. We say that (C, σ) and (C', σ') are isomorphic involutive coalgebras if there exists $f : C \to C'$, an involutive coalgebra morphism, which is, moreover, an isomorphism.

The Clifford process for coalgebras asserts to an involutive coalgebra (C, σ) and to a fixed $q \in k^*$ another involutive coalgebra $(\overline{C}, \overline{\sigma})$ of twice dimension, providing C finite-dimensional. More precisely we have the following.

Proposition 3.1. *Let (C, σ) be an involutive coalgebra and $q \in k^*$ fixed arbitrary. Then $\overline{C} = C \times C$ is an involutive coalgebra via the structure defined by*

$$\overline{\Delta}(c, 0) = \frac{1}{2}\big((c_1, 0) \otimes (c_2, 0) + q^{-1}(0, c_1) \otimes \big(0, \sigma(c_2)\big)\big), \qquad \bar{\varepsilon}\big((c, 0)\big) = 2\varepsilon(c),$$

$$\overline{\Delta}(0, c) = \frac{1}{2}\big((c_1, 0) \otimes (0, c_2) + (0, c_1) \otimes \big(\sigma(c_2), 0\big)\big), \qquad \bar{\varepsilon}\big((0, c)\big) = 0,$$

$$\overline{\sigma}\big(c, c'\big) = \big(\sigma(c), -\sigma\big(c'\big)\big),$$

for all $c, c' \in C$.

Proof. All we have to prove is that $\bar{\varepsilon}$ is a counit for $\overline{\Delta}$, that $\overline{\sigma}$ behaves well with respect to the comultiplication of $\overline{\Delta}$, and that $\overline{\sigma}^2 = \mathrm{Id}_{\overline{C}}$.

Since ε is a counit for Δ it follows that $\bar{\varepsilon}$ is a counit for $\overline{\Delta}$. We next compute

$$(\overline{\sigma} \otimes \overline{\sigma})\overline{\Delta}(c, 0) = \frac{1}{2}\big(\big(\sigma(c_1), 0\big) \otimes \big(\sigma(c_2), 0\big) + q^{-1}\big(0, \sigma(c_1)\big) \otimes (0, c_2)\big) = \overline{\Delta}\big(\overline{\sigma}(c, 0)\big),$$

because σ is a coalgebra morphism and, similarly,

$$(\overline{\sigma} \otimes \overline{\sigma})\overline{\Delta}(0, c) = -\frac{1}{2}\big(\big(\sigma(c_1), 0\big) \otimes \big(0, \sigma(c_2)\big) + \big(0, \sigma(c_1)\big) \otimes (c_2, 0)\big) = \overline{\Delta}\big(\overline{\sigma}(0, c)\big),$$

because $\overline{\sigma}((0, c)) = -(0, \sigma(c))$, σ is a coalgebra morphism and $\sigma^2 = \mathrm{Id}_C$. Finally, $\overline{\sigma}^2(c, 0) = \overline{\sigma}(\sigma(c), 0) = (\sigma^2(c), 0) = (c, 0)$, for all $c \in C$. Likewise, $\overline{\sigma}^2(0, c) = -\overline{\sigma}(0, \sigma(c)) = (0, \sigma^2(c)) = (0, c)$, for all $c \in C$, finishing the proof. □

For (C, σ) an involutive coalgebra and $q \in k^*$ we call the involutive coalgebra $(\overline{C}, \overline{\sigma})$ the Clifford coalgebra obtained from (C, σ) and q through Clifford process for coalgebras. As in the algebra case, we identify $(c, 0) \equiv c$ and denote $c\mathbf{v} = (0, c)$, for all $c \in C$, and so $\overline{C} = C \oplus C\mathbf{v}$. We then have

$$\overline{\Delta}(c) = \frac{1}{2}\big(c_1 \otimes c_2 + q^{-1}c_1\mathbf{v} \otimes \sigma(c_2)\mathbf{v}\big), \qquad \bar{\varepsilon}(c) = 2\varepsilon(c), \tag{3.1}$$

$$\overline{\Delta}(c\mathbf{v}) = \frac{1}{2}\big(c_1 \otimes c_2\mathbf{v} + c_1\mathbf{v} \otimes \sigma(c_2)\big), \qquad \bar{\varepsilon}(c\mathbf{v}) = 0, \tag{3.2}$$

for all $c \in C$. The involutive coalgebra automorphism $\overline{\sigma}$ of $\overline{C}$ transforms to $\overline{\sigma}(c + c'\mathbf{v}) = \sigma(c) - \sigma(c')\mathbf{v}$, for all $c, c' \in C$. Clearly, if $\mathbb{B}$ is a basis for C then $\{c, c\mathbf{v} \mid c \in \mathbb{B}\}$ is a basis for $\overline{C}$.

Proposition 3.2. *If C is a coassociative k-coalgebra then so is $\overline{C}$.*

Proof. For all $c \in C$ we compute

$$
\begin{aligned}
(\overline{\Delta} \otimes \mathrm{Id}_{\overline{C}})\overline{\Delta}(c) &= \frac{1}{2}\big(\overline{\Delta}(c_1) \otimes c_2 + q^{-1}\overline{\Delta}(c_1\mathbf{v}) \otimes \sigma(c_2)\mathbf{v}\big) \\
&= \frac{1}{4}\big(c_1 \otimes c_2 \otimes c_3 + q^{-1}c_1\mathbf{v} \otimes \sigma(c_2)\mathbf{v} \otimes c_3 + q^{-1}c_1 \otimes c_2\mathbf{v} \otimes \sigma(c_3)\mathbf{v} \\
&\quad + q^{-1}c_1\mathbf{v} \otimes \sigma(c_2) \otimes \sigma(c_3)\mathbf{v}\big) \\
&= \frac{1}{2}\big(c_1 \otimes \overline{\Delta}(c_2) + q^{-1}c_1\mathbf{v} \otimes \overline{\Delta}\big(\sigma(c_2)\mathbf{v}\big)\big) \\
&= (\mathrm{Id}_{\overline{C}} \otimes \overline{\Delta})\overline{\Delta}(c),
\end{aligned}
$$

where we freely use the fact that σ is an involutive coalgebra morphism of C.

Similar arguments show that $(\overline{\Delta} \otimes \mathrm{Id}_{\overline{C}})\overline{\Delta}(c\mathbf{v}) = (\mathrm{Id}_{\overline{C}} \otimes \overline{\Delta})\overline{\Delta}(c\mathbf{v})$, for all $c \in C$. So the proof is complete. □

We next see that all the coalgebras obtained through the Clifford process for coalgebras from the input data (k, Id_k) are of the form $k^F[G]$, where G is a finite direct sum of copies of $\mathbb{Z}_2$ and F is the 2-cocycle on it defined exactly as in the Clifford algebra process. Recall that, in general, for G a finite group such that $|G| \neq 0$ in k and $F \in (G \times G)^*$ a 2-cochain on it $k^F[G]$ denotes the k-vector space $k[G]$ equipped with the comultiplication and counit given by

$$
\Delta_F(x) = \frac{1}{|G|}\sum_{u \in G} F\big(u, u^{-1}x\big)^{-1} u \otimes u^{-1}x \quad \text{and} \quad \varepsilon_F(x) = |G|\delta_{x,e},
$$

for all $x \in G$, where $\delta_{x,e}$ is the Kronecker's delta. By [5, Proposition 3.2] we know that $k^F[G]$ is a G-graded quasicoalgebra with associator $\Delta_2(F^{-1})$, this means a coalgebra in the monoidal category $\mathrm{Vect}^G_{\Delta_2(F^{-1})}$. Furthermore, if G is abelian then $k^F[G]$ is cocommutative in $\mathrm{Vect}^G_{\Delta_2(F^{-1}), \mathcal{R}_{F^{-1}}}$.

Clearly (k, Id_k) is an involutive coassociative k-coalgebra, so all the coalgebras obtained from it by using the Clifford process for coalgebras will be coassociative as well. The second coalgebra in this chain, i.e. the one obtained from (k, Id_k) and q, is precisely $C(q)$, the Clifford (co)algebra associated to the isometry class of the one-dimensional quadratic space $(k\mathbf{e}, \mathbf{q}(\alpha\mathbf{e}) = \alpha^2 q)$. As we will see, up to an isomorphism it has the form $k^F[G]$.

Proposition 3.3. *Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the cyclic group of order two written additively, $q \in k^*$ and $F : \mathbb{Z}_2 \times \mathbb{Z}_2 \to k^*$ the 2-cochain defined by $F(x, y) = q^{xy}$, for all $x, y \in \mathbb{Z}_2$, where we also made use of the multiplication of $\mathbb{Z}_2$. Then $\overline{(k, \mathrm{Id}_k)}$, the Clifford coalgebra obtained from (k, Id_k) and q by Clifford process for coalgebras, is isomorphic to $(k^F[\mathbb{Z}_2], \sigma)$ as involutive coalgebras, where $k^F[\mathbb{Z}_2]$ is the twisted $\mathbb{Z}_2$-graded quasicoalgebra with associator $\Delta_2(F^{-1})$ described above, and where $\sigma(x) = (-1)^x x$, for all $x \in G$.*

Proof. Observe that (F, σ) is the same as in Proposition 2.1. Thus F is indeed a 2-cochain on $\mathbb{Z}_2$ (and, moreover, a 2-cocycle). One can easily check that σ is an involutive coalgebra morphism for $k^F[\mathbb{Z}_2]$, so $(k^F[\mathbb{Z}_2], \sigma)$ is indeed an involutive coalgebra.

Consider $\mu : \overline{(k, \mathrm{Id}_k)} \to k^F[\mathbb{Z}_2]$ given by $\mu(1) = \bar{0}$ and $\mu(\mathbf{v}) = \bar{1}$, and extended by linearity. We claim that μ is an involutive coalgebra isomorphism. For this observe that the k-coalgebra structure of $\overline{(k, \mathrm{Id}_k)} = k \oplus k\mathbf{v}$ is given by

$$
\overline{\Delta}(1) = \frac{1}{2}\big(1 \otimes 1 + q^{-1}\mathbf{v} \otimes \mathbf{v}\big), \qquad \bar{\varepsilon}(1) = 2, \tag{3.3}
$$

$$
\overline{\Delta}(\mathbf{v}) = \frac{1}{2}(1 \otimes \mathbf{v} + \mathbf{v} \otimes 1), \qquad \bar{\varepsilon}(\mathbf{v}) = 0, \tag{3.4}
$$

while its involutive coalgebra automorphism is given by $\overline{\sigma}(\alpha+\beta\mathbf{v})=\alpha-\beta\mathbf{v}$, the conjugation. On the other hand, the coalgebra structure of $k^F[\mathbb{Z}_2]$ is given by

$$\overline{\Delta}(\bar{0})=\frac{1}{2}\big(F(\bar{0},\bar{0})^{-1}\bar{0}\otimes\bar{0}+F(\bar{1},\bar{1})^{-1}\bar{1}\otimes\bar{1}\big)=\frac{1}{2}\big(\bar{0}\otimes\bar{0}+q^{-1}\bar{1}\otimes\bar{1}\big),$$

$$\overline{\Delta}(\bar{1})=\frac{1}{2}\big(F(\bar{0},\bar{1})^{-1}\bar{0}\otimes\bar{1}+F(\bar{1},\bar{0})^{-1}\bar{1}\otimes\bar{0}\big)=\frac{1}{2}(\bar{0}\otimes\bar{1}+\bar{1}\otimes\bar{0}),$$

and $\varepsilon(\bar{0})=2$ and $\bar{\varepsilon}(\bar{1})=0$. It then follows that indeed μ is a coalgebra isomorphism. We also have $\sigma\mu(1)=\sigma(\bar{0})=\bar{0}=\mu(1)=\mu\overline{\sigma}(1)$ and $\sigma\mu(\mathbf{v})=\sigma(\bar{1})=-\bar{1}=-\mu(\mathbf{v})=\mu\overline{\sigma}(\mathbf{v})$, as required. □

One can easily see that if C is $\mathbb{Z}_2$-graded then so is $\overline{C}$ with $\overline{C}_0=C_0\oplus C_1\mathbf{v}$ and $\overline{C}_1=C_1\oplus C_0\mathbf{v}$.

Corollary 3.4. *The Clifford algebra $C(q)$ corresponding to the isometry class of the one-dimensional quadratic space $(k\mathbf{e},\mathbf{q}(\alpha\mathbf{e})=\alpha^2q)$ has a $\mathbb{Z}_2$-graded k-coalgebra structure.*

Proof. Follows from Proposition 2.1 and the previous result since $C(q)$, $k_F[\mathbb{Z}_2]$ and $k^F[\mathbb{Z}_2]$ are all isomorphic to $k[\mathbb{Z}_2]$ as k-vector spaces. Thus $C(q)\cong k\oplus k\mathbf{v}$ is $\mathbb{Z}_2$-graded and has a k-algebra structure determined by $\mathbf{e}^2=q1$, and a k-coalgebra structure determined by the relations (3.3)-(3.4), with $\mathbf{v}$ replaced by $\mathbf{e}$. By the above considerations we obtain that via these structures $C(q)$ is a commutative algebra and a cocommutative coalgebra in $\mathrm{Vect}^{\mathbb{Z}_2}_{\mathfrak{e},\mathcal{R}}$, the strict monoidal category of $\mathbb{Z}_2$-graded spaces endowed with the braiding defined by $\mathcal{R}(x,y)=(-1)^{xy}$, for all $x,y\in\mathbb{Z}_2$. □

Remarks 3.5. (1) For $q=-1$ we have $C(-1)=k[\mathbf{i}]$, the algebra of complex numbers over k. In this case we reobtain the $\mathbb{Z}_2$-graded k-algebra and k-coalgebra structures on $k[\mathbf{i}]$ obtained via the Cayley–Dickson process for algebras and coalgebras, see [1, Proposition 4.4] and [5, Lemma 3.6]. Note that $k[\mathbf{i}]_{\bar{0}}=k1$ and $k[\mathbf{i}]_{\bar{1}}=k\mathbf{i}$.

(2) For $q=1$ we have $C(1)=k[\mathbb{Z}_2]$ as $\mathbb{Z}_2$-graded k-algebra. The $\mathbb{Z}_2$-graded k-coalgebra structure of it is defined by

$$\overline{\Delta}(\bar{0})=\frac{1}{2}(\bar{0}\otimes\bar{0}+\bar{1}\otimes\bar{1}),\qquad \overline{\Delta}(\bar{1})=\frac{1}{2}(\bar{0}\otimes\bar{1}+\bar{1}\otimes\bar{0}),\qquad \bar{\varepsilon}(\bar{0})=2\quad\text{and}\quad\bar{\varepsilon}(\bar{1})=0.$$

At the ungraded level, we have that $k[\mathbb{Z}_2]\cong k\times k$, as k-algebras, via the identifications $(0,1)\equiv e_-:=\frac{1}{2}(\bar{0}-\bar{1})$ and $(1,0)\equiv e_+:=\frac{1}{2}(\bar{0}+\bar{1})$. So $C(1)$ can be viewed as the unital k-algebra generated by $e_\pm$ with relations $e_\pm^2=e_\pm$ and $e_-e_+=e_+e_-$. Then the k-coalgebra structure is given by

$$\begin{aligned}\Delta(e_\pm)&\equiv\frac{1}{2}\big(\overline{\Delta}(\bar{0})\pm\overline{\Delta}(\bar{1})\big)=\frac{1}{4}\big((\bar{0}\otimes\bar{0}+\bar{1}\otimes\bar{1})\pm(\bar{0}\otimes\bar{1}+\bar{1}\otimes\bar{0})\big)\\&=\frac{1}{4}\big(\bar{0}\otimes(\bar{0}\pm\bar{1})+\bar{1}\otimes(\bar{1}\pm\bar{0})\big)=\frac{1}{4}(\bar{0}\pm\bar{1})\otimes(\bar{0}\pm\bar{1})\equiv e_\pm\otimes e_\pm,\end{aligned}$$

and $\varepsilon(e_\pm)=1$. Note that the unit of $C(1)$ can be expressed as $1=e_+-e_-$.

From now on when we refer to $k^F[G]$ it is implicitly understood that G is a finite group. Furthermore, by analogy with the algebra case, for G abelian and F a 2-cochain on G we call $k^F[G]$ a standard involutive coalgebra if $k^F[G]$ is an involutive coalgebra modulo an involutive coalgebra automorphism σ having the form $\sigma(x)=s(x)^{-1}x$, for all $x\in G$, for a suitable map $s:G\to k^*$. If this is the case we then denote σ by σ_s. Notice that the involutive coalgebra automorphism of $k^F[\mathbb{Z}_2]$ in Proposition 3.3 is standard since $\sigma_s(x)=s(x)^{-1}x$, where $s:G\to k^*$ is given by $s(x)=(-1)^x$, for all $x\in G$.

Remark 3.6. σ_s is an involutive coalgebra automorphism of $k^F[G]$ if and only if σ_s is an involutive algebra automorphism of $k_F[G]$, this means, if and only if $s(x)^2=1$, $s(e)=1$ and $s(xy)=s(x)s(y)$, for all $x,y\in G$.

Indeed, $\sigma_s^2=\mathrm{Id}_{k^F[G]}$ and $\varepsilon_F\sigma_s=\varepsilon_F$ if and only if $s(x)^2=1$, for all $x\in G$, and $s(e)=1$, respectively. Also, $(\sigma_s\otimes\sigma_s)\Delta_F(x)=\Delta_F(\sigma_s(x))$ if and only if

$$\frac{s(x)^{-1}}{|G|}\sum_{u\in G}F\big(u,u^{-1}x\big)^{-1}u\otimes u^{-1}x=\frac{1}{|G|}\sum_{u\in G}F\big(u,u^{-1}x\big)^{-1}s(u)^{-1}s\big(u^{-1}x\big)^{-1}u\otimes u^{-1}x,$$

so if and only if $s(x)^{-1}=s(u)^{-1}s(u^{-1}x)^{-1}$, for all $u,x\in G$ or, equivalently, $s(xy)=s(x)s(y)$, for all $x,y\in G$. Observe now that these three necessary and sufficient conditions are precisely the ones required for s in order to have σ_s an involutive algebra automorphism of $k_F[G]$.

We shall prove that through the Clifford process for coalgebras to a standard involutive coalgebra of the form $(k^F[G],\sigma_s)$ corresponds a standard involutive coalgebra having a similar form.

Proposition 3.7. *Let G be a finite abelian group, k a field such that $2|G|\neq 0$ in k, $q\in k^*$ fixed arbitrary and F a 2-cochain on G such that $k^F[G]$ admits a standard involutive coalgebra structure via, say, $s:G\to k^*$. If $\overline{G}=G\times\mathbb{Z}_2$ and $\overline{F}$ and $\bar{s}$ are the 2-cochain, respectively the map, defined as in Proposition 2.2 then $(k^{\overline{F}}[\overline{G}],\sigma_{\bar{s}})$ is a standard involutive coalgebra isomorphic to $\overline{(k^F[G],\sigma_s)}$, the involutive coalgebra obtained from $(k^F[G],\sigma_s)$ and q by Clifford process for coalgebras.*

Proof. From the proof of Proposition 2.2 we know that $\sigma_{\bar{s}}$ is an involutive algebra automorphism of $k_{\overline{F}}[\overline{G}]$, and so by Remark 3.6 it follows that $\sigma_{\bar{s}}$ is an involutive coalgebra automorphism of $k^{\overline{F}}[\overline{G}]$.

To see that $\overline{(k^F[G],\sigma_s)}$ and $(k^{\overline{F}}[\overline{G}],\sigma_{\bar{s}})$ are isomorphic as involutive coalgebras define $\nu:\overline{(k^F[G],\sigma_s)}\to(k^{\overline{F}}[\overline{G}],\sigma_{\bar{s}})$ exactly as in the proof of Proposition 2.2. Namely, $\nu(x)=(x,\bar{0})$ and $\nu(x\mathbf{v})=(x,\bar{1})$, for all $x\in G$, extended by linearity. We shall prove that ν is a coalgebra morphism as well. We compute,

$$\begin{aligned}
\Delta_{\overline{F}}\nu(x)=\Delta_{\overline{F}}\big((x,\bar{0})\big)&=\frac{1}{2|G|}\bigg(\sum_{u\in G}\overline{F}\big((u,\bar{0}),\big(u^{-1}x,\bar{0}\big)\big)^{-1}(u,\bar{0})\otimes\big(u^{-1}x,\bar{0}\big)\\
&\quad+\overline{F}\big((u,\bar{1}),\big(u^{-1}x,\bar{1}\big)\big)^{-1}(u,\bar{1})\otimes\big(u^{-1}x,\bar{1}\big)\bigg)\\
&=\frac{1}{2|G|}\bigg(\sum_{u\in G}F\big(u,u^{-1}x\big)^{-1}(u,\bar{0})\otimes\big(u^{-1}x,\bar{0}\big)\\
&\quad+q^{-1}\sum_{u\in G}s\big(u^{-1}x\big)^{-1}F\big(u,u^{-1}x\big)^{-1}(u,\bar{1})\otimes\big(u^{-1}x,\bar{1}\big)\bigg)\\
&=\frac{1}{2}(\nu\otimes\nu)\bigg(\frac{1}{|G|}\bigg(\sum_{u\in G}F\big(u,u^{-1}x\big)^{-1}u\otimes u^{-1}x+q^{-1}\sum_{u\in G}F\big(u,u^{-1}x\big)^{-1}u\mathbf{v}\otimes\sigma_s\big(u^{-1}x\big)\mathbf{v}\bigg)\bigg)\\
&=(\nu\otimes\nu)\big(x_1^F\otimes x_2^F+q^{-1}x_1^F\mathbf{v}\otimes\sigma_s\big(x_2^F\big)\mathbf{v}\big)\overset{(3.1)}{=}(\nu\otimes\nu)\overline{\Delta_F}(x),
\end{aligned}$$

for all $x\in G$, where we have denoted $\Delta_F(x)=x_1^F\otimes x_2^F$, and where $\overline{\Delta_F}$ is the comultiplication of $\overline{(k^F[G],\sigma_s)}$ obtained through the Clifford process for coalgebras.

A similar computation shows that $\Delta_{\overline{F}}\nu(x\mathbf{v})=(\nu\otimes\nu)\overline{\Delta_F}(x\mathbf{v})$, for all $x\in G$, as needed. Moreover, one can easily see that $\varepsilon_{\overline{F}}\nu(x)=\varepsilon_{\overline{F}}(x,\bar{0})=2|G|\delta_{x,e}=2\varepsilon_F(x)=\overline{\varepsilon_F}(x)$, and that $\varepsilon_{\overline{F}}\nu(x\mathbf{v})=\varepsilon_{\overline{F}}(x,\bar{1})=$

$0 = \overline{\varepsilon_F}(x\mathbf{v})$, for all $x \in G$. Hence ν is a coalgebra isomorphism. Finally, ν respects the involutive coalgebra automorphisms since

$$\sigma_{\bar{s}}\nu(x) = \sigma_{\bar{s}}(x, \bar{0}) = \bar{s}(x, \bar{0})^{-1}(x, \bar{0}) = s(x)^{-1}\nu(x) = \nu\big(\sigma_s(x)\big) = \nu\overline{\sigma_s}(x),$$

and

$$\sigma_{\bar{s}}\nu(x\mathbf{v}) = \sigma_{\bar{s}}(x, \bar{1}) = \bar{s}(x, \bar{1})^{-1}(x, \bar{1}) = -s(x)^{-1}\nu(x\mathbf{v}) = -\nu\big(\sigma_s(x)\mathbf{v}\big) = \nu\overline{\sigma_s}(x\mathbf{v}),$$

for all $x \in G$. So the proof is finished. □

By Proposition 3.3 and Proposition 3.7 it follows that the $\mathbb{Z}_2$-graded k-coalgebras obtained from the input data (k, Id_k) through the Clifford process for coalgebras are all standard involutive coalgebras of the form $(k^F[(\mathbb{Z}_2)^n], \sigma_s)$, where $n \in \mathbb{N}$ and F is a 2-cocycle on $(\mathbb{Z}_2)^n$. Furthermore, since in the inductive process $(\overline{F}, \bar{s})$ is obtained from (F, s) as in the algebra case it follows that at the nth step of the process the involutive k-coalgebra that we get is $(k^F[(\mathbb{Z}_2)^n], \sigma_s)$ with F the 2-cocycle on $(\mathbb{Z}_2)^n$ defined in (2.1) and σ_s defined by $s(x) = (-1)^{\rho(x)}$, for all $x \in (\mathbb{Z}_2)^n$. Consequently, when we start with the input data (k, Id_k), viewed as a $\mathbb{Z}_2$-graded k-algebra or k-coalgebra, either if we use the Clifford process for algebras or for coalgebras, at the nth step of the process we get, up to an isomorphism, the same triple $(k[(\mathbb{Z}_2)^n], F, s)$ with F and s as above. But $k[(\mathbb{Z}_2)^n]$ is isomorphic, as k-vector space, with the underlying vector space of a Clifford algebra of the type $C(q_1, \ldots, q_n)$. Thus a Clifford algebra of this type has both involutive k-algebra and k-coalgebra structures that are commutative, respectively cocommutative, within the braided strict monoidal category $\mathrm{Vect}^{(\mathbb{Z}_2)^n}_{\mathfrak{e}, \mathcal{R}_{F^{-1}}}$.

Corollary 3.8. *The Clifford algebras $C(q_1, \ldots, q_n)$ are cocommutative coalgebras in $\mathrm{Vect}^{(\mathbb{Z}_2)^n}_{\mathfrak{e}, \mathcal{R}_{F^{-1}}}$ via the structure*

$$\Delta(e_x) = \frac{1}{2^n}\sum_{u \in (\mathbb{Z}_2)^n} F(u, u \oplus x)^{-1} e_u \otimes e_{u \oplus x}, \qquad \varepsilon(e_x) = 2^n \delta_{x,0}, \tag{3.5}$$

for all $x \in (\mathbb{Z}_2)^n$, where, once more, F is the normalized 2-cocycle on $(\mathbb{Z}_2)^n$ defined in (2.1), and where $\{e_x \mid x \in (\mathbb{Z}_2)^n\}$ with $e_x = e_1^{x_1} \cdots e_n^{x_n}$, for all $x = (x_1, \ldots, x_n) \in (\mathbb{Z}_2)^n$, is the canonical basis of $C(q_1, \ldots, q_n)$ obtained from the orthogonal basis $\{e_1, \ldots, e_n\}$ of the quadratic space (V, q).

Proof. Follows from the above comments. We have to identify, as a k-vector space, $C(q_1, \ldots, q_n)$ with $k[(\mathbb{Z}_2)^n]$ via $e_x \mapsto x$ and then to transport through this isomorphism the coalgebra structure of $k^F[(\mathbb{Z}_2)^n]$ to $C(q_1, \ldots, q_n)$. We also point out that the braided monoidal structure of $\mathrm{Vect}^{(\mathbb{Z}_2)^n}$ is the strict monoidal one, with the braiding defined by $\mathcal{R}_{F^{-1}}$ from (2.4). □

The result below can be viewed as the coalgebra version of [2, Corollary 3.6]. It says that the coalgebra structure of a Clifford algebra can be obtained inductively from the one of k.

Proposition 3.9. *Take $q_1, \ldots, q_{n+1} \in k^*$. Then the standard involutive $\mathbb{Z}_2$-graded k-coalgebra $C(q_1, \ldots, q_{n+1})$ is isomorphic to the standard involutive $\mathbb{Z}_2$-graded k-coalgebra obtained from $C(q_1, \ldots, q_n)$ and q_{n+1} by Clifford process for coalgebras.*

Proof. Let F be the 2-cocycle on $(\mathbb{Z}_2)^n$ corresponding to the Clifford coalgebra $C(q_1, \ldots, q_n)$. If $\overline{F}$ is the 2-cocycle associated to F as in Proposition 3.7 then, as in the algebra case, it can be checked that it coincides with the 2-cocycle on $(\mathbb{Z}_2)^{n+1}$ corresponding to the Clifford coalgebra $C(q_1, \ldots, q_{n+1})$. These facts allow us to conclude that $C(q_1, \ldots, q_{n+1}) \cong k^{\overline{F}}[(\mathbb{Z}_2)^{n+1}] \cong \overline{k^F[(\mathbb{Z}_2)^n]} \cong \overline{C(q_1, \ldots, q_n)}$, as involutive coalgebras, as needed.

A direct approach is the following. Denote a vector from $(\mathbb{Z}_2)^{n+1}$ by $\bar{x} = (x, x_{n+1})$, where $x = (x_1, \dots, x_n)$ is a certain vector of $(\mathbb{Z}_2)^n$. Also, by $\{e_x \mid x \in (\mathbb{Z}_2)^n\}$ and $\{\bar{e}_{\bar{x}} \mid \bar{x} \in (\mathbb{Z}_2)^{n+1}\}$ denote the canonical bases of $C(q_1, \dots, q_n)$ and $C(q_1, \dots, q_{n+1})$, respectively. We then can prove that $e_x \mapsto \bar{e}_{(x,\bar{0})}$ and $e_x\mathbf{v} \mapsto \bar{e}_{(x,\bar{1})}$ produce an involutive coalgebra isomorphism from $\overline{C(q_1, \dots, q_n)}$ to $C(q_1, \dots, q_{n+1})$. We leave the verification of all these details to the reader. Remind only that, in general, the standard involutive coalgebra structure of $C(q_1, \dots, q_n)$ is produced by the map $s(e_x) = (-1)^{\rho(x)}$, $x \in (\mathbb{Z}_2)^n$. □

A Clifford algebra is a $\mathbb{Z}_2$-graded algebra as well and since $\mathrm{Vect}^{\mathbb{Z}_2}$ has several braided monoidal structures many categorical constructions can be applied to them. For instance, we can consider the tensor product algebra structure produced by two Clifford algebras which, according to [2, Corollary 2.6], is again a Clifford algebra. Our next aim is to show that a similar result works in the coalgebra setting, too.

Recall that the braided monoidal structures on $\mathrm{Vect}^{\mathbb{Z}_2}$ are completely determined by the elements $\nu \in k^*$ such that $\nu^4 = 1$, see [10,11]. If ν is such an element then the monoidal structure is produced by the 3-cocycle on $\mathbb{Z}_2$ given by, $x, y, z \in \mathbb{Z}_2$,

$$\phi_{\nu^2}(x, y, z) = \begin{cases} \nu^2 & \text{if } x,\ y \text{ and } z \text{ are all odd,} \\ 1 & \text{otherwise.} \end{cases}$$

The braiding in this case is given by $\mathcal{R}_\nu(x, y) = \nu^{xy}$, for all $x, y \in \mathbb{Z}_2$.

In what follows we consider $\nu = -1$, so we will deal with the strict monoidal category $\mathrm{Vect}^{\mathbb{Z}_2}$ endowed with the braided structure given by $c_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v$, for all homogeneous elements $v \in V \in \mathrm{Vect}^{\mathbb{Z}_2}$ and $w \in W \in \mathrm{Vect}^{\mathbb{Z}_2}$. Let us denote this braided monoidal category by $\mathrm{Vect}^{\mathbb{Z}_2}_{-1}$.

Consequently, if A, A' are $\mathbb{Z}_2$-graded algebras then the tensor product algebra structure of $A \otimes A'$ in $\mathrm{Vect}^{\mathbb{Z}_2}_{-1}$ is determined by

$$(a \otimes a')(b \otimes b') = (-1)^{|a'||b|} ab \otimes a'b',$$

for all homogeneous elements $a, b \in A$ and $a', b' \in A'$, cf. [15]. In order to distinguish this tensor product algebra structure on $A \otimes A'$ we will denote it by $A \widehat{\otimes} A'$. Also, the elements of $A \widehat{\otimes} A'$ will be denoted by $a \widehat{\otimes} a'$, where $a \in A$ and $a' \in A'$.

Likewise, if C, D are coalgebras in $\mathrm{Vect}^{\mathbb{Z}_2}$ we then denote by $C \widehat{\otimes} D$ the tensor product coalgebra of C and D in $\mathrm{Vect}^{\mathbb{Z}_2}_{-1}$. More precisely, by [15] the comultiplication Δ of $C \widehat{\otimes} D$ is

$$\Delta(c \widehat{\otimes} d) = (-1)^{|c_2||d_1|}(c_1 \widehat{\otimes} d_1) \widehat{\otimes} (c_2 \widehat{\otimes} d_2),$$

and its counit is $\varepsilon(c \widehat{\otimes} d) = \varepsilon(c)\varepsilon(d)$, for all $c \in C$ and $d \in D$.

Since $\mathrm{Vect}^{\mathbb{Z}_2}_{-1}$ is strict monoidal all the (co)algebras that lie inside of it are (co)associative in the usual sense.

Proposition 3.10. *For any $q_1, q_2, \dots, q_{n+1} \in k^*$ we have*

$$C(q_1, \dots, q_{n+1}) \cong C(q_1, \dots, q_n) \widehat{\otimes} C(q_{n+1}),$$

as $\mathbb{Z}_2$-graded k-coalgebras.

Proof. As before, denote a vector of $(\mathbb{Z}_2)^{n+1}$ by $\bar{x} = (x, x_{n+1})$, where x is a vector of $(\mathbb{Z}_2)^n$. The canonical bases of $C(q_1, \dots, q_{n+1})$, $C(q_1, \dots, q_n)$ and $C(q_{n+1})$ will be then denoted by $\{\bar{e}_{\bar{x}} \mid \bar{x} \in (\mathbb{Z}_2)^{n+1}\}$, $\{e_x \mid x \in (\mathbb{Z}_2)^n\}$ and $\{1, \mathbf{e}\}$, respectively. We also consider the 2-cocycles F and $\bar{F}$ associated to $C(q_1, \dots, q_n)$, and respectively to $C(q_1, \dots, q_{n+1})$, and remind that they are connected via (2.2).

We claim that $\zeta : C(q_1, \ldots, q_n) \mathbin{\widehat{\otimes}} C(q_{n+1}) \to C(q_1, \ldots, q_{n+1})$ given by

$$\zeta(e_x \mathbin{\widehat{\otimes}} 1) = \bar{e}_{(x,0)} \quad \text{and} \quad \zeta(e_x \mathbin{\widehat{\otimes}} \mathbf{e}) = \bar{e}_{(x,1)}, \quad \forall x \in (\mathbb{Z}_2)^n,$$

is a $\mathbb{Z}_2$-graded k-coalgebra isomorphism. Indeed, if Δ is the comultiplication of the $\mathbb{Z}_2$-graded coalgebra $C(q_1, \ldots, q_n) \mathbin{\widehat{\otimes}} C(q_{n+1})$ as it was defined above then

$$\begin{aligned}
(\zeta \otimes \zeta)\Delta(e_x \mathbin{\widehat{\otimes}} 1) &\overset{(3.3),(3.5)}{=} \frac{1}{2^{n+1}}(\zeta \otimes \zeta)\Bigg(\sum_{u \in (\mathbb{Z}_2)^n} F(u, u \oplus x)^{-1}(e_u \mathbin{\widehat{\otimes}} 1) \otimes (e_{u \oplus x} \mathbin{\widehat{\otimes}} 1) \\
&\quad + q_{n+1}^{-1}(-1)^{\rho(u \oplus x)} \sum_{u \in (\mathbb{Z}_2)^n} F(u, u \oplus x)^{-1}(e_u \mathbin{\widehat{\otimes}} \mathbf{e}) \otimes (e_{u \oplus x} \mathbin{\widehat{\otimes}} \mathbf{e})\Bigg) \\
&= \frac{1}{2^{n+1}}\Bigg(\sum_{u \in (\mathbb{Z}_2)^n} F(u, u \oplus x)^{-1}\bar{e}_{(u,0)} \otimes \bar{e}_{(u \oplus x,0)} \\
&\quad + q_{n+1}^{-1}(-1)^{\rho(u \oplus x)} \sum_{u \in (\mathbb{Z}_2)^n} F(u, u \oplus x)^{-1}\bar{e}_{(u,1)} \otimes \bar{e}_{(u \oplus x,1)}\Bigg) \\
&= \frac{1}{2^{n+1}}\Bigg(\sum_{u \in (\mathbb{Z}_2)^n} \overline{F}\big((u,0),(u \oplus x,0)\big)^{-1}\bar{e}_{(u,0)} \otimes \bar{e}_{(u \oplus x,0)} \\
&\quad + \sum_{u \in (\mathbb{Z}_2)^n} \overline{F}\big((u,1),(u \oplus x,1)\big)^{-1}\bar{e}_{(u,1)} \otimes \bar{e}_{(u \oplus x,1)}\Bigg) \\
&\overset{(3.5)}{=} \Delta(\bar{e}_{(x,0)}) = \Delta\zeta(e_x \mathbin{\widehat{\otimes}} 1),
\end{aligned}$$

for all $x \in (\mathbb{Z}_2)^n$. In a similar manner we can prove that $(\zeta \otimes \zeta)\Delta(e_x \mathbin{\widehat{\otimes}} \mathbf{e}) = \Delta\zeta(e_x \mathbin{\widehat{\otimes}} \mathbf{e})$, for all $x \in (\mathbb{Z}_2)^n$. Also, a simple verification shows that ζ respects the counits and the $\mathbb{Z}_2$-gradings, so ζ is a $\mathbb{Z}_2$-graded k-coalgebra isomorphism, as desired. □

The next result is the coalgebra version of [2, Corollary 2.6].

Corollary 3.11. *For $q_1, \ldots, q_m, q'_1, \ldots, q'_n \in k^*$ we have*

$$C\big(q_1, \ldots, q_m, q'_1, \ldots, q'_n\big) \cong C(q_1, \ldots, q_m) \mathbin{\widehat{\otimes}} C\big(q'_1, \ldots, q'_n\big),$$

as $\mathbb{Z}_2$-graded k-coalgebras.

Proof. We proceed by mathematical induction on n. If $n = 1$ then the assertion follows from the above result. For $n \geqslant 2$ we have

$$\begin{aligned}
C(q_1, \ldots, q_m) \mathbin{\widehat{\otimes}} C\big(q'_1, \ldots, q'_n\big) &\cong C(q_1, \ldots, q_m) \mathbin{\widehat{\otimes}} \big(C\big(q'_1, \ldots, q'_{n-1}\big) \mathbin{\widehat{\otimes}} C\big(q'_n\big)\big) \\
&\cong \big(C(q_1, \ldots, q_m) \mathbin{\widehat{\otimes}} C\big(q'_1, \ldots, q'_{n-1}\big)\big) \mathbin{\widehat{\otimes}} C\big(q'_n\big) \\
&\cong C\big(q_1, \ldots, q_m, q'_1, \ldots, q'_{n-1}\big) \mathbin{\widehat{\otimes}} C\big(q'_n\big) \\
&\cong C\big(q_1, \ldots, q_m, q'_1, \ldots, q'_n\big),
\end{aligned}$$

as $\mathbb{Z}_2$-graded k-coalgebras. Note that the first and the last isomorphisms follow from Proposition 3.10, the second one from the associativity of the tensor product coalgebra structure [15], and the third one from the induction hypothesis, respectively. □

4. The coalgebra structure of some Clifford algebras

In this section we will present the $\mathbb{Z}_2$-graded coalgebra structure of some Clifford algebras. Actually, we will completely describe the $\mathbb{Z}_2$-graded coalgebra structure of the Clifford algebra of generalized quaternions and of the Clifford algebras of the type $C(q_1, q_2, q_3)$.

Let $V = \langle q_1\rangle \perp \langle q_2\rangle$ be a two-dimensional quadratic space with $q(e_1) = q_1$ and $q(e_2) = q_2$, where $\{e_1, e_2\}$ is an orthogonal basis for V. So $q(\alpha e_1 + \beta e_2) = q_1\alpha^2 + q_2\beta^2$, for all $\alpha, \beta \in k$. Then $C(q_1, q_2) = C(\langle q_1\rangle \perp \langle q_2\rangle)$ is the k-algebra generated by e_1, e_2 with relations

$$e_1e_2 = -e_2e_1, \qquad e_1^2 = q_1 1, \qquad e_2^2 = q_2 1.$$

We will denote it by $(\frac{q_1,q_2}{k})$ and call it the generalized quaternion algebra over k. Note that, in the case where $q_1 = q_2 = -1$, $(\frac{-1,-1}{k}) = \mathbb{H}$, the k-algebra of quaternions, and this justifies our terminology. Notice also that its $\mathbb{Z}_2$-grading is given by $(\frac{q_1,q_2}{k})_{\bar 0} = k1 \oplus k\mathbf{k}$ and $(\frac{q_1,q_2}{k})_{\bar 1} = k\mathbf{i} \oplus k\mathbf{j}$.

For further use record that when $q_1 = q_2 = 1$ we have $(\frac{1,1}{k})$ isomorphic to the matrix algebra $M_2(k)$ via the identifications $e_1 \equiv \left(\begin{smallmatrix} 1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and $e_2 \equiv \left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)$. The same we get in the case when $q_1 = -1$ and $q_2 = 1$, this time via the identifications $e_1 \equiv \left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)$ and $e_2 \equiv \left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)$. Therefore, we have algebra isomorphisms

$$C(-1,-1) = \mathbb{H} \quad \text{and} \quad C(1,1) \cong C(-1,1) \cong M_2(k).$$

Proposition 4.1. *The space of generalized quaternions $C(q_1, q_2) = (\frac{q_1,q_2}{k})$ has a k-coalgebra structure given by*

$$\Delta(1) = \frac{1}{4}\big(1\otimes 1 + q_1^{-1}e_1\otimes e_1 + q_2^{-1}e_2\otimes e_2 - (q_1q_2)^{-1}e_1e_2\otimes e_1e_2\big),$$
$$\Delta(e_1) = \frac{1}{2}\big(e_1\otimes 1 + 1\otimes e_1 + q_2^{-1}e_1e_2\otimes e_2 - q_2^{-1}e_2\otimes e_1e_2\big),$$
$$\Delta(e_2) = \frac{1}{4}\big(e_2\otimes 1 + 1\otimes e_2 + q_1^{-1}e_1\otimes e_1e_2 - q_1^{-1}e_1e_2\otimes e_1\big),$$
$$\Delta(e_1e_2) = \frac{1}{4}(e_1e_2\otimes 1 + 1\otimes e_1e_2 + e_1\otimes e_2 - e_2\otimes e_1),$$

and $\varepsilon(e_1^{x_1}e_2^{x_2}) = 4\delta_{x_1,0}\delta_{x_2,0}$, for all $x_1, x_2 \in \{0,1\}$. Furthermore, with this structure $(\frac{q_1,q_2}{k})$ becomes a $\mathbb{Z}_2$-graded k-coalgebra and a braided cocommutative $\mathbb{Z}_2\times\mathbb{Z}_2$-graded k-coalgebra.

Proof. $C(q_1, q_2)$ can be identified with $k^F[(\mathbb{Z}_2)^2]$, where

$$F(x, y) = (-1)^{x_2y_1}q_1^{x_1y_1}q_2^{x_2y_2},$$

for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $(\mathbb{Z}_2)^2$. By (3.5) we then have

$$\begin{aligned}\Delta(1)=\Delta(e_{(0,0)})&=\frac{1}{4}\sum_{u\in(\mathbb{Z}_2)^2}F(u,u)^{-1}e_u\otimes e_u\\&=\frac{1}{4}\big(F\big((0,0),(0,0)\big)^{-1}e_{(0,0)}\otimes e_{(0,0)}+F\big((1,0),(1,0)\big)^{-1}e_{(1,0)}\otimes e_{(1,0)}\\&\quad+F\big((0,1),(0,1)\big)^{-1}e_{(0,1)}\otimes e_{(0,1)}+F\big((1,1),(1,1)\big)^{-1}e_{(1,1)}\otimes e_{(1,1)}\big)\\&=\frac{1}{4}\big(1\otimes 1+q_1^{-1}e_1\otimes e_1+q_2^{-1}e_2\otimes e_2-(q_1q_2)^{-1}e_1e_2\otimes e_1e_2\big).\end{aligned}$$

The other formulas can be deduced in a similar way, we leave the verification to the reader. □

As we will see in some particular cases Clifford (co)algebras can be identified with matrix (co)algebras. In order to express these isomorphisms as isomorphisms in $\mathrm{Vect}^{\mathbb{Z}_2}$, the categories where all Clifford (co)algebras reside, we have to point out the $\mathbb{Z}_2$-grading of a such matrix (co)algebra. Namely, it will always be of the following form.

If A is a $\mathbb{Z}_2$-graded algebra then we define a $\mathbb{Z}_2$-graded algebra structure on $\widehat{M}_r(A)$ out of $M_r(A)$ by taking

$$\widehat{M}_r(A)_0:=\begin{pmatrix}A_0 & A_1 & \\ A_1 & A_0 & \\ & & \ddots\end{pmatrix}\quad\text{and}\quad \widehat{M}_r(A)_1:=\begin{pmatrix}A_1 & A_0 & \\ A_0 & A_1 & \\ & & \ddots\end{pmatrix},$$

where $A=A_0\oplus A_1$ is the $\mathbb{Z}_2$-grading of A. We will refer to it as being the checkerboard grading of $M_r(A)$. Then the classical algebra isomorphism $M_n(k)\otimes A\cong M_n(A)$ extends to $\mathbb{Z}_2$-graded algebras. The same is valid for the isomorphism $M_m(k)\otimes M_n(k)\cong M_{mn}(k)$, providing now that m or n is even. All these considerations can be adapted for coalgebras as follows.

Let C be a k-coalgebra and denote by $M_n(C)$ the set on $n\times n$-matrices with entries in C. One can easily see that C induces a k-vector space structure on $M_n(C)$, and that $\dim_k M_n(C)=n^2\dim_k C$. Actually, for $1\leqslant i,j\leqslant n$ and $c\in C$ denote by $E_{ij}(c)$ the $n\times n$-matrix having c on position (i,j) and 0 elsewhere. Then it can be easily checked that $\{E_{ij}(c_\lambda)\mid 1\leqslant i,j\leqslant n,\ \lambda\in\Lambda\}$ is a basis for $M_n(C)$, providing that $(c_\lambda)_{\lambda\in\Lambda}$ is a basis of C. Now a routine computation shows that

$$\Delta\big(E_{ij}(c)\big)=\sum_{s=1}^{n}E_{is}(c_1)\otimes E_{sj}(c_2)\quad\text{and}\quad \varepsilon\big(E_{ij}(c)\big)=\delta_{i,j}\varepsilon(c),$$

defined for all $1\leqslant i,j\leqslant n$ and c running on a basis of C, and extended by linearity, endow $M_n(C)$ with a k-coalgebra structure, called in what follows the matrix coalgebra over C. When $C=k$ we recover the comatrix coalgebra structure of $M_n(k)$.

Lemma 4.2. *Let C be a $\mathbb{Z}_2$-graded k-coalgebra. Then $M_n(C)$ is a $\mathbb{Z}_2$-graded k-coalgebra with the k-coalgebra structure defined above and via the checkerboard grading (the same grading presented before for an algebra A; now in place of A we have to take C). It will be denoted by $\widehat{M}_n(C)$. We then have $\widehat{M}_n(k)\otimes C\cong\widehat{M}_n(C)$, as $\mathbb{Z}_2$-graded k-coalgebras. Furthermore, if either m or n is even then we also have $\widehat{M}_m(\widehat{M}_n(k))\cong\widehat{M}_{mn}(k)$, as $\mathbb{Z}_2$-graded k-coalgebras.*

Proof. Let $\{E_{ij}(c)\mid 1\leqslant i,j\leqslant n,\ c\in\mathbb{B}\}$ be the canonical basis of $M_n(C)$ (here $\mathbb{B}$ denotes a basis of C). By the definition of the checkerboard grading we have $|E_{ij}(c)|$ equals to 0 if $i-j$ is even, and 1 otherwise, providing $|c|=0$. In the case when $|c|=1$ we have $|E_{ij}(c)|$ equals to 1, if $i-j$ is even, and 0 otherwise. From these facts we conclude that

$$\left|E_{ij}(c)\right| = \begin{cases} |c| & \text{if } i-j \in 2\mathbb{Z}, \\ 1-|c| & \text{if } i-j \in 2\mathbb{Z}+1. \end{cases} \tag{4.1}$$

Recall that $\Delta(E_{ij}(c)) = \sum_{s=1}^{n} E_{is}(c_1) \otimes E_{sj}(c_2)$. So when $i-j$ is even we must have $i-s$ and $s-j$ of the same parity, for all $1 \leqslant s \leqslant n$. This implies that $E_{is}(c_1) \otimes E_{sj}(c_2)$ has degree $|c_1| \oplus |c_2| = |c| = |E_{ij}(c)|$. Similarly, when $i-j$ is odd we should have $i-s$ and $s-j$ of different parities, hence $E_{is}(c_1) \otimes E_{sj}(c_2)$ has degree $1-(|c_1| \oplus |c_2|) = 1-|c| = |E_{ij}(c)|$. Since $\varepsilon(E_{ij}(c)) = \varepsilon(c)\delta_{i,j}$ it follows then that $M_n(C)$ is a $\mathbb{Z}_2$-graded k-coalgebra.

Now it can be easily checked that the canonical isomorphism $M_n(k) \otimes C \ni E_{ij} \otimes c \mapsto E_{ij}(c) \in M_n(C)$ is a morphism of k-coalgebras (here $\{E_{ij} \mid 1 \leqslant i, j \leqslant n\}$ is the canonical basis of $M_n(k)$ and c is taken from a basis $\mathbb{B}$ of C) and, moreover, it preserves the $\mathbb{Z}_2$-gradings. The details are left to the reader.

When m is even the canonical isomorphism

$$M_n(k) \otimes M_m(k) \ni E^n_{ij} \otimes E^m_{uv} \mapsto E^{nm}_{(i-1)m+u,(j-1)m+v} \in M_{nm}(k),$$

extended by linearity, is a $\mathbb{Z}_2$-graded coalgebra isomorphism. (Here we have denoted by $(E^n_{ij})_{1\leqslant i,j\leqslant n}$, $(E^m_{uv})_{1\leqslant u,v\leqslant m}$ and $(E^{nm}_{pq})_{1\leqslant p,q\leqslant nm}$ the canonical bases of $M_n(k)$, $M_m(k)$ and $M_{nm}(k)$, respectively.) But $\widehat{M}_n(k) \otimes \widehat{M}_m(k) \cong \widehat{M}_n(\widehat{M}_m(k))$, from the first isomorphism, so we are done. Likewise, if n is even then

$$M_n(k) \otimes M_m(k) \ni E^n_{ij} \otimes E^m_{uv} \mapsto E^{nm}_{i+(n-1)u,j+(n-1)v} \in M_{nm}(k)$$

is a $\mathbb{Z}_2$-graded k-coalgebra isomorphism, and so we can continue as above. □

Remarks 4.3. (1) One can easily see that the correspondences $1 \mapsto 1$, $e_1 \mapsto E_2$, $e_2 \mapsto E_1$ and $e_1e_2 \mapsto -E_1E_2$ define a $\mathbb{Z}_2$-graded k-algebra isomorphism between $(\frac{q_1,q_2}{k})$ and $(\frac{q_2,q_1}{k})$, where $\{1, e_1, e_2, e_1e_2\}$ and $\{1, E_1, E_2, E_1E_2\}$ are the canonical bases of $(\frac{q_1,q_2}{k})$ and $(\frac{q_2,q_1}{k})$, respectively. A simple verification shows that the same correspondences define a $\mathbb{Z}_2$-graded k-coalgebra isomorphism, too.

(2) For $q_1 = q_2 = -1$ we have $(\frac{-1,-1}{k}) = \mathbb{H}$ and in this way we reobtain the k-coalgebra structure of $\mathbb{H}$ obtained through the Cayley–Dickson process for coalgebras in [5].

In addition, if $\iota = \sqrt{-1} \in k$ then $\mathbb{H} \cong \widehat{M}_2(k)$. In terms of the canonical basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ of $M_2(k)$, the isomorphism is produced by

$$E_{11} \leftrightarrow \frac{1}{2}(1-\iota\mathbf{k}), \qquad E_{12} \leftrightarrow \frac{1}{2}(\mathbf{i}-\iota\mathbf{j}), \qquad E_{21} \leftrightarrow -\frac{1}{2}(\mathbf{i}+\iota\mathbf{j}), \qquad E_{22} \leftrightarrow \frac{1}{2}(1+\iota\mathbf{k}).$$

A direct computation shows that it is a $\mathbb{Z}_2$-graded algebra and coalgebra isomorphism, where $M_2(k)$ has the comatrix k-coalgebra structure, shifted by $\frac{1}{2}$. This means that

$$\Delta(E_{ij}) = \frac{1}{2}\sum_{s=1}^{2} E_{is} \otimes E_{sj} \quad \text{and} \quad \varepsilon(E_{ij}) = 2\delta_{i,j},$$

for all $1 \leqslant i, j \leqslant 2$. Clearly it is isomorphic, as a coalgebra, to the usual comatrix coalgebra structure of $M_n(k)$.

(3) When $q_1 = 1$ and $q_2 = -1$ we pointed out that the identifications $e_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $e_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ give rise to a k-algebra isomorphism between $(\frac{1,-1}{k})$ and $M_2(k)$. If we make use of the canonical basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ of $M_2(k)$ then

$$E_{11} \leftrightarrow \frac{1}{2}(1-e_1e_2), \qquad E_{12} \leftrightarrow \frac{1}{2}(e_1+e_2), \qquad E_{21} \leftrightarrow \frac{1}{2}(e_1-e_2), \qquad E_{22} \leftrightarrow \frac{1}{2}(1+e_1e_2).$$

A simple inspection shows that this isomorphism is $\mathbb{Z}_2$-graded, so $(\frac{1,-1}{k}) \cong \widehat{M}_2(k)$ as $\mathbb{Z}_2$-graded algebras. Furthermore, transporting the $\mathbb{Z}_2$-graded coalgebra structure from $(\frac{1,-1}{k})$ to $M_2(k)$ we obtain, again, the comatrix coalgebra structure of $M_2(k)$, shifted by $\frac{1}{2}$. Therefore, $(\frac{1,-1}{k}) \cong \widehat{M}_2(k)$ as $\mathbb{Z}_2$-graded k-coalgebras, too.

(4) We have seen that $(\frac{1,1}{k}) \cong M_2(k)$ but the isomorphism described at the beginning of this section does not preserve the $\mathbb{Z}_2$-gradings because e_1, an element of degree 1, is mapped to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which is an element of degree zero in $\widehat{M}_2(k)$. Nevertheless, if $\iota = \sqrt{-1} \in k$ then $(\frac{1,1}{k}) \cong \widehat{M}_2(k)$ as $\mathbb{Z}_2$-graded algebras and coalgebras, the isomorphism being produced by the identifications $e_1 \equiv \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}$ and $e_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proposition 4.4. *For $q_1, q_2, q_3 \in k^*$ the Clifford algebra $C(q_1, q_2, q_3)$ has a $\mathbb{Z}_2$-graded k-coalgebra structure given by*

$$\Delta(1) = \frac{1}{8}\left(1 \otimes 1 + \sum_{i=1}^{3} q_i^{-1} e_i \otimes e_i - \sum_{1 \leqslant i < j \leqslant 3} (q_i q_j)^{-1} e_i e_j \otimes e_i e_j - (q_1 q_2 q_3)^{-1} e_1 e_2 e_3 \otimes e_1 e_2 e_3\right),$$

$$\begin{aligned}
\Delta(e_t) = \frac{1}{8}\big(&e_t \otimes 1 + 1 \otimes e_t + q_{t+1}^{-1}(e_t e_{t+1} \otimes e_{t+1} - e_{t+1} \otimes e_t e_{t+1}) \\
&+ q_{t+2}^{-1}(e_t e_{t+2} \otimes e_{t+2} - e_{t+2} \otimes e_t e_{t+2}) \\
&- (q_{t+1} q_{t+2})^{-1}(e_{t+1} e_{t+2} \otimes e_t e_{t+1} e_{t+2} + e_t e_{t+1} e_{t+2} \otimes e_{t+1} e_{t+2})\big),
\end{aligned}$$

$$\begin{aligned}
\Delta(e_t e_{t+1}) = \frac{1}{8}\big(&e_t e_{t+1} \otimes 1 + 1 \otimes e_t e_{t+1} + e_t \otimes e_{t+1} - e_{t+1} \otimes e_t \\
&+ q_{t+2}^{-1}(e_{t+1} e_{t+2} \otimes e_t e_{t+2} - e_t e_{t+2} \otimes e_{t+1} e_{t+2}) \\
&+ q_{t+2}^{-1}(e_t e_{t+1} e_{t+2} \otimes e_{t+2} + e_{t+2} \otimes e_t e_{t+1} e_{t+2})\big),
\end{aligned}$$

$$\Delta(e_1 e_2 e_3) = \frac{1}{8}\left(e_1 e_2 e_3 \otimes 1 + 1 \otimes e_1 e_2 e_3 + \sum_{i=1}^{3} (e_i \otimes e_{i+1} e_{i+2} + e_{i+1} e_{i+2} \otimes e_i)\right),$$

for all $1 \leqslant t \leqslant 3$, where we always reduce the indices of e's modulo 3. The counit is $\varepsilon(e_1^{x_1} e_2^{x_2} e_3^{x_3}) = 8\delta_{x_1,0}\delta_{x_2,0}\delta_{x_3,0}$, for all $x_i \in \{0,1\}$, $1 \leqslant i \leqslant 3$, where, as usual,

$$\{1, e_1, e_2, e_3, e_1 e_2, e_1 e_3, e_2 e_3, e_1 e_2 e_3\}$$

is the canonical basis of $C(q_1, q_2, q_3)$. Furthermore, with this structure $C(q_1, q_2, q_3)$ becomes a braided cocommutative coalgebra in $\mathrm{Vect}_{\mathfrak{e}, \mathcal{R}_{F^{-1}}}^{(\mathbb{Z}_2)^3}$.

Proof. $C(q_1, q_2, q_3)$ identifies as a $(\mathbb{Z}_2)^3$-graded k-coalgebra with $k^F[(\mathbb{Z}_2)^3]$, where

$$F(x, y) = (-1)^{x_2 y_1 + x_3(y_1 + y_2)} q_1^{x_1 y_1} q_2^{x_2 y_2} q_3^{x_3 y_3},$$

for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in (\mathbb{Z}_2)^3$. Using this explicit form for F, as in the proof of Proposition 4.1 we can compute the values of Δ on the elements of the canonical basis of $C(q_1, q_2, q_3)$. We leave these computations to the reader. □

Other descriptions for $C(q_1, q_2, q_3)$ are the following.

Proposition 4.5. *For any $q_1, q_2, q_3 \in k^*$ we have*

$$\begin{aligned} C(q_1, q_2, q_3) &\cong C(q_1, q_2) \mathbin{\widehat{\otimes}} C(q_3) \cong C(q_1) \mathbin{\widehat{\otimes}} C(q_2) \mathbin{\widehat{\otimes}} C(q_3) \\ &\cong C_{\mathrm{tr}}(-q_1q_3, -q_2q_3) \mathbin{\widehat{\otimes}} C(-q_1q_2q_3) \equiv C(-q_1q_3, -q_2q_3) \otimes C(-q_1q_2q_3), \end{aligned}$$

as $\mathbb{Z}_2$-graded algebras and coalgebras, where $C_{\mathrm{tr}}(-q_1q_3, -q_2q_3)$ is the k-vector space $C(-q_1q_3, -q_2q_3)$ with the trivial $\mathbb{Z}_2$-grading (the one for which all the elements are of degree zero).

Proof. For the algebra case, the first two isomorphisms are consequences of [2, Corollary 2.6] while in the coalgebra case they are consequences of Corollary 3.11. Also, in the algebra case the last isomorphism can be produced as follows (see [18, Chapter 9, Lemma 2.9]). Suppose that for each $1 \leqslant i \leqslant 3$, $C(q_i)$ is generated by e_i, so that $e_i^2 = q_i 1$. If $A = C(q_1) \mathbin{\widehat{\otimes}} C(q_2) \mathbin{\widehat{\otimes}} C(q_3)$ then

$$\{\mathbf{1} := 1 \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} 1,\ I := e_1 \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} e_3,\ J := 1 \mathbin{\widehat{\otimes}} e_2 \mathbin{\widehat{\otimes}} e_3,\ K := -q_3 e_1 \mathbin{\widehat{\otimes}} e_2 \mathbin{\widehat{\otimes}} 1\}$$

is a basis for the even part of A, A_0. Since the multiplication of A is given by

$$(a \mathbin{\widehat{\otimes}} b \mathbin{\widehat{\otimes}} c)\big(a' \mathbin{\widehat{\otimes}} b' \mathbin{\widehat{\otimes}} c'\big) = (-1)^{|a'|(|b|+|c|)+|c||b'|} aa' \mathbin{\widehat{\otimes}} bb' \mathbin{\widehat{\otimes}} cc',$$

for any homogeneous elements a, b, c, a', b', c' of A, we compute that

$$I^2 = -q_1q_3\mathbf{1}, \qquad J^2 = -q_2q_3\mathbf{1}, \qquad IJ = -JI = K,$$

and therefore A_0 identifies with $C(-q_1q_3, -q_2q_3)$ as k-algebra. Furthermore, the embedding $A_0 = C_{\mathrm{tr}}(-q_1q_3, -q_2q_3) \hookrightarrow A$ is a $\mathbb{Z}_2$-graded algebra morphism.

On the other hand, if $E := e_1 \mathbin{\widehat{\otimes}} e_2 \mathbin{\widehat{\otimes}} e_3$ then $E^2 = -q_1q_2q_3\mathbf{1}$, and so $C(-q_1q_2q_3)$ embeds into A as a $\mathbb{Z}_2$-graded algebra, too. In addition, the element E commutes elementwise with A_0, thus by the universal property of the graded tensor product (see [18, Chapter 9, Section 1]) we have a well defined $\mathbb{Z}_2$-graded algebra morphism $h : C_{\mathrm{tr}}(-q_1q_3, -q_2q_3) \mathbin{\widehat{\otimes}} C(-q_1q_2q_3) \to C(q_1) \mathbin{\widehat{\otimes}} C(q_2) \mathbin{\widehat{\otimes}} C(q_3)$ defined by the following correspondences,

$$\begin{gathered} 1 \mapsto 1, \qquad I \mathbin{\widehat{\otimes}} E \mapsto IE = q_1q_3 1 \mathbin{\widehat{\otimes}} e_2 \mathbin{\widehat{\otimes}} 1, \qquad J \mathbin{\widehat{\otimes}} E \mapsto JE = -q_2q_3 e_1 \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} 1, \qquad 1 \mathbin{\widehat{\otimes}} E \mapsto E, \\ K \mathbin{\widehat{\otimes}} E \mapsto KE = q_1q_2q_3 1 \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} e_3, \qquad I \mathbin{\widehat{\otimes}} 1 \mapsto I, \qquad J \mathbin{\widehat{\otimes}} 1 \mapsto J, \qquad K \mathbin{\widehat{\otimes}} 1 \mapsto K. \end{gathered}$$

From here we easily conclude that h is a $\mathbb{Z}_2$-graded algebra isomorphism.

Obviously, $C_{\mathrm{tr}}(-q_1q_3, -q_2q_3) \mathbin{\widehat{\otimes}} C(-q_1q_2q_3) = C(-q_1q_3, -q_2q_3) \otimes C(-q_1q_2q_3)$ as algebras, where the latest is the tensor product algebra made in the category of vector spaces. The induced $\mathbb{Z}_2$-grading on it has for the component of degree zero the basis $\{1, I \otimes 1, J \otimes 1, K \otimes 1\}$, and for the component of degree one the basis $\{1 \otimes E, I \otimes E, J \otimes E, K \otimes E\}$, respectively.

We next prove that h is a coalgebra morphism as well, and this would finish the proof. Actually, using the universal property of the tensor graded coproduct we will build a $\mathbb{Z}_2$-graded coalgebra morphism from $C(q_1) \mathbin{\widehat{\otimes}} C(q_2) \mathbin{\widehat{\otimes}} C(q_3)$ to $C_{\mathrm{tr}}(-q_1q_3, -q_2q_3) \mathbin{\widehat{\otimes}} C(-q_1q_2q_3)$ that will came out to be precisely h^{-1}, the bijective inverse of h defined above.

As an algebra $A = C(q_1) \mathbin{\widehat{\otimes}} C(q_2) \mathbin{\widehat{\otimes}} C(q_3)$ identifies with $C(q_1, q_2, q_3)$ by abbreviating the tensor product in A by juxtaposition. For instance $I \equiv e_1e_3$, $J \equiv e_2e_3$, $K \equiv -q_3e_1e_2$, $E \equiv e_1e_2e_3$, and so on. Using these identifications and the coalgebra structure of $C(q_1, q_2, q_3)$ described in Proposition 4.4 we get for free the coalgebra structure of A.

Define now $p_1 : A \to A_0 = C_{\mathrm{tr}}(-q_1q_3, -q_2q_3)$ by $p_1(a) = 2a_0$, for all $a \in A$, that is p_1 is the projection of $A = A_0 \oplus A_1$ on A_0, multiplied by 2. Denote by (Δ, ε) the coalgebra structure of $A \equiv C(q_1, q_2, q_3)$ and by (Δ', ε') the one of $C_{\mathrm{tr}}(-q_1q_3, -q_2q_3)$ from Proposition 4.1. We then claim that p_1 is a $\mathbb{Z}_2$-graded coalgebra morphism. Indeed, since $A_0 = C_{\rm tr}(-q_1q_3, -q_2q_3)$ we get that p_1 preserves the gradings. We also have

$$\begin{aligned}(p_1 \otimes p_1)\Delta(1) &= \frac{1}{2}\left(1 \otimes 1 - \sum_{1 \leqslant i<j \leqslant 3}(q_iq_j)^{-1}e_ie_j \otimes e_ie_j\right) \\ &= \frac{1}{2}\big(1 \otimes 1 - (-q_1q_3)^{-1}(-q_2q_3)^{-1}K \otimes K + (-q_1q_3)^{-1}I \otimes I + (-q_2q_3)^{-1}J \otimes J\big) \\ &= 2\Delta'(1) = \Delta'\big(p_1(1)\big).\end{aligned}$$

Likewise, $(p_1 \otimes p_1)\Delta(x) = \Delta'(p_1(x))$, for all $x \in \{I, J, K, E, IE, JE, KE\}$. This together with $\varepsilon' p_1(1) = 2\varepsilon'(1) = 4 = \varepsilon(1)$ and $\varepsilon' p_1(x) = 0 = \varepsilon(x)$, for all $x \in \{I, J, K, E, IE, JE, KE\}$, implies that p_1 is a $\mathbb{Z}_2$-graded coalgebra morphism.

We next define $p_2 : A \to C(-q_1q_2q_3)$ as being the linear projection of A on the subspace $C(-q_1q_2q_3)$, multiplied by 4. In other words p_2 maps 1 to 4, E to $4E$ and the remaining elements of the canonical basis of A to zero. We have that p_2 is a $\mathbb{Z}_2$-graded coalgebra morphism as well. To this end observe first that 1 has the same degree, zero, either if it viewed as an element of A or $C(-q_1q_2q_3)$, and E has the same degree, one, either if it considered an element of A or $C(-q_1q_2q_3)$. Hence p_2 is a $\mathbb{Z}_2$-graded morphism. It respects the comultiplications Δ and Δ'' of A and respectively of $C(-q_1q_2q_3)$ because, for instance,

$$(p_2 \otimes p_2)\Delta(1) = 2\big(1 \otimes 1 + (-q_1q_2q_3)^{-1}E \otimes E\big) = 4\Delta''(1) = \Delta''\big(p_2(1)\big);$$

the remaining details are left to the reader. Now, if ε'' is the counit of $C(-q_1q_2q_3)$ then $\varepsilon'' p_2(1) = 4\varepsilon''(1) = 8 = \varepsilon(1)$ and $\varepsilon'' p_2(x) = 0 = \varepsilon(x)$, for $x \in \{e_t, e_te_{t+1} \mid 1 \leqslant t \leqslant 3\}$ or $x = e_1e_2e_3$, so p_2 is a $\mathbb{Z}_2$-graded coalgebra morphism. It remains to verify that p_1 and p_2 satisfy the equality $(p_2 \mathbin{\widehat{\otimes}} p_1)\Delta = c_{A,A}(p_1 \mathbin{\widehat{\otimes}} p_2)\Delta$, where c is the braiding of $\mathrm{Vect}^{\mathbb{Z}_2}_{-1}$. A direct computation leads us to the following equalities:

$$\begin{gathered}(p_2 \otimes p_1)\Delta(1) = c_{A,A}(p_1 \otimes p_2)\Delta(1) = 1 \otimes 1, \\ (p_2 \otimes p_1)\Delta(e_t) = c_{A,A}(p_1 \otimes p_2)\Delta(e_t) = -(q_{t+1}q_{t+2})^{-1}E \otimes e_{t+1}e_{t+2}, \\ (p_2 \otimes p_1)\Delta(e_te_{t+1}) = c_{A,A}(p_1 \otimes p_2)\Delta(e_te_{t+1}) = 1 \otimes e_te_{t+1},\end{gathered}$$

and

$$(p_2 \otimes p_1)\Delta(e_1e_2e_3) = c_{A,A}(p_1 \otimes p_2)\Delta(e_1e_2e_3) = E \otimes 1,$$

for all $1 \leqslant t \leqslant 3$, where we have identified A with $C(q_1, q_2, q_3)$ in the way explained above. Thus we are in position to apply the universal property of the tensor graded coproduct. It guarantees that $h^{-1} : A \to C_{\rm tr}(-q_1q_3, -q_2q_3) \mathbin{\widehat{\otimes}} C(-q_1q_2q_3)$ defined by $h^{-1} = (p_1 \mathbin{\widehat{\otimes}} p_2)\Delta$ is a $\mathbb{Z}_2$-graded coalgebra morphism. Since

$$\begin{gathered}h^{-1}(1) = 1 \mathbin{\widehat{\otimes}} 1, \qquad h^{-1}(IE) = q_1q_3h^{-1}(e_2) = -e_3e_1 \mathbin{\widehat{\otimes}} E = e_1e_3 \mathbin{\widehat{\otimes}} E = I \mathbin{\widehat{\otimes}} E, \\ h^{-1}(JE) = -q_2q_3h^{-1}(e_1) = e_2e_3 \mathbin{\widehat{\otimes}} E = J \mathbin{\widehat{\otimes}} E, \\ h^{-1}(KE) = q_1q_2q_3h^{-1}(e_3) = -q_3e_1e_2 \mathbin{\widehat{\otimes}} E = K \mathbin{\widehat{\otimes}} E, \\ h^{-1}(E) = h^{-1}(e_1e_2e_3) = 1 \mathbin{\widehat{\otimes}} E, \qquad h^{-1}(I) = h^{-1}(e_1e_3) = e_1e_3 \mathbin{\widehat{\otimes}} 1 = I \mathbin{\widehat{\otimes}} 1, \\ h^{-1}(J) = h^{-1}(e_2e_3) = e_2e_3 \mathbin{\widehat{\otimes}} 1 = J \mathbin{\widehat{\otimes}} 1\end{gathered}$$

and

$$h^{-1}(K) = -q_3 h^{-1}(e_1 e_2) = -q_3 e_1 e_2 \mathbin{\widehat{\otimes}} 1 = K \mathbin{\widehat{\otimes}} 1,$$

it follows that h^{-1} is the bijective inverse of h. Thus our proof is complete. □

As a conclusion, using the techniques presented above, tedious but straightforward computations give us the coalgebra structure of any Clifford algebra of the form $C(q_1, \dots, q_n)$. In the next section we will see that even if the two structures are associative, and respectively coassociative in the usual sense all together do not define a ordinary Hopf algebra structure on $C(q_1, \dots, q_n)$. As in [5] they will determine a weak braided Hopf algebra structure on a such Clifford algebra.

5. The weak braided Hopf algebra structure of some Clifford algebras

For any non-zero $n \in \mathbb{N}$ and $q_1, \dots, q_n \in k^*$ we have seen that $C(q_1, \dots, q_n)$ can be viewed either as an algebra or coalgebra as a deformation of $k[(\mathbb{Z}_2)^n]$ by a 2-cocycle F on $(\mathbb{Z}_2)^n$. Keeping the presentation of [5], for G an abelian group and F a 2-cochain on it by $k_F^F[G]$ we denote the k-linear space $k[G]$ equipped with the algebra structure of $k_F[G]$ and coalgebra structure of $k^F[G]$. It has been proved in [5, Proposition 4.7] that $k_F^F[G]$ is a commutative and cocommutative weak braided Hopf algebra in the symmetric monoidal category $\mathrm{Vect}^G_{\Delta_2(F^{-1}), \mathcal{R}_{F^{-1}}}$ with the antipode defined by the identity morphism of $k[G]$. We invite the reader to consult [3–5] for the definition of a weak braided Hopf algebra within in a symmetric monoidal category. For further use record only that the comultiplication Δ_F of $k_F^F[G]$ is multiplicative (cf. [5, Corollary 4.2]), in the sense that

$$\Delta_F(x \bullet y) = \Delta_F(x) \bullet \Delta_F(y), \quad \forall x, y \in G,$$

where the first $\bullet$ is the multiplication of $k_F[G]$ and the second one is the multiplication of the tensor product algebra made inside of the symmetric monoidal category $\mathrm{Vect}^G_{\Delta_2(F^{-1}), \mathcal{R}_{F^{-1}}}$. More precisely, according to [5, Lemma 4.1] the second $\bullet$ is given by

$$(x \otimes y) \bullet (z \otimes t) = \frac{F(z,t)F(xy,zt)F(x,y)}{F(xz,yt)} xz \otimes yt, \tag{5.1}$$

for all $x, y, z, t \in G$.

As a consequence of all these facts we obtain the following result.

Theorem 5.1. *All the algebras or, equivalently, all the coalgebras obtained through the Clifford process for algebras, respectively for coalgebras, from the input data (k, Id_k) are commutative and cocommutative weak braided Hopf algebras in a suitable category of graded spaces endowed with a symmetric monoidal structure given by a certain coboundary abelian 3-cocycle defined by a normalized 2-cocycle on G.*

Proof. Similar to the one given in the Cayley–Dickson case, see [5, Theorem 4.8]. If the two Clifford processes have the same input data (k, Id_k) then they produce the same k-vector spaces, and all these vector spaces are group algebras corresponding to a finite direct sum of a family of copies of $\mathbb{Z}_2$. So they are of the form $k_F^F[G]$, for a certain abelian group G and a twist $F \in (G \times G)^*$ on G, which we have seen that are commutative and cocommutative weak braided Hopf algebras.

So all we have to prove is the fact that in the inductive Clifford process the comultiplication of $\overline{A}$ behaves well with respect to the relations in (1.1) that define the multiplication of $\overline{A}$. Therefore, we shall prove that

$$F(x,y)\Delta_{\overline{F}}\big((xy,\bar{1})\big)=\Delta_{\overline{F}}\big((x,\bar{0})\big)\bullet\Delta_{\overline{F}}\big((y,\bar{1})\big),$$
$$s(y)F(x,y)\Delta_{\overline{F}}\big((xy,\bar{1})\big)=\Delta_{\overline{F}}\big((x,\bar{1})\big)\bullet\Delta_{\overline{F}}\big((y,\bar{0})\big),$$

and

$$qs(y)F(x,y)\Delta_{\overline{F}}\big((xy,\bar{0})\big)=\Delta_{\overline{F}}\big((x,\bar{1})\big)\bullet\Delta_{\overline{F}}\big((y,\bar{1})\big),$$

for all $x, y\in G$. The first equality follows because of

$$\begin{aligned}
&\Delta_{\overline{F}}\big((x,\bar{0})\big)\bullet\Delta_{\overline{F}}\big((y,\bar{1})\big)\\
&\quad=\frac{1}{4|G|^2}\sum_{u,v\in G}\big(\overline{F}\big((u,\bar{0}),\big(u^{-1}x,\bar{0}\big)\big)^{-1}(u,\bar{0})\otimes\big(u^{-1}x,\bar{0}\big)\\
&\qquad+\overline{F}\big((u,\bar{1}),\big(u^{-1}x,\bar{1}\big)\big)^{-1}(u,\bar{1})\otimes\big(u^{-1}x,\bar{1}\big)\big)\bullet\big(\overline{F}\big((v,\bar{0}),\big(v^{-1}y,\bar{1}\big)\big)^{-1}(v,\bar{0})\otimes\big(v^{-1}y,\bar{1}\big)\\
&\qquad+\overline{F}\big((v,\bar{1}),\big(v^{-1}y,\bar{0}\big)\big)^{-1}(v,\bar{1})\otimes\big(v^{-1}y,\bar{0}\big)\big)\\
&\quad=\frac{1}{4|G|^2}\sum_{u,v\in G}F\big(u,u^{-1}x\big)^{-1}F\big(v,v^{-1}y\big)^{-1}\big(\big((u,\bar{0})\otimes\big(u^{-1}x,\bar{0}\big)\big)\bullet\big((v,\bar{0})\otimes\big(v^{-1}y,\bar{1}\big)\big)\\
&\qquad+s\big(v^{-1}y\big)^{-1}\big((u,\bar{0})\otimes\big(u^{-1}x,\bar{0}\big)\big)\bullet\big((v,\bar{1})\otimes\big(v^{-1}y,\bar{0}\big)\big)\\
&\qquad+q^{-1}s\big(u^{-1}x\big)^{-1}\big((u,\bar{1})\otimes\big(u^{-1}x,\bar{1}\big)\big)\bullet\big((v,\bar{0})\otimes\big(v^{-1}y,\bar{1}\big)\big)\\
&\qquad+q^{-1}s\big(u^{-1}x\big)^{-1}s\big(v^{-1}y\big)^{-1}\big((u,\bar{1})\otimes\big(u^{-1}x,\bar{1}\big)\big)\bullet\big((v,\bar{1})\otimes\big(v^{-1}y,\bar{0}\big)\big)\big)\\
&\quad=\frac{F(x,y)}{2|G|^2}\sum_{u,v\in G}\big(F\big(uv,(uv)^{-1}xy\big)^{-1}(uv,\bar{0})\otimes\big((uv)^{-1}xy,\bar{1}\big)\\
&\qquad+s\big((uv)^{-1}xy\big)^{-1}F\big(uv,(uv)^{-1}xy\big)^{-1}(uv,\bar{1})\otimes\big((uv)^{-1}xy,\bar{0}\big)\big)\\
&\quad=\frac{F(x,y)}{2|G|^2}\sum_{\theta\in G}\sum_{uv=\theta}\big(F\big(\theta,\theta^{-1}xy\big)^{-1}(\theta,\bar{0})\otimes\big(\theta^{-1}xy,\bar{1}\big)\\
&\qquad+s\big(\theta^{-1}xy\big)^{-1}F\big(\theta,\theta^{-1}xy\big)^{-1}(\theta,\bar{1})\otimes\big(\theta^{-1}xy,\bar{0}\big)\big)\\
&\quad=\frac{F(x,y)}{2|G|}\sum_{\theta\in G}\big(\overline{F}\big((\theta,\bar{0}),\big(\theta^{-1}xy,\bar{1}\big)\big)^{-1}(\theta,\bar{0})\otimes\big(\theta^{-1}xy,\bar{1}\big)\\
&\qquad+\overline{F}\big((\theta,\bar{1}),\big(\theta^{-1}xy,\bar{0}\big)\big)^{-1}(\theta,\bar{1})\otimes\big(\theta^{-1}xy,\bar{0}\big)\big)=F(x,y)\Delta_{\overline{F}}\big((xy,\bar{1})\big),
\end{aligned}$$

as needed. In the computation above we used the definition of $\Delta_{\overline{F}}$ from Section 3, the definition of $\overline{F}$ from Proposition 2.2, the properties of s from Remark 3.6 and the multiplication of the tensor product algebra $k_{\overline{F}}[\overline{G}]\otimes k_{\overline{F}}[\overline{G}]$ described in (5.1).

The other two relations can be verified in a similar manner, we leave the details to the reader. □

As we next see the $\mathbb{Z}_2$-graded algebra and coalgebra structures of $C(q)$ and $C(q_1,q_2)$ determine completely the weak braided Hopf algebra structure of any $C(q_1,\dots,q_n)$. More precisely, we have the following structure theorem.

Theorem 5.2. *Let n be a non-zero natural number and $q_1,\dots,q_n$ non-zero scalars of k. Then the following assertions hold*:

(i) *If $n = 2m$ for some $m \in \mathbb{N}$ then*

$$C(q_1, \ldots, q_{2m}) \cong \widehat{\bigotimes_{k=\overline{1,m-1}}} C_{\mathrm{tr}}\big((-1)^k q_1 \cdots q_{2k-1} q_{2k+1}, -q_{2k} q_{2k+1}\big) \widehat{\otimes} \mathbb{H}_m$$

with $\mathbb{H}_m = C((-1)^{m-1} q_1 \cdots q_{2m-1}, q_{2m})$, the isomorphism being between $\mathbb{Z}_2$-graded algebras and coalgebras. That is, $C(q_1, \ldots, q_{2m})$ is the $\mathbb{Z}_2$-graded algebra and coalgebra tensor product of $m-1$ trivially $\mathbb{Z}_2$-graded algebras and coalgebras of generalized quaternions with one last factor $\mathbb{H}_m$, this time a non-trivially $\mathbb{Z}_2$-graded algebra and coalgebra of generalized quaternions. Here we assumed $m \geqslant 2$, for otherwise we have $C(q_1, q_2) = \mathbb{H}_1$, a tautology.

(ii) *If $n = 2m + 1$ for a certain $m \in \mathbb{N}$ then*

$$C(q_1, \ldots, q_{2m+1}) \cong \widehat{\bigotimes_{k=\overline{1,m}}} C_{\mathrm{tr}}\big((-1)^k q_1 \cdots q_{2k-1} q_{2k+1}, -q_{2k} q_{2k+1}\big) \widehat{\otimes} C(\delta_m)$$

with $\delta_m = (-1)^m q_1 \cdots q_{2m+1}$, again an isomorphism between $\mathbb{Z}_2$-graded algebras and coalgebras. So in this case $C(q_1, \ldots, q_{2m+1})$ is the $\mathbb{Z}_2$-graded algebra and coalgebra tensor product of $m-1$ trivially $\mathbb{Z}_2$-graded algebras and coalgebras of generalized quaternions with one last factor $C(\delta_m)$.

Proof. We proceed by mathematical induction on m. Using the results in Proposition 4.5, [2, Corollary 2.6] and Corollary 3.11 we get that

$$C(q_1, q_2, q_3) \cong C_{\mathrm{tr}}(-q_1 q_3, -q_2 q_3) \widehat{\otimes} C(-q_1 q_2 q_3)$$

and

$$\begin{aligned} C(q_1, q_2, q_3, q_4) &\cong C(q_1, q_2, q_3) \widehat{\otimes} C(q_4) \cong C_{\mathrm{tr}}(-q_1 q_3, -q_2 q_3) \widehat{\otimes} C(-q_1 q_2 q_3) \widehat{\otimes} C(q_4) \\ &\cong C_{\mathrm{tr}}(-q_1 q_3, -q_2 q_3) \widehat{\otimes} C(-q_1 q_2 q_3, q_4), \end{aligned}$$

as $\mathbb{Z}_2$-graded algebras and coalgebras, as required. For the induction step assume first that n is even of the form $n = 2m + 2$. Then

$$\begin{aligned} C(q_1, \ldots, q_{2m+2}) &\cong C(q_1, \ldots, q_{2m}) \widehat{\otimes} C(q_{2m+1}) \widehat{\otimes} C(q_{2m+2}) \\ &\cong \widehat{\bigotimes_{k=\overline{1,m-1}}} C_{\mathrm{tr}}\big((-1)^k q_1 \cdots q_{2k-1} q_{2k+1}, -q_{2k} q_{2k+1}\big) \widehat{\otimes} \mathbb{H}_m \widehat{\otimes} C(q_{2m+1}) \widehat{\otimes} C(q_{2m+2}), \end{aligned}$$

and since

$$\begin{aligned} &\mathbb{H}_m \widehat{\otimes} C(q_{2m+1}) \widehat{\otimes} C(q_{2m+2}) \\ &\quad \cong C\big((-1)^{m-1} q_1 \cdots q_{2m-1}\big) \widehat{\otimes} C(q_{2m}) \widehat{\otimes} C(q_{2m+1}) \widehat{\otimes} C(q_{2m+2}) \\ &\quad \cong C_{\mathrm{tr}}\big((-1)^m q_1 \cdots q_{2m-1} q_{2m+1}, -q_{2m} q_{2m+1}\big) \widehat{\otimes} C\big((-1)^m q_1 \cdots q_{2m+1}\big) \widehat{\otimes} C(q_{2m+2}) \\ &\quad \cong C_{\mathrm{tr}}\big((-1)^m q_1 \cdots q_{2m-1} q_{2m+1}, -q_{2m} q_{2m+1}\big) \widehat{\otimes} \mathbb{H}_{m+1}, \end{aligned}$$

it follows that

$$C(q_1,\ldots,q_{2m+2})\cong \widehat{\bigotimes_{k=\overline{1,m}}} C_{\rm tr}\big((-1)^kq_1\cdots q_{2k-1}q_{2k+1},-q_{2k}q_{2k+1}\big)\,\widehat{\otimes}\,\mathbb{H}_{m+1},$$

as $\mathbb{Z}_2$-graded algebras and coalgebras, as desired. In case that n is odd and $n=2m+3$ then by similar arguments to the ones above we obtain that

$$\begin{aligned}
C(q_1,\ldots,q_{2m+3})&\cong \widehat{\bigotimes_{k=\overline{1,m}}} C_{\rm tr}\big((-1)^kq_1\cdots q_{2k-1}q_{2k+1},-q_{2k}q_{2k+1}\big)\\
&\quad\widehat{\otimes}\,C\big((-1)^mq_1\cdots q_{2m+1}\big)\,\widehat{\otimes}\,C(q_{2m+2})\,\widehat{\otimes}\,C(q_{2m+3})\\
&\cong \widehat{\bigotimes_{k=\overline{1,m}}} C_{\rm tr}\big((-1)^kq_1\cdots q_{2k-1}q_{2k+1},-q_{2k}q_{2k+1}\big)\\
&\quad\widehat{\otimes}\,C_{\rm tr}\big((-1)^{m+1}q_1\cdots q_{2m+1}q_{2m+3},-q_{2m+2}q_{2m+3}\big)\,\widehat{\otimes}\,C\big((-1)^{m+1}q_1\cdots q_{2m+3}\big)\\
&= \widehat{\bigotimes_{k=\overline{1,m+1}}} C_{\rm tr}\big((-1)^kq_1\cdots q_{2k-1}q_{2k+1},-q_{2k}q_{2k+1}\big)\,\widehat{\otimes}\,C(\delta_{m+1}),
\end{aligned}$$

again as $\mathbb{Z}_2$-graded algebras and coalgebras. So we are done. □

Thus, corroborated with the results in [5], so far we have obtained two classes of weak braided Hopf algebras. The first one is the class of Cayley–Dickson algebras that are not, in general, algebras and coalgebras in the usual sense (i.e., in ${}_k\mathcal{M}$) but are algebras and coalgebras in categories of graded vector spaces, and all together weak braided Hopf algebras. The second class is given by the Clifford algebras $C(q_1,\ldots,q_n)$ that are algebras and coalgebras in the usual sense and weak Hopf algebras in the braided sense. We next show that both classes contain only selfdual objects (i.e., any such object is isomorphic to its left and right categorical duals, as weak braided Hopf algebra). In fact, more generally, we will prove that $k_F^F[G]$ is a selfdual weak braided Hopf algebra in a categorical sense that we shall explain below.

Let G be a group and $\phi\in H^3(G,k^*)$ a normalized 3-cocycle on G with coefficients in k^*. We have seen in Section 2.1 that Vect_ϕ^G is isomorphic to $\mathcal{M}^{k_\phi[G]}$ as monoidal category. Since the category of finite-dimensional right comodules over a dual quasi-Hopf algebra with bijective antipode is rigid we get that so is vect_ϕ^G, the category of finite-dimensional G-graded spaces equipped with the monoidal structure given by ϕ. (We invite the reader to consult [12,16] for the definition of a rigid monoidal category.) Concretely, if V is a finite-dimensional G-graded vector space with a basis $\{{}_iv\}_i$ and corresponding dual basis $\{{}^iv\}_i$ then the left dual of V is $V^*=\mathrm{Hom}_k(V,k)$, the linear dual of V, having the homogeneous component of degree $g\in G$ defined by

$$V_g^*=\big\{\big(v\mapsto v^*\big(v^{g^{-1}}\big)\big)\ \big|\ v^*\in V^*\big\}=\big\{v^*\in V^*\ \big|\ v^*_{|V_\sigma}=0,\ \forall\sigma\neq g^{-1}\big\},$$

where by $v^{g^{-1}}$ we have denoted the component of degree g^{-1} of $v\in V$. The evaluation and coevaluation maps are given by

$$\mathrm{ev}_V:V^*\otimes V\to k,\qquad \mathrm{ev}_V\big(v^*\otimes v\big)=v^*(v),\quad \forall v^*\in V^*,\ v\in V,$$

and

$$\mathrm{coev}_V:k\to V\otimes V^*,\qquad \mathrm{coev}_V(1)=\sum_{i;\,g\in G}\phi\big(g,g^{-1},g\big)^{-1}({}_iv)^g\otimes{}^iv,$$

respectively. The right dual ${}^*V$ of V coincides, as a G-graded vector space, with V^* described above, but the evaluation and coevaluation maps are now defined by

$$\mathrm{ev}'_V : V \otimes {}^*V \to k, \qquad \mathrm{ev}'_V\big(v \otimes {}^*v\big) = {}^*v(v), \quad \forall {}^*v \in {}^*V,\ v \in V,$$

and

$$\mathrm{coev}'_V : k \to {}^*V \otimes V, \qquad \mathrm{coev}'_V(1) = \sum_{i;g\in G} \phi\big(g^{-1}, g, g^{-1}\big)^{-1}\, {}^i v \otimes ({}_i v)^g,$$

respectively. This follows because $S^{-1}(g) = S(g) = g^{-1}$, for all $g \in G$, and because of the dual quasi-Hopf algebra structure of $k_\phi[G]$.

If A is a finite-dimensional algebra in $\mathcal{M}^{k_\phi[G]}$ then A^*, the left categorical dual of A, has a coalgebra structure in $\mathcal{M}^{k_\phi[G]}$ that will be called the co-opposite dual coalgebra of the algebra A. It will be denoted by $A^\star$ and has the comultiplication and counit given by

$$\underline{\Delta}_{A^\star} : A^* \xrightarrow{\underline{m}_A^*} (A \otimes A)^* \xrightarrow{\lambda_{A,A}^{-1}} A^* \otimes A^*$$

and

$$\underline{\varepsilon}_{A^\star} : A^* \xrightarrow{r_{A^*}^{-1}} A^* \otimes \underline{1} \xrightarrow{\mathrm{Id}_{A^*} \otimes \underline{\eta}_A} A^* \otimes A \xrightarrow{\mathrm{ev}_A} \underline{1},$$

where $\underline{m}_A^*$ is the transpose morphism of the multiplication $\underline{m}_A$ of A, $\underline{\eta}_A$ is the unit morphism of A and, in general, for any finite-dimensional objects $X, Y \in \mathcal{M}^{k_\phi[G]}$, $\lambda_{X,Y} : (X \otimes Y)^* \to Y^* \otimes X^*$ denotes the isomorphism defined by

$$\begin{aligned}\lambda_{X,Y} = {} & \big((\mathrm{ev}_{X\otimes Y} \otimes \mathrm{Id}_{Y^*}) \otimes \mathrm{Id}_{X^*}\big) \circ \big(a^{-1}_{(X\otimes Y)^*, X\otimes Y, Y^*} \otimes \mathrm{Id}_{X^*}\big) \\ & \circ \big(\big(\mathrm{Id}_{(X\otimes Y)^*} \otimes a^{-1}_{X,Y,Y^*}\big) \otimes \mathrm{Id}_{X^*}\big) \circ \big(\big(\mathrm{Id}_{(X\otimes Y)^*} \otimes (\mathrm{Id}_X \otimes \mathrm{coev}_Y)\big) \otimes \mathrm{Id}_{X^*}\big) \\ & \circ a^{-1}_{(X\otimes Y)^*, X, X^*} \circ (\mathrm{Id}_{(X\otimes Y)^*} \otimes \mathrm{coev}_X).\end{aligned}$$

When $\mathcal{M}^{k_\phi[G]}$ has a braided monoidal structure given by the braiding c then A^* with the comultiplication $\Delta_{A^*} = c_{A^*,A^*} \circ \Delta_{A^\star}$ and the counit of $A^\star$ is a coalgebra in $\mathcal{M}^{k_\phi[G]}$, too. It is simply denoted by A^* and called the dual coalgebra of the algebra A.

The isomorphism $\lambda_{X,Y}$ was computed explicitly in the proof of [6, Proposition 4.2] (there it was denoted by $\phi^{*-1}_{N,M}$) in the situation when the objects X and Y are finite-dimensional left modules over a quasi-Hopf algebra. If we dualize it we then get that

$$\lambda_{X,Y}(\mu) = f^{-1}({}_ix_{(1)}, {}_jy_{(1)})\mu({}_ix_{(0)} \otimes {}_jy_{(0)})^j y \otimes {}^i x,$$

for any finite-dimensional right comodules X and Y over a dual quasi-Hopf algebra H. Here $\mu \in (X \otimes Y)^*$, $\{{}_ix, {}^ix\}_i$ and $\{{}_jy, {}^jy\}_j$ are dual bases in X and X^*, respectively in Y and Y^*, $M \ni m \mapsto m_{(0)} \otimes m_{(1)} \in M \otimes H$ is the sigma notation for the coaction of H on a vector space M, and f^{-1} is the inverse of the Drinfeld twist that is defined as follows. Suppose that φ is the reassociator of H (the convolution invertible element in $(H \otimes H \otimes H)^*$ that controls, up to conjugation, the associativity of the multiplication of H) and α, β are the distinguished elements of H^* that come together with the antipode S of H. If we define $\delta \in (H \otimes H)^*$ by

$$\delta(h, g) = \varphi\big(h_1g_1, S(g_5), S(h_4)\big)\beta(h_3)\varphi^{-1}\big(h_2, g_2, S(g_4)\big)\beta(g_3),$$

for all $h, g \in H$, then f^{-1} is given, for all $h, g \in H$, by

$$f^{-1}(h, g) = \varphi^{-1}\big(S(h_1g_1), h_3g_3, S(g_5)S(h_5)\big)\alpha(h_2g_2)\delta(h_4, g_4).$$

Proposition 5.3. *Let G be a finite abelian group and $F \in (G \times G)^*$ a 2-cochain on G. If $k_F[G]$ is the G-graded quasialgebra with associator $\Delta_2(F^{-1})$ built on $k[G]$ then $k_F[G]^\star = k_F[G]^* \cong k^F[G]$, as monoidal coalgebras, where $k^F[G]$ is the G-graded quasicoalgebra with associator $\Delta_2(F^{-1})$ constructed from $k[G]$ and F.*

Proof. Recall that $k_F[G]$, respectively $k^F[G]$, is a monoidal algebra, respectively coalgebra, within $\mathrm{Vect}^G_{\Delta_2(F^{-1}),\mathcal{R}_{F^{-1}}}$. Recall also that $\mathcal{R}_{F^{-1}}(x, y) = \frac{F(x,y)}{F(y,x)}$, for all $x, y \in G$.

If we take $H = k_\phi[G]$ with $\phi := \Delta_2(F^{-1})$ we then have

$$\phi(x, y, z) = \frac{F(xy, z)F(x, y)}{F(y, z)F(x, yz)} \quad \text{and} \quad \beta(x) = \phi\big(x, x^{-1}, x\big)^{-1} = \frac{F(x^{-1}, x)}{F(x, x^{-1})},$$

for all $x, y, z \in G$. Thus, by the above definition of δ we get that

$$\begin{aligned}\delta(x, y) &= \phi\big(xy, y^{-1}, x^{-1}\big)\beta(x)\phi^{-1}\big(x, y, y^{-1}\big)\beta(y)\\ &= \frac{F(x, x^{-1})F(xy, y^{-1})}{F(y^{-1}, x^{-1})F(xy, y^{-1}x^{-1})} \cdot \frac{F(x^{-1}, x)}{F(x, x^{-1})} \cdot \frac{F(y, y^{-1})}{F(xy, y^{-1})F(x, y)} \cdot \frac{F(y^{-1}, y)}{F(y, y^{-1})}\\ &= \frac{F(x^{-1}, x)F(y^{-1}, y)}{F(y^{-1}, x^{-1})F(xy, y^{-1}x^{-1})F(x, y)},\end{aligned}$$

and therefore

$$\begin{aligned}f^{-1}(x, y) &= \phi^{-1}\big((xy)^{-1}, xy, y^{-1}x^{-1}\big)\delta(x, y)\\ &= \frac{F(xy, y^{-1}x^{-1})}{F(y^{-1}x^{-1}, xy)} \cdot \frac{F(x^{-1}, x)F(y^{-1}, y)}{F(y^{-1}, x^{-1})F(xy, y^{-1}x^{-1})F(x, y)}\\ &= \frac{F(x^{-1}, x)F(y^{-1}, y)}{F(y^{-1}x^{-1}, xy)F(y^{-1}, x^{-1})F(x, y)},\end{aligned}$$

for all $x, y \in G$. If $\{P_\theta \mid \theta \in G\}$ is the basis in $k_F[G]^*$ dual to the basis $\{\theta \mid \theta \in G\}$ of $k_F[G]$ then the comultiplication of the co-opposite dual coalgebra structure of the algebra $k_F[G]$ is given, for all $\theta \in G$, by

$$\begin{aligned}\underline{\Delta}_{k_F[G]^\star}(P_\theta) &= \lambda_{k_F[G],k_F[G]} \circ \underline{m}^*_{k_F[G]}(P_\theta) = \lambda_{k_F[G],k_F[G]}(P_\theta \circ \underline{m}_{k_F[G]})\\ &= \sum_{\sigma,\tau\in G} f^{-1}(\sigma, \tau)P_\theta(\sigma \bullet \tau)P_\tau \otimes P_\sigma\\ &= \sum_{\sigma,\tau\in G} F(\sigma, \tau)\frac{F(\sigma^{-1}, \sigma)F(\tau^{-1}, \tau)}{F(\tau^{-1}\sigma^{-1}, \sigma\tau)F(\tau^{-1}, \sigma^{-1})F(\sigma, \tau)}P_\theta(\sigma\tau)P_\tau \otimes P_\sigma\\ &= F\big(\theta^{-1}, \theta\big)^{-1} \sum_{\sigma\tau=\theta} \frac{F(\sigma^{-1}, \sigma)F(\tau^{-1}, \tau)}{F(\tau^{-1}, \sigma^{-1})}P_\tau \otimes P_\sigma.\end{aligned}$$

Its counit is $\underline{\varepsilon}_{k_F[G]^\star}(P_\theta) = \mathrm{ev}_{k_F[G]}(P_\theta \otimes e) = P_\theta(e) = \delta_{\theta,e}$, where e is the neutral element of G.

We claim now that $\Xi : k^F[G] \to k_F[G]^\star$ defined by $\Xi(\theta) = |G|F(\theta, \theta^{-1})P_{\theta^{-1}}$, for all $\theta \in G$ and extended by linearity, is an isomorphism of G-graded quasicoalgebras with associator $\Delta_2(F^{-1})$. Toward this end we compute

$$
\begin{aligned}
(\Xi \otimes \Xi)\Delta_F(\theta) &= \frac{1}{|G|}\sum_{\sigma\tau=\theta}(\Xi \otimes \Xi)\big(F(\sigma,\tau)^{-1}\sigma \otimes \tau\big) \\
&= \frac{|G|^2}{|G|}\sum_{\sigma\tau=\theta}\frac{F(\sigma,\sigma^{-1})F(\tau,\tau^{-1})}{F(\sigma,\tau)}P_{\sigma^{-1}} \otimes P_{\tau^{-1}} \\
&= |G|\sum_{\tau^{-1}\sigma^{-1}=\theta^{-1}}\frac{F(\tau,\tau^{-1})F(\sigma,\sigma^{-1})}{F(\sigma,\tau)}P_{\sigma^{-1}} \otimes P_{\tau^{-1}} \\
&= |G|F\big(\theta,\theta^{-1}\big)\underline{\Delta}_{k_F[G]^\star}(P_{\theta^{-1}}) = \underline{\Delta}_{k_F[G]^\star}\big(\Xi(P_\theta)\big),
\end{aligned}
$$

and $\underline{\varepsilon}_{k_F[G]^\star} \circ \Xi(\theta) = |G|F(\theta,\theta^{-1})\underline{\varepsilon}_{k_F[G]^\star}(P_{\theta^{-1}}) = |G|\delta_{\theta,e} = \varepsilon_F(\theta)$, for all $\theta \in G$, as needed.

Since $P_{\theta^{-1}}$ has degree θ in $k_F[G]^*$ it follows that Ξ is a morphism in $\mathrm{Vect}^G_{\Delta_2(F^{-1})}$. Moreover, one can easily see that it is an isomorphism with inverse given by $\Xi^{-1}(P_\theta) = \frac{1}{|G|F(\theta^{-1},\theta)}\theta^{-1}$, for all $\theta \in G$.

Finally, the category $\mathrm{Vect}^G_{\Delta_2(F^{-1}),\mathcal{R}_{F^{-1}}}$ is symmetric monoidal, so $k_F[G]^* = k_F[G]^\star$ as coalgebras in $\mathrm{Vect}^G_{\Delta_2(F^{-1})}$ since G is abelian and, for all $\theta \in G$,

$$
\begin{aligned}
\Delta_{k_F[G]^*}(P_\theta) &= c_{k_F[G]^*,k_F[G]^*} \circ \underline{\Delta}_{k_F[G]^\star}(P_\theta) \\
&= \frac{1}{F(\theta^{-1},\theta)}\sum_{\sigma\tau=\theta}\frac{F(\sigma^{-1},\sigma)F(\tau^{-1},\tau)}{F(\tau^{-1},\sigma^{-1})}c_{k_F[G]^*,k_F[G]^*}(P_\tau \otimes P_\sigma) \\
&= \frac{1}{F(\theta^{-1},\theta)}\sum_{\sigma\tau=\theta}\frac{F(\sigma^{-1},\sigma)F(\tau^{-1},\tau)}{F(\tau^{-1},\sigma^{-1})}\cdot\frac{F(\tau^{-1},\sigma^{-1})}{F(\sigma^{-1},\tau^{-1})}P_\sigma \otimes P_\tau \\
&= \frac{1}{F(\theta^{-1},\theta)}\sum_{\tau\sigma=\theta}\frac{F(\sigma^{-1},\sigma)F(\tau^{-1},\tau)}{F(\sigma^{-1},\tau^{-1})}P_\sigma \otimes P_\tau = \underline{\Delta}_{k_F[G]^\star}(P_\theta).
\end{aligned}
$$

Notice that in the above computation we used again the fact that, in general, P_x has degree x^{-1} in $k_F[G]^*$, for any $x \in G$. So our proof is complete. □

We focus now on the algebra version of the above result. If C is a coalgebra in $\mathcal{M}^{k_\phi[G]}$ then C^* admits an algebra structure in $\mathcal{M}^{k_\phi[G]}$. The multiplication is given by

$$
C^* \otimes C^* \xrightarrow{\lambda^{-1}_{C,C}} (C \otimes C)^* \xrightarrow{\underline{\Delta}^*} C^*,
$$

where $\underline{\Delta}^*$ is the transpose of the comultiplication $\underline{\Delta}$ of C and $\lambda^{-1}_{X,Y} : Y^* \otimes X^* \to (X \otimes Y)^*$ is the bijective inverse of $\lambda_{X,Y}$ used above. In general, it is defined by

$$
\begin{aligned}
\lambda^{-1}_{X,Y} &= (\mathrm{ev}_Y \otimes \mathrm{Id}_{(X\otimes Y)^*}) \circ \big(\big(\mathrm{Id}_{Y^*} \otimes (\mathrm{ev}_X \otimes \mathrm{Id}_N)\big) \otimes \mathrm{Id}_{(X\otimes Y)^*}\big) \\
&\quad \circ \big(\mathrm{Id}_{Y^*} \otimes a^{-1}_{X^*,X,Y} \otimes \mathrm{Id}_{(X\otimes Y)^*}\big) \circ (a_{Y^*,X^*,X\otimes Y} \otimes \mathrm{Id}_{(X\otimes Y)^*}) \\
&\quad \circ a^{-1}_{Y^*\otimes X^*,X\otimes Y,(X\otimes Y)^*} \circ (\mathrm{Id}_{Y^*\otimes X^*} \otimes \mathrm{coev}_{X\otimes Y}).
\end{aligned}
$$

The unit of C^* is

$$k \xrightarrow{\mathrm{coev}_C} C \otimes C^* \xrightarrow{\underline{\varepsilon} \otimes \mathrm{Id}_{C^*}} k \otimes C^* \cong C^*,$$

where $\underline{\varepsilon}_C$ is the counit of C. C^* with this monoidal algebra structure is denoted by $C^\star$ and called the opposite dual algebra of the coalgebra C. When $\mathcal{M}^{k_\phi[G]}$ has a braided structure produced by the braiding c then C^* is also a unital algebra in $\mathcal{M}^{k_\phi[G]}$ via the same unit as that of $C^\star$ and multiplication $c^{-1}_{C^*,C^*}\underline{m}_{C^\star}$. In this case we say that C^* is the dual algebra of the coalgebra C, and we denote it by $\check{C}^*$.

The explicit form of $\lambda^{-1}_{X,Y}$ when X, Y are right finite-dimensional comodules over a dual quasi-Hopf algebra H is

$$\lambda^{-1}_{X,Y}\big(y^* \otimes x^*\big)(x \otimes y) = f(x_{(1)}, y_{(1)})x^*(x_{(1)})y^*(y_{(1)}), \tag{5.2}$$

for all $x^* \in X^*$, $y^* \in Y^*$, $x \in X$ and $y \in Y$, where f is the convolution inverse of f^{-1}. When $H = k_\phi[G]$ we have that f is the pointwise inverse of f^{-1} computed in the proof of Proposition 5.3.

Proposition 5.4. *Let G be a finite abelian group and $F \in (G \times G)^*$ a 2-cochain on G. Then $k^F[G]^\star = k^F[G]^* \cong k_F[G]$ as G-graded quasialgebras with associator $\Delta_2(F^{-1})$.*

Proof. We show that the map Ξ defined in the proof of Proposition 5.3, viewed now as a map from $k_F[G]$ to $k^F[G]^\star$, is an isomorphism of G-graded quasialgebras with associator $\Delta_2(F^{-1})$. To this end we first compute the opposite dual algebra structure of the coalgebra $k^F[G]$. The multiplication $\diamond$ of $k^F[G]^\star$ is given by

$$\begin{aligned}
(P_g \diamond P_h)(\theta) &= \lambda^{-1}_{k^F[G],k^F[G]}(P_g \otimes P_h)\big(\Delta_F(\theta)\big) \\
&= \frac{1}{|G|}\sum_{\sigma\tau=\theta} F(\sigma,\tau)^{-1}\lambda^{-1}_{k^F[G],k^F[G]}(P_g \otimes P_h)(\sigma \otimes \tau) \\
&= \frac{1}{|G|}\sum_{\sigma\tau=\theta} F(\sigma,\tau)^{-1} f(\sigma,\tau)P_h(\sigma)P_g(\tau) \\
&= \frac{1}{|G|} \cdot \frac{1}{F(h,g)} \cdot \frac{F(g^{-1}h^{-1},hg)F(g^{-1},h^{-1})F(h,g)}{F(h^{-1},h)F(g^{-1},g)}\delta_{hg,\theta} \\
&= \frac{1}{|G|} \cdot \frac{F(g^{-1}h^{-1},hg)F(g^{-1},h^{-1})}{F(h^{-1},h)F(g^{-1},g)}P_{hg}(\theta),
\end{aligned}$$

and so

$$P_g \diamond P_h = \frac{1}{|G|} \cdot \frac{F(g^{-1}h^{-1},hg)F(g^{-1},h^{-1})}{F(h^{-1},h)F(g^{-1},g)}P_{hg}, \quad \forall g,h \in G.$$

The unit of $k^F[G]^\star$ is $\sum_{g\in G}\beta(g)\varepsilon_F(g)P_g = \beta(e)|G|P_e = |G|P_e$, where all the notations are as in the proof of Proposition 5.3.

We are now in position to show that $\Xi : k_F[G] \to k^F[G]^\star$, $\Xi(\theta) = |G|F(\theta,\theta^{-1})P_{\theta^{-1}}$, for all $\theta \in G$, is an isomorphism of G-graded quasialgebras with associator $\Delta_2(F^{-1})$. We already have seen that Ξ is an isomorphism in $\mathrm{Vect}^G_{\Delta_2(F^{-1})}$, hence we have only to prove that Ξ is an algebra morphism. We compute

$$\begin{aligned}\Xi(\sigma)\diamond\Xi(\tau) &= |G|^2F\big(\sigma,\sigma^{-1}\big)F\big(\tau,\tau^{-1}\big)P_{\sigma^{-1}}\diamond P_{\tau^{-1}}\\ &= |G|F\big(\sigma,\sigma^{-1}\big)F\big(\tau,\tau^{-1}\big)\cdot\frac{F(\sigma\tau,\tau^{-1}\sigma^{-1})F(\sigma,\tau)}{F(\tau,\tau^{-1})F(\sigma,\sigma^{-1})}P_{\tau^{-1}\sigma^{-1}}\\ &= |G|F\big(\sigma\tau,\tau^{-1}\sigma^{-1}\big)F(\sigma,\tau)P_{(\sigma\tau)^{-1}}\\ &= F(\sigma,\tau)\Xi(\sigma\tau) = \Xi(\sigma\bullet\tau),\end{aligned}$$

for all $\sigma,\tau\in G$, and $\Xi(e)=|G|P_e$, as required. Finally, when G is abelian one can easily verify that the multiplication $\diamond'$ of $k^F[G]^*$ coincides with that of $k^F[G]^\star$. Actually,

$$\begin{aligned}P_g\diamond' P_h &= \frac{F(g^{-1},h^{-1})}{F(h^{-1},g^{-1})}P_h\diamond P_g = \frac{1}{|G|}\cdot\frac{F((gh)^{-1},gh)F(g^{-1},h^{-1})}{F(g^{-1},g)F(h^{-1},h)}P_{gh}\\ &= \frac{1}{|G|}\cdot\frac{F((hg)^{-1},hg)F(g^{-1},h^{-1})}{F(g^{-1},g)F(h^{-1},h)}P_{hg} = P_g\diamond P_h,\end{aligned}$$

for all $g,h\in G$, so the proof is finished. □

If $\mathcal{C}$ is a rigid braided monoidal category and H is a weak braided Hopf algebra in $\mathcal{C}$ then H^* and ${}^*H$ (the left and right categorical duals of H) are weak braided Hopf algebras in $\mathcal{C}$, too. Their structures are defined by the dual algebra and coalgebra structures of H. If, moreover, $\mathcal{C}$ is symmetric monoidal then H^* and ${}^*H$ are isomorphic as weak braided Hopf algebras.

When $H=k^F_F[G]$ the weak braided Hopf algebra isomorphism mentioned above is produced by the map $\Theta_{k^F_F[G]}: k^F_F[G]^*\to {}^*k^F_F[G]$ defined by $\Theta_{k^F_F[G]}(P_g)=\frac{F(g,g^{-1})}{F(g^{-1},g)}P_g$, for all $g\in G$. Note that the weak braided Hopf algebra structure of ${}^\star k^F_F[G]={}^*k^F_F[G]$ is given by

$$P_g\diamond P_h = \frac{1}{|G|}\cdot\frac{F(hg,g^{-1}h^{-1})F(g^{-1},h^{-1})}{F(g,g^{-1})F(h,h^{-1})}P_{hg},\qquad 1_{{}^*k^F_F[G]}=|G|P_e;$$

$$\underline{\Delta}_{{}^*k^F_F[G]}(P_g) = \frac{1}{F(g,g^{-1})}\sum_{\sigma\tau=g}\frac{F(\tau,\tau^{-1})F(\sigma,\sigma^{-1})}{F(\tau^{-1},\sigma^{-1})}P_\tau\otimes P_\sigma,\qquad \underline{\varepsilon}_{{}^*k^F_F[G]}(P_g)=\delta_{g,e},$$

for all $g,h\in G$. Its antipode is given by the identity morphism.

Corollary 5.5. *If G is a finite abelian group and F a 2-cochain on it then the weak braided Hopf algebra $k^F_F[G]$ is selfdual. Consequently, all the Cayley–Dickson, and respectively Clifford, weak braided Hopf algebras obtained from (k,Id_k) by successive applications of the Cayley–Dickson, respectively Clifford, processes for algebras and coalgebras, are selfdual weak braided Hopf algebras in certain symmetric monoidal categories of graded vector spaces.*

Proof. The isomorphism $k^F_F[G]^*\cong k^F_F[G]$ is produced by the map Ξ which we have seen in the last two results that is an isomorphism of G-graded quasialgebras and quasicoalgebras with associator $\Delta_2(F^{-1})$. So Ξ is an isomorphism of weak braided Hopf algebras, as needed. The last assertion in the statement is a consequence of the comments that we made after the proof of Theorem 5.2. □

6. Cayley–Dickson and Clifford (co)algebras are monoidal (co)Frobenius (co)algebras

The main aim of this section is to show that when G is a finite abelian group and F a 2-cochain on it the isomorphism $\Xi: k^F_F[G]^*\cong k^F_F[G]$ of weak braided Hopf algebras in $\mathrm{Vect}^G_{\Delta_2(F^{-1}),\mathcal{R}_{F^{-1}}}$ constructed in the previous section induce the (co)Frobenius property on $k^F_F[G]$. As we pointed out several times, the Cayley–Dickson objects considered in [5] and the Clifford objects subject of this paper are weak braided Hopf algebras of this form, so we will obtain that they are monoidal (co)Frobenius (co)algebras. To this end we need first some preliminary results.

Let A be an algebra in a monoidal category $\mathcal{C}$ and suppose that A has a left dual object A^*. Then A^* is a right A-module via the structure morphism

$$A^* \otimes A \xrightarrow{r^{-1}_{A^*\otimes A}} \big(A^* \otimes A\big) \otimes \underline{1} \xrightarrow{\mathrm{Id}_{A^*\otimes A} \otimes \mathrm{coev}_A} \big(A^* \otimes A\big) \otimes \big(A \otimes A^*\big) \xrightarrow{a_{A^*,A,A\otimes A^*}} A^* \otimes \big(A \otimes \big(A \otimes A^*\big)\big)$$
$$\xrightarrow{\mathrm{Id}_{A^*} \otimes a^{-1}_{A,A,A^*}} A^* \otimes \big((A \otimes A) \otimes A^*\big) \xrightarrow{\mathrm{Id}_{A^*} \otimes (\underline{m}_A \otimes \mathrm{Id}_{A^*})} A^* \otimes \big(A \otimes A^*\big) \xrightarrow{a^{-1}_{A^*,A,A^*}} \big(A^* \otimes A\big) \otimes A^*$$
$$\xrightarrow{\mathrm{ev}_A \otimes \mathrm{Id}_{A^*}} \underline{1} \otimes A^* \xrightarrow{l_{A^*}} A^*.$$

If A has a right dual ${}^*A$ in $\mathcal{C}$ then ${}^*A$ is a left A-module in $\mathcal{C}$ via an action similar to the one above. When A has both left and right dual objects in $\mathcal{C}$ we say that A is a Frobenius algebra if A and A^* are isomorphic as right A-modules or, equivalently, if A and ${}^*A$ are isomorphic as left A-modules.

In the first part of this section we show that the algebra $k_F[G]$ in $\mathcal{M}^{k_\phi[G]}$ is Frobenius. We are familiar with the computations required by the quasi-Hopf algebra framework and this is why we prefer to work in this setting rather than the dual one. Then the results that we obtain for quasi-Hopf algebras will be dualized and then specialized for $k_\phi[G]$.

Recall that a quasi-bialgebra is a unital algebra H together with a comultiplication Δ, counit ε and invertible element $\Phi \in H \otimes H \otimes H$, called the reassociator, such that

$$(\mathrm{Id}_H \otimes \Delta)\big(\Delta(h)\big) = \Phi(\Delta \otimes \mathrm{Id}_H)\big(\Delta(h)\big)\Phi^{-1}, \tag{6.1}$$

$$(\mathrm{Id}_H \otimes \varepsilon)\big(\Delta(h)\big) = h, \qquad (\varepsilon \otimes \mathrm{Id}_H)\big(\Delta(h)\big) = h, \tag{6.2}$$

for all $h \in H$. Furthermore, Φ has to be a normalized 3-cocycle, in the sense that

$$(1 \otimes \Phi)(\mathrm{Id}_H \otimes \Delta \otimes \mathrm{Id}_H)(\Phi)(\Phi \otimes 1) \tag{6.3}$$

$$= (\mathrm{Id}_H \otimes \mathrm{Id}_H \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{Id}_H \otimes \mathrm{Id}_H)(\Phi), \tag{6.4}$$

$$(\mathrm{Id}_H \otimes \varepsilon \otimes \mathrm{Id}_H)(\Phi) = 1 \otimes 1. \tag{6.5}$$

One can easily see that the identities (6.2), (6.4) and (6.5) also imply that

$$(\varepsilon \otimes \mathrm{Id}_H \otimes \mathrm{Id}_H)(\Phi) = (\mathrm{Id}_H \otimes \mathrm{Id}_H \otimes \varepsilon)(\Phi) = 1 \otimes 1. \tag{6.6}$$

We denote $\Delta(h) = h_1 \otimes h_2$, but since Δ is only quasi-coassociative we adopt the further convention (summation understood):

$$(\Delta \otimes \mathrm{Id}_H)\big(\Delta(h)\big) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \qquad (\mathrm{Id}_H \otimes \Delta)\big(\Delta(h)\big) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely

$$\Phi = X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = V^1 \otimes V^2 \otimes V^3 = \cdots,$$
$$\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = v^1 \otimes v^2 \otimes v^3 = \cdots.$$

A quasi-bialgebra H is called a quasi-Hopf algebra if there exist an anti-algebra endomorphism S of H, called the antipode, and elements $\alpha, \beta \in H$ such that, for all $h \in H$, we have

$$S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad h_1\beta S(h_2) = \varepsilon(h)\beta, \tag{6.7}$$

$$X^1\beta S\big(X^2\big)\alpha X^3 = 1 \quad \text{and} \quad S\big(x^1\big)\alpha x^2\beta S\big(x^3\big) = 1. \tag{6.8}$$

For quasi-Hopf algebras the antipode is an anti-coalgebra morphism up to conjugation by an invertible element $f \in H \otimes H$, called the Drinfeld twist. More exactly, define $\gamma, \delta \in H \otimes H$ by

$$\gamma = S\big(X^2x^1_2\big)\alpha X^3x^2 \otimes S\big(X^1x^1_1\big)\alpha x^3 \overset{(6.4),(6.6)}{=} S\big(x^1X^2\big)\alpha x^2X^3_1 \otimes S\big(X^1\big)\alpha x^3X^3_2, \tag{6.9}$$

$$\delta = X^1_1x^1\beta S\big(X^3\big) \otimes X^1_2x^2\beta S\big(X^2x^3\big) \overset{(6.4),(6.6)}{=} x^1\beta S\big(x^3_2X^3\big) \otimes x^2X^1\beta S\big(x^3_1X^2\big), \tag{6.10}$$

and then

$$f = (S \otimes S)\big(\Delta^{\mathrm{cop}}\big(x^1\big)\big)\gamma\Delta\big(x^2\beta S\big(x^3\big)\big) \tag{6.11}$$

and

$$f^{-1} = \Delta\big(S\big(x^1\big)\alpha x^2\big)\delta(S \otimes S)\big(\Delta^{\mathrm{cop}}\big(x^3\big)\big). \tag{6.12}$$

By [8] f and f^{-1} are each others inverses, and the following relations hold, $h \in H$,

$$f\Delta(h)f^{-1} = (S \otimes S)\big(\Delta^{\mathrm{cop}}(h)\big), \qquad \gamma = f\Delta(\alpha) \quad \text{and} \quad \delta = \Delta(\beta)f^{-1}. \tag{6.13}$$

One can prove now the following result.

Lemma 6.1. *Let H be a quasi-Hopf algebra and A a finite-dimensional algebra in ${}_H\mathcal{M}$. Then A^* is a right A-module in ${}_H\mathcal{M}$ via the action*

$$a^* \cdot a = \sum_i \big\langle a^*, \big(g^1S\big(X^2\big)\alpha X^3 \cdot a\big) \diamond \big(g^2S\big(X^1\big) \cdot a_i\big)\big\rangle a^i,$$

where $\cdot$ denotes the left action of H on A, $\diamond$ is the multiplication of A in ${}_H\mathcal{M}$, and $\{a_i\}$ is a basis in A with corresponding dual basis $\{a^i\}$ in A^*.

Proof. We specialize the general right action of A on A^* in $\mathcal{C}$ for the case when $\mathcal{C} = {}_H\mathcal{M}$, H a quasi-Hopf algebra, and A is a finite-dimensional algebra in ${}_H\mathcal{M}$. Recall first that ${}_H\mathcal{M}$ is monoidal with $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $X, Y, Z \in {}_H\mathcal{M}$, given by

$$a_{X,Y,Z}\big((x \otimes y) \otimes z\big) = \Phi \cdot \big(x \otimes (y \otimes z)\big), \quad \forall x \in X,\ y \in Y,\ z \in Z,$$

and that the left dual of A is its k-linear dual A^* equipped with the left H-module structure given by $(h \cdot a^*)(a) = a^*(S(h) \cdot a)$, for all $a^* \in A^*$, $h \in H$ and $a \in A$, and with the evaluation and coevaluation morphisms given by

$$\mathrm{ev}_A : A^* \otimes A \to k, \qquad \mathrm{ev}_A\big(a^* \otimes a\big) = a^*(\alpha \cdot a), \quad \forall a^* \in A^*,\ a \in A,$$

$$\mathrm{coev}_A : k \to A \otimes A^*, \qquad \mathrm{coev}_A(1) = \sum_i \beta \cdot a_i \otimes a^i,$$

where $\{a_i\}$ and $\{a^i\}$ are dual bases in A and A^*.

We then obtain the following right A-module structure on A^*:

$$a^* \cdot a = \sum_i \langle y^1 X^1 \cdot a^*, y^2 \cdot \big[\big(x^1 X^2 \cdot a\big) \diamond \big(x^2 X^3_1 \beta \cdot a_i\big)\big]\rangle y^3 x^3 X^3_2 \cdot a^i.$$

To land at the formula claimed in the statement we use the H-module structure of A^* and the equality

$$h \cdot (a \diamond b) = (h_1 \cdot a) \diamond (h_2 \cdot b), \quad \forall h \in H,\ a, b \in A$$

that expresses the H-linearity of the multiplication $\diamond$ of A to compute

$$\begin{aligned}
a^* \cdot a &= \sum_i \langle a^*, S\big(y^1 X^1\big)\alpha y^2 \cdot \big[\big(x^1 X^2 \cdot a\big) \diamond \big(x^2 X^3_1 \beta S\big(y^3 x^3 X^3_2\big) \cdot a_i\big)\big]\rangle a^i \\
&\overset{(6.7),(6.6)}{=} \sum_i \langle a^*, S\big(y^1\big)\alpha y^2 \cdot \big[\big(x^1 \cdot a\big) \diamond \big(x^2 \beta S\big(y^3 x^3\big) \cdot a_i\big)\big]\rangle a^i \\
&= \sum_i \langle a^*, \big(S\big(y^1\big)_1 \alpha_1 y^2_1 x^1 \cdot a\big) \diamond \big(S\big(y^1\big)_2 \alpha_2 y^2_2 x^2 \beta S\big(y^3 x^3\big)\big)\rangle a^i \\
&= \sum_i \langle a^*, \big(g^1 S\big(y^1_2\big)\gamma^1 y^2_1 x^1 \cdot a\big) \diamond \big(g^2 S\big(y^1_1\big)\gamma^2 y^2_2 x^2 \beta S\big(y^3 x^3\big) \cdot a_i\big)\rangle a^i \\
&= \sum_i \langle a^*, \big(g^1 S\big(X^2 z^1_2 y^1_2\big)\alpha X^3 z^2 y^2_1 x^1 \cdot a\big) \diamond \big(g^2 S\big(X^1 z^1_1 y^1_1\big)\alpha z^3 y^2_2 x^2 \beta S\big(y^3 x^3\big) \cdot a_i\big)\rangle a^i \\
&\overset{(6.4)}{=} \sum_i \langle a^*, \big(g^1 S\big(X^2 y^1_{(1,2)} x^1_2\big)\alpha X^3 y^1_2 x^2 \cdot a\big) \diamond \big(g^2 S\big(X^1 y^1_{(1,1)} x^1_1\big)\alpha y^2 x^3_1 \beta S\big(y^3 x^3_2\big) \cdot a_i\big)\rangle a^i \\
&\overset{(6.1),(6.7)}{=} \sum_i \langle a^*, \big(g^1 S\big(X^2\big)\alpha X^3 \cdot a\big) \diamond \big(g^2 S\big(y^1 X^1\big)\alpha y^2 \beta S\big(y^3\big) \cdot a_i\big)\rangle a^i \\
&\overset{(6.8)}{=} \sum_i \langle a^*, \big(g^1 S\big(X^2\big)\alpha X^3 \cdot a\big) \diamond \big(g^2 S\big(X^1\big) \cdot a_i\big)\rangle a^i,
\end{aligned}$$

as needed. Note that $g^1 \otimes g^2$ is the inverse of the Drinfeld twist, $\gamma^1 \otimes \gamma^2$ is the element defined in (6.9) and the fourth equality follows because of the relations stated in (6.13). □

Proposition 6.2. *Let G be a group and F a 2-cochain on G. Then the G-graded quasialgebra $k_F[G]$ with associator $\Delta_2(F^{-1})$ is a Frobenius algebra in $\mathrm{Vect}^G_{\Delta_2(F^{-1})}$. Consequently, all the Cayley–Dickson and Clifford algebras obtained through their corresponding processes for algebras from the input data (k, Id_k) are monoidal Frobenius algebras in some suitable symmetric monoidal categories of graded vector spaces.*

Proof. We show that the isomorphism Ξ constructed in Proposition 5.3, but viewed now as a morphism from $k_F[G]$ to $k_F[G]^*$, is right $k_F[G]$-linear in the sense considered above. To this end we need the dual of the result proved in Lemma 6.1. Namely, if A is a finite-dimensional algebra in the category $\mathcal{M}^H$, H a dual quasi-Hopf algebra, then A^* has a right A-module structure in $\mathcal{M}^H$ modulo the action

$$\big(a^* \cdot a\big)(b) = f^{-1}(a_{(1)}, b_{(1)})\varphi\big(S(b_{(2)}), S(a_{(2)}), a_{(4)}\big)\alpha(a_{(3)})a^*(a_{(0)} \diamond b_{(0)}),$$

where the notations are from Section 5. Thus, if we consider $H = k_\phi[G]$ and $A = k_F[G]$ we then get that $k_F[G]^*$ is a right $k_F[G]$-module via the action

$$\begin{aligned}(P_\theta\cdot\sigma)(\tau)&=f^{-1}(\sigma,\tau)\phi\big(\tau^{-1},\sigma^{-1},\sigma\big)\varepsilon(\sigma)P_\theta(\sigma\bullet\tau)\\&=\frac{F(\sigma^{-1},\sigma)F(\tau^{-1},\tau)}{F(\tau^{-1},\sigma^{-1})F(\sigma,\tau)F(\tau^{-1}\sigma^{-1},\sigma\tau)}\cdot\frac{F(\tau^{-1}\sigma^{-1},\sigma)F(\tau^{-1},\sigma^{-1})}{F(\sigma^{-1},\sigma)}\cdot F(\sigma,\tau)P_\theta(\sigma\tau)\\&=\frac{F(\tau^{-1},\tau)F(\tau^{-1}\sigma^{-1},\sigma)}{F(\tau^{-1}\sigma^{-1},\sigma\tau)}\delta_{\theta,\sigma\tau}\\&=\frac{F(\theta^{-1}\sigma,\sigma^{-1}\theta)F(\theta^{-1},\sigma)}{F(\theta^{-1},\theta)}P_{\sigma^{-1}\theta}(\tau),\end{aligned}$$

where the notations are the same as in the proof of Proposition 5.3. We then have

$$\begin{aligned}\Xi(\theta)\cdot\sigma&=|G|F\big(\theta,\theta^{-1}\big)P_{\theta^{-1}}\cdot\sigma=|G|F\big(\theta,\theta^{-1}\big)\frac{F(\theta\sigma,\sigma^{-1}\theta^{-1})F(\theta,\sigma)}{F(\theta,\theta^{-1})}P_{\sigma^{-1}\theta^{-1}}\\&=|G|F\big(\theta\sigma,\sigma^{-1}\theta^{-1}\big)F(\theta,\sigma)P_{\sigma^{-1}\theta^{-1}}=F(\theta,\sigma)\Xi(\theta\sigma)=\Xi(\theta\bullet\sigma),\end{aligned}$$

for all $\theta,\sigma\in G$, and this shows that Ξ is right $k_F[G]$-linear, as claimed. □

We move now to the coFrobenius notion. Dual to the algebra case, any coalgebra C in a monoidal category $\mathcal{C}$ with right duality is a right ${}^\star C$-module in $\mathcal{C}$ via

$$C\otimes{}^\star C\xrightarrow{\underline{\Delta}\otimes\mathrm{Id}_{{}^\star C}}(C\otimes C)\otimes{}^\star C\xrightarrow{a_{C,C,{}^\star C}}C\otimes\big(C\otimes{}^\star C\big)\xrightarrow{\mathrm{Id}_C\otimes\mathrm{ev}'_C}C\otimes\underline{1}\xrightarrow{r_C}C,$$

where $\underline{1}$ is the unit object of $\mathcal{C}$ and r is the right unit constraint of $\mathcal{C}$.

Likewise, it can be proved that C is a left $C^\star$-module in $\mathcal{C}$ via an action similar to the one defined above, providing that C^* exists. We then say that C is a coFrobenius coalgebra in $\mathcal{C}$ if C and ${}^\star C$ are isomorphic as right ${}^\star C$-modules or, equivalently, if C and $C^\star$ are isomorphic as left $C^\star$-modules.

Proposition 6.3. *For any finite abelian group G and any 2-cochain $F\in(G\times G)^*$ the G-graded quasicoalgebra $k^F[G]$ with associator $\Delta_2(F^{-1})$ is a monoidal coFrobenius coalgebra. Consequently, all the Cayley–Dickson coalgebras and all the Clifford coalgebras obtained from the input data (k,Id_k) through the coalgebra processes that carry out their names are monoidal coFrobenius coalgebras.*

Proof. $k^F[G]$ is a monoidal coFrobenius coalgebra if and only if $k^F[G]^\star$ is a monoidal Frobenius algebra. But $k^F[G]^\star\cong k_F[G]$ as monoidal algebras, cf. Proposition 5.4, and so everything follows because of Proposition 6.2. □

Remark 6.4. One can prove directly that $k^F[G]$ is a monoidal coFrobenius coalgebra. Actually, for H a quasi-Hopf algebra and C a coalgebra in $\mathcal{C}={}_H\mathcal{M}$ (not necessarily finite-dimensional) the linear dual space of C, C^*, has a unital algebra structure in $\mathcal{C}$ given by

$$c^*d^*(c)=c^*\big(f^2\cdot c_2\big)d^*\big(f^1\cdot c_1\big)\quad\text{and}\quad 1_{C^*}=\varepsilon_C,$$

for all $c^*,d^*\in C^*$ and $c\in C$, where $f=f^1\otimes f^2$ is the Drinfeld twist defined in (6.11). Furthermore, C is a left C^*-module in $\mathcal{C}$ via the action

$$c^*\rightharpoonup' c:=c^*\big(S\big(x^1\big)\alpha x^2\cdot c_1\big)x^3\cdot c_2.$$

Dualizing this result for the dual quasi-Hopf algebra $k_{\Delta_2(F^{-1})}[G]$ we get that $k^F[G]$ is a left $k^F[G]^\star$-

module via the action

$$P_\theta \rightharpoonup' g = \frac{1}{|G|} \cdot \frac{F(\theta^{-1}, g)}{F(\theta^{-1}, \theta)} \theta^{-1} g.$$

Then the isomorphism

$$\Xi : K^F[G] \to k^F[G]^\star, \qquad \Xi(\theta) = |G| F\big(\theta, \theta^{-1}\big) P_{\theta^{-1}}, \quad \forall \theta \in G,$$

often used so far, is left $k^F[G]^\star$-linear as well, and so $k^F[G]$ is a monoidal coFrobenius coalgebra. We leave to the reader the verification of all these details.

7. Periodicity properties for some Clifford weak braided Hopf algebras

Let k be a field of characteristic different from 2. By $C^{p,q}$ we denote the Clifford algebra $C(-1, \dots, -1, 1, \dots, 1)$ having p of -1's and q of 1's. The aim of this section is to calculate all the Clifford algebras of the form $C^{p,q}$ with $p, q \in \mathbb{N}$. By convention we set $C^{0,0} = k$. To be more precise, we shall prove that the descriptions of $C^{p,q}$'s that exist in the algebra case are valid also in the coalgebra case, so that we will be able to describe the Clifford algebras of this type at the level of weak braided Hopf algebras.

Let us start with the following technical result.

Lemma 7.1. *Take $q_1, \dots, q_n \in k^*$ and let $\{e_x \mid x \in (\mathbb{Z}_2)^n\}$ be the canonical basis of the Clifford coalgebra $C(q_1, \dots, q_n)$. If F is the normalized 2-cocycle on $(\mathbb{Z}_2)^n$ associated to $C(q_1, \dots, q_n)$ as in* (2.1) *we then have*

$$F(1, x) F(u, u \oplus x \oplus 1)^{-1} F(1, u \oplus x)^{-1} = (-1)^{(n-1)\rho(u)} F(u, u \oplus x)^{-1}, \tag{7.1}$$

$$F(u, u \oplus x)^{-1} F(1, u) F(1, u \oplus x) = (-1)^{\frac{n(n-1)}{2} + (n-1)\rho(u)} F(u \oplus 1, u \oplus 1 \oplus x)^{-1} \prod_{i=1}^{n} q_i, \tag{7.2}$$

$$\Delta\big((e_1 \cdots e_n)^t e_x\big) = \frac{1}{2^n} \sum_{u \in G} F(u, u \oplus x)^{-1} (-1)^{(n-1)t\rho(u)} e_u \otimes (e_1 \cdots e_n)^t e_{u \oplus x}, \tag{7.3}$$

for all $u, x \in (\mathbb{Z}_2)^n$ and $t \in \mathbb{N}$, where, as usual, $\rho(x) = \sum_{i=1}^{n} x_i$.

Proof. Using the definition of F and the equalities $u_i(u_i \oplus x_i \oplus 1) = u_i x_i$, $u_i \oplus x_i = u_i + x_i - 2u_i x_i$ and $u_i(u_i \oplus x_i) = u_i(1 - x_i)$ we obtain (7.1). Similarly, the definition of F and $(u_i \oplus 1)(u_i \oplus 1 \oplus x_i) = (1 - u_i)(1 - x_i)$, for all $1 \leqslant i \leqslant n$, imply (7.2).

To show (7.3) we use the formula

$$(e_1 \cdots e_n)^2 = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} q_i 1 \tag{7.4}$$

that follows easily from (2.3) by identifying $e_1 \cdots e_n \equiv e_{(1,\dots,1)}$, where $(1, \dots, 1) \in (\mathbb{Z}_2)^n$; note that in several places it has been simply written 1. Now, we have

$$\Delta\big((e_1 \cdots e_n)^t e_x\big) = \begin{cases} \frac{\theta^{\frac{t}{2}}}{2^n} \sum_{u \in (\mathbb{Z}_2)^n} F(u, u \oplus x)^{-1} e_u \otimes e_{u \oplus x} & \text{if } t \in 2\mathbb{N}, \\ \frac{\theta^{\frac{t-1}{2}}}{2^n} \sum_{u \in (\mathbb{Z}_2)^n} F(u, u \oplus x \oplus 1)^{-1} F(1, x) e_u \otimes e_{u \oplus x \oplus 1} & \text{if } t \in 2\mathbb{N} + 1 \end{cases}$$

$$\overset{(2.3)}{=}\frac{1}{2^n}\begin{cases}\sum_{u\in(\mathbb{Z}_2)^n}F(u,u\oplus x)^{-1}e_u\otimes e^te_{u\oplus x} & \text{if } t\in 2\mathbb{N},\\ \sum_{u\in(\mathbb{Z}_2)^n}\frac{F(1,x)}{F(u,u\oplus x\oplus 1)F(1,u\oplus x)}e_u\otimes e^te_{u\oplus x} & \text{if } t\in 2\mathbb{N}+1\end{cases}$$
$$\overset{(7.1)}{=}\frac{1}{2^n}\sum_{u\in(\mathbb{Z}_2)^n}F(u,u\oplus x)^{-1}(-1)^{(n-1)t\rho(u)}e_u\otimes e^te_{u\oplus x},$$

as claimed, where, for simplicity, we have denoted $\theta=(-1)^{\frac{n(n-1)}{2}}q_1\cdots q_n$ and $e=e_1\cdots e_n$. This finishes the proof. □

The result below is essential in the proof of some periodicity properties for the Clifford algebras $C^{p,q}$.

Theorem 7.2. *Let (V,q) and (V',q') be quadratic spaces of dimensions n and $2m$, respectively, and write $C(V,q)=C(q_1,\dots,q_n)$ and $C(V',q')=C(q'_1,\dots,q'_{2m})$, for some $q_1,\dots,q_n,q'_1,\dots,q'_{2m}\in k^*$. Take F be the 2-cocycle on $(\mathbb{Z}_2)^n$ associated to $C(V,q)$ as in (2.1) and define $\mathcal{F}:(\mathbb{Z}_2)^n\times(\mathbb{Z}_2)^n\to k^*$ by*

$$\mathcal{F}(x,y)=F(x,y)\gamma^{-x\cdot y},\quad \text{where } \gamma:=(-1)^{m(2m-1)}q'_1\cdots q'_{2m},$$

and where $x\cdot y$ denotes the dot product of the $\mathbb{Z}_2$-valued vectors x and y. Then

$$C(V,q)\,\widehat{\otimes}\,C\big(V',q'\big)\cong k^{\mathcal{F}}_{\mathcal{F}}\big[(\mathbb{Z}_2)^n\big]\otimes C\big(V',q'\big),$$

as $\mathbb{Z}_2$-graded k-algebras and coalgebras. Furthermore, if γ has a squareroot in k^* then

$$C(V,q)\,\widehat{\otimes}\,C\big(V',q'\big)\cong C(V,q)\otimes C\big(V',q'\big),$$

as $\mathbb{Z}_2$-graded k-algebras and coalgebras.

Proof. Observe first that $\mathcal{F}$ is a 2-cocycle on $(\mathbb{Z}_2)^n$, too. This follows from the fact that F is a 2-cocycle on $(\mathbb{Z}_2)^n$ and since $\gamma^{-x\cdot y}\gamma^{-(x+y)\cdot z}=\gamma^{-y\cdot z}\gamma^{-x\cdot(y+z)}$, for all $x,y,z\in(\mathbb{Z}_2)^n$. Thus $k^{\mathcal{F}}_{\mathcal{F}}[(\mathbb{Z}_2)^n]$ is a k-algebra and a k-coalgebra, and so the claimed isomorphism is between k-algebras and coalgebras, as stated. For further use consider also F', the 2-cocycle on $(\mathbb{Z}_2)^{2m}$ associated to $C(V',q')$.

Define now $\zeta:C(V,q)\,\widehat{\otimes}\,C(V',q')\to k^{\mathcal{F}}_{\mathcal{F}}[(\mathbb{Z}_2)^n]\otimes C(V',q')$ on the canonical basis of $C(V,q)\,\widehat{\otimes}\,C(V',q')$, obtained from the canonical bases $\{e_x\mid x\in(\mathbb{Z}_2)^n\}$ and $\{e'_{x'}\mid x'\in(\mathbb{Z}_2)^{2m}\}$ of $C(V,q)$ and $C(V',q')$, respectively, by

$$\zeta\big(e_x\otimes e'_{x'}\big)=x\otimes\big(e'_1\cdots e'_{2m}\big)^{\rho(x)}e'_{x'}$$

and extend it by linearity. By [2, Proposition 3.9] we know that ζ is a $\mathbb{Z}_2$-graded algebra isomorphism, so it is suffices to show that ζ is a graded coalgebra morphism, too. We compute

$$\begin{aligned}\Delta\zeta\big(e_x\,\widehat{\otimes}\,e'_{x'}\big)&\overset{(7.3)}{=}\frac{1}{2^{n+2m}}\sum_{u\in(\mathbb{Z}_2)^n,\ u'\in(\mathbb{Z}_2)^{2m}}\mathcal{F}(u,u\oplus x)^{-1}F'\big(u',u'\oplus x'\big)^{-1}(-1)^{\rho'(u')\rho(x)}\big(u\otimes e'_{u'}\big)\\&\quad\otimes\big((u\oplus x)\otimes\big(e'_1\cdots e'_{2m}\big)^{\rho(x)}e'_{u'\oplus x'}\big)\\&=\frac{1}{2^{n+2m}}\sum_{u\in(\mathbb{Z}_2)^n,\ u'\in(\mathbb{Z}_2)^{2m}}F(u,u\oplus x)^{-1}F'\big(u',u'\oplus x'\big)^{-1}(-1)^{\rho'(u')\rho(x)}\gamma^{\rho(u)-u\cdot x}\\&\quad\times\big(u\otimes e'_{u'}\big)\otimes\big((u\oplus x)\otimes\big(e'_1\cdots e'_{2m}\big)^{\rho(x)}e'_{u'\oplus x'}\big),\end{aligned}$$

where we used the fact that $u_i(u_i \oplus x_i) = u_i(1 - x_i)$, and so $u \cdot (u \oplus x) = \rho(u) - u \cdot x$. On the other hand,

$$
\begin{aligned}
&(\zeta \otimes \zeta)\Delta\big(e_x \widehat{\otimes} e'_{x'}\big)\\
&= \frac{1}{2^{n+2m}} \sum_{u\in(\mathbb{Z}_2)^n,\ u'\in(\mathbb{Z}_2)^{2m}} F(u, u \oplus x)^{-1} F'\big(u', u' \oplus x'\big)^{-1} (-1)^{\rho(u\oplus x)\rho'(u')} \big(u \otimes \big(e'_1 \cdots e'_{2m}\big)^{\rho(u)} e'_{u'}\big)\\
&\quad \otimes \big((u \oplus x) \otimes \big(e'_1 \cdots e'_{2m}\big)^{\rho(u\oplus x)} e'_{u'\oplus x'}\big)\\
&= \frac{1}{2^{n+2m}} \sum_{u\in(\mathbb{Z}_2)^n,\ u'\in(\mathbb{Z}_2)^{2m}} F(u, u \oplus x)^{-1} F'\big(u', u' \oplus x'\big)^{-1} \gamma^{-u\cdot x} (-1)^{\rho(u\oplus x)\rho'(u')}\\
&\quad \times \big(u \otimes \big(e'_1 \cdots e'_{2m}\big)^{\rho(u)} e'_{u'}\big) \otimes \big((u \oplus x) \otimes \big(e'_1 \cdots e'_{2m}\big)^{\rho(x)} \big(e'_1 \cdots e'_{2m}\big)^{\rho(u)} e'_{u'\oplus x'}\big),
\end{aligned}
$$

this time because of $\rho(u \oplus x) = \rho(u) + \rho(x) - 2u \cdot x$, and of (7.4). Hence ζ respects the comultiplications if and only if

$$
\begin{aligned}
&\sum_{u'\in(\mathbb{Z}_2)^{2m}} F'\big(u', u' \oplus x'\big)^{-1} \gamma^{\rho(u)-u\cdot x} (-1)^{\rho'(u')\rho(x)} e'_{u'} \otimes \big(e'_1 \cdots e'_{2m}\big)^{\rho(x)} e'_{u'\oplus x'}\\
&\quad = \sum_{u'\in(\mathbb{Z}_2)^{2m}} F'\big(u', u' \oplus x'\big)^{-1} \gamma^{-u\cdot x} (-1)^{\rho(u\oplus x)\rho'(u')} \big(e'_1 \cdots e'_{2m}\big)^{\rho(u)} e'_{u'}\\
&\quad \otimes \big(e'_1 \cdots e'_{2m}\big)^{\rho(x)} \big(e'_1 \cdots e'_{2m}\big)^{\rho(u)} e'_{u'\oplus x'},
\end{aligned}
\tag{7.5}
$$

for all $u \in (\mathbb{Z}_2)^n$. Now, observe that

$$
\begin{aligned}
&F'\big(u', u' \oplus x'\big)^{-1} \big(e'_1 \cdots e'_{2m}\big)^{\rho(u)} e'_{u'} \otimes \big(e'_1 \cdots e'_{2m}\big)^{\rho(u)} e'_{u'\oplus x'}\\
&\overset{(2.3)}{=} F'\big(u', u' \oplus x'\big)^{-1} \begin{cases} \gamma^{\rho(u)} e'_{u'} \otimes e'_{u'\oplus x'} & \text{if } \rho(u) \in 2\mathbb{N},\\ \gamma^{\rho(u)-1} F'(1, u') F'(1, u' \oplus x') e'_{u'\oplus 1} \otimes e'_{u'\oplus 1\oplus x'} & \text{if } \rho(u) \in 2\mathbb{N}+1 \end{cases}\\
&\overset{(7.2)}{=} \gamma^{\rho(u)} \begin{cases} F'(u', u' \oplus x')^{-1} e'_{u'} \otimes e'_{u'\oplus x'} & \text{if } \rho(u) \in 2\mathbb{N},\\ F'(u' \oplus 1, u' \oplus 1 \oplus x')^{-1} (-1)^{\rho'(u')} e'_{u'\oplus 1} \otimes e_{u'\oplus 1\oplus x'} & \text{if } \rho(u) \in 2\mathbb{N}+1. \end{cases}
\end{aligned}
$$

Therefore (7.5) holds if and only if, for all $u' \in (\mathbb{Z}_2)^{2m}$, we have

$$(-1)^{\rho'(u')\rho(x)} = (-1)^{\rho(u\oplus x)\rho'(u')}, \quad \forall u \in (\mathbb{Z}_2)^n \text{ with } \rho(u) \in 2\mathbb{N},$$

and

$$(-1)^{\rho'(u'\oplus 1)\rho(x)} = (-1)^{\rho(u\oplus x)\rho'(u')+\rho'(u')}, \quad \forall u \in (\mathbb{Z}_2)^n \text{ with } \rho(u) \in 2\mathbb{N}+1.$$

It is trivial to see that the two conditions above are always satisfied, so ζ indeed respects the comultiplications. One can easily see that ζ respects the counits as well, so it is a $\mathbb{Z}_2$-graded k-coalgebra isomorphism, as we claimed.

The second assertion can be proved exactly as in [2, Proposition 3.6], by using now coalgebra arguments. In fact, assume that there exists $\alpha \in k^*$ such that $\gamma = \alpha^2$. If we define $s : (\mathbb{Z}_2)^n \to k^*$ by $s(x) = \alpha^{-\rho(x)}$, for all $x \in (\mathbb{Z}_2)^n$, we then have

$$s(x \oplus y) = \alpha^{-\rho(x \oplus y)} = \alpha^{-\rho(x)-\rho(y)+2x\cdot y} = s(x)s(y)\gamma^{x\cdot y},$$

for all $x, y \in (\mathbb{Z}_2)^n$, and so $\mathcal{F}(x, y) = F(x, y)s(x)s(y)s(x \oplus y)^{-1}$, for all $x, y \in (\mathbb{Z}_2)^n$. This shows that $\mathcal{F}$ and F are cohomologous 2-cocycles. Hence, by [2, Proposition 3.9] the map $\varphi : k_{\mathcal{F}}[G] \to k_F[G]$ defined by $\varphi(x) = s(x)x$, for all $x \in G$, is a $\mathbb{Z}_2$-graded algebra isomorphism. If we view φ as a map from $k^{\mathcal{F}}[G]$ to $k^F[G]$ then φ is a $\mathbb{Z}_2$-graded coalgebra isomorphism as well. We have

$$\begin{aligned}(\varphi \otimes \varphi)\Delta_{\mathcal{F}}(x) &= (\varphi \otimes \varphi)\left(\frac{1}{|G|}\sum_{u\in G}\mathcal{F}\big(u, u^{-1}x\big)^{-1}u \otimes u^{-1}x\right)\\ &= \frac{1}{|G|}\sum_{u\in G}\mathcal{F}\big(u, u^{-1}x\big)^{-1}s(u)s\big(u^{-1}x\big)u \otimes u^{-1}x\\ &= \frac{1}{|G|}\sum_{u\in G}F\big(u, u^{-1}x\big)^{-1}s(x)u \otimes u^{-1}x\\ &= s(x)\Delta_F(x) = \Delta_F\varphi(x).\end{aligned}$$

In addition, $\varepsilon_F\varphi(x) = s(x)\varepsilon_F(x) = |G|s(x)\delta_{x,e} = |G|\delta_{x,e} = \varepsilon_{\mathcal{F}}(x)$, for all $x \in G$, and so φ is a k-coalgebra isomorphism, as desired.

We known that $k_F^F[(\mathbb{Z}_2)^n]$ and $C(V, q)$ are isomorphic as $\mathbb{Z}_2$-graded algebras and coalgebras, and therefore $k_{\mathcal{F}}^{\mathcal{F}}[(\mathbb{Z}_2)^n]$ and $C(V, q)$ are isomorphic as $\mathbb{Z}_2$-graded algebras and coalgebras, too. Thus the second isomorphism follows from the first one. □

Corollary 7.3. *For all $p \in \mathbb{N}$ we have $C^{p,p} \cong \widehat{M}_{2^p}(k)$, as $\mathbb{Z}_2$-graded algebras and coalgebras.*

Proof. Take in Theorem 7.2 $m = 1$, $q'_1 = 1$ and $q'_2 = -1$, so that $C(V', q') = C^{1,1}$. We have $\gamma = 1$, so $C(V, q) \mathbin{\widehat{\otimes}} C^{1,1} \cong C(V, q) \otimes C^{1,1}$, as $\mathbb{Z}_2$-graded algebras and coalgebras, for any quadratic space (V, q). Since $C^{1,1} \cong \widehat{M}_2(k)$ as $\mathbb{Z}_2$-graded algebras and coalgebras, see Remarks 4.3(3), we obtain that $C(V, q) \mathbin{\widehat{\otimes}} C^{1,1} \cong \widehat{M}_2(C(V, q))$, cf. Lemma 4.2 and its algebraic version. Hence [2, Corollary 2.6] and Corollary 3.11 give

$$C^{p+1,p+1} \cong C^{p,p} \mathbin{\widehat{\otimes}} C^{1,1} \cong \widehat{M}_2\big(C^{p,p}\big), \quad \forall p \in \mathbb{N},$$

as $\mathbb{Z}_2$-graded algebras and coalgebras. We prove now the assertion by mathematical induction on p. For $p = 0$ we have $C^{0,0} = k \cong \widehat{M}_1(k)$, and for $p = 1$ we have seen that $C^{1,1} \cong \widehat{M}_2(k)$, both isomorphisms being of $\mathbb{Z}_2$-graded algebras and coalgebras. If we assume $C^{p,p} \cong \widehat{M}_{2^p}(k)$ then applying again Lemma 4.2 and its algebraic version we deduce that $C^{p+1,p+1} \cong \widehat{M}_2(\widehat{M}_{2^p}(k)) \cong \widehat{M}_{2^{p+1}}(k)$, as $\mathbb{Z}_2$-graded algebras and coalgebras. So our proof is complete. □

Another important result related to the computation of all $C^{p,q}$'s is the following.

Corollary 7.4. *For all $p, q, m \in \mathbb{N}$ we have $C^{p+m,q+m} \cong \widehat{M}_{2^m}(C^{p,q})$, as $\mathbb{Z}_2$-graded algebras and coalgebras.*

Proof. Take in Theorem 7.2 $q'_1 = \cdots = q'_m = 1$ and $q'_{m+1} = \cdots = q'_{2m} = -1$, so that $C(V', q') = C^{m,m}$. We have $\gamma = 1$, and thus

$$C(V, q) \mathbin{\widehat{\otimes}} C^{m,m} \cong C(V, q) \otimes C^{m,m} \cong C(V, q) \otimes \widehat{M}_{2^m}(k) \cong \widehat{M}_{2^m}\big(C(V, q)\big),$$

as $\mathbb{Z}_2$-graded algebras and coalgebras, for any quadratic space (V, q). Specializing for $C(V, q) = C^{p,q}$ and using again [2, Corollary 2.6] and Corollary 3.11 we deduce that

$$C^{p+m,q+m} \cong C^{p,q} \mathbin{\widehat{\otimes}} C^{m,m} \cong \widehat{M}_{2^m}\big(C^{p,q}\big),$$

as $\mathbb{Z}_2$-graded algebras and coalgebras, as stated. Hence the proof is finished. □

Corollary 7.5. *Keep the hypothesis of Theorem 7.2 and assume, in addition, that $\gamma = -1$. Then $C(V,q) \mathbin{\widehat{\otimes}} C(V',q') \cong C(V,-q) \otimes C(V',q')$, as $\mathbb{Z}_2$-graded algebras and coalgebras, where $-q$ denotes the opposite quadratic form associated to q, i.e., $C(V,-q) = C(-q_1,\dots,-q_n)$, providing $C(V,q) = C(q_1,\dots,q_n)$. Consequently,*

$$C^{p+2,q} \cong C^{q,p} \otimes \mathbb{H} \quad \textit{and} \quad C^{p,q+2} \cong C^{q,p} \otimes \left(\frac{1,1}{k}\right),$$

for all $p,q \in \mathbb{N}$, and from here it follows that $C^{4,0} \cong C^{0,4} \cong \mathbb{H} \otimes (\frac{1,1}{k})$ as $\mathbb{Z}_2$-graded algebras and coalgebras.

Proof. If $\gamma = -1$ then $\mathcal{F}(x,y) = F(x,y)(-1)^{x\cdot y}$, for all $x,y \in (\mathbb{Z}_2)^n$. A direct computation shows that $\mathcal{F}$ is precisely the 2-cocycle on $(\mathbb{Z}_2)^n$ associated to $C(V,-q)$, as in (2.1), and so $k_{\mathcal{F}}^{\mathcal{F}}[(\mathbb{Z}_2)^n] \cong C(V,-q)$, as $\mathbb{Z}_2$-graded algebras and coalgebras. Thus Theorem 7.2 implies

$$C(V,q) \mathbin{\widehat{\otimes}} C\big(V',q'\big) \cong k_{\mathcal{F}}^{\mathcal{F}}\big[(\mathbb{Z}_2)^n\big] \otimes C\big(V',q'\big) \cong C(V,-q) \otimes C\big(V',q'\big),$$

proving the first $\mathbb{Z}_2$-graded algebra and coalgebra isomorphism.

The above isomorphism applies when m is odd and all the elements q'_i are equal to either 1 or -1. Hence

$$C(V,q) \mathbin{\widehat{\otimes}} C^{0,4t+2} \cong C(V,-q) \otimes C^{0,4t+2} \quad \text{and} \quad C(V,q) \mathbin{\widehat{\otimes}} C^{4t+2,0} \cong C(V,-q) \otimes C^{4t+2,0},$$

for any quadratic space (V,q) and for all $t \in \mathbb{N}$. In particular,

$$C^{p+2,q} \cong C^{p,q} \mathbin{\widehat{\otimes}} C^{2,0} \cong C^{q,p} \otimes \mathbb{H} \quad \text{and} \quad C^{p,q+2} \cong C^{p,q} \mathbin{\widehat{\otimes}} C^{0,2} \cong C^{q,p} \otimes \left(\frac{1,1}{k}\right),$$

for all $p,q \in \mathbb{N}$, as needed. From here we get that $C^{4,0} \cong C^{0,2} \otimes \mathbb{H} \cong (\frac{1,1}{k}) \otimes \mathbb{H}$, as $\mathbb{Z}_2$-graded algebras and coalgebras. Similarly, $C^{0,4} \cong C^{2,0} \otimes (\frac{1,1}{1}) \cong \mathbb{H} \otimes (\frac{1,1}{1})$, as $\mathbb{Z}_2$-graded algebras and coalgebras. □

Another substantial result for the computation of all $C^{p,q}$'s is the following.

Proposition 7.6 *("Periodicity 8"). For all $p,q \in \mathbb{N}$ we have $C^{p+8,q} \cong C^{p,q+8} \cong \widehat{M}_{16}(C^{p,q})$, as $\mathbb{Z}_2$-graded algebras and coalgebras.*

Proof. By Corollary 7.5 and Corollary 7.3 we have

$$C^{8,0} \cong C^{4,0} \mathbin{\widehat{\otimes}} C^{4,0} \cong C^{4,0} \mathbin{\widehat{\otimes}} C^{0,4} \cong C^{4,4} \cong \widehat{M}_{16}(k).$$

Similarly, $C^{0,8} \cong C^{0,4} \mathbin{\widehat{\otimes}} C^{0,4} \cong C^{0,4} \mathbin{\widehat{\otimes}} C^{4,0} \cong C^{4,4} \cong \widehat{M}_{16}(k)$. Now, for $C^{8,0}$ we have $m = 4$ and $q'_1 = \dots = q'_8 = -1$, so $\gamma = 1$. By Theorem 7.2 we obtain that

$$C^{p+8,q} \cong C^{p,q} \mathbin{\widehat{\otimes}} C^{8,0} \cong C^{p,q} \otimes C^{8,0} \cong C^{p,q} \otimes \widehat{M}_{16}(k) \cong \widehat{M}_{16}\big(C^{p,q}\big).$$

Likewise one can show that $C^{p,q+8} \cong \widehat{M}_{16}(C^{p,q})$, we leave the verification to the reader. Note that all the above isomorphisms are at the level of $\mathbb{Z}_2$-graded algebras and coalgebras. □

The above results reduce the computation of all $C^{p,q}$'s to the computations of $C^{p,0}$ and $C^{0,p}$, for $1 \leqslant p \leqslant 7$. All these $\mathbb{Z}_2$-graded algebras and coalgebras can be expressed in terms of

$$X := k[\mathbf{i}], \qquad Y = \mathbb{H}, \qquad Z = k[\mathbb{Z}_2] \quad \text{and} \quad W := C^{0,2} = \left(\frac{1,1}{k}\right),$$

as follows.

Theorem 7.7. *The Clifford weak braided Hopf algebras $C^{p,0}$ and $C^{0,p}$ with $1 \leqslant p \leqslant 7$ are of the form*

n	0	1	2	3	4	5	6	7
$C^{n,0}$	k	X	Y	$Y \otimes Z$	$Y \otimes W$	$\widehat{M}_2(X \otimes W)$	$\widehat{M}_4(W)$	$\widehat{M}_8(Z)$
$C^{0,n}$	k	Z	W	$X \otimes W$	$Y \otimes W$	$\widehat{M}_2(Y \otimes Z)$	$\widehat{M}_4(Y)$	$\widehat{M}_8(X)$

Proof. We have already seen that $C^{0,0} = k$, $C^{1,0} = k[\mathbf{i}] = X$, $C^{2,0} = \mathbb{H} = Y$, $C^{0,1} = k[\mathbb{Z}_2] = Z$, $C^{0,2} = (\frac{1,1}{k}) = W$, and that $C^{0,4} \cong C^{4,0} \cong \mathbb{H} \otimes (\frac{1,1}{k}) = Y \otimes W$, as $\mathbb{Z}_2$-graded algebras and coalgebras.

For $C^{3,0}$ and $C^{0,3}$ we use Corollary 7.5. Namely,

$$C^{3,0} \cong C^{0,1} \otimes \mathbb{H} = Z \otimes Y \cong Y \otimes Z \quad \text{and} \quad C^{0,3} \cong C^{1,0} \otimes \left(\frac{1,1}{k}\right) = X \otimes W,$$

respectively. For the remaining situations we use the following isomorphisms,

$$C^{p+4,0} \cong C^{p,0} \mathbin{\widehat{\otimes}} C^{4,0} \cong C^{p,0} \mathbin{\widehat{\otimes}} C^{0,4} \cong C^{p,4} \cong \widehat{M}_{2^p}\big(C^{0,4-p}\big)$$

and

$$C^{0,p+4} \cong C^{0,p} \mathbin{\widehat{\otimes}} C^{0,4} \cong C^{0,p} \mathbin{\widehat{\otimes}} C^{4,0} \cong C^{4,p} \cong \widehat{M}_{2^p}\big(C^{4-p,0}\big)$$

that are consequences of Corollary 7.4 and of the fact that $C^{4,0} \cong C^{0,4}$ as $\mathbb{Z}_2$-graded algebras and coalgebras. We then compute

$$C^{5,0} \cong \widehat{M}_2\big(C^{0,3}\big) \cong \widehat{M}_2(X \otimes W), \qquad C^{6,0} \cong \widehat{M}_4\big(C^{0,2}\big) \cong \widehat{M}_4(W),$$
$$C^{7,0} \cong \widehat{M}_8\big(C^{0,1}\big) \cong \widehat{M}_8(Z), \qquad C^{0,5} \cong \widehat{M}_2\big(C^{3,0}\big) \cong \widehat{M}_2(Y \otimes Z),$$
$$C^{0,6} \cong \widehat{M}_4\big(C^{2,0}\big) \cong \widehat{M}_4(Y), \qquad C^{0,7} \cong \widehat{M}_8\big(C^{1,0}\big) \cong \widehat{M}_8(X),$$

all the isomorphism being of $\mathbb{Z}_2$-graded algebras and coalgebras. Thus our proof is complete. □

We end by considering the case when k contains a primitive fourth root of unit. In this case we have proved in Remarks 4.3 that $\mathbb{H} \cong M_2(k)$ and $C^{1,1} \cong M_2(k)$, as algebras and coalgebras. As we will explain below we have $k[\mathbf{i}] \cong k[\mathbb{Z}_2] \cong k \times k$ as algebras and coalgebras, too.

Lemma 7.8. *If C, D are k-coalgebras then so is $C \times D$ with*

$$\Delta(c,d) = (c_1, 0) \otimes (c_2, 0) + (0, d_1) \otimes (0, d_2) \quad \textit{and} \quad \varepsilon(c,d) = \varepsilon(c) + \varepsilon(d),$$

for all $c \in C$, $d \in D$. Then for all $n \in \mathbb{N}$ we have $M_n(C \times D) \cong M_n(C) \times M_n(D)$, as k-coalgebras.

Proof. One can easily verify that with the above structure $C \times D$ is indeed a k-coalgebra. We have $M_n(C \times D) \cong M_n(C) \times M_n(D)$, as k-coalgebras, an isomorphism being defined by

$$\xi : M_n(C \times D) \ni E_{ij}(c,d) \mapsto \big(E_{ij}(c), E_{ij}(d)\big) \in M_n(C) \times M_n(D),$$

where $1 \leqslant i, j \leqslant n$, and where c, d run over a basis of C and D, respectively. Actually, we have

$$\begin{aligned}
\Delta\xi\big(E_{ij}(c,d)\big) &= \Delta\big(E_{ij}(c), E_{ij}(d)\big) \\
&= \sum_{s=1}^{n}\big(E_{is}(c_1), 0\big) \otimes \big(E_{sj}(c_2), 0\big) + \sum_{s=1}^{n}\big(0, E_{is}(d_1)\big) \otimes \big(0, E_{sj}(d_2)\big) \\
&= (\xi \otimes \xi)\left(\sum_{s=1}^{n} E_{is}(c_1, 0) \otimes E_{sj}(c_2, 0) + \sum_{s=1}^{n} E_{is}(0, d_1) \otimes E_{sj}(0, d_2)\right) \\
&= (\xi \otimes \xi)\Delta\big(E_{ij}(c,d)\big),
\end{aligned}$$

and $\varepsilon\xi(E_{ij}(c,d)) = \varepsilon(E_{ij}(c), E_{ij}(d)) = \varepsilon(E_{ij}(c)) + \varepsilon(E_{ij}(d)) = \delta_{i,j}(\varepsilon(c) + \varepsilon(d)) = \delta_{i,j}\varepsilon(c,d) = \varepsilon(E_{ij}(c,d))$, as required. □

Corollary 7.9. *We have that $C(1) \equiv k[\mathbb{Z}_2] \cong k \times k$ as algebras and coalgebras. Furthermore, if $\iota := \sqrt{-1} \in k$ we then have that $C(-1) \equiv k[\mathbf{i}] \cong k \times k$ as algebras and coalgebras, too.*

Proof. Consider $\{e_\pm\}$ the basis of $k[\mathbb{Z}_2]$ defined by the orthogonal idempotent elements $e_\pm = \frac{1}{2}(\bar{0} \pm \bar{1})$. We have computed in Remarks 3.5(2) that $\Delta(e_\pm) = e_\pm \otimes e_\pm$ and $\varepsilon(e_\pm) = 1$. It is then clear that the algebra isomorphism defined by $e_- \mapsto (1,0)$ and $e_+ \mapsto (0,1)$, extended by linearity, is a k-coalgebra isomorphism as well.

If $\iota := \sqrt{-1} \in k$ then $1 \mapsto (1,1)$ and $\mathbf{i} \mapsto (\iota, -\iota)$, extended by linearity, defines a k-algebra isomorphism between $k[\mathbf{i}]$ and $k \times k$ that will be denoted by χ. We have

$$\begin{aligned}
(\chi \otimes \chi)\Delta(1) &= \frac{1}{2}(\chi \otimes \chi)(1 \otimes 1 - \mathbf{i} \otimes \mathbf{i}) \\
&= \frac{1}{2}\big((1,1) \otimes (1,1) - (\iota, -\iota) \otimes (\iota, -\iota)\big) \\
&= \frac{1}{2}\Big(\big((1,0) + (0,1)\big) \otimes \big((1,0) + (0,1)\big) + \Big(\big((1,0) - (0,1)\big) \otimes \big((1,0) - (0,1)\big)\Big)\Big) \\
&= (1,0) \otimes (1,0) + (0,1) \otimes (0,1) = \Delta(1,1) = \Delta\chi(1),
\end{aligned}$$

and, similarly, $(\chi \otimes \chi)\Delta(\mathbf{i}) = \Delta\chi(\mathbf{i})$. Since $\varepsilon\chi(1) = \varepsilon(1,1) = \varepsilon(1) + \varepsilon(1) = 2$ and $\varepsilon\chi(\mathbf{i}) = \varepsilon(\iota) - \varepsilon(\iota) = 0$ we conclude that χ is a k-coalgebra isomorphism, as needed. □

So when $\iota := \sqrt{-1} \in k$ we can make in Theorem 7.7 the following identifications: $X = Z = k \times k$ and $Y = W = M_2(k)$, respectively (see Remarks 4.3). Hence the table in Theorem 7.7 simplifies as follows.

Corollary 7.10. *If k contains a root of -1 we then have $C^{p,0} \cong C^{0,p}$, for all $0 \leqslant p \leqslant 7$. Namely, we have the following algebra and coalgebra isomorphisms,*

$$C^{0,0}=k,\qquad C^{1,0}\cong C^{0,1}\cong k\times k,\qquad C^{2,0}\cong C^{0,2}\cong M_2(k),$$
$$C^{3,0}\cong C^{0,3}\cong M_2(k)\times M_2(k),\qquad C^{4,0}\cong C^{0,4}\cong M_4(k),$$
$$C^{5,0}\cong C^{0,5}\cong M_4(k)\times M_4(k),\qquad C^{6,0}\cong C^{0,6}\cong M_8(k),$$

and $C^{7,0}\cong C^{0,7}\cong M_8(k)\times M_8(k)$, *respectively.*

Proof. The isomorphisms tabulated in the statement can be derived from the above considerations and from the table obtained in Theorem 7.7. For instance,

$$C^{3,0}\cong \mathbb{H}\otimes k[\mathbb{Z}_2]\cong M_2(k)\otimes k[\mathbb{Z}_2]\cong M_2\big(k[\mathbb{Z}_2]\big)\cong M_2(k\times k)\cong M_2(k)\times M_2(k),$$
$$C^{0,3}\cong k[\mathbf{i}]\otimes C^{0,2}\cong k[\mathbf{i}]\otimes M_2(k)\cong M_2\big(k[\mathbf{i}]\big)\cong M_2(k\times k)\cong M_2(k)\times M_2(k),$$

and, similarly,

$$C^{5,0}\cong M_2\big((k\times k)\otimes M_2(k)\big)\cong M_2\big(M_2(k\times k)\big)\cong M_4(k)\times M_4(k),$$
$$C^{0,5}\cong M_2\big(M_2(k)\otimes (k\times k)\big)\cong M_4(k\times k)\cong M_4(k)\times M_4(k),\quad \text{etc.}$$

This ends the proof. □

Acknowledgment

I would like to express my sincere gratitude to Sorin Dăscălescu for various comments during the development of this paper and to the referee for the helpful comments.

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