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New results on $\mathcal{N}=4$ super-Yang–Mills theory

L. Baulieu [a,b,*], G. Bossard [a]

[a] LPTHE, CNRS and Universités Paris VI and Paris VII, 4 place Jussieu, F-75252 Paris cedex 05, France
[b] Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA

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Abstract

The $\mathcal{N}=4$ super-Yang–Mills theory is covariantly determined by two scalar and one vector BRST topological symmetry operators. This determines an off-shell closed sector of $\mathcal{N}=4$ super-Yang–Mills, with 6 generators, which is big enough to fully determine the theory, in a Lorentz-covariant way. This reduced algebra derives from horizontality conditions in four dimensions. The horizontality conditions only depend on the geometry of the Yang–Mills fields. They also descend from a genuine horizontality condition in eight dimensions. When the four-dimensional manifold is hyper-Kähler, one can perform a twist operation that defines the $\mathcal{N}=4$ supersymmetry and a $SL(2,\mathbb{H})$ intern symmetry (the "Euclidean version" of the $SU(4)$ R-symmetry in Minkowski space). These results extend in a covariant way the light-cone property that the $\mathcal{N}=4$ super-Yang–Mills theory is actually determined by only 8 independent generators, instead of the 16 generators that occur in the physical representation of the super-Poincaré algebra. The topological construction disentangles the off-shell closed sector of the (twisted) maximally supersymmetric theory from the sector that closes only modulo equations of motion. It allows one to escape the question of auxiliary fields in $\mathcal{N}=4$ super-Yang–Mills theory.

1. Introduction

Recently, we have constructed the genuine $\mathcal{N}=2$ supersymmetric algebra in four and eight dimensions in their twisted form, directly from extended horizontality conditions [1]. The new features were the geometrical construction of both scalar and vector topological BRST symmetries. A remarkable property is that the supersymmetry Yang–Mills algebra, both with 8 and 16 generators, contain an off-shell closed sector, with 5 and 9 generators, respectively, which is big enough to completely determine the theory. The key of the geometrical construction is the understanding that one must determine the scalar topological BRST symmetry in a way that is explicitly consistent with reparametrization symmetry. This yields the vector topological BRST symmetry in a purely geometrical way.

Here we will extend the result to the case of the maximally supersymmetric $\mathcal{N}=4$ algebra in four dimensions, with its 16 supersymmetric spinorial generators and $SL(2,\mathbb{H})$ internal symmetry.[1] The most determining phenomenon occurs when one computes the dimensional reduction from eight to seven dimensions, which provides an $SL(2,\mathbb{R})$ symmetry. This explains the organization of the Letter and the eventual obtaining of four-dimensional horizontality conditions, which determine the $\mathcal{N}=4$ algebra in a twisted form. We will also discuss the possible other twists of the supersymmetric theory.

As for the physical application of our construction, having obtained an off-shell closed algebra allows one to escape the question of auxiliary fields in $\mathcal{N}=4$ super-Yang–Mills theory. In a separate publication, using this algebra and its consequences, we will give an improved demonstration of the renormalization and finiteness of the $\mathcal{N}=4$ super-Yang–Mills theory [2].

* Corresponding author.
E-mail addresses: baulieu@lpthe.jussieu.fr (L. Baulieu), bossard@lpthe.jussieu.fr (G. Bossard).

[1] The internal symmetry group $SU(4)$ of $\mathcal{N}=4$ super-Yang–Mills, defined on a Minkowski space, must be replaced on a Euclidean one by $SL(2,\mathbb{H})\sim SO(5,1)$. This is implied by the fact that $\mathcal{N}=4$ super-Yang–Mills is the dimensional reduction of the ten-dimensional $\mathcal{N}=1$ super-Yang–Mills theory, which is only defined on Minkowski space.

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2. From the $D=8$ to the $D=7$ topological Yang–Mills theory

2.1. Determination of both topological scalar symmetries in seven dimensions

The $D=8$ topological Yang–Mills theory relies on the following horizontality equation, completed with its Bianchi identity [1]:

$$\begin{aligned}&(d+s+\delta-i_\kappa)\big(A+c+|\kappa|\bar c\big)+\big(A+c+|\kappa|\bar c\big)^2\\&\quad=F+\Psi+g(\kappa)\eta+i_\kappa\chi+\Phi+|\kappa|^2\bar\Phi.\end{aligned}\tag{1}$$

One has the closure relations:

$$s^2=\delta^2=0,\qquad \{s,\delta\}=\mathcal{L}_\kappa.\tag{2}$$

We refer to [1] for a detailed explanation of these formula and the twisted fields that they involve. Ψ is a 1-form topological ghost and χ is an antiselfdual 2-form in eight dimensions, with 7 independent components. Selfduality exists in eight dimensions when the manifold has a holonomy group included in $Spin(7)$. The octonionic invariant 4-form of such a manifold allows one to define selfduality, by the decomposition of a 2-form as $\mathbf{28}=\mathbf{7}\oplus\mathbf{21}$. κ is a covariantly constant vector, which exists if the holonomy group is included in $G_2\subset Spin(7)$. Φ and $\bar\Phi$ are, respectively, a topological scalar ghost of ghost and an antighost for antighost. c and $\bar c$ can be interpreted as the Faddeev–Popov ghost and antighost of the eight-dimensional Yang–Mills field A. s and δ are the scalar and vector topological BRST operators. In flat space, one can write $\delta+|\kappa|\delta_{\text{gauge}}(\bar c)=\kappa^\mu Q_\mu$, and we understand that $Q=s+\delta_{\text{gauge}}(c)$ and Q_μ count for 9 independent generators, giving an off-shell closed sector of $\mathcal{N}=2$, $D=8$ twisted supersymmetry, which fully determines the theory [1].

To determine the topological symmetry in seven dimensions, we start from a manifold in eight dimensions that is reducible, $M_8=N_7\times S^1$, where N is a G_2-manifold. We can chose κ as a tangent vector to S^1. Thus, the 8-dimensional vector symmetry along the circle reduces to a scalar one in seven dimensions, $\bar s$, and the reduction of the antiselfdual 2-form χ gives a seven-dimensional 1-form $\bar\Psi$. So, the dimensional reduction of the horizontality condition (1) is

$$\begin{aligned}&(d+s+\bar s)(A+c+\bar c)+(A+c+\bar c)^2\\&\quad=F+\Psi+\bar\Psi+\Phi+L+\bar\Phi.\end{aligned}\tag{3}$$

Indeed, with our choice of κ, $L=i_\kappa A=A_8$ in eight dimensions. Eq. (3) can be given a different interpretation than Eq. (1). It contains no vector symmetry and looks like a standard 7-dimensional BRST–antiBRST equation. The transformation of L, the origin of which is A_8, is now given by the Bianchi identity of Eq. (3). In fact, after dimensional reduction, L is understood as a curvature. Using the convenient pyramidal diagrammatic description of ghost–antighost structures, we can rewrite the field description, as follows:

$$\begin{array}{ccccc} & & A,L & & \\ & \Psi,\bar\eta & & \bar\Psi & \\ \Phi & & & & \bar\Phi \\ & & & \eta & \end{array}\ \longrightarrow\ \begin{array}{ccccc} & & A & & \\ & \Psi & & \bar\Psi & \\ \Phi & & L & & \bar\Phi \\ & \bar\eta & & \eta & \end{array}\tag{4}$$

As compared to the asymmetrical diagram on the left-hand side, the one on the right-hand side exhibits an $SL(2,\mathbb{R})$ symmetry, which counts the ghost–antighost numbers. Indeed, each line of this diagram corresponds to an irreducible representation of $SL(2,\mathbb{R})$, namely, a completely symmetrical $SL(2,\mathbb{R})$-spinorial tensor with components $\phi^{g,G-g}$, where g and $G-g$ are, respectively, the (positive) ghost and antighost numbers of ϕ, and $2g-G$ is the effective ghost number of ϕ. This $SL(2,\mathbb{R})$ symmetry actually applies to the covariant ghost–antighost spectrum of a p-form gauge field ϕ_p, $\tilde\phi_p=\sum_{0\leqslant G\leqslant p}\sum_{0\leqslant g\leqslant G}\phi^{g,G-g}_{p-G}$.

In fact, the fields $(\Psi,\bar\Psi)$ and $(\eta,\bar\eta)$ can be identified as $SL(2,\mathbb{R})$ doublets, Ψ^α and η^α, $\alpha=1,2$, and the three scalar fields $(\Phi,L,\bar\Phi)$ as a $SL(2,\mathbb{R})$ triplet, Φ^i, $i=1,2,3$. The index α and i are, respectively, raised and lowered by the volume form $\varepsilon_{\alpha\beta}$ of $SL(2,\mathbb{R})$ and the Minkowski metric η_{ij} of signature $(2,1)$. Both BRST and antiBRST operators can be assembled into a $SL(2,\mathbb{R})$ doublet $s^\alpha=(s,\bar s)$.

The horizontality condition (3) can be solved, with the introduction of three 0-form Lagrange multipliers, η, $\bar\eta$, b and a 1-form T:

$$\begin{aligned}
&sA=\Psi-d_Ac, && \bar sA=\bar\Psi-d_A\bar c,\\
&s\Psi=-d_A\Phi-[c,\Psi], && \bar s\Psi=-T-d_AL-[\bar c,\Psi],\\
&s\Phi=-[c,\Phi], && \bar s\Phi=-\bar\eta-[\bar c,\Phi],\\
&s\bar\Phi=\eta-[c,\bar\Phi], && \bar s\Phi=-[\bar c,\bar\Phi],\\
&s\eta=[\Phi,\bar\Phi]-[c,\eta], && \bar s\eta=-[\bar\Phi,L]-[\bar c,\eta],\\
&sL=\bar\eta-[c,L], && \bar sL=-\eta-[\bar c,L],\\
&s\bar\eta=[\Phi,L]-[c,\bar\eta], && \bar s\bar\eta=[\Phi,\bar\Phi]-[\bar c,\bar\eta],\\
&s\bar\Psi=T-[c,\bar\Psi], && \bar s\bar\Psi=-d_A\bar\Phi-[\bar c,\bar\Psi],\\
&sT=[\Phi,\bar\Psi]-[c,T], && \bar sT=-d_A\eta+[L,\bar\Psi]-[\bar\Phi,\Psi]\\
& && \qquad-[\bar c,T],\\
&sc=\Phi-c^2, && \bar sc=L-b,\\
&s\bar c=b-[c,\bar c], && \bar s\bar c=\bar\Phi-\bar c^2,\\
&sb=[\Phi,\bar c]-[c,\bar c], && \bar sb=\eta+[\bar c,L].
\end{aligned}\tag{5}$$

These equations are not $SL(2,\mathbb{R})$-covariant because of our simplest choice of the transformation of antighosts transformations, like $s\bar c=b-[c,\bar c]$. By suitable redefinitions of auxiliary fields, one can get, however, a manifestly $SL(2,\mathbb{R})$-covariant formulation of the symmetry, as follows:

$$\begin{aligned}
&s^\alpha A=\Psi^\alpha-d_Ac^\alpha,\\
&s^\alpha\eta_\beta=-2\sigma^{ij}{}_\beta{}^\alpha[\Phi_i,\Phi_j]-\big[c^\alpha,\eta_\beta\big],\\
&s^\alpha\Psi_\beta=\delta^\alpha_\beta T-\sigma^i{}_\beta{}^\alpha d_A\Phi_i-\big[c^\alpha,\Psi_\beta\big],\\
&s^\alpha T=\frac{1}{2}d_A\eta^\alpha+\sigma^{i\alpha\beta}[\Phi_i,\Psi_\beta]-\big[c^\alpha,T\big],
\end{aligned}$$

$$
\begin{aligned}
s^\alpha \Phi_i &= \frac{1}{2}\sigma_i{}^{\alpha\beta}\eta_\beta - [c^\alpha, \Phi_i],\\
s^\alpha c_\beta &= -\delta^\alpha_\beta b + \sigma^i{}_\beta{}^\alpha \Phi_i - \frac{1}{2}[c^\alpha, c_\beta],\\
s^\alpha b &= -\frac{1}{2}\eta^\alpha + \frac{1}{2}\sigma^{i\,\alpha\beta}[\Phi_i, c_\beta] + \frac{1}{12}[c^\beta, [c_\beta, c^\alpha]]\\
&\quad - \frac{1}{2}[c^\alpha, b].
\end{aligned}
\tag{6}
$$

The Cartan algebra (that we will denote by the subindex $_{(c)}$) is obtained by adding gauge transformations with parameters c and $\bar{c}$, from s and $\bar{s}$, respectively. It reads:

$$
\begin{aligned}
s^\alpha_{(c)} A &= \Psi^\alpha,\\
s^\alpha_{(c)} \eta_\beta &= -2\sigma^{ij}{}_\beta{}^\alpha [\Phi_i, \Phi_j],\\
s^\alpha_{(c)} \Psi_\beta &= \delta^\alpha_\beta T - \sigma^i{}_\beta{}^\alpha d_A \Phi_i,\\
s^\alpha_{(c)} T &= \frac{1}{2} d_A \eta^\alpha + \sigma^{i\alpha\beta}[\Phi_i, \Psi_\beta],\\
s^\alpha_{(c)} \Phi_i &= \frac{1}{2}\sigma_i{}^{\alpha\beta}\eta_\beta.
\end{aligned}
\tag{7}
$$

The equivariant (Cartan) algebra is the one that will match by twist with the relevant part of the twisted supersymmetry algebra. Its closure is only modulo gauge transformations, with parameters that are ghosts of ghosts.

2.2. *Determination of the equivariant part of both topological vector symmetries in seven dimensions*

We have produced by dimensional reduction a new scalar (antiBRST) topological symmetry operator. However, we have apparently lost the rest of the vector symmetry in eight dimensions, since we have chosen κ along the circle of dimensional reduction. The freedom of choosing this circle generates an automorphism of the seven-dimensional symmetry. It allows one to obtain two vector topological symmetries, in seven dimensions, in a $SL(2,\mathbb{R})$ symmetric way.

In the genuine theory in eight dimensions, the definition of a chosen $Spin(7)$ structure is actually arbitrary. The different choices of a $Spin(7)$ structure on the eight-dimensional Riemann manifold can be parametrized by the transformations of $SO(8)\setminus Spin(7)$. However, such transformations cannot be represented on eight-dimensional fields such as the antiselfdual 2-form $\chi_{\mu\nu}$. They are generated by antiselfdual 2-form infinitesimal parameters, which can be parametrized by seven-dimensional vectors, after dimensional reduction to 7 dimensions. One can thus define the following (commuting) 7-dimensional derivation $\boldsymbol{\gamma}$, which depends on a covariantly constant seven-dimensional vector κ^μ (the indices $\mu, \nu, \ldots$ are seven-dimensional), and acts as follows on the 7-dimensional fermion fields:

$$
\begin{aligned}
\boldsymbol{\gamma}\Psi^\alpha_\mu &= -\kappa_\mu \eta^\alpha - C_{\mu\nu}{}^\sigma \kappa^\nu \Psi^\alpha_\sigma,\\
\boldsymbol{\gamma}\eta^\alpha &= \kappa^\mu \Psi^\alpha_\mu.
\end{aligned}
\tag{8}
$$

The action of $\boldsymbol{\gamma}$ is zero on the bosonic field of the equivariant BRST algebra, but on the Lagrange multiplier field T. (We will shortly define $\boldsymbol{\gamma}T$, by consistency.) $C_{\mu\nu\sigma}$ is the seven-dimensional G_2-invariant octonionic 3-form and its dual is the 4-form $C^\star_{\mu\nu\sigma\rho}$. We will use the notation $i_\kappa \star C^\star w_1 = -C_{\mu\nu}{}^\sigma \kappa^\nu w_\sigma\, dx^\mu$. The derivation $\boldsymbol{\gamma}$ expresses the arbitrariness in the choice of an eighth component, in order to perform the dimensional reduction. To each constant vector on N, one can assign an $U(1)$ group, which is subset of $SO(8)\setminus Spin(7)$. Since $\{s^\alpha_{(c)}, s^\beta_{(c)}\} = \sigma^{i\alpha\beta}\delta_{\text{gauge}}(\Phi_i)$, one can verify on all fields:

$$
\{e^{t\boldsymbol{\gamma}} s^\alpha_{(c)} e^{-t\boldsymbol{\gamma}}, e^{t\boldsymbol{\gamma}} s^\beta_{(c)} e^{-t\boldsymbol{\gamma}}\} = \{s^\alpha_{(c)}, s^\beta_{(c)}\}.
\tag{9}
$$

One defines the vector operator:

$$
\delta^\alpha_{(c)} \equiv [s^\alpha_{(c)}, \boldsymbol{\gamma}].
\tag{10}
$$

Since $[[s^\alpha_{(c)}, \boldsymbol{\gamma}], \boldsymbol{\gamma}] = -s^\alpha_{(c)}$, one has:

$$
e^{t\boldsymbol{\gamma}} s^\alpha_{(c)} e^{-t\boldsymbol{\gamma}} = \cos t\, s^\alpha_{(c)} - \sin t\, \delta^\alpha_{(c)}.
\tag{11}
$$

To ensure the validity of this formula on T, one defines the transformation of the auxiliary field T as follows:

$$
\boldsymbol{\gamma}T = -i_\kappa F - 2 i_\kappa \star C^\star T.
\tag{12}
$$

Computing the commutators of $s^\alpha_{(c)}$ and $\boldsymbol{\gamma}$, one gets the action of $\delta^\alpha_{(c)}$:

$$
\begin{aligned}
\delta^\alpha_{(c)} A &= g(\kappa)\eta^\alpha + i_\kappa \star C^\star \Psi^\alpha,\\
\delta^\alpha_{(c)} \Psi_\beta &= \delta^\alpha_\beta i_\kappa (F + \star C^\star T) + \sigma^i{}_\beta{}^\alpha i_\kappa \star C^\star d_A \Phi_i\\
&\quad - 2g(\kappa)\sigma^{ij}{}_\beta{}^\alpha [\Phi_i, \Phi_j],\\
\delta^\alpha_{(c)} \Phi_i &= -\frac{1}{2}\sigma_i{}^{\alpha\beta} i_\kappa \Psi_\beta,\\
\delta^\alpha_{(c)} \eta_\beta &= -\delta^\alpha_\beta i_\kappa T + \sigma^i{}_\beta{}^\alpha \mathcal{L}_\kappa \Phi_i,\\
\delta^\alpha_{(c)} T &= \frac{1}{2} d_A i_\kappa \Psi^\alpha - i_\kappa \star C^\star d_A \eta^\alpha + g(\kappa)\sigma^{i\alpha\beta}[\Phi_i, \eta_\beta]\\
&\quad - \sigma^{i\alpha\beta} i_\kappa \star C^\star [\Phi_i, \Psi_\beta] - \mathcal{L}_\kappa \Psi^\alpha.
\end{aligned}
\tag{13}
$$

$\delta^\alpha_{(c)}$ is a pair of two vector symmetries in seven dimensions, which transform as an $SL(2,\mathbb{R})$-doublet. This completes the scalar doublet $s^\alpha_{(c)}$.

Then, one can verify that the anticommutation relations for the vector operator $\delta^\alpha_{(c)}$ are:

$$
\begin{aligned}
\{s^\alpha_{(c)}, \delta^\beta_{(c)}\} &= \varepsilon^{\alpha\beta}(\mathcal{L}_\kappa + \delta_{\text{gauge}}(i_\kappa A)),\\
\{\delta^\alpha_{(c)}, \delta^\beta_{(c)}\} &= 2\sigma^{i\alpha\beta}\delta_{\text{gauge}}(\Phi_i).
\end{aligned}
\tag{14}
$$

Reciprocally, these closure relations uniquely determine $\delta^\alpha_{(c)}$, from the knowledge of $s^\alpha_{(c)}$.

In the next section, we will rederive these transformation laws, from horizontality equations. Moreover, we will extend them as nilpotent transformations, by including gauge transformations.

It is instructive to check the expression we have just obtained for $\delta^\alpha_{(c)}$, by starting from Eq. (1) in 8 dimensions (with the notational change $\bar{c} \to \gamma$), and computing the dimensional reduction with a vector κ along N, instead of along the circle. This gives:

$$
\delta\bar{\Phi} = -[|\kappa|\gamma, \bar{\Phi}],
$$

$$\begin{aligned}
\delta A &= g(\kappa)\eta + i_\kappa \star C^\star \bar{\Psi} - |\kappa| d_A \gamma, \\
\delta \eta &= \mathcal{L}_\kappa \bar{\Phi} - \big[|\kappa|\gamma, \eta\big], \\
\delta \Psi &= i_\kappa (F - \star C^\star T) + g(\kappa)[\Phi, \bar{\Phi}] - \big[|\kappa|\gamma, \Psi\big], \\
\delta L &= i_\kappa \bar{\Psi} - \big[|\kappa|\gamma, L\big], \\
\delta \Phi &= i_\kappa \Psi - \big[|\kappa|\gamma, \Phi\big], \\
\delta \bar{\eta} &= i_\kappa (T + d_A L) - \big[|\kappa|\gamma, \bar{\eta}\big], \\
\delta \bar{\Psi} &= i_\kappa \star C^\star d_A \bar{\Phi} - g(\kappa)[\bar{\Phi}, L] - \big[|\kappa|\gamma, \bar{\Psi}\big], \\
\delta T &= \mathcal{L}_\kappa \bar{\Psi} + i_\kappa \star C^\star \big(d_A \eta + [\bar{\Phi}]\big) + g(\kappa)[L, \eta] \\
&\quad - g(\kappa)[\bar{\Phi}, \bar{\eta}] - \big[|\kappa|\gamma, T\big].
\end{aligned} \tag{15}$$

One can then verify:

$$[\bar{s}, \boldsymbol{\gamma}] = \delta \tag{16}$$

with the modified definition that

$$\begin{aligned}
&\boldsymbol{\gamma} T = i_\kappa F - 2 i_\kappa \star C^\star T - i_\kappa \star C^\star d_A L, \\
&\boldsymbol{\gamma} \bar{c} = -|\kappa|\gamma, \qquad \boldsymbol{\gamma}\gamma = \frac{1}{|\kappa|}\bar{c}.
\end{aligned} \tag{17}$$

The difference between this expression of $\delta + |\kappa|\delta_{\text{gauge}}(\gamma)$ and that of a component of the $SL(2,\mathbb{R})$-covariant vector symmetry operators $\delta^\alpha_{(c)}$ is just a field redefinition.

2.3. *The complete Faddeev–Popov ghost dependent vector and scalar topological symmetries in seven dimensions*

We now directly construct the scalar and vector BRST topological operators s^α and δ^α, the equivariant analogs of which are $s^\alpha_{(c)}$ and $\delta^\alpha_{(c)}$. One needs scalar Faddeev–Popov ghosts, c, $\bar{c}$, γ, $\bar{\gamma}$, which are associated to the equivariant BRST operators $s_{(c)}$, $\bar{s}_{(c)}$, $\delta_{(c)}$, $\bar{\delta}_{(c)}$, respectively.

The relations (14) suggests the following horizontality condition, with $(d + s + \bar{s} + \delta + \bar{\delta})^2 = 0$:

$$\begin{aligned}
&(d + s + \bar{s} + \delta + \bar{\delta})\big(A + c + \bar{c} + |\kappa|\gamma + |\kappa|\bar{\gamma}\big) \\
&\quad + \big(A + c + \bar{c} + |\kappa|\gamma + |\kappa|\bar{\gamma}\big)^2 \\
&\qquad = F + \Psi + \bar{\Psi} + g(\kappa)(\eta + \bar{\eta}) + i_\kappa \star C^\star(\Psi + \bar{\Psi}) \\
&\qquad\quad + \big(1 + |\kappa|^2\big)(\Phi + L + \bar{\Phi}).
\end{aligned} \tag{18}$$

It is $SL(2,\mathbb{R})$- and $\boldsymbol{\gamma}$-invariant. By construction, this equation has the following indetermination:

$$\{s, \delta\} + \{\bar{s}, \bar{\delta}\} = 0. \tag{19}$$

This degeneracy is raised, owing to the introduction of the constant vector κ, with

$$\{s, \delta\} = \mathcal{L}_\kappa, \qquad \{\bar{s}, \bar{\delta}\} = -\mathcal{L}_\kappa. \tag{20}$$

This relation is fulfilled by completing Eq. (18) by the following ones:

$$\begin{aligned}
&(d + s + \delta - i_\kappa)\big(A + c + |\kappa|\gamma\big) + \big(A + c + |\kappa|\gamma\big)^2 \\
&\quad = F + \Psi + g(\kappa)\eta + i_\kappa \star C^\star(\bar{\Psi}) + \big(\Phi + |\kappa|^2 \bar{\Phi}\big),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&(d + \bar{s} + \bar{\delta} + i_\kappa)\big(A + \bar{c} + |\kappa|\bar{\gamma}\big) + \big(A + \bar{c} + |\kappa|\bar{\gamma}\big)^2 \\
&\quad = F + \bar{\Psi} + g(\kappa)\bar{\eta} + i_\kappa \star C^\star(\Psi) + \big(\bar{\Phi} + |\kappa|^2 \Phi\big).
\end{aligned} \tag{22}$$

The $SL(2,\mathbb{R})$- and $\boldsymbol{\gamma}$-invariant equations (18), (21), (22) are consistent, but not independent. Only one of Eqs. (21) or (22) is needed to complete Eq. (18). One introduces the b field as usual in order to solve equations of the type $s\bar{c} + \bar{s}c + \cdots = 0$. One can verify that, by expansion in ghost number, the horizontality conditions reproduce the transformations (5) and (15).

We note that the number of symmetries carried by the generators $s^\alpha_{(c)}$ and $\delta^\alpha_{(c)}$ is $1 + 1 + 7 + 7 = 16$. They yield, by untwisting in flat space, the complete set of Poincaré supersymmetry generators (we do not reproduce here this lengthy computation). We can understand the seven generators determined by $\bar{\delta}$ as the dimensional reduction of the twisted antiselfdual generator of $\mathcal{N} = 2$, $D = 8$ supersymmetry, and $\bar{s}$ and δ as the dimensional reduction of the twisted vector supersymmetry in 8 dimensions. Only a maximum of $9 = 1 + 1 + 7$ generators, among the 16 ones that are determined by s, $\bar{s}$, δ, $\bar{\delta}$, build an off-shell closed algebra, since both operators $\delta^\alpha_{(c)}$ depend on a single vector κ. In fact, the commutation relations of the vector operators Q_μ and $\bar{Q}_\nu$, where $\delta_{(c)} = \kappa^\mu Q_\mu$ and $\bar{\delta}_{(c)} = \kappa^\mu \bar{Q}_\mu$, yield non-closure terms for their antisymmetric part in μ, ν. One can choose Q, $\bar{Q}$, Q_μ as such a maximal subalgebra.

Dimensional reduction therefore transforms the $\mathcal{N}_T = 1$ eight-dimensional theory into a $\mathcal{N}_T = 2$ theory, with an $SL(2,\mathbb{R})$ internal symmetry, and a $G_2 \subset Spin(7)$ Lorentz symmetry. As a matter of fact, this algebra gives the $SL(2,\mathbb{R})$-invariant twisted supersymmetry transformations [4] in the limit of flat manifold.

2.4. *Seven-dimensional invariant action*

The most general gauge-invariant topological gauge function $\boldsymbol{\Psi}$, which yields a δ-invariant action $S = s\boldsymbol{\Psi} - \frac{1}{2}\int_M C_\wedge \operatorname{Tr} F_\wedge F$, is:

$$\begin{aligned}
\boldsymbol{\Psi} = 2 \int_M &\operatorname{Tr}\big(C^\star_\wedge \bar{\Psi}_\wedge F + \bar{\Psi} \star (d_A L + T) \\
&+ \Psi \star d_A \bar{\Phi} + \star\eta[\Phi, \bar{\Phi}] + \star\bar{\eta}[\bar{\Phi}, L]\big).
\end{aligned} \tag{23}$$

This function turns out to be $\bar{s}$-exact and thus $\bar{s}$-invariant. Moreover, S is $\bar{\delta}$-invariant. By using the $SL(2,\mathbb{R})$-covariant form of the algebra, we can compute the gauge function $\boldsymbol{\Psi}$ in a manifestly invariant way, as a component of a doublet $\boldsymbol{\Psi}_\alpha$. One has $\sigma^{i\alpha\beta}\delta_\alpha \boldsymbol{\Psi}_\beta = 0$ and $\boldsymbol{\Psi}_\alpha = \delta_\alpha \mathcal{G}$. $S = s^\alpha \boldsymbol{\Psi}_\alpha - \frac{1}{2}\int_M C_\wedge \operatorname{Tr} F_\wedge F$, and, with our conventions, $\boldsymbol{\Psi} \propto \boldsymbol{\Psi}_1$. The automorphism generated by $\boldsymbol{\gamma}$ leaves invariant neither the gauge function, nor the action, since its action breaks the $Spin(7)$-structure of the 8-dimensional theory. These properties will remain analogous in 4 dimensions, and we will give more details in this case.

3. Reduction to four dimensions and $\mathcal{N} = 4$ theory

The process of dimensional reduction can be further done, from 7 dimensions. The ghost–antighost symmetry that has appeared when going down from 8 to 7 dimensions will continue to hold true, and therefore, one remains in the framework of an $\mathcal{N}_T = 2$ theory, with $SL(2,\mathbb{R})$ invariance.

One is concerned by going down from seven to four dimensions. The $SO(7)$ symmetry is decomposed into $SO(4) \times$

$SO(3) \sim SU(2) \times SU(2) \times SU(2)$, and insides this decomposition, the G_2 symmetry is decomposed into $SU(2) \times \mathrm{diag}(SU(2) \times SU(2))$. So, the (twisted) 4-dimensional theory has a $Spin(4)$ Lorentz symmetry, with an $SL(2,\mathbb{R})$ internal symmetry.

It is useful to use a hyper-Kähler structure to simplify the form of the equations. For instance, given the 3 constant hyper-Kähler 2-forms $J^I_{\mu\nu}$, an antiselfdual 2-form $h_{\mu\nu}$ can be written as $h_{\mu\nu} = h_I J^I_{\mu\nu}$, where h_I is a $SU(2)$ triplet made of scalars. Capital indices as I are devoted to the adjoint representation of the chiral $SU(2)$ factor of the Lorentz symmetry (which leaves invariant self-dual 2-forms), the scalars h^I correspond to A_7, A_6, A_5, etc. This allows simplified expressions for scalars, such as, for instance, $\varepsilon_{IJK} h^I h^J h^K$ instead of $h_\mu{}^\nu h_\nu{}^\sigma h_\sigma{}^\mu$. Moreover, one is interested in obtaining by twist the $\mathcal{N}=4$ super-Yang–Mills theory. This is a justification of the restricted choice of a hyper-Kähler manifold, since two constant spinors are needed to perform the twist operation and eventually to map the topological ghosts on spinors. In the untwisted theory, the bosonic fields h^I is in the representation of the $SU(2) \subset SL(2,\mathbb{H})$ R-symmetry.

3.1. Equivariant scalar and vector algebra in four dimensions

By dimensional reduction of the seven-dimensional equations of Section 2, one can compute the Cartan $SL(2,\mathbb{R})$ doublet of scalar topological BRST operators for the topological symmetry in four dimensions:

$$\begin{aligned}
&s^\alpha_{(c)} A = \Psi^\alpha, \\
&s^\alpha_{(c)} \Psi_\beta = \delta^\alpha_\beta T - \sigma^i{}_\beta{}^\alpha d_A \Phi_i, \\
&s^\alpha_{(c)} h^I = \chi^{\alpha I}, \\
&s^\alpha_{(c)} \Phi_i = \frac{1}{2}\sigma_i{}^{\alpha\beta}\eta_\beta, \\
&s^\alpha_{(c)} \chi^I_\beta = \delta^\alpha_\beta H^I + \sigma^i{}_\beta{}^\alpha\big[\Phi_i, h^I\big], \\
&s^\alpha_{(c)} \eta_\beta = -2\sigma^{ij}{}_\beta{}^\alpha[\Phi_i, \Phi_j], \\
&s^\alpha_{(c)} H^I = \frac{1}{2}\big[\eta^\alpha, h^I\big] + \sigma^{i\alpha\beta}\big[\Phi_i, \chi^I_\beta\big], \\
&s^\alpha_{(c)} T = \frac{1}{2} d_A \eta^\alpha + \sigma^{i\alpha\beta}[\Phi_i, \Psi_\beta].
\end{aligned} \tag{24}$$

One has the closure relation $s^{\{\alpha}_{(c)} s^{\beta\}}_{(c)} = \sigma^{i\alpha\beta}\delta_{\mathrm{gauge}}(\Phi_i)$. The Cartan vector algebra is:

$$\begin{aligned}
&\delta^\alpha_{(c)} A = g(\kappa)\eta^\alpha + g(J_I\kappa)\chi^{\alpha I}, \\
&\delta^\alpha_{(c)} \Psi_\beta = \delta^\alpha_\beta\big(i_\kappa F - g(J_I\kappa)H^I\big) + \sigma^i{}_\beta{}^\alpha g(J_I\kappa)\big[\Phi_i, h^I\big] \\
&\qquad - 2\sigma^{ij}{}_\beta{}^\alpha g(\kappa)[\Phi_i, \Phi_j], \\
&\delta^\alpha_{(c)} \Phi_i = -\frac{1}{2}\sigma_i{}^{\alpha\beta} i_\kappa \Psi_\beta, \\
&\delta^\alpha_{(c)} \eta_\beta = -\delta^\alpha_\beta i_\kappa T + \sigma^i{}_\beta{}^\alpha \mathcal{L}_\kappa \Phi_i, \\
&\delta^\alpha_{(c)} T = \frac{1}{2} d_A i_\kappa \Psi^\alpha - g(J_I\kappa)\big(\big[\eta^\alpha, h^I\big] + \sigma^{i\alpha\beta}\big[\Phi_i, \chi^I_\beta\big]\big) \\
&\qquad + g(\kappa)\sigma^{i\alpha\beta}[\Phi_i, \eta_\beta] - \mathcal{L}_\kappa \Psi^\alpha, \\
&\delta^\alpha_{(c)} h^I = -i_{J^I\kappa}\Psi^\alpha, \\
&\delta^\alpha_{(c)} \chi^I_\beta = \delta^\alpha_\beta\big(\mathcal{L}_\kappa h^I + i_{J^I\kappa} T\big) + \sigma^i{}_\beta{}^\alpha \mathcal{L}_{J^I\kappa}\Phi_i, \\
&\delta^\alpha_{(c)} H^I = \frac{1}{2}\big[i_\kappa \Psi^\alpha, h^I\big] + \mathcal{L}_{J^I\kappa}\eta^\alpha + \sigma^{i\alpha\beta}[\Phi_i, i_{J^I\kappa}\Psi_\beta] \\
&\qquad - \mathcal{L}_\kappa \chi^{\alpha I}.
\end{aligned} \tag{25}$$

One has, $\delta^{\{\alpha}_{(c)}\delta^{\beta\}}_{(c)} = |\kappa|^2\sigma^{i\alpha\beta}\delta_{\mathrm{gauge}}(\Phi_i)$, and $\delta^\alpha_{(c)}$ anticommute with $s^\alpha_{(c)}$, as follows[2]:

$$\big\{s^\alpha_{(c)}, \delta^\beta_{(c)}\big\} = \varepsilon^{\alpha\beta}\big(\mathcal{L}_\kappa + \delta_{\mathrm{gauge}}(i_\kappa A)\big). \tag{26}$$

The four-dimensional vector operators are s^α-exact, as in seven dimensions, $\delta^\alpha_{(c)} = [s^\alpha_{(c)}, \boldsymbol{\gamma}]$, where the non-zero component of $\boldsymbol{\gamma}$ are

$$\begin{aligned}
&\boldsymbol{\gamma}\Psi_\alpha = -g(\kappa)\eta_\alpha - g(J_I\kappa)\chi^I_\alpha, \\
&\boldsymbol{\gamma} T = -i_\kappa F + 2g(J_I\kappa)H^I, \\
&\boldsymbol{\gamma}\eta_\alpha = i_\kappa \Psi_\alpha, \\
&\boldsymbol{\gamma} H^I = -\mathcal{L}_\kappa h^I - 2 i_{J^I\kappa} T, \\
&\boldsymbol{\gamma}\chi^I_\alpha = i_{J^I\kappa}\Psi_\alpha.
\end{aligned} \tag{27}$$

As in seven dimensions, one has a $U(1)$ automorphism of the algebra, which is not a symmetry of the theory, with $e^{t\boldsymbol{\gamma}} s^\alpha_{(c)} e^{-t\boldsymbol{\gamma}} = \cos t\, s^\alpha_{(c)} - \sin t\, \delta^\alpha_{(c)}$ and $e^{t\boldsymbol{\gamma}} \delta^\alpha_{(c)} e^{-t\boldsymbol{\gamma}} = \cos t\, \delta^\alpha_{(c)} + \sin t\, s^\alpha_{(c)}$.

3.2. Invariant action

There are two gauge functions which fit in a fundamental multiplet of $SL(2,\mathbb{R})$, and satisfy:

$$\sigma^{i\alpha\beta}\delta_{(c)\alpha}\boldsymbol{\Psi}_\beta = 0. \tag{28}$$

The action is defined as:

$$S = -\frac{1}{2}\int_M \mathrm{Tr}\, F_\wedge F + s^\alpha_{(c)}\boldsymbol{\Psi}_\alpha. \tag{29}$$

Eq. (28) completely constrains the gauge function (up to a global scale factor), as follows:

$$\begin{aligned}
\boldsymbol{\Psi}_\alpha = \int_M \mathrm{Tr}\Big(&\star\chi^I_\alpha H_I + \chi^I_\alpha J_I \star F - \Psi_\alpha \star T + J^I \star \Psi_{\alpha\wedge} d_A h_I \\
&- \sigma^i{}_\alpha{}^\beta \Psi_\beta \star d_A \Phi_i - 2\star\sigma^{ij}{}_\alpha{}^\beta \eta_\beta[\Phi_i, \Phi_j] \\
&- \star\sigma^i{}_\alpha{}^\beta \chi^I_\beta[\Phi_i, h_I] + \frac{1}{2}\star\varepsilon_{IJK}\chi^I_\alpha\big[h^J, h^K\big]\Big).
\end{aligned} \tag{30}$$

The action (29) is δ^α- and s^α-invariant. Indeed, one can check that it verifies:

$$S = -\frac{1}{2}\int_M \mathrm{Tr}\, F_\wedge F + s^\alpha_{(c)} s_{(c)\alpha}\mathcal{F} = -\frac{1}{2}\int_M \mathrm{Tr}\, F_\wedge F + s^\alpha_{(c)}\delta_{(c)\alpha}\mathcal{G} \tag{31}$$

[2] Note that, the existence of a covariantly constant vector field on a hyper-Kähler manifold M implies that M is flat. This explains the possible use of a complete quaternionic base of the tangent space TM: κ, $J^I\kappa$ for the description of the vector symmetry.

with

$$\mathcal{F}=\int_M \mathrm{Tr}\bigg(\star h_I H^I + h_I J^I\star F+\frac{1}{3}\varepsilon_{IJK}h^I h^J h^K - \frac{1}{2}\Psi^\alpha\star\Psi_\alpha+\frac{1}{2}\star\eta^\alpha\eta_\alpha\bigg), \tag{32}$$

$$\mathcal{G}=\int_M \mathrm{Tr}\bigg(-\frac{1}{2}g(\kappa)_\wedge\Big((A-\overset{\circ}{A})_\wedge(F+\overset{\circ}{F})-\frac{1}{3}(A-\overset{\circ}{A})^3\Big) - \frac{1}{2}\star\varepsilon_{IJK}h^I\mathcal{L}_{J^J\kappa}h^K+\star s^\alpha_{(c)}\delta_{(c)\alpha}\Big(\frac{1}{2}h_I h^I-\frac{2}{3}\Phi^i\Phi_i\Big)\bigg). \tag{33}$$

These facts remind us that we are in the context of a $\mathcal{N}_T=2$ theory. The critical points of the Morse function $\mathcal{F}$ in the field space are given by the equations

$$\begin{aligned}&J^I\star F+\frac{1}{2}\star\varepsilon^I{}_{JK}\big[h^J,h^K\big]=0,\\ &d_A\star h_I J^I=0.\end{aligned} \tag{34}$$

Eqs. (34) are the dimensional reduction of selfduality equations in 7 dimensions. Refs. [5–7] display analogous equations, corresponding to the dimensional reduction of the selfduality equation in 8 dimensions. Ref. [3] indicates that the moduli problems defined by both equations are equivalent.[3]

Expanding the action S, and integrating out T and H^I, reproduces the $\mathcal{N}=4$ action in its twisted form [5,6][4]

$$\begin{aligned}S\approx\int_M \mathrm{Tr}\bigg(&-\frac{1}{2}F\star F+\frac{1}{4}d_A h_I\star d_A h^I+2d_A\Phi^i\star d_A\Phi_i\\ &-2\chi_I^\alpha J^I\star d_A\Psi_\alpha+2\Psi^\alpha\star d_A\eta_\alpha\\ &+2\star\eta^\alpha\big[h_I,\chi^I_\alpha\big]+J_I\star\Psi^\alpha\big[h^I,\Psi_\alpha\big]\\ &+\star\varepsilon_{IJK}\chi^{\alpha I}\big[h^J,\chi^K_\alpha\big]-2\star\sigma^{i\alpha\beta}\chi_{\alpha I}\big[\Phi_i,\chi^I_\beta\big]\\ &-2\star\sigma^{i\alpha\beta}\eta_\alpha[\Phi_i,\eta_\beta]-2\sigma^{i\alpha\beta}\Psi_\alpha\star[\Phi_i,\Psi_\beta]\\ &-\frac{1}{8}\star[h_I,h_J]\big[h^I,h^J\big]-2\star\big[\Phi^i,h_I\big]\big[\Phi_i,h^I\big]\\ &-4\star\big[\Phi^i,\Phi^j\big][\Phi_i,\Phi_j]\bigg).\end{aligned} \tag{35}$$

In fact, it is not necessary to ask $SL(2,\mathbb{R})$-invariance from the beginning. Rather, looking for a δ-, s- and $\bar{s}$-invariant action, with ghost number zero, determines a unique action, Eq. (29). This action has the additional $SL(2,\mathbb{R})$ and $\bar{\delta}$ invariances. Thus the $\mathcal{N}=4$ supersymmetric action is determined by the invariance under the action of only 6 generators s, $\bar{s}$, δ, with a much smaller internal symmetry than the $SL(2,\mathbb{H})$ R-symmetry, namely, the ghost number symmetry.

[3] In fact, to solve the complete moduli problem [3], one must add the following equations for the curvatures: $d_A\Phi_i=0$, $[\Phi_i,h^I]=0$, $[\Phi_i,\Phi_j]=0$.

[4] As a matter of fact, if the manifold on which the theory is defined is not hyper-Kähler, the J^I cannot be considered as constant. In this case the scalar fields h^I acquires a “mass” term linear in the selfdual part of the Riemann tensor.

3.3. Horizontality condition in four dimensions

The algebra is not contained in a single horizontality condition, as in the seven-dimensional case. In fact, one has splitted conditions for the Yang–Mills field, and for the scalar fields h^I. (This gives the possibility of building matter multiplets, by relaxing the condition that h^I is in the adjoint representation of the gauge group.) They are:

$$\begin{aligned}&(d+s+\bar{s}+\delta+\bar{\delta})\big(A+c+\bar{c}+|\kappa|\gamma+|\kappa|\bar{\gamma}\big)\\ &\quad+\big(A+c+\bar{c}+|\kappa|\gamma+|\kappa|\bar{\gamma}\big)^2\\ &\qquad=F+\Psi+\bar{\Psi}+g(\kappa)(\eta+\bar{\eta})+g(J_I\kappa)\big(\chi^I+\bar{\chi}^I\big)\\ &\qquad\quad+\big(1+|\kappa|^2\big)(\Phi+L+\bar{\Phi}),\\ &(d_A+s_{(c)}+\bar{s}_{\bar{c}}+\delta_\gamma+\bar{\delta}_{\bar{\gamma}})h^I\\ &\quad=d_A h^I+\bar{\chi}^I-\chi^I+i_{J^I\kappa}(\bar{\Psi}-\Psi),\end{aligned} \tag{36}$$

$$\begin{aligned}&(d+s+\delta-i_\kappa)\big(A+c+|\kappa|\gamma\big)+\big(A+c+|\kappa|\gamma\big)^2\\ &\quad=F+\Psi+g(\kappa)\eta+g(J_I\kappa)\chi^I+\Phi+|\kappa|^2\bar{\Phi},\\ &(d_A+s_{(c)}+\delta_\gamma-i_\kappa)h^I=d_A h^I+\bar{\chi}^I+i_{J^I\kappa}\bar{\Psi}.\end{aligned} \tag{37}$$

These equations and their Bianchi identities fix the action of s, $\bar{s}$, δ and $\bar{\delta}$, by expansion in ghost number, up to the introduction of auxiliary fields that are needed for solving the indeterminacies of the form “s-antighost $+\,\bar{s}$-ghost”. These indeterminacies introduce auxiliary fields in the equivariant part of the algebra, T and H^I, as well as in the Faddeev–Popov sector. The latter does not affect the equivariant, that is, gauge-invariant, sector. To be more precise about the number of auxiliary fields, all the actions given by a symmetrized product of operators on the four Faddeev–Popov ghosts are determined by the closure relations of the algebra. There is one indeterminacy for each antisymmetrized product of operators. To close the algebra in the Faddeev–Popov sector, $11=6+4+1$ Lagrange multiplier fields must be introduced, with the standard technique. We do not give here the complete algebra for these fields, which we postpone for a further paper, devoted to a new demonstration of the finiteness of the $\mathcal{N}=4$, $D=4$ theory.

We can gauge-fix the action in s- and $\bar{s}$-invariant way and/or in a s- and δ-invariant way. In the former case, one uses an $s\bar{s}$-exact term which gauge-fixes the connection A. This $s\bar{s}$-exact term eliminates all fields of the Faddeev–Popov sector, but c, $\bar{c}$, and $b\equiv s\bar{c}$. In the former case, one uses an $s\delta$-exact term. In this case, $\bar{c}$ is replaced by γ.

4. Different twists of $\mathcal{N}=4$ super-Yang–Mills

As noted in [6,8] there are three non-equivalent twists of $\mathcal{N}=4$ super-Yang–Mills, corresponding to the different possible choices of an $SU(2)$ in the R-symmetry group $SL(2,\mathbb{H})$. When one defines the symmetry by horizontality conditions, these 3 different possibilities correspond to different representations of the matter fields. These matter fields are, respectively, organized in an $SL(2,\mathbb{R})$ Majorana–Weyl spinor, a vector field and a quaternion. The latter is the one studied in the previous section,[5] and, as a matter of fact, the most studied in the literature [3,5,7,10]. It is the only case that can be understood as a dimensional reduction of the eight-dimensional topological theory. We will shortly see that the two other cases have scalar and vector symmetries that are not big enough for a determination of the action.

4.1. Spinor representation

The first twist gives an $\mathcal{N}_T = 1$ theory. It is obtained by breaking $Spin(4) \otimes SL(2,\mathbb{H})$ into $Spin(4) \otimes SU(2) \otimes SL(2,\mathbb{R}) \otimes U(1)$, and then taking the diagonal of the chiral $SU(2)$ of $Spin(4)$ with the $SU(2)$ of the previous decomposition of $SL(2,\mathbb{H})$. The bosonic matter field h^α is then a chiral $SL(2,\mathbb{R})$ Majorana–Weyl spinor. The ghost λ_+^α and antighost λ_-^α of the matter field are, respectively, chiral and antichiral $SL(2,\mathbb{R})$ Majorana–Weyl spinors. The horizontality condition reads:

$$(d_A + s_{(c)} + \delta_{(c)} - i_\kappa) h^\alpha = d_A h^\alpha + \lambda_+^\alpha + \not\kappa \lambda_-^\alpha. \tag{38}$$

Introducing the auxiliary antichiral $SL(2,\mathbb{R})$ Majorana–Weyl spinors D^α, one gets:

$$\begin{aligned}
&s_{(c)} h^\alpha = \lambda_+^\alpha, && \delta_{(c)} h^\alpha = \not\kappa \lambda_-^\alpha,\\
&s_{(c)} \lambda_+^\alpha = [\Phi, h^\alpha], && \delta_{(c)} \lambda_+^\alpha = \not\kappa D^\alpha + \mathcal{L}_\kappa h^\alpha,\\
&s_{(c)} \lambda_-^\alpha = D^\alpha, && \delta_{(c)} \lambda_-^\alpha = \not\kappa [\bar\Phi, h^\alpha],\\
&s_{(c)} D^\alpha = [\Phi, \lambda_-^\alpha], && \delta_{(c)} D^\alpha = \not\kappa [\eta, h^\alpha] + \not\kappa [\bar\Phi, \lambda_+^\alpha]\\
& && \qquad + \mathcal{L}_\kappa \lambda_-^\alpha.
\end{aligned} \tag{39}$$

With this definition of the twist, there is no other scalar or vector charge, which leaves us with a $\mathcal{N}_T = 1$ theory. The action is not completely determined by these two symmetries.

4.2. Vector representation

One breaks $SL(2,\mathbb{H})$ into $Spin(4) \otimes SO(1,1)$ and then takes the diagonal of $Spin(4) \otimes Spin(4)$ [9]. The matter horizontality condition involves a vector field $V^\mu \equiv h^\mu$, its vector ghost $\bar\Psi$ and antighosts scalar $\bar\eta$ and selfdual 2-form $\bar\chi$,

$$(d_A + s_{(c)} + \delta_{(c)} - i_\kappa) V = d_A V + \bar\Psi + g(\kappa)\bar\eta + i_\kappa \bar\chi. \tag{40}$$

This give the following transformations of the fields:

$$\begin{aligned}
&s_{(c)} V = \bar\Psi, && \delta_{(c)} V = g(\kappa)\bar\eta + i_\kappa \bar\chi,\\
&s_{(c)} \bar\Psi = [\Phi, V], && \delta_{(c)} \bar\Psi = g(\kappa) h + i_\kappa \bar H + \mathcal{L}_\kappa V,\\
&s_{(c)} \bar\eta = h, && \delta_{(c)} \bar\eta = -[\bar\Phi, i_\kappa V],\\
&s_{(c)} h = [\Phi, \bar\eta], && \delta_{(c)} h = [\eta, i_\kappa V] - [\bar\Phi, i_\kappa \bar\Psi] + \mathcal{L}_\kappa \bar\eta,\\
&s_{(c)} \bar\chi = \bar H, && \delta_{(c)} \bar\chi = -2[\bar\Phi, (g(\kappa)V)^+],\\
&s_{(c)} \bar H = [\Phi, \bar\chi], && \delta_{(c)} \bar H = 2[\eta, (g(\kappa)V)^+]\\
& && \qquad - 2[\bar\Phi, (g(\kappa)\bar\Psi)^+]\\
& && \qquad + \mathcal{L}_\kappa \bar\chi.
\end{aligned} \tag{41}$$

This corresponds to a $\mathcal{N}_T = 2$ theory. However, the mirror operators $\bar s_{(c)}$ and $\bar\delta_{(c)}$ have the same ghost number as the primary ones. The internal symmetry in this case is $\mathbb{Z}_2$ instead of $SL(2,\mathbb{R})$ [6]. As a matter of fact, the four symmetries are not enough to fix the action and do not give an algebra which can be closed off-shell without the introduction of an infinite set of fields.

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