New convergence results for continued fractions generated by four-term recurrence relations

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Received 15 December 1982
Revised 21 March 1983

Abstract: In the study of simultaneous rational approximation of functions using rational functions with a common denominator (which can be viewed as the 'German polynomial' problem in simultaneous Padé approximation, cf. [1]) the quest for convergence results leads to the study of generalized continued fractions, a type of Jacobi–Perron algorithm [2,3]. It then becomes important to exploit the connection between the convergence of the generalized continued fraction and the solutions of the associated difference equation (cf. [4,5]).

Keywords: Generalized continued fractions, recurrence relations, convergence properties, rational approximation to pairs of power series.

AMS (MOS) Subject Classification: Primary 10F20, 40A15, secondary 10A35, 30B70.

1. Introduction

For the sake of simplicity this paper is restricted to the case of four-term recurrence relations, a situation closely related to the simultaneous rational approximation of two functions as will be indicated in the sequel. Consider three sequences of complex numbers (which might depend upon a complex variable)

\[ (a_v^{(1)})_{v=1}^\infty, \quad (a_v^{(2)})_{v=1}^\infty, \quad (b_v)_{v=1}^\infty \quad \text{with} \quad a_v^{(1)} \neq 0, \quad b_v \neq 0, \quad v \geq 1. \]  

\[ (0) \]

Definition 1.1. A so-called 2-fraction is given by two sequences of approximants \((A_v^{(1)}/B_v)_{v=1}^\infty, (A_v^{(2)}/B_v)_{v=1}^\infty\) where the numerators and denominators all satisfy the same recurrence relation

\[ X_v = b_v X_{v-1} + a_v^{(2)} X_{v-2} + a_v^{(1)} X_{v-3}, \quad v \geq 1 \]  

with initial values

\[ \begin{pmatrix} A_2^{(1)} & A_1^{(1)} & A_0^{(1)} \\ A_2^{(2)} & A_1^{(2)} & A_0^{(2)} \\ B_{-2} & B_{-1} & B_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]  

\[ (2) \]

The 2-fraction will be denoted by

\[ \left( \begin{array}{c} a_v^{(1)} \\ a_v^{(2)} \\ b_v \end{array} \right)_{v=1}^\infty \quad \text{or} \quad \left( \begin{array}{c} a_1^{(1)} \cdots a_v^{(1)} \cdots \\ a_1^{(2)} \cdots a_v^{(2)} \cdots \\ b_1 \cdots b_v \end{array} \right). \]  

\[ (3) \]
Remark 1.2. This is already a slight simplification compared to the general situation. In case $A^{(1)}_0 = b^{(1)}_0$, $A^{(2)}_0 = b^{(2)}_0$, the zeroth column in (3) has to be replaced by the column $(b^{(1)}_0, b^{(2)}_0)^T$; convergence properties are not influenced by this change (cf. [2,3]). It follows from the theory of difference equations that the recurrence relation (1) has as set of solutions a 3-dimensional subspace of the linear space of all infinite sequences. The sequences $(A^{(1)}_c)_{c=1}^\infty$, $(A^{(2)}_c)_{c=1}^\infty$ and $(B_c)_{c=1}^\infty$ obviously form a basis. Thus the 2-fraction can be interpreted as being given by quotients of special solutions of the recurrence relation (1).

On the other hand, the simultaneous Padé table for a pair of functions $f^{(1)}, f^{(2)}$ from the field of formal power series, of the form $f^{(i)}(z) = \sum_{i=0}^\infty f^{(i)}_iz^i$ ($i = 1, 2$), gives rise to sequences of approximants satisfying a recurrence relation of the form (1) in which the coefficients are simple functions of the variable $z$ (polynomials, etc., cf. [2,3]).

To be more precise (but without going into much detail): the Padé-2-table is the configuration of points $(\rho_0, \rho_1, \rho_2)$ with non-negative coordinates in which at each point a triple of polynomials $(A^{(1)}, A^{(2)}, B)$ is situated satisfying

\begin{align*}
\deg A^{(i)} &\leq \rho_0 + \rho_1 + \rho_2 - \rho_i, \quad i = 1, 2, \\
\text{ord}(B^{(i)} - A^{(i)}) &\geq \rho_0 + \rho_1 + \rho_2 + 1, \quad i = 1, 2.
\end{align*}

Under suitable conditions on the functions $f^{(1)}, f^{(2)}$ the sequences of approximants $(A^{(i)}_c/B_c)_{c=1}^\infty$ for $i = 1, 2$ on a generalized step line (starting from the point $(\rho_0, 0, 0)$ each of the coordinates is increased by 1 in a cyclic order starting with the first coordinate) satisfy a recurrence relation (1) with

\begin{align*}
a^{(1)}_1 &= a_{1,1}z, & a^{(1)}_v &= a_{1,v}z^v, & v \geq 2, \\
a^{(2)}_v &= a_{2,v}z, & b_v &= 1, & v \geq 1
\end{align*}

while the initial values $(A^{(1)}_0, A^{(2)}_0, B_0)$ are the polynomials at the point $(\rho_0, 0, 0)$.

2. Convergence

It is now obvious that convergence of the 2-fraction, i.e. the existence of

$$\lim_{v \to \infty} A^{(i)}_v/B_v, \quad i = 1, 2$$

can be of extreme importance and one might be interested in the tools and methods to be used. There are several possibilities some of which are listed below.

(a) The recurrence relation (1) has constant coefficients. In this case the numerators and denominators can be calculated explicitly in terms of the zeros of the auxiliary polynomial $a^{(1)}z^3 + a^{(2)}z^2 + bz - 1$ (which actually is the denominator of the generating function $\sum_{i=0}^\infty X^i z^i$).

Convergence is then easily decided upon.

(b) The recurrence relation (1) has special linearly independent solutions. The two linearly independent solutions $(f^{(1)}_v)_v$ and $(f^{(2)}_v)_v$ satisfy

(i) $(f^{(i)}_v)_v$ is dominated by all solutions $(y_v)_v \in \text{span}((f^{(1)}_v)_v, (f^{(2)}_v)_v)$, i.e.$\lim_{v \to \infty} f^{(i)}_v/y_v = 0, \quad i = 1, 2$;

(ii) $f^{(1)}_1f^{(2)}_2 - f^{(1)}_2f^{(2)}_1 = 0$.

This is equivalent to convergence of the 2-fraction; this is a special case of the generalization of Pincherle’s theorem by van der Cruyssen [4,5] to recurrence relations of arbitrarily (fixed) length.

(c) Interpretation as successive linear fractional transformation application. The approximants can be viewed as the image of the origin under successive application of linear fractional transformations of a special type (cf. [3]; this viewpoint of the algorithmic nature of generalized continued fractions is elaborated from the numerical performance-angle in [4,5]). Then apply an inclusion technique for value- and convergence regions (cf. [3]; in [6] an extensive treatment of this method for ordinary continued fractions is given).
(d) Interpretation as a sequence of Padé approximants. Sometimes the 2-fraction can be viewed as the Jacobi–Perron type algorithm connected with a certain path in the Padé-2-table for a pair of functions (the generalized stepline has been touched upon in section 1; there are, however, other sequences of points that give rise to a so-called ‘regular algorithm’). Convergence then might follow from results in the theory of simultaneous Padé approximation.

This method can of course be used the other way around: convergence of the 2-fraction might lead to convergence in the Padé-2-table!

(e) Brute force. Last, but not least as will become clear later on, there is the method of ‘brute force’ using rough estimates on the expressions to be studied. This method obviously leaves no hopes for optimality of the results.

An example of a result using (e) will be given in the next section.

3. Main result and examples

Theorem 3.1. The 2-fraction (3) converges if there exist \( a, b \in \mathbb{R}; \ a, b \geq 0 \) with

\[
|a^{(1)}_v/b_v| = \frac{b}{1 + a + b^2}, \tag{4}
\]

\[
\sup_{v \geq 3} |a^{(1)}_v/b_v| \leq \frac{b}{1 + a + b^3}, \tag{5}
\]

\[
\sup_{v \geq 2} |a^{(2)}_v/b_v| \leq \frac{a}{1 + a + b^2}. \tag{6}
\]

Remark 3.2. On some occasions (for instance if we are dealing with a C-2-fraction cf. [2,3]) the conditions can be modified in the following manner: delete (4) and replace \( \sup \) by \( \lim \) in (5) and (6).

After treating some examples, the proof of the theorem will be given in the next section.

Example 3.3. (C-2-fractions). Consider 2-fractions with

\[
a^{(1)}_v = a_{1,v} z, \quad a^{(1)}_v = a_{1,v} z^2, \quad v \geq 2,
\]

\[
a^{(2)}_v = a_{2,v} z, \quad b_v = 1, \quad v \geq 1
\]

where the \( a \)'s and \( z \) are complex and \( a_{1,v} \neq 0 \) (\( v > 1 \)) when also \( a_{2,v} \neq 0 \) (\( v > 1 \)) we have a regular C-2-fraction cf. [2,3].

(a) \( |a_{1,v}| \leq 1/(16p^2), \ |a_{2,v}| \leq 1/(32p), \ v \geq 2, \) where \( p > 0 \) is a constant. The following circular convergence regions are known:

\[
|z| \leq (\sqrt{3} - 1) p = 0.732 p \quad \text{(see [3])}
\]

\[
|z| \leq \sqrt{1.3} p = 1.140 p \quad \text{(see [4]).}
\]

Using the main theorem we find

\[
|z| \leq 1.384 p
\]

\[
c := a + b = \frac{129}{257 \times 256} + \frac{127}{256} + \left( \frac{129}{257 \times 256} + \frac{127}{256} \right)^2 - \frac{63}{257},
\]

\[
b = c + \frac{1 + c}{128} - \left( \left( c + \frac{1 + c}{128} \right)^2 - c^2 \right)^{1/2}.
\]

(b) The main stepline \( (\rho_0 = 0) \) in the Padé-2-table for \((1 - z)^{1/2}, (1 - z)^{1/4}\) leads to a regular C-2-frac-
tion with \( \lim_{v \to \infty} a_{1,v} = \frac{1}{2}, \lim_{v \to \infty} a_{2,v} = \frac{1}{3} \) from which a circular region of the form
\[
|z| < \frac{3}{4} (\sqrt{3} - 1) \approx 0.5490
\]
can be derived on which the \( C \)-2-fraction converges to the functions it has been derived from (cf. [3]).

Using the same method of 'cutting off' the \( C \)-2-fraction as in [3] the main theorem leads to convergence to the original functions for
\[
|z| < 7(2\sqrt{2} - 1)/(11 + 6\sqrt{2}) \approx 0.6568
\]
(use \( c := a + b = \frac{1}{2}(2 + 3\sqrt{2}), b = \frac{1}{4}(5c + 3 - ((7c + 3)(3c + 3))^{1/2}). \)

Note the gap between this result and what one minimally expects: convergence for \( |z| < 1 \).

**Example 3.4 (Ordinary 2-fractions).**

(a) The following result is due to van der Cruyssen [5]. A 2-fraction with \( h_v = 1 \) \( (v \geq 1) \) converges in case

\[
(i) \quad |a_v^{(1)}| \leq 0.0625, \quad |a_v^{(2)}| \leq 0.125 \quad v \geq 1
\]

and, moreover, either (ii) is satisfied for all \( v \geq 1 \) or (iii) for all \( v \geq 1 

\[
(ii) \quad |3 + 4a_v^{(2)}| > \frac{1}{13},
\]

\[
(iii) \quad |\arg a_v^{(2)}| < \frac{1}{2} \pi.
\]

The main theorem leads to

1. (i) and (iii) can be omitted and (i) is needed for \( v \geq 2 \) only (\( a = b = 0.5 \));
2. with \( a = 0.398925781 \) and \( b = 0.351074219 \) the main theorem shows that the 2-fraction is convergent if the following condition is satisfied
\[
|a_v^{(1)}| < 0.0655, \quad |a_v^{(2)}| < 0.1302, \quad v \geq 2.
\]

(b) Consider a 2-fraction satisfying
\[
|a_v^{(1)}|, |a_v^{(2)}| < p/\alpha, \quad |b_v| > p + \beta, \quad \alpha > 0, \beta, \ p \geq 0.
\]
This 2-fraction converges for

\[
(i) \quad p \geq 2, \quad \beta = \frac{1}{2}: \quad \alpha \geq 2.56,
\]

\[
(ii) \quad p \geq 2, \quad \alpha = 8: \quad \beta \geq 0,
\]

\[
(iii) \quad \alpha = 8, \quad \beta = \frac{1}{2}: \quad p \geq 0,
\]

\[
(iv) \quad p = \beta = \frac{1}{2}: \quad \alpha \geq 5.5451
\]

(optimize the missing parameter in
\[
\frac{1}{\alpha} \frac{p}{(p + \beta)^2} \leq \frac{b}{(1 + a + b)^2}, \quad \frac{1}{\alpha} \frac{p}{(p + \beta)^3} \leq \frac{b}{(1 + a + b)^3}, \quad \frac{1}{\alpha} \frac{p}{(p + \beta)^4} \leq \frac{a}{(1 + a + b)^4}
\]

selecting an appropriate choice for \( a \) and \( b \).

Case (ii) improves upon a result in [4] (there the condition on the \( h_v \) is \( \text{Re} \ b_v > p + \frac{1}{2} \)) and case (iv) shows that the brute force method can lead to a better result than the convergence region method which starts from
\[
|a_v^{(1)}| \leq A, \quad |a_v^{(2)}| \leq A,
\]
then the conditions become
\[
A \leq \frac{1}{2} \xi, \quad A \leq \frac{1}{4} - \xi, \quad \xi \geq 0;
\]

optimal for \( \xi = \frac{1}{2} : |a_v^{(1)}|, |a_v^{(2)}| < \frac{1}{12}. \)
4. Proof of the main result

Because of the conditions \( b_v = 0 \) the convergence of the 2-fraction (1), (2) is equivalent to the convergence of the 2-fraction that is derived from it using multiplicative constants \( p_v = (1 + a + b)/b_v \) (\( v \geq 1 \)), \( p_0 = \min(1, b/a) \) (cf. [2,3]):

\[
\begin{pmatrix}
\tilde{a}_v^{(1)} \\
0 \\
\tilde{a}_v^{(2)} \\
0 \\
1 + a + b)
\end{pmatrix}_{v=1}^\infty
\]

(7)

where the bounds (4), (5), (6) can be translated into

\[
|\tilde{a}_v^{(1)}| \leq b, \quad |\tilde{a}_v^{(2)}| \leq a, \quad v \geq 1
\]

(8)

(for \( v = 1 \) this follows from the value of \( p_0 \)).

The numerators \( C_v \) (\( i = 1, 2 \)) and denominators \( D_v \) of (7) satisfy

\[
C_v^{(i)}/D_v = p_v A_v^{(i)}/B_v, \quad i = 1, 2, \quad v > 0.
\]

From the recurrence relation connected with (7)

\[
D_v = (1 + a + b)D_{v-1} + \tilde{a}_v^{(2)}D_{v-2} + \tilde{a}_v^{(1)}D_{v-3}, \quad v \geq 1
\]

and the bounds (8), the triangle inequality implies

\[
|D_v| \geq (1 + a + b)|D_{v-1}| - a|D_{v-2}| - b|D_{v-3}|, \quad v \geq 1.
\]

Introducing the quantities

\[
\Delta_v := |D_v| - (a + b)|D_{v-1}| - b|D_{v-2}|, \quad v \geq 1
\]

the estimates (9) directly lead to

\[
\Delta_v \geq \Delta_{v-1} \geq \Delta_{v-2} \geq \cdots \geq \Delta_2 \geq \Delta_1 = 1.
\]

(10)

Now it is obvious that the \( (D_v) \) can be recovered from the sequence \( (\Delta_v)_v \), induction and (10) then imply

\[
|D_v| \geq \sum_{j=0}^{v} k_j, \quad v \geq 0
\]

where

\[
k_0 = 1, \quad k_1 = a + b, \quad k_j = (a + b)k_{j-1} + bk_{j-2}, \quad j \geq 2.
\]

(11)

Application of the recurrence relation connected with (7) to the quantities

\[
\delta^{(i)} : = C_v^{(i)}D_{v-1} - C_v^{(i-1)}D_v, \quad \delta^{(i)} : = C_v^{(i)}D_{v-2} - C_v^{(i-2)}D_v
\]

elimination of \( \delta^{(i)}_{v+1} \) leads to

\[
\delta^{(i)}_{v-1} = -\tilde{a}_v^{(2)}\delta^{(i)}_{v-1,1} - \tilde{a}_v^{(1)}\delta^{(i)}_{v-1,2,1} + \tilde{a}_v^{(1)}\delta^{(i)}_{v-1,1,1} \delta^{(i)}_{v-1,1} \delta^{(i)}_{v-1,1,1} \delta^{(i)}_{v-1,1}, \quad i = 1, 2, \quad v \geq 2
\]

with initial values

\[
\delta^{(i)}_{1,1} = 1, \quad \delta^{(i)}_{-1,1} = 0, \quad i = 1, 2.
\]

\[
\delta^{(i)}_{0,1} = \delta^{(i)}_{0,1} = 0, \quad i = 1, 2.
\]

The inequalities (8) then imply

\[
|\delta^{(i)}_{v,1}| \leq b|\delta^{(i)}_{v-1,1}| + b(1 + a + b)|\delta^{(i)}_{v-2,1}| + b^2|\delta^{(i)}_{v-3,1}|, \quad i = 1, 2, \quad v \geq 2.
\]

\[
|\delta^{(i)}_{1,1}| \leq b, \quad |\delta^{(i)}_{0,1}| = |\delta^{(i)}_{-1,1}| = 0, \quad |\delta^{(i)}_{1,1}| \leq a, \quad |\delta^{(i)}_{0,1}| = 1, \quad |\delta^{(i)}_{-1,1}| = 0.
\]
Define the quantities $\Delta_v^{(i)} (i = 1, 2, v \geq -1)$ by

\[
\Delta_v^{(i)} = a\Delta_{v-1}^{(i)} + b(1 + a + b)\Delta_{v-2}^{(i)} + b^2\Delta_{v-3}^{(i)}, \quad i = 1, 2, \quad v \geq 2,
\]

\[
\Delta_v^{(1)} = b, \quad \Delta_v^{(0)} = \Delta_{v-1}^{(1)} = 0, \quad \Delta_v^{(2)} = a, \quad \Delta_v^{(0)} = 1, \quad \Delta_{v-1}^{(2)} = 0.
\]

It is now obvious that $|\delta_v^{(i)}| \leq \Delta_v^{(i)} (i = 1, 2, v \geq -1)$ and because of the identity

\[
C_v^{(i)} / D_v = \sum_{k=1}^{v} \left( C_k^{(i)} / D_k - C_{k-1}^{(i)} / D_{k-1} \right), \quad i = 1, 2, \quad v \geq 1
\]

convergence of the 2-fraction follows if we establish convergence of the majorizing series

\[
\sum_{v=1}^{\infty} \frac{\Delta_v^{(i)}}{v-1} - \frac{1}{i-1, 2}.
\]

From the recurrence relation and initial values (11) for the $k_j$, we conclude

\[
\{bT^2 + (a + b)T - 1\} \sum_{j=0}^{\infty} k_j T^j = -k_0 + ((a + b)k_0 - k_1)T + \sum_{j=2}^{\infty} \{bk_{j-2} + (a + b)k_{j-1} - k_j\}T^j = -1
\]

and we find after division by $bT^2 + (a + b)T - 1$

\[
\sum_{j=0}^{\infty} k_j T^j = \frac{-b}{(bT)^2 + (a + b)(bT) - b}
\]

where the fraction on the right-hand side has been multiplied by $b/b$.

In a similar way (12) implies

\[
\sum_{j=0}^{\infty} \frac{\Delta^{(1)}_v}{v} = \frac{bT}{bT^2 + (a + b)(bT) - b}.
\]

Let $\sigma = \frac{1}{2}(a + b)^2 + 4b + a + b)$ and $\tau = \frac{1}{2}(a + b)^2 + 4b - a - b)$, then obviously $\sigma > \tau > 0$ and moreover

\[
(bT^2 + (a + b)(bT) - b = (bT + \sigma)(bT - \tau).
\]

The partial fraction decomposition for (14) then follows using $\sigma \tau = b$:

\[
\sum_{j=0}^{\infty} k_j T^j = \frac{1}{\sqrt{(a + b)^2 + 4b}} \left( \frac{\sigma}{1 - \sigma T} + \frac{\tau}{1 + \tau T} \right),
\]

which leads to

\[
k_j = \frac{1}{\sqrt{(a + b)^2 + 4b}} \left( \sigma^{j+1} + (-1)^j \tau^{j+1} \right), \quad j = 0, 1, \ldots
\]

The asymptotic behaviour of $\sum_{j=0}^{v} k_j$ is now perfectly clear

\[
\sigma < 1 \quad (\Leftrightarrow a + 2b < 1): \quad \lim_{v \to \infty} \sum_{j=0}^{v} k_j = \sigma / \left( (1 - \sigma)\sqrt{(a + b)^2 + 4b} \right).
\]
\[ \sigma = 1 \quad (\Leftrightarrow \quad a + 2b = 1): \quad \lim_{v \to \infty} \frac{1}{v} \sum_{j=0}^{v} k_j = \frac{1}{\sqrt{(a+b)^2 + 4b}}. \] (17)

\[ \sigma > 1 \quad (\Leftrightarrow \quad a + 2b > 1): \quad \lim_{v \to \infty} \frac{1}{\sigma^v} \sum_{j=0}^{v} k_j = \sigma^2 / \left\{ (\sigma - 1)\sqrt{(a+b)^2 + 4b} \right\}. \]

In a similar manner one would like to determine the asymptotics for the \( \Delta^{(i)} \); this time, however, one must be careful. The denominator in (15) and (16) has the zeros \(-1/b, -\sigma/b, \tau/b\). This shows the partial fraction decomposition to be dependent on the value of \( \sigma \): for \( \sigma = 1 \) there is a double zero, for \( \sigma \neq 1 \) we have three different zeros.

The case \( \sigma \neq 1 \) leads, after some simple calculations, to

\[ \sum_{j=0}^{\infty} \Delta^{(1)}_j T^j = -\frac{b}{a + 2b - 1} \cdot \frac{1}{1 + bT} + \frac{b}{(\tau + 1)\sqrt{(a+b)^2 + 4b}} \cdot \frac{1}{1 - \sigma T} \]

\[ + \frac{b}{(\sigma - 1)\sqrt{(a+b)^2 + 4b}} \cdot \frac{1}{1 + \tau T}. \]

\[ \sum_{j=0}^{\infty} \Delta^{(2)}_j T^j = \frac{b}{a + 2b - 1} \cdot \frac{1}{1 + bT} + \frac{\sigma}{(\tau + 1)\sqrt{(a+b)^2 + 4b}} \cdot \frac{1}{1 - \sigma T} \]

\[ - \frac{\tau}{(\sigma - 1)\sqrt{(a+b)^2 + 4b}} \cdot \frac{1}{1 + \tau T}. \] (18)

The case \( \sigma = 1 \) (i.e. \( a + 2b = 1 \)) gives rise to the following decompositions (using the fact that \( \sigma \tau = b \) implies \( \tau = b \)):

\[ \sum_{j=0}^{\infty} \Delta^{(1)}_j T^j = \frac{h^2}{(b+1)^2} \cdot \frac{1}{1 + bT} - \frac{h}{b+1} \cdot \frac{1}{(1+bT)^2} + \frac{h}{(b+1)^2} \cdot \frac{1}{1 - T}. \]

\[ \sum_{j=0}^{\infty} \Delta^{(2)}_j T^j = \frac{b}{(b+1)^2} \cdot \frac{1}{1 + bT} + \frac{b}{(1+bT)^2} + \frac{b}{(b+1)^2} \cdot \frac{1}{1 - T}. \] (19)

From (18) and (19) the asymptotics for the \( \Delta^{(i)}_c \) follow at once: there exist non-zero constants \( c_1, c_2, \ldots, c_6 \), such that

\[ \sigma < 1: \quad \lim_{v \to \infty} \Delta^{(i)}_c \sigma^{-v} = c_i, \quad i = 1, 2, \]

\[ \sigma = 1: \quad \lim_{v \to \infty} \Delta^{(i)}_c = c_{2+i}, \quad i = 1, 2, \] (20)

\[ \sigma > 1: \quad \lim_{v \to \infty} \Delta^{(i)}_c \sigma^{-v} = c_{4+i}, \quad i = 1, 2. \]

Combination of (17) and (20) shows that the behaviour of (13) turns out to be that of the series \( \Sigma \sigma^v, \Sigma v^{-2} \) and \( \Sigma \sigma^{-v} \), in the cases \( \sigma < 1, \sigma = 1 \) and \( \sigma > 1 \), respectively. Therefore the 2-fraction is convergent; moreover we see the convergence is uniform with respect to parameters – on which the coefficients \( a^{(i)}_c, b_v \) might depend – restricted to the subset of the parameter space induced by the inequalities (4), (5) and (6).

\[ \square \]

References
