JOURNAL OF ALGEBRA 45, 295–305 (1977)

Subalgebras of Finitely Presented Solvable Lie Algebras

GILBERT BAUMSLAG*

City College of CUNY, Convent Avenue at 138 Street, New York, New York 10031

Communicated by N. Jacobson

Received November 6, 1975

1. INTRODUCTION

1.1. In 1961, Higman [6] proved that a finitely generated group is a subgroup of a finitely presented group if and only if it is recursively presentable. Recently, Bokut [4] established an analogous, somewhat weaker, result for lie algebras. The main thrust of this paper is not so much to remedy the deficits in Bokut's paper but to provide some precise information about the subalgebras of finitely presented solvable lie algebras.

1.2. In order to explain we recall that a Lie algebra L (over a commutative field $\mathbf{k}$) is termed metabelian if its derived lie algebra L' is abelian, i.e., if $L'' = 0$ (see, e.g., [7, p. 23]). Now it follows from a theorem of Amayo and Stewart [1] that every finitely generated metabelian lie algebra is recursively presentable. However, Bokut's theorem does not apply to such finitely generated metabelian lie algebras. In fact, it is not too hard to prove that finitely generated metabelian lie algebras are embeddable in finitely presented lie algebras. Our objective here is to prove the less obvious

THEOREM 1. *A finitely generated metabelian lie algebra over a field of characteristic $p \neq 2$ can be embedded in a finitely presented metabelian lie algebra.*

Notice that characteristic $p = 0$ is included in Theorem 1.

Theorem 1 has both a group-theoretic and an associative algebra analog [2, 3].

On combining the ideas involved in the proof of Theorem 1 with an analog of an idea of Thomson [9] we shall also prove

THEOREM 2. *Let $\mathbf{k}$ be a field of characteristic $p \neq 2$ and let K be an associative commutative $\mathbf{k}$-algebra. Let, further, L be a finitely generated lie algebra over $\mathbf{k}$ of lower triangular matrices with coefficients in K. Then L can be embedded in a finitely presented lie algebra of lower triangular matrices (with coefficients again in K).*

* Support from the National Science Foundation is gratefully acknowledged.

ISSN 0021-8693

1.3. The arrangement of this papeı is as follows. First we shall discuss the notation and all the other preliminaries in Section 2. The proof of Theorem 1 is begun in Section 3 and concluded in Section 4. Finally, Theorem 2 is proved in Section 5.

2. Preliminaries

2.1. As usual we denote the product of two elements a, b in the lie algebra L by $[a, b]$. If $x_1, \ldots, x_n \in L$ then we define $[x_1, \ldots, x_n]$ inductively by

$$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n] \qquad (n > 2).$$

2.2. We make frequent use of the following simple lemmas in the remainder of this paper.

Lemma 1. *Let x, y, z be elements of a Lie algebra. Then the following hold:*

(i) *If* $[x, y] = 0$, *then* $[x, [y, z]] = [y, [x, z]]$.

(ii) *If* $[y, z] = 0$, *then* $[x, y, z] = [x, z, y]$.

Proof. (i) $[x, y, z] + [y, z, x] + [z, x, y] = 0$; hence $[y, z, x] = [x, z, y]$ or $[x, [y, z]] = [y, [x, z]]$, as required.

(ii) $[x, y, z] + [y, z, x] + [z, x, y] = 0$; hence $[x, y, z] = [x, z, y]$, as required.

In the next lemma we need some notation that will be useful throughout. Specifically, we denote the ideal generated by a subset X of a given lie algebra by $\mathrm{id}(X)$ and the lie subalgebra generated by X by $\mathrm{la}(X)$. Notice that if a lie algebra L is abelian and $L = \mathrm{la}(X)$, then X actually spans L, qua vector space.

Lemma 2. *Suppose that the lie algebra W is the direct sum of its abelian ideal B and its abelian lie subalgebra T:*

$$W = B \oplus T.$$

Furthermore suppose $B = \mathrm{id}(a_1, \ldots, a_m)$ *and* $T = \mathrm{la}(t_1, \ldots, t_n)$. *Then*

$$B = \mathrm{la}([a_l, t_{j_1}, \ldots, t_{j_s}]; 1 \leqslant l \leqslant m, \{j_1, j_2, \ldots, j_s\} \subseteq \{1, 2, \ldots, n\}).$$

Proof. The assertion follows inductively on noting that both B and T are abelian.

2.3. Let T be a lie algebra and let B be a right T-module. Then we denote the split extension W of B by T by $B \,]\, T$:

$$W = B \,]\, T.$$

We recall that vector space wise $W = B \oplus T$ and

$$[b_1 + t_1, b_2 + t_2] = (b_1t_2 - b_2t_1) + [t_1, t_2](b_i \in B, t_i \in T).$$

We are concerned here with a particular case of this construction. Thus, suppose A is an abelian lie algebra and that T is any lie algebra. Notice that A may be viewed as a vector space with the zero multiplication. Let $\{a_i \mid i \in I\}$ be a basis for A. Now let U be the universal enveloping algebra of T and let B be a free right U-module with the elements a_i as module basis $(i \in I)$. As usual, then, B is also a lie algebra module for T and hence we can form the split extension $W = B\,]\,T$. We shall view A as a lie subalgebra of W and observe that $W =$ la(A, T). W is independent of the choice of the vector space basis for A and we term it the *wreath product*[1] of the lie algebras A and T and denote it by

$$W = A \wr T.$$

The ideal B of W is called the *base ideal* of W.

Our interest in such wreath products stems from a theorem of Lewin [8] which we reformulate here as

LEMMA 3. *Let F be a free lie algebra freely generated by the elements $\{x_i \mid i \in I\}$ and let R be an ideal of F. Furthermore, let $T = F/R$, $t_i = x_i + R\ (i \in I)$, and let A be an abelian lie algebra with basis $\{a_i \mid i \in I\}$. Then the kernel J of the homomorphism θ of F into $W = A \wr T$ defined by*

$$x_i \mapsto a_i + t_i \qquad (i \in I)$$

is R':

$$J = R'.$$

The point of Lemma 3 is that F/R' is naturally embedded in $W = A \wr F/R$. We have recorded Lemma 3 explicitly here in order to be able to prove

LEMMA 4. *Let M be a finitely generated metabelian lie algebra. Then there exist finite-dimensional abelian lie algebras A and T such that M can be embedded in a quotient W/N of W, where W is the wreath product of A by T and N is contained in the base ideal B of W.*

Proof. Let $c_1, \ldots, c_l$ be a finite set of generators of M, let F be the free lie algebra on $x_1, \ldots, x_l$, and let φ be the homomorphism of F onto M defined by

$$\varphi\colon x_i \mapsto c_i \qquad (i = 1, 2, \ldots, l).$$

[1] The first explicit use of wreath products in lie algebras is due, I believe to A. L. Smelkin (in a talk).

Let A be an abelian lie algebra of dimension l with basis $a_1,\ldots, a_l$. Furthermore let $R = \varphi^{-1}M'$. Finally let $T = F/R$, $t_i = x_i + R$ $(i = 1, 2,\ldots, l)$, and let $W = A \wr T$.

Now by Lemma 3 the kernel of the homomorphism θ of F into W defined by

$$x_i \mapsto a_i + t_i \qquad (i = 1, 2,\ldots, l)$$

is R'. Consider next the kernel K of the homomorphism φ. Obviously $K \leqslant R$, and since $R/K \cong M'$ is abelian, $R' \leqslant K$. Notice now that if L is any ideal of F contained in R then $L\theta \leqslant B$. So, in particular, $K\theta \leqslant B$. Since K is an ideal of F, $K\theta$ is an ideal of $F\theta = \mathrm{la}(a_i + t_i \mid i = 1, 2,\ldots, l)$. But $[K\theta, B] = 0$ and so

$$[K\theta, t_i] = [K\theta, a_i + t_i] \leqslant K\theta,$$

which implies that $K\theta$ is actually an ideal of W.

In conclusion we note that $F/K \cong M$ and since $K \geqslant R'$, $F\theta/K\theta \cong M$. On putting $K\theta = N$ we therefore find that M is isomorphic to the lie subalgebra $F\theta/N$ of W/N, where of course $N \leqslant B$, as required.

3. Embedding Metabelian Wreath Products in Finitely Presented Metabelian Lie Algebras

3.1. Suppose A and T are finite-dimensional abelian lie algebras. The main step in the proof of Theorem 1 is the proof that $W = A \wr T$ can be embedded in a finitely presented metabelian lie algebra. The key step in this proof is the following:

Lemma 5. *Let L be a lie algebra of characteristic $p \neq 2$. Suppose a, b, t, u are elements of L and suppose*

$$[a, b] = [a, t, b] = [b, t, a] = [t, u] = 0$$

and

$$[a, u] = [a, t, t], \ [b, u] = [b, t, t].$$

Then

$$[[a, \underbrace{t,\ldots, t}_{i}], [b, \underbrace{t,\ldots, t}_{j}]] = 0$$

for every $i \geqslant 0, j \geqslant 0$.

Proof. Put

$$a_i = [a, \underbrace{t,\ldots, t}_{i}], \qquad b_i = [b, \underbrace{t,\ldots, t}_{i}] \quad (i = 0, 1,\ldots).$$

The proof that $[a_i, b_j] = 0$ whenever $i \geqslant 0, j \geqslant 0$ will be by induction. Thus we suppose

$$[a_i, b_j] = 0 \qquad (0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n).$$

It suffices then to prove that

$$[a_i, b_j] = 0 \qquad (0 \leqslant i \leqslant n+1, 0 \leqslant j \leqslant n+1).$$

Notice first that it follows from Lemma 1(ii) that

$$[a_i, u] = a_{i+2}, \qquad [b_i, u] = b_{i+2} \qquad (i = 0, 1, \ldots).$$

Moreover since $[a_i, b_j] = 0$ $(0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n)$ we find by Lemma 1(i) with $a_i = x$, $b_j = y$, and $t = z$, that

$$[a_i, b_{n+1}] = [b_n, a_{i+1}] = 0 \qquad (0 \leqslant i \leqslant n-1).$$

A similar application of Lemma 1(i) yields

$$[b_j, a_{n+1}] = 0 \qquad (0 \leqslant j \leqslant n-1).$$

It remains only to verify that

$$[a_n, b_{n+1}] = [a_{n+1}, b_{n+1}] = [b_n, a_{n+1}] = 0.$$

Now $[b_{n-1}, a_{n+1}] = 0$; so applying Lemma 1(i) with $b_{n-1} = x$, $a_{n+1} = y$, $t = z$ we obtain

$$[b_{n-1}, a_{n+2}] = [a_{n+1}, b_n]. \tag{1}$$

Similarly, $[b_{n-1}, a_n] = 0$; so applying Lemma 1(i) with $b_{n-1} = x$, $a_n = y$, $u = z$, we obtain

$$[b_{n-1}, a_{n+2}] = [a_n, b_{n+1}]. \tag{2}$$

Furthermore $[a_n, b_n] = 0$; so again applying Lemma 1(i) with $a_n = x$, $b_n = y$, $t = z$ we obtain

$$[a_n, b_{n+1}] = [b_n, a_{n+1}]. \tag{3}$$

On putting (2) and (3) together we find

$$[b_{n-1}, a_{n+2}] = [b_n, a_{n+1}]. \tag{4}$$

Thus by (1) and (4)

$$[b_n, a_{n+1}] = [b_{n-1}, a_{n+2}] = -[b_n, a_{n+1}].$$

Hence $2[b_n, a_{n+1}] = 0$. Since L is of characteristic $p \neq 2$

$$[b_n, a_{n+1}] = 0. \tag{5}$$

Similarly it follows that

$$[a_n, b_{n+1}] = 0. \tag{6}$$

We now go through three further applications of Lemma 1(i). First $[a_n, b_{n+1}] = 0$ [by (6)]; so Lemma 1(i) applies with $a_n = x$, $b_{n+1} = y$, $t = z$:

$$[a_n, b_{n+2}] = [b_{n+1}, a_{n+1}]. \tag{7}$$

Similarly, $[a_n, b_n] = 0$; so again Lemma 1(i) applies with $a_n = x$, $b_n = y$, $u = t$:

$$[a_n, b_{n+2}] = [b_n, a_{n+2}]. \tag{8}$$

By (5), $[a_{n+1}, b_n] = 0$; again Lemma 1(i) applies with $a_{n+1} = x$, $b_n = y$, $t = z$ to give us

$$[a_{n+1}, b_{n+1}] = [b_n, a_{n+2}]. \tag{9}$$

Putting (7), (8), and (9) together we find

$$[a_{n+1}, b_{n+1}] = [b_n, a_{n+2}] = [a_n, b_{n+2}] = [b_{n+1}, a_{n+1}].$$

Therefore, again remembering $p \neq 2$, it follows that

$$[a_{n+1}, b_{n+1}] = 0. \tag{10}$$

Equations (5), (6), and (10) complete the inductive step and therefore also the proof of Lemma 5.

3.2. Now let A and T be finite-dimensional abelian lie algebras and let $W = A \wr T$. Our objective is to embed W in a finitely presented metabelian lie algebra. To this end, let $a_1, \ldots, a_m$ be a basis for A and let $t_1, \ldots, t_n$ be a basis for T.

Let U be the universal enveloping algebra of T. Notice that U may be viewed as the associative $\mathbf{k}$-algebra of polynomials in $t_1, \ldots, t_n$ where $\mathbf{k}$ is the underlying ground field of characteristic $p \neq 2$ (see, e.g., [7, p. 163]). Let B be the free (right) U-module with $a_1, \ldots, a_m$ as module basis for B. As we remarked in Section 2.2, B may be viewed as a lie algebra module for T where the action of T on B is simply the action of T, qua subset of U, on the U-module B. It is worth noting explicitly that U can be turned into a lie algebra by using the vector space structure already present in U and defining a new multiplication in U by

$$[u, v] = uv - vu.$$

In the situation under discussion, of course, U is simply an abelian lie algebra when viewed in this light and T is a lie subalgebra of this lie algebra U. We shall need a slightly larger lie algebra then T in order to embed W in a finitely presented metabelian lie algebra. To this end let

$$T^+ = \mathrm{la}(t_1, \ldots, t_n, u_1, \ldots, u_n)$$

be the lie subalgebra of the lie algebra U generated by $t_1, \ldots, t_n, u_1, \ldots, u_n$ where

$$u_i = t_i^2 \qquad (i = 1, 2, \ldots, n).$$

Clearly T^+ is a $2n$-dimensional abelian lie algebra. We now form

$$W^+ = B \,]\, T^+$$

the split extension W^+ of B by T^+ where again the action of T^+ is simply the action of T^+ thought of as a subset of U. Notice that in W^+

$$[a_i, u_j] = a_i t_j^2 = [a_i, t_j, t_j] \tag{11}$$

for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. Moreover,

$$[t_j, t_k] = [t_j, u_k] = [u_j, u_k] = 0, \tag{12}$$

for $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n$. Relations (11) and (12) will enable us to invoke Lemma 5 to prove that W^+ is finitely presented.

It is easy to find a presentation for W^+.

LEMMA 6. *W^+ can be presented on the generators*

$$a_1, \ldots, a_m, t_1, \ldots, t_n, u_1, \ldots, u_n$$

subject to the relations

$$[[a_k, t_{i_1}, \ldots, t_{i_r}], [a_l, t_{j_1}, \ldots, t_{j_s}]] = 0,$$

$(1 \leqslant k \leqslant m,\ 1 \leqslant l \leqslant m,\ \{i_1, \ldots, i_r, j_1, \ldots, j_s\} \subseteq \{1, 2, \ldots, n\},\ r \geqslant 0,\ s \geqslant 0)$, and

$$[t_i, t_j] = [t_i, u_j] = [u_i, u_j] = 0 \qquad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n)$$
$$[a_k, u_l] = [a_k, t_l, t_l] \qquad (1 \leqslant k \leqslant n,\ 1 \leqslant l \leqslant n).$$

Lemma 6 follows easily on combining (11) and (12) with Lemma 2 and Lemma 1(ii) (cf. the proof of Lemma 7 below).

On putting Lemma 5 and Lemma 6 together we can now prove

LEMMA 7. *W^+ can be presented on the generators*

$$a_1, \ldots, a_m, t_1, \ldots, t_n, u_1, \ldots, u_n,$$

subject to the finitely many relations

$$[a_k, a_l] = [a_k, t_j, a_l] = 0 \qquad (1 \leqslant k \leqslant m,\ 1 \leqslant l \leqslant m,\ 1 \leqslant j \leqslant n),$$
$$[t_i, t_j] = [t_i, u_j] = [u_i, u_j] = 0 \qquad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n),$$
$$[a_k, u_l] = [a_k, t_l, t_l] \qquad (1 \leqslant k \leqslant n,\ 1 \leqslant l \leqslant n).$$

Proof. We note first that it follows from Lemma 1(ii) that

$$[a_k, x_1, x_2, \dots, x_s] = [a_k, x_{i_1}, x_{i_2}, \dots, x_{i_s}] \tag{13}$$

if $1 \leqslant k \leqslant m$, $\{x_1, x_2, \dots, x_s\} \subseteq \{t_1, \dots, t_n, u_1, \dots, u_n\}$ and $i_1, i_2, \dots, i_s$ is any permutation of $1, 2, \dots, s$. In view of this identity we find

$$[a_k, t_{j_1}, t_{j_2}, \dots, t_{j_l}, u_i] = [a_k, t_{j_1}, t_{j_2}, \dots, t_{j_l}, t_i, t_i] \tag{14}$$

if

$$1 \leqslant k \leqslant m, \{j_1, j_2, \dots, j_l, i\} \subseteq \{1, 2, \dots, n\}.$$

Since

$$[a_i, a_j] = [a_i, t_k, a_j] = [t_k, u_l] = 0$$

and

$$[a_i, u_l] = [a_i, t_l, t_l], [a_j, u_l] = [a_j, t_l, t_l],$$

we can apply Lemma 5 here. The result is that

$$[[a_i, \underbrace{t_l, t_l, \dots, t_l}_{r}], [a_j, \underbrace{t_l, t_l, \dots, t_l}_{s}]] = 0$$

whenever $r \geqslant 0$, $s \geqslant 0$. Consequently, by making use of (14) and (13) and then repeatedly applying Lemma 5, it finally follows that the given relations suffice to define W^+.

In conclusion we note that W^+ is obviously metabelian and since W is embedded in W^+ we have indeed embedded W in the finitely presented metabelian lie algebra W^+, as required.

4. The Proof of Theorem 1

4.1. Suppose M is a finitely generated metabelian lie algebra over a field $\mathbf{k}$ of characteristic $p \neq 2$. Then, by Lemma 3, M can be embedded in W/N where $W = A \wr T$ is the wreath product of two finite-dimensional abelian lie algebras and N is contained in the base ideal B of W.

We turn now to the lie algebra $W^+ = B \wr T^+$ of Section 3.2. Notice that if $c \in N$ then $[c, u_i] = [c, t_i, t_i] \in N$ $(1 \leqslant i \leqslant n)$. It follows that N is also an ideal of W^+; hence W^+/N makes sense. Clearly, M is embedded in W^+/N. So it only remains to prove that W^+/N is finitely presented. Now a finitely generated metabelian lie algebra satisfies the maximum condition for ideals [1]. Consequently, the class of finitely presented metabelian lie algebras is closed under homomorphic images. In particular, W^+/N is finitely presented, since W^+ is finitely presented by Lemma 7. This completes the proof of Theorem 1.

5. The Proof of Theorem 2

5.1. We shall need the following analog of a lemma of Hall [5].

Lemma 8. *An extension of one finitely presented lie algebra by another is finitely presented.*

Lemma 8 can be proved by copying the corresponding proof for groups by Hall in [5] and is therefore omitted.

5.2. Throughout the rest of this section we shall assume that $\mathbf{k}$ is a field of characteristic $p \neq 2$ and that K is an associative commutative $\mathbf{k}$-algebra. We denote by $\mathrm{tr}(l, K)$ the $\mathbf{k}$-lie algebra of all lower $l \times l$ triangular matrices over K. Thus if $F, G \in \mathrm{tr}(l, K)$, then as usual

$$[F, G] = FG - GF.$$

We shall need some additional notation. The $l \times l$ matrix with y in the (i, j)th place and zeros elsewhere is denoted by $E_{i,j}(y)$; notice that there is no explicit reference to l which always will be clearly understood. Similarly, $D_i(x)$ is the $l \times l$ matrix with x in the (i, i)th place. If $\mathscr{X}$ is any set of matrices we denote the set of those elements of K which are coefficients of matrices in $\mathscr{X}$ by $\mathscr{C}(\mathscr{X})$. We term $\mathscr{X}$ *adequate* if (i) $\mathscr{X}$ is finite, and (ii) $E_{i,j}(c) \in \mathscr{X}$ whenever $c \in \mathscr{C}(\mathscr{X})$ and $j \leqslant i$. Clearly, every finite set $\mathscr{X}$ is contained in a unique minimal adequate set $\bar{\mathscr{X}}$. Moreover, if $\mathscr{X} \subseteq \mathrm{tr}(l, K)$ then $\bar{\mathscr{X}} \subseteq \mathrm{tr}(l, K)$.

5.3. Suppose now that L is a finitely generated lie algebra of lower triangular $l \times l$ matrices over K. Our objective is to embed L in a finitely presented lie subalgebra of $\mathrm{tr}(l, K)$. It follows from the remarks above that we may assume without loss of generality that L has an adequate set $\mathscr{X}$ of generators. The main step in the proof of Theorem 2 is then the following

Lemma 9. *Let L be a lie algebra of lower triangular matrices over K with an adequate set $\mathscr{X}$ of generators. Then*

$$L^* = \mathrm{la}(\mathscr{X} \cup \{D_i(c^2) \mid i = 1, 2, ..., l, c \in \mathscr{C}(\mathscr{X}))$$

is finitely presented.

Notice that $L^* \leqslant \mathrm{tr}(l, K)$. So the proof of Theorem 2 is complete once we have established Lemma 9.

The proof of Lemma 9 is by induction on l. If $l = 1$, L^* is finite dimensional and hence it is certainly finitely presented. Thus we assume $l > 1$.

Consider now the homomorphism θ of $\mathrm{tr}(l, K)$ into $\mathrm{tr}(l-1, K)$ defined by

$$\theta: \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 0 \\ a_{l1} & a_{l2} & \cdots & a_{ll} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 0 \\ a_{l-1,1} & a_{l-1,2} & \cdots & a_{l-1,l-1} \end{pmatrix}.$$

Let φ denote the restriction of θ to L^*. Observe now that $L\varphi$ is a lie algebra of $l-1 \times l-1$ lower triangular matrices K and that $\mathscr{X}\varphi$ is an adequate set of generators of $L\varphi$. So, by induction, $L^*\varphi$ is finitely presented. It suffices therefore, in view of Lemma 8, to prove that the kernel $\overline{W}^+$ of φ is finitely presented.

To this end let us relabel the matrices $E_{l,i}(c)$ $(1 \leqslant i < l, c \in \mathscr{C}(\mathscr{X}))$ as $A_1, \ldots, A_m$ and the matrices $D_l(c)$ $(c \in \mathscr{C}(\mathscr{X}))$ as $T_1, \ldots, T_n$. Furthermore, if $T_i = D_l(c)$ we define $U_i = D_l(c^2)$. Then it follows easily from the adequacy of $\mathscr{X}$ and the way in which matrices multiply that

LEMMA 10. $\overline{W}^+ = \mathrm{la}(A_1, \ldots, A_m, T_1, \ldots, T_n, U_1, \ldots, U_n)$.

Now it is an easy matter to check that $\overline{W}^+$ is metabelian. In fact, $\overline{W}^+$ is actually a homomorphic image of the lie algebra W^+ of Section 3.2. In order to verify this we recall that

$$W^+ = \mathrm{la}(a_1, \ldots, a_m, t_1, \ldots, t_n, u_1, \ldots, u_n).$$

Consider the mapping σ from the generators of W^+ to the generators of $\overline{W}^+$ defined by

$$\sigma: a_i \mapsto A_i \ (i = 1, \ldots, m),\ t_i \mapsto T_i,\ u_i \mapsto U_i \qquad (i = 1, \ldots, n).$$

Now all of the finitely many relations used to define W^+ in Lemma 7 go over into relations in $\overline{W}^+$ on replacing the generators of W^+ by their images under σ. So σ defines a homomorphism, which we again denote by σ, of W^+ onto $\overline{W}^+$. Hence $\overline{W}^+$ is a homomorphic image of a finitely presented metabelian lie algebra and is therefore itself finitely presented. This finally completes the proof of Theorem 2.

REFERENCES

1. R. K. AMAYO AND I. STEWART, Finitely generated lie algebras, *J. London Math. Soc.* **5** (1972), 697–703.
2. GILBERT BAUMSLAG, Subgroups of finitely presented metabelian groups, *J. Aust. Math. Soc.* **XVI** (1973), 98–110.

3. G. Baumslag, On the subalgebras of certain finitely presented algebras, *Bull. Amer. Math. Soc.*, in press.
4. L. A. Bokut, Unsolvability of the word problem and subalgebras of finitely presented lie algebras, *Izv. Akad. Nauk. SSSR Ser. Mat.* **36** (1972), 1173–1219.
5. P. Hall, Finiteness conditions for solvable groups, *Proc. London Math. Soc.* (3) **4** (1954), 419–436.
6. G. Higman, Subgroups of finitely presented groups, *Proc. Roy. Soc. London* **262** (1961), 455–475.
7. N. Jacobson, "Lie Algebras," Interscience Publishers, New York, 1962.
8. J. Lewin, A matrix representation for associative algebras, II, *Trans. Amer. Math. Soc.* **188** (1974), 309–317.
9. M. Thomson, Embedding theorems for finitely generated solvable groups, Ph.D. thesis, City University of New York Graduate School 1976.