Torsion theories and Galois coverings of topological groups

Marino Gran\textsuperscript{a,\ast}, Valentina Rossi\textsuperscript{b}

\textsuperscript{a} Lab. Math. Pures et Appliquées, Université du Littoral Côte d’Opale, 50 Rue F. Buisson, 62228 Calais, France
\textsuperscript{b} Università degli Studi di Udine, Dipartimento di Matematica e Informatica, Via delle Scienze 206, 33100 Udine, Italy

Received 14 April 2005; received in revised form 11 November 2005
Available online 18 January 2006
Communicated by J. Adámek

Abstract

For any torsion theory in a homological category, one can define a categorical Galois structure and try to describe the corresponding Galois coverings. In this article we provide several characterizations of these coverings for a special class of torsion theories, which we call quasi-hereditary. We describe a new reflective factorization system that is induced by any quasi-hereditary torsion theory. These results are then applied to study various examples of torsion theories in the category of topological groups.

MSC: 18E40; 18C20; 22A99

0. Introduction

The study of semi-abelian categories and of homological categories is becoming a fruitful new subject in categorical algebra (see, for instance, the founding articles \[7,22\] by Bourn, Janelidze, Márki and Tholen and the reference book \[4\] by Borceux and Bourn). Among many other things, in these categories it is possible to define a natural notion of torsion theory that extends the classical one introduced by Dickson in any abelian category \[14\]. The introduction of the notion of torsion theory in this more general context makes it possible to look for new examples of torsion theories in the categories of groups, crossed modules and topological groups.

The idea of studying torsion theories in a homological category was first considered by Bourn and Gran in \[10\]. In this article some new examples of torsion theories in the categories of topological groups and of topological semi-abelian algebras (introduced by Borceux and Clementino in \[5\]) were also investigated. This new approach opened the way to other interesting investigations, for instance by Clementino, Dikranjan and Tholen, who further extended the context to the so-called normal categories \[13\].

In the present paper we study the categorical Galois structure associated with a torsion theory in a homological category, and we provide a complete characterization of the corresponding Galois coverings for a special class of torsion theories, which we call quasi-hereditary. Recall that a torsion theory in a homological category \(\mathcal{C}\) is a pair \((\mathcal{T}, \mathcal{X})\) of full replete subcategories of \(\mathcal{C}\) such that

\[\ast\] Corresponding author.

E-mail addresses: gran@lmpa.univ-littoral.fr (M. Gran), rossi@dimi.uniud.it (V. Rossi).

0022-4049/$ - see front matter \copyright$ 2005 Elsevier B.V. All rights reserved.
(1) the only arrow $f: T \to X$ from $T \in T$ to $X \in \mathcal{X}$ is the zero arrow 0;
(2) for any object $A$ in $\mathcal{C}$ there exists a short exact sequence
\[ 0 \longrightarrow T \longrightarrow A \longrightarrow X \longrightarrow 0 \]
with $T \in T$ and $X \in \mathcal{X}$.

We say that a torsion theory $(T, \mathcal{X})$ is quasi-hereditary when its torsion subcategory $T$ is closed in $\mathcal{C}$ under regular subobjects. Of course, when $\mathcal{C}$ is an abelian category, any quasi-hereditary torsion theory in $\mathcal{C}$ is hereditary, so that the distinction between these two notions vanishes. However, in a general homological category, this distinction is meaningful: for instance, the torsion theory given by the categories $(\text{Grp}(\text{Ind}), \text{Grp}(\text{Haus}))$ of indiscrete groups and of Hausdorff groups in the category $\text{Grp}(\text{Top})$ of topological groups is quasi-hereditary, but it is not hereditary.

Any torsion theory gives rise to an admissible Galois structure in the sense of Janelidze [18], and our Theorem 4.5 solves the problem of characterizing the coverings that correspond to a quasi-hereditary torsion theory in a homological category. This result is then used to give a precise description of a corresponding reflective factorization system (Theorem 4.7), which combines the ideas of localization/stabilization of Carboni, Janelidze, Kelly and Paré [12] and of the concordant/dissonant factorization system of Janelidze and Tholen [27].

In the special case of the torsion theory $(\text{Grp}(\text{Ind}), \text{Grp}(\text{Haus}))$, we find out that the coverings are exactly the open surjective homomorphisms with a Hausdorff kernel. It is interesting to remark that, in this homological context, there are coverings that are not trivial coverings. The above characterization is also useful to study another torsion theory in the category of topological groups. The pair of full subcategories $(\text{Grp}(\text{Conn}), \text{Grp}(\text{TotDis}))$ of connected groups and of totally disconnected groups is indeed a torsion theory, but it is not a quasi-hereditary torsion theory. For this reason, the characterization of the coverings relative to this second torsion theory is more delicate. An important ingredient helping us to solve this problem comes from a result of Arkhangel’ski˘ı [1] in the full subcategory $\text{Grp}(\text{Haus})$ of $\text{Grp}(\text{Top})$: any Hausdorff group is a regular quotient of a totally disconnected group. We can then show that the coverings relative to $(\text{Grp}(\text{Conn}), \text{Grp}(\text{TotDis}))$ are exactly the open surjective homomorphisms with a totally disconnected kernel, a notion already considered in the category of connected compact Hausdorff groups by Berestovskij and Plaut [3].

Finally, we would like to mention that our work is related to the work of Janelidze, Márki and Tholen in [23], where these authors solved the problem of classifying the coverings with respect to general radical theories. The main difference with their approach is that, in the present article, we want to get free of the heavy requirement that any object of $\mathcal{C}$ is a regular quotient of an object in $\mathcal{X}$. The present article also provides some new examples of “locally semi-simple coverings” in the non-exact category of topological groups; again, this can be considered as a complement to their work in the exact case (see the last remark in [23]).

The paper is structured as follows:

(1) Effective descent morphisms.
(2) Torsion theories and homological categories.
(3) Quasi-hereditary torsion theories.
(4) Galois structure of torsion theories.
(5) Applications: coverings of topological groups.

In the first section we revise the basic notions of descent theory and prove that the open surjective homomorphisms are effective for descent in any category of topological Maltsev algebras [30,29]. This fact will be useful later on to define a Galois structure in the context of topological algebras. We then recall some properties of homological categories and torsion theories in the second section [8,10]. The third section is devoted to the study of quasi-hereditary torsion theories. We analyse an example of a quasi-hereditary torsion theory which is not hereditary in the category of topological semi-abelian algebras [5]. Necessary and sufficient conditions for a torsion theory to be quasi-hereditary are given in terms of the associated reflection and coreflection. In Section 4 we introduce the Galois structure induced by a torsion theory in a homological category, and we characterize the coverings relative to a quasi-hereditary torsion theory. We analyse the reflective factorization system induced by the coverings. In the last section we consider the coverings corresponding to various torsion theories in the category of topological groups.
1. Effective descent morphisms

In this section we recall some basic definitions and results on descent theory (we refer to [24] for more details). We then explain why the open surjective homomorphisms are effective for descent in $\text{Grp}(\text{Top})$ and, much more generally, in any category of topological Maltsev algebras.

Let $\mathcal{C}$ be a finitely complete category. For any object $B$ in $\mathcal{C}$, we write $\mathcal{C} \downarrow B$ for the comma category over $B$. Given an arrow $p: E \to B$ in $\mathcal{C}$ and an object $(A, f)$ in $\mathcal{C} \downarrow B$, we write $p^*: \mathcal{C} \downarrow B \to \mathcal{C} \downarrow E$ for the usual pullback functor along $p$: it is defined on objects by $p^*(A, f) = (E \times_B A, \pi_E)$, where $E \times_B A$ is given by the pullback

$$
\begin{array}{ccc}
E \times_B A & \to & A \\
\downarrow \pi_A & & \downarrow f \\
E & \to & B.
\end{array}
$$

Any arrow $p: E \to B$ determines an internal equivalence relation in $\mathcal{C}$, given by the kernel pair $R[p] \xrightarrow{\pi_1} E$ of $p$.

By taking the kernel pairs of the projection $\pi_A: E \times_B A \to A$ and of $p: E \to B$ in the pullback (1), one obtains a discrete fibration over $R[p]$, represented by the following diagram:

$$
\begin{array}{ccc}
R[\pi_A] & \xrightarrow{\pi_1} & E \times_B A \\
\downarrow \pi_E \times \pi_E & & \downarrow \pi_E \\
R[p] & \xrightarrow{\pi_2} & E.
\end{array}
$$

Recall that, when one says that an internal functor is a discrete fibration, it means that the two commutative squares in the given diagram are pullbacks. For any morphism $p: E \to B$, let $\text{DiscFib}(R[p])$ be the category of discrete fibrations over $R[p]$. Let $K^p: \mathcal{C} \downarrow B \to \text{DiscFib}(R[p])$ be the functor sending the object $(A, f)$ to the discrete fibration (2) over $R[p]$ (with obvious definition on morphisms).

**Definition 1.1.** Let $p: E \to B$ be a morphism in $\mathcal{C}$. The category $\text{DiscFib}(R[p])$ is called the category of descent data for $p$, and is denoted by $\text{Des}_\mathcal{C}(p)$. The morphism $p$ is said to be:

1. a descent morphism if the functor $K^p: \mathcal{C} \downarrow B \to \text{Des}_\mathcal{C}(p)$ is full and faithful;
2. an effective descent morphism if the functor $K^p: \mathcal{C} \downarrow B \to \text{Des}_\mathcal{C}(p)$ is a category equivalence.

Now let $\mathcal{C}$ be a finitely complete regular category. This means that: (a) any arrow $f: A \to B$ has a factorization $f = i \cdot p$, where $p: A \to I$ is a regular epi and $i: I \to B$ is a monomorphism; and (b) these factorizations are pullback-stable.

In particular, in the pullback (1), the second projection $\pi_A: E \times_B A \to A$ is a regular epi whenever $p$ is a regular epi. This is the key property needed to show that the descent morphisms are exactly the regular epimorphisms in any regular category [25]. However, there are various examples of regular categories where regular epimorphisms fail to be effective descent morphisms (see [26], section 2.7).

Let us then recall that a variety of universal algebras is a Maltsev variety [32] if its theory $\text{Th}$ has a ternary term $p(x, y, z)$ satisfying the axioms $p(x, y, y) = x$ and $p(x, y, x) = y$; in this case $\text{Th}$ is called a Maltsev theory. For instance, when $\text{Th}$ is the theory of groups, a Maltsev operation is given by the term $p(x, y, z) = x \cdot y^{-1} \cdot z$. We shall write $\text{Th}(\text{Top})$ for any category of topological Maltsev algebras, i.e. the models of a Maltsev theory in the category $\text{Top}$ of topological spaces. For these categories, there is a simple characterization of the effective descent morphisms, which essentially follows from the results of Johnstone and Pedicchio in [29].

**Lemma 1.2.** Let $\text{Th}(\text{Top})$ be a category of topological Maltsev algebras. For an arrow $p: E \to B$ in $\text{Th}(\text{Top})$, the following conditions are equivalent:

- $p$ is a descent morphism,
- there is a ternary term $p(x, y, z)$ in $\text{Th}$ such that $p(x, y, y) = x$ and $p(x, y, x) = y$, and
- for any arrow $g: A \to B$ in $\text{Top}$, the canonical homomorphism $\text{Des}_\text{Th}(g) \to \text{Des}_\text{Th}(p)$ is an isomorphism.
(1) \( p \) is a regular epimorphism;
(2) \( p \) is an open surjective homomorphism;
(3) \( p \) is an effective descent morphism.

**Proof.** (1) \( \Rightarrow \) (2) This fact is well known, and it was proved in [29] (Corollary 2.6).

(2) \( \Rightarrow \) (3) When \( p: E \to B \) is an open surjective homomorphism in \( Th(Top) \), the functor \( K^p \) is clearly fully faithful. In order to show that it is also essentially surjective on objects, consider the solid diagram below, which represents a discrete fibration \((R, r)\) over \( R[p] \) in \( Th(Top) \):

\[
\begin{array}{ccc}
R & \xrightarrow{\pi_1} & F \\
\downarrow r \times r & & \downarrow r \\
R[p] & \xrightarrow{\pi_2} & E \xrightarrow{\pi_2} B.
\end{array}
\]

Since \( p \) is an open surjection, and thus an effective descent morphism in \( Top \) [31], there is a unique (up to isomorphism) object \((A, f)\) in \( Top \downarrow B \) with the property that \( K^p(A, f) = (R, r) \), where \((R, r)\) is seen as a discrete fibration over \( R[p] \) in \( Top \). The product of open surjections is an open surjection, thus \( p^n: E^n \to B^n \) is an effective descent morphism in \( Top \) (for any natural number \( n \geq 1 \)). Therefore, we get a similar diagram for any \( n \geq 1 \),

\[
\begin{array}{ccc}
R^n & \xrightarrow{\pi_1^n} & F^n \\
\downarrow r^n \times r^n & & \downarrow r^n \\
R[p^n] & \xrightarrow{\pi_2^n} & E^n \xrightarrow{\pi_2^n} B^n.
\end{array}
\]

where the left-hand squares determine a discrete fibration (in \( Top \)), the right-hand square is a pullback (in \( Top \)), and \( R^n \) is the kernel pair of \( s^n \). The fact that \( s^n \) is a regular epimorphism implies that, for any fundamental \( n \)-ary operation \( \tau \) in the theory \( Th \), there is a corresponding arrow \( \tau_A: A^n \to A \) induced by the corresponding \( n \)-ary operations \( \tau_R: R^n \to R \) and \( \tau_F: F^n \to F \). When the theory \( Th \) contains an equation \( \lambda = \mu \), where \( \lambda \) and \( \mu \) are derived operations with the same arity \( n \), this equation will be satisfied by \( R \) and \( F \), since these objects belong to \( Th(Top) \). But then the same equation will hold true also for \( A \), since \( s^n: F^n \to A^n \) is an epimorphism. It follows that \( A \) is a topological algebra in \( Th(Top) \), then \( K^p(A, f) = (R, r) \) in \( Th(Top) \), and \( K^p: Th(Top) \downarrow B \to Des_{Th(Top)}(p) \) is a category equivalence.

(3) \( \Rightarrow \) (1) This implication is well known, and can be found in [24]. \( \square \)

2. Torsion theories and homological categories

In this section we recall some basic properties of torsion theories in a homological category, and we give some examples.

A category \( C \) is said to be **pointed** if it has a zero object 0.

**Definition 2.1.** A homological category [4] is a regular pointed category \( C \) satisfying the following property [7]: given a commutative diagram

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
D & \to & E
\end{array}
\]

where the dotted vertical arrow is a regular epi, the left-hand square and the whole rectangle are pullbacks, then the right-hand square is also a pullback.
A nice property of homological categories is that any regular epi \( p: A \to B \) is the cokernel of its kernel \([7]\), so that the class of regular epimorphisms coincides with the class of normal epimorphisms. In a homological category, a short exact sequence

\[
0 \longrightarrow K \xrightarrow{k} A \xrightarrow{p} B \longrightarrow 0
\]

is then a zero sequence \((p \cdot k = 0, \text{with } 0 \text{ the zero arrow})\) such that \( p \) is a regular epi and \( k \) is the kernel of \( p \).

Homological categories are known to provide a nice setting where some important aspects of the homological algebra on the model of the category of groups can be developed. The classical basic diagram lemmas, such as the Short Five Lemma, the 3 × 3-Lemma, the Snake Lemma and the long exact homology sequence associated with a chain complex, hold true in any homological category \([4,8,16]\).

When a homological category is also exact \([2]\) and has binary coproducts, it is called semi-abelian \([22]\). A characterization of the algebraic theories with the property that the corresponding category of algebras is semi-abelian was obtained by Bourn and Janelidze:

**Theorem 2.2 \([11]\).** A variety of universal algebras \( V \) is semi-abelian if and only if its theory \( \text{Th} \) has a unique constant \( 0 \), binary terms \( t_1, t_2, \ldots, t_n \) and a \((n + 1)\)-ary term \( \tau \) satisfying the identities \( \tau(x, t_1(x, y), t_2(x, y), \ldots, t_n(x, y)) = y \) and \( t_i(x, x) = 0 \) for each \( i = 1, \ldots, n \).

Classical examples of semi-abelian varieties are groups, rings, Lie algebras, commutative algebras, crossed modules and Heyting semilattices \([28]\).

From now on, \( \text{Th}(\text{Top}) \) will always denote a category of topological semi-abelian algebras, i.e. a category of models of a semi-abelian theory \( \text{Th} \) in the category \( \text{Top} \) of topological spaces. In general, when \( \text{Th} \) is a semi-abelian theory, the category \( \text{Th}(\text{Top}) \) is a homological category, as proved by Borceux and Clementino in \([5]\). Similarly, the categories \( \text{Th}(\text{Haus}) \) of Hausdorff semi-abelian algebras and \( \text{Th}(\text{TotDis}) \) of totally disconnected semi-abelian algebras are homological (recall that a space is totally disconnected if and only if the connected component of any point is reduced to that point).

Let us then recall the following property, due to Bourn, needed in the following:

**Proposition 2.3 \([8]\).** In any homological category \( \mathcal{C} \), let us consider the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K' \xrightarrow{k'} X' \xrightarrow{f'} Y' \longrightarrow 0 \\
\downarrow{u} & & \downarrow{v} & \downarrow{w} \\
0 & \longrightarrow & K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 0
\end{array}
\]

where the two rows are exact. Then:

1. \( u \) is an isomorphism if and only if the right-hand square is a pullback;
2. \( w \) is a monomorphism if and only if the left-hand square is a pullback.

**Proof.** (1) Given the following commutative diagram

\[
\begin{array}{ccc}
K' & \xrightarrow{k'} & X' & \xrightarrow{v} & X \\
\downarrow{(1)} & & \downarrow{f'} & \downarrow{(2)} \\
0 & \longrightarrow & Y' \xrightarrow{w} & Y
\end{array}
\]

one has that (1) is a pullback by construction. But \( \nu k' = ku \), so that the whole rectangle (1) + (2) is a pullback whenever \( u \) is an isomorphism. From the fact that \( \mathcal{C} \) is homological and \( f' \) is a regular epimorphism, it follows that (2) is a pullback.
Conversely, if we assume that the square (2) is a pullback, so is the following rectangle:

\[
\begin{array}{ccc}
K' & \xrightarrow{u} & K \\
\downarrow & & \downarrow \\
k & \xrightarrow{k} & X \\
\end{array}
\]

Since the right-hand square is a pullback by construction, it follows that the left-hand square is a pullback as well, and \(u\) is an isomorphism.

(2) The non-trivial implication essentially follows from the fact that, in a homological category, an arrow is a monomorphism if its kernel is 0 [7]. □

The following definition extends the classical one by Dickson in the context of abelian categories [14]. It was introduced and studied in the context of homological categories in [10], where the relationship between torsion theories and some closure operators on kernels was also clarified.

**Definition 2.4.** Let \(C\) be a homological category. A *torsion theory* in \(C\) is a pair \((T, \mathcal{X})\) of full replete subcategories of \(C\) such that:

1. the only arrow \(f: T \to X\) from \(T \in T\) to \(X \in \mathcal{X}\) is the zero arrow 0;
2. for any object \(A\) in \(C\) there exists a short exact sequence

\[
0 \longrightarrow T \xrightarrow{\iota_A} A \xrightarrow{\eta_A} X \longrightarrow 0
\]

with \(T \in T\) and \(X \in \mathcal{X}\).

It turns out that, besides the classical examples of torsion theories in the abelian categories, there are some interesting new examples in the (non-abelian) context of topological groups.

**Example 2.5.** (1) In the category \(Ab\) of abelian groups there is the classical torsion theory \((Ab_t, Ab_{tf})\), where \(Ab_t\) is the category of torsion abelian groups and \(Ab_{tf}\) is the category of torsion-free abelian groups. Another classical torsion theory in \(Ab\) is given by \((Div, Red)\), where \(Div\) denotes the category of divisible abelian groups and \(Red\) is the category of reduced abelian groups. Much more generally, any torsion theory in an abelian category is a torsion theory according to the **Definition 2.4**.

(2) Given a semi-abelian theory \(Th\), let \(Th(Ind)\) be the category of indiscrete algebras and let \(Th(Conn)\) be the category of connected algebras. It has been proved in [10] that the pairs \((Th(Ind), Th(Haus))\) and \((Th(Conn), Th(TotDis))\) are torsion theories in \(Th(Top))\).

We now recall some properties of torsion theories in homological categories that will be needed in the following:

**Lemma 2.6** ([10]). Let \((T, \mathcal{X})\) be a torsion theory in a homological category \(C\). Then:

1. for any \(A\) in \(C\) there exists exactly one (up to isomorphism) short exact sequence

\[
0 \longrightarrow T \xrightarrow{\iota_A} A \xrightarrow{\eta_A} X \longrightarrow 0
\]

with \(T \in T\) and \(X \in \mathcal{X}\);
2. \(T \cap \mathcal{X} = 0\);
3. \(\mathcal{X}\) is closed in \(C\) under subobjects;
4. \(\mathcal{X}\) is closed in \(C\) under extensions.

**Proof.** (1) Let us consider two short exact sequences

\[
\begin{array}{ccc}
0 & \xrightarrow{\iota_A} & T \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\iota'_A} & T'
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{\eta_A} & A \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\eta'_A} & A'
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{\iota_A} & T \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\eta_A} & X \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{\iota'_A} & T' \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\eta'_A} & X'
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{\iota_A} & A \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\eta_A} & X \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{\iota'_A} & A' \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\eta'_A} & X'
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{\iota_A} & T \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\eta_A} & X \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{\iota'_A} & T' \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\eta'_A} & X'
\end{array}
\]
where \( T, T' \) are in \( \mathcal{T} \) and \( X, X' \) are in \( \mathcal{X} \). The arrows \( \eta_A' \cdot t_A \) and \( \eta_A \cdot t_A' \) are the zero arrows. There are then two factorizations \( \alpha: T \rightarrow T' \) and \( \beta: T' \rightarrow T \) such that \( t_A' \cdot \alpha = t_A \) and \( t_A \cdot \beta = t_A' \). It follows that \( \beta \cdot \alpha = 1_T \) and \( \alpha \cdot \beta = 1_{T'} \), hence \( T \cong T' \). Since, in a homological category, a regular epi is the coproduct of its kernel, this also implies that \( X \cong X' \), as desired.

(2) If \( A \in \mathcal{T} \cap \mathcal{X} \), then the identity arrow \( 1_A: A \rightarrow A \) is the zero map, and \( A = 0 \).

(3) If \( f: A \rightarrow B \) is a monomorphism and \( B \) belongs to \( \mathcal{X} \), by considering the canonical exact sequence

\[
0 \rightarrow T \rightarrow A \rightarrow X \rightarrow 0
\]

one has that \( f \cdot t_A = 0 \). There then exists a unique arrow \( g: X \rightarrow B \) with \( g \cdot \eta_A = f \), hence \( \eta_A \) is an isomorphism and \( A \) is in \( \mathcal{X} \).

(4) Let us consider an exact sequence

\[
0 \rightarrow F_1 \xrightarrow{a} A \xrightarrow{b} F_2 \rightarrow 0
\]

where both \( F_1 \) and \( F_2 \) are in \( \mathcal{X} \). Then, by taking the canonical exact sequence

\[
0 \rightarrow T \xrightarrow{\eta_A} A \rightarrow X \rightarrow 0
\]

one has that \( b \cdot t_A = 0 \), because \( T \) is in \( \mathcal{T} \) and \( F_2 \) is in \( \mathcal{X} \); then there is a unique \( i: T \rightarrow F_1 \) with \( a \cdot i = t_A \), whence \( i \) is a mono. Accordingly, \( T \) belongs to \( \mathcal{X} \) by (3); so \( T = 0 \), and \( A \cong X \) is in \( \mathcal{X} \). □

The following property will have a useful consequence concerning the effective descent morphisms:

**Lemma 2.7.** Let \( (\mathcal{T}, \mathcal{X}) \) be a torsion theory in a homological category \( \mathcal{C} \). Given any pullback \( E \times_B A \) in \( \mathcal{C} \) where \( E, B \) and \( E \times_B A \) are in \( \mathcal{X} \), then \( A \) is in \( \mathcal{X} \).

**Proof.** Let us first remark that, by the uniqueness of the exact sequence in Definition 2.4, an object \( C \) belongs to \( \mathcal{X} \) if and only if its canonical exact sequence is given by

\[
0 \rightarrow 0 \rightarrow C \rightarrow C \rightarrow 0.
\]

Consequently, by taking the canonical exact sequences determined by the torsion theory, one gets the following commutative diagram (where the back square is given by the corresponding objects in \( \mathcal{T} \)):

\[
\begin{array}{ccc}
0 & \rightarrow & T \\
\downarrow & & \downarrow \\
E \times_B A & \xrightarrow{\pi_A} & A \\
\downarrow & & \downarrow \\
E & \xrightarrow{f} & B \\
\end{array}
\]

The universal property of the front pullback implies that there is a unique arrow \( \phi: T \rightarrow E \times_B A \) with \( \pi_A \cdot \phi = t_A \) and \( \pi_E \cdot \phi = 0 \). Such an arrow is necessarily a monomorphism, since \( t_A \) is a monomorphism. Since \( E \times_B A \) is in \( \mathcal{X} \), it follows that \( T \) belongs to \( \mathcal{X} \), and then to \( \mathcal{T} \cap \mathcal{X} = \{0\} \). This shows that \( T = 0 \), and \( A \) belongs to \( \mathcal{X} \), as desired. □

**Corollary 2.8.** Let \( \mathcal{C} \) be a homological category in which the regular epimorphisms are effective for descent, and let \( (\mathcal{T}, \mathcal{X}) \) be a torsion theory in \( \mathcal{C} \). Then the regular epimorphisms are effective for descent in \( \mathcal{X} \).

**Proof.** It follows from the previous lemma and Corollary 2.7 in [26]. □
3. Quasi-hereditary torsion theories

In this section we introduce the notion of quasi-hereditary torsion theory and we give an example of a quasi-hereditary torsion theory which is not hereditary. We then prove a useful characterization of quasi-hereditary torsion theories in the homological categories.

A full replete subcategory $\mathcal{X}$ of a homological category $\mathcal{C}$ is called a torsion-free subcategory if there is a full replete subcategory $\mathcal{T}$ with the property that $(\mathcal{T}, \mathcal{X})$ is a torsion theory in $\mathcal{C}$. It has been proved in [10] that a torsion-free subcategory $\mathcal{X}$ of a homological category $\mathcal{C}$ is always reflective in $\mathcal{C}$:

$$
\begin{array}{ccc}
\mathcal{X} & \xleftarrow{F} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{U} & \mathcal{C}
\end{array}
$$

Since $\mathcal{X}$ is also closed in $\mathcal{C}$ under subobjects, $\mathcal{X}$ is itself homological. The universal arrow of the reflection $F: \mathcal{C} \rightarrow \mathcal{X}$ of an object $A$ in $\mathcal{C}$ is given by the arrow $\eta_A$ in the canonical short exact sequence

$$
0 \rightarrow T \xrightarrow{i_A} A \xrightarrow{\eta_A} X \rightarrow 0
$$

of Definition 2.4. In order to mention explicitly the functor $F$, from now on we shall denote this canonical exact sequence by

$$
0 \rightarrow K(A) \xrightarrow{k_A} A \xrightarrow{\eta_A} F(A) \rightarrow 0
$$

where the arrow $k_A$ is defined as the kernel of $\eta_A$.

The assignment sending $A$ to $K(A)$ determines a functor $K: \mathcal{C} \rightarrow \mathcal{T}$. Moreover, the full replete subcategory $\mathcal{T}$ consists precisely of the objects $A$ in $\mathcal{C}$ such that $F(A) \simeq 0$ (or, equivalently, by those $A$ with $K(A) \simeq A$). It is then easy to check that the torsion subcategory $\mathcal{T}$ of $\mathcal{C}$ is coreflective in $\mathcal{C}$:

$$
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{K} & \mathcal{T} \\
\downarrow & & \downarrow \\
\mathcal{T} & \xrightarrow{J} & \mathcal{C}
\end{array}
$$

When a torsion theory $(\mathcal{T}, \mathcal{X})$ in a homological category $\mathcal{C}$ has the property that $\mathcal{T}$ is closed in $\mathcal{C}$ under subobjects, it is called hereditary [10]. The main reason for this terminology comes from the fact that the hereditary torsion theories correspond to the hereditary closure operators on kernels (as it is the case in the abelian case).

In the homological categories there are examples of torsion subcategories $\mathcal{T}$ that are closed in $\mathcal{C}$ only under regular subobjects, and not under subobjects. It seems then natural to introduce the following definition:

**Definition 3.1.** A torsion theory $(\mathcal{T}, \mathcal{X})$ in a homological category $\mathcal{C}$ is quasi-hereditary if $\mathcal{T}$ is closed in $\mathcal{C}$ under regular subobjects. This means that, for every equalizer $e: E \rightarrow T$ in $\mathcal{C}$ with $T$ in $\mathcal{T}$, one has that $E$ is in $\mathcal{T}$ as well.

Of course, in the abelian context a torsion theory is hereditary if and only if it is quasi-hereditary, so that the distinction just introduced vanishes. However, here there is an example showing the importance of this distinction in the more general homological context:

**Proposition 3.2.** When $\text{Th}$ is a semi-abelian theory, then the torsion theory $(\text{Th}(\text{Ind}), \text{Th}(\text{Haus}))$ in $\text{Th}(\text{Top})$ is quasi-hereditary. Moreover, it is not hereditary in general.

**Proof.** Let $e: (E, \tau_E) \rightarrow (T, \tau_T)$ be an equalizer in $\text{Th}(\text{Top})$ with $(T, \tau_T)$ in $\text{Th}(\text{Ind})$. Since $e$ is an equalizer, one has that $\tau_E$ is the topology induced on $E$ by $\tau_T$, and $\tau_E$ is then indiscrete.

On the other hand, given an arbitrary subobject $e: (X, \tau_X) \rightarrow (T, \tau_T)$ of an indiscrete algebra $(T, \tau_T)$, there is no reason for $\tau_X$ to be the indiscrete topology. For instance, consider the subobject given by the identity $Id_T: (T, \tau_{dis}) \rightarrow (T, \tau_T)$, where $\tau_{dis}$ denotes the discrete topology on $T$. \[\square\]

In the case of a homological category $\mathcal{C}$ it is possible to characterize the quasi-hereditary torsion theories $(\mathcal{T}, \mathcal{X})$ in $\mathcal{C}$ in terms of some properties of the functors $F: \mathcal{C} \rightarrow \mathcal{X}$ and $J \cdot K: \mathcal{C} \rightarrow \mathcal{T} \rightarrow \mathcal{C}$.
Theorem 3.3. Let \((T, \mathcal{X})\) be a torsion theory in a homological category \(\mathcal{C}\). Then the following conditions are equivalent:

1. \((T, \mathcal{X})\) is quasi-hereditary;
2. \(J \cdot K: \mathcal{C} \rightarrow T \rightarrow \mathcal{C}\) preserves finite limits;
3. \(J \cdot K: \mathcal{C} \rightarrow T \rightarrow \mathcal{C}\) preserves equalizers;
4. for every regular subobject \(e: E \rightarrow A\) in \(\mathcal{C}\), \(F(e): F(E) \rightarrow F(A)\) is a monomorphism in \(\mathcal{X}\).

Proof. (1) \(\Rightarrow\) (2) The functor \(J: T \rightarrow \mathcal{C}\) preserves equalizers because it is the inclusion of a full subcategory closed under equalizers, and \(K: \mathcal{C} \rightarrow T\) preserves equalizers because it is a coreflection. The same is true for finite products — having in mind that \(T\) is closed in \(\mathcal{C}\) under finite products, since \(T\) is a torsion subcategory and the canonical “product injections” are jointly epimorphic in any homological category.

(2) \(\Rightarrow\) (3) Trivial.

(3) \(\Rightarrow\) (4) Given an equalizer \(e: E \rightarrow A\) in \(\mathcal{C}\) of a pair of morphisms \(u, v: A \rightarrow B\), let us consider the following diagram

\[
\begin{array}{ccc}
K(E) & \xrightarrow{k_E} & E \\
\downarrow \psi & & \downarrow e \\
K(A) \times_A E & \xrightarrow{p_1} & K(A) \\
\downarrow p_2 & & \downarrow k_A \\
K(\varepsilon) & \xrightarrow{\varepsilon} & A
\end{array}
\]

where \(k_A = \ker(\eta_A), k_E = \ker(\eta_E)\), \((K(A) \times_A E, p_1, p_2)\) is the pullback of \(e\) along \(k_A\), and \(\psi\) is the comparison arrow towards the pullback. The fact that \(K(e): K(E) \rightarrow K(A)\) is the equalizer of \((K(u), K(v))\) in \(\mathcal{C}\) implies that the arrow \(\psi\) is an isomorphism, and then \(K(E) \simeq K(A) \times_A E\). Therefore, in the following commutative diagram of exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & K(E) & \xrightarrow{k_E} & E & \xrightarrow{\eta_E} & F(E) & \rightarrow & 0 \\
\downarrow K(e) & & \downarrow e & & \downarrow F(e) & & \downarrow \eta_A \\
0 & \rightarrow & K(A) & \xrightarrow{k_A} & A & \xrightarrow{\eta_A} & F(A) & \rightarrow & 0
\end{array}
\]

the left-hand square is a pullback in \(\mathcal{C}\). From Proposition 2.3(2), it follows that \(F(e): F(E) \rightarrow F(A)\) is a monomorphism in \(\mathcal{X}\).

(4) \(\Rightarrow\) (1) Given an equalizer \(e: E \rightarrow T\) in \(\mathcal{C}\) with \(T \in T\), by (4) one has that \(F(e): F(E) \rightarrow F(T)\) is a monomorphism in \(\mathcal{X}\). But if \(T \in T\), then \(F(T) \simeq 0\); therefore \(F(E) \simeq 0\) and \(E\) is in \(T\). □

4. Galois structure of torsion theories

In this section we will show that every torsion theory gives rise to a Galois structure in the sense of Janelidze [18, 19]. In particular, when the torsion theory is quasi-hereditary, it will be possible to characterize the coverings of the induced Galois structure.

Let \((T, \mathcal{X})\) be a torsion theory in a homological category \(\mathcal{C}\). As we mentioned before, \(\mathcal{X}\) is a full reflective subcategory of \(\mathcal{C}\):

\[
\mathcal{X} \xrightarrow{F} U \xrightarrow{U} \mathcal{C}.
\]

We denote by \(\eta\) the unit of this adjunction, and by \(\mathcal{E}\) and \(\mathcal{Z}\) the classes of regular epimorphisms in \(\mathcal{C}\) and in \(\mathcal{X}\), respectively.

Proposition 4.1. \(((\mathcal{C}, \mathcal{E}), (\mathcal{X}, \mathcal{Z}), F \dashv U)\) is a Galois structure.
Proof. As required in the definition of Galois structure, the categories \( \mathcal{C} \) and \( \mathcal{X} \) have pullbacks. Furthermore, both classes \( \mathcal{E} \) and \( \mathcal{Z} \) clearly contain all the isomorphisms, and are stable under composition and under pullbacks; moreover, \( F(\mathcal{E}) \subseteq \mathcal{Z} \) and \( U(\mathcal{Z}) \subseteq \mathcal{E} \). The counit of the adjunction above is an isomorphism, because \( \mathcal{X} \) is a full reflective subcategory of \( \mathcal{C} \), while every component \( \eta_A : A \to F(A) \) of the unit of the adjunction is an arrow in \( \mathcal{E} \). \( \square \)

We shall write \( \mathcal{E}(B) \) for the full subcategory of \( \mathcal{C} \downarrow B \) whose objects are arrows \( p : E \to B \) in \( \mathcal{E} \), i.e. the regular epimorphisms of \( \mathcal{C} \) with codomain \( B \). Given an arrow \( p : E \to B \) in \( \mathcal{E} \), there is the composition functor \( W^p : \mathcal{E}(E) \to \mathcal{E}(B) \) defined by \( (A, f) \mapsto (A, p \cdot f) \). Under our assumptions, \( W^p \) has a right adjoint \( G^p : \mathcal{E}(B) \to \mathcal{E}(E) \), the pullback functor along \( p \), defined by the assignment:

\[
(A, f) \mapsto (E \times_B A, \pi_E).
\]

Given an object \( E \in \mathcal{C} \), we call \( F^E : \mathcal{C} \downarrow E \to \mathcal{X} \downarrow F(E) \) the functor induced by \( F \), and \( U^E : \mathcal{X} \downarrow F(E) \to \mathcal{C} \downarrow E \) its right adjoint, which corresponds to the pullback functor along \( \eta_E \). We denote by \( F^{E, \Gamma} : \mathcal{E}(E) \to \mathcal{Z}(F(E)) \) and \( U^{E, \Gamma} : \mathcal{Z}(F(E)) \to \mathcal{E}(E) \) the adjoint functors induced by \( F^E \) and \( U^E \), respectively.

Remark that the counit \( \epsilon^{E, \Gamma} : F^{E, \Gamma} U^{E, \Gamma} \to 1_{\mathcal{Z}(F(E))} \) of this latter adjunction is an isomorphism, since \( \mathcal{X} \) is a semi-left exact reflective subcategory of \( \mathcal{C} \) [10]. This means that this particular Galois structure is admissible in the sense of the categorical Galois theory.

Convention:

From now on we shall only consider the special Galois structure

\[
\Gamma = ((\mathcal{C}, \mathcal{E}), (\mathcal{X}, \mathcal{Z}), F \dashv U)
\]

of Proposition 4.1. Furthermore, we shall always assume that \( \mathcal{C} \) is a homological category with the property that the regular epimorphisms are effective for descent.

**Definition 4.2.** Let \((A, f)\) be an object in \(\mathcal{E}(B)\). We say that \((A, f)\) is a trivial covering (with respect to \(\Gamma\)) if the canonical commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & F(A) \\
\downarrow f & & \downarrow F(f) \\
B & \xrightarrow{\eta_B} & F(B)
\end{array}
\]

is a pullback.

**Definition 4.3.** Let \((A, f)\) and \((E, p)\) be two objects in \(\mathcal{E}(B)\). We say that \((A, f)\) is split over \((E, p)\) (with respect to \(\Gamma\)) if the canonical commutative diagram

\[
\begin{array}{ccc}
E \times_B A & \xrightarrow{\eta_E \times_B \eta_A} & F(E \times_B A) \\
\downarrow \pi_E & & \downarrow F(\pi_E) \\
E & \xrightarrow{\eta_E} & F(E)
\end{array}
\]

is a pullback.

We shall write \(\text{Spl}(E, p)\) for the full subcategory of \(\mathcal{E}(B)\) whose objects are split over \((E, p)\).

Thanks to the assumption that regular epimorphisms are effective for descent in \(\mathcal{C}\), the definitions given in [18] can be simplified as follows:

**Definition 4.4.** Let \((A, f)\) be in \(\mathcal{E}(B)\).

1. \((A, f)\) is a \(\Gamma\)-covering if there exists a \((E, p)\) in \(\mathcal{E}(B)\) with the property that \((A, f)\) is split over \((E, p)\).
2. \((A, f)\) is a normal covering if \((A, f)\) is split over \((A, f)\).

If the torsion theory \((T, \mathcal{X})\) is quasi-hereditary, it is possible to give several characterizations of the \(\Gamma\)-coverings:

**Theorem 4.5.** Let \((A, f)\) be an object in \(\mathcal{E}(B)\). If the torsion theory \((T, \mathcal{X})\) in \(\mathcal{C}\) is quasi-hereditary, then the following conditions are equivalent:

1. \((A, f)\) is a \(\Gamma\)-covering;
2. \(\ker(f) \in \mathcal{X}\);
3. \(K(f): K(A) \rightarrow K(B)\) is a monomorphism;
4. \((A, f)\) is a normal covering.

**Proof.** (1) \(\Rightarrow\) (2) When \((A, f)\) is a \(\Gamma\)-covering, there exists a regular epimorphism \(p: E \rightarrow B\) in \(\mathcal{C}\) such that \((A, f)\) is \((E, p)\)-split. Let us consider the following diagram

\[
\begin{array}{ccc}
F(E \times_B A) & \xrightarrow{\eta_{E \times_B A}} & E \times_B A \\
\downarrow F(\pi_E) & & \downarrow \pi_E \\
F(E) & \xleftarrow{\eta_E} & E \\
\downarrow p & & \downarrow f \\
& & B,
\end{array}
\]

where both commutative squares are pullbacks by assumption. Since \(\mathcal{X}\) is closed in \(\mathcal{C}\) under subobjects, the kernel \(K\) of \(F(\pi_E)\) lies in \(\mathcal{X}\). Clearly, \(K\) is isomorphic to the kernel of \(\pi_E\), and then to the kernel of \(f\), which then lies in \(\mathcal{X}\), as desired. Let us observe that this first implication does not depend on the assumption that the torsion theory \((T, \mathcal{X})\) is quasi-hereditary.

(2) \(\Rightarrow\) (3) Let us consider the following commutative diagram of short exact sequences:

\[
\begin{array}{ccc}
0 & \rightarrow & K(\ker(f)) \xrightarrow{k_{\ker(f)}} \ker(f) \xrightarrow{\eta_{\ker(f)}} F(\ker(f)) \rightarrow 0 \\
K(\ker(f)) & \downarrow & \ker(f) \downarrow & \rightarrow F(\ker(f)) \\
0 & \rightarrow & K(A) \xrightarrow{k_A} A \xrightarrow{\eta_A} F(A) \rightarrow 0.
\end{array}
\]

One has that

\(\ker(f) \in \mathcal{X} \iff K(\ker(f)) \cong 0 \iff K(f)\) is a monomorphism

where the last equivalence follows from the fact that \(K(\ker(f)) = \ker(K(f))\) by Theorem 3.3(2).

(3) \(\Rightarrow\) (4) Let \((A \times_B A, \pi_1, \pi_2)\) be the pullback of \(f\) along itself. In order to show that the square

\[
\begin{array}{ccc}
A \times_B A & \xrightarrow{\eta_{A \times_B A}} & F(A \times_B A) \\
\downarrow \pi_1 & & \downarrow F(\pi_1) \\
A & \xrightarrow{\eta_A} & F(A)
\end{array}
\]

is a pullback, it suffices to prove that \(K(\pi_1): K(A \times_B A) \rightarrow K(A)\) is an isomorphism (Proposition 2.3(1)). This follows from the fact that the square

\[
\begin{array}{ccc}
K(A \times_B A) & \xrightarrow{K(\pi_2)} & K(A) \\
K(\pi_1) & \downarrow & \downarrow K(f) \\
K(A) & \xrightarrow{K(f)} & K(B)
\end{array}
\]

is a pullback in \(\mathcal{C}\) (by Theorem 3.3) and \(K(f)\) is a monomorphism by assumption.

(4) \(\Rightarrow\) (1) It follows by Definition 4.4. \(\square\)
When the torsion theory \((T, \mathcal{X})\) is quasi-hereditary, the \(\Gamma\)-coverings have the nice property that they give rise to a factorization system, which also turns out to be functorial. For this, we need a further description of the \(\Gamma\)-coverings:

**Lemma 4.6.** Let \((A, f)\) be an object in \(E(B)\). If the torsion theory \((T, \mathcal{X})\) in \(C\) is quasi-hereditary, then \((A, f)\) is a \(\Gamma\)-covering if and only if \(K(A) \cap \ker(f) = 0\).

**Proof.** Since \((T, \mathcal{X})\) is a quasi-hereditary torsion theory, the arrow \(F(\ker(f))\) is a monomorphism. From Proposition 2.3 it follows that the left-hand square of the diagram \((\star)\) in the proof of the previous theorem is a pullback. Accordingly, \(K(A) \cap \ker(f) = K(\ker(f))\), yielding the desired equivalence. \(\square\)

**Theorem 4.7.** Let \((T, \mathcal{X})\) be a quasi-hereditary torsion theory in \(C\). Then, for every regular epimorphism \(f: A \rightarrow B\) in \(C\), there exists a pair \((e, m)\) of morphisms in \(C\) such that:

1. \(f = m \cdot e;\)
2. \(m\) is a \(\Gamma\)-covering;
3. \(F(e)\) is an isomorphism.

Moreover, the construction of \((e, m)\) is functorial.

**Proof.** If we consider a regular epimorphism \(f: A \rightarrow B\) in \(C\), one trivially has that \(K(A) \cap \ker(f) = \ker(\eta_A) \cap \ker(f) = \ker((\eta_A, f))\), where \(\eta_A, f): A \rightarrow F(A) \times B\) is the canonical arrow to the product. We write \(I = K(A) \cap \ker(f)\), and we then define \(k = \ker((\eta_A, f))\): \(I \rightarrow A\) and \(e = \coker(k): A \rightarrow A_T\). Since \(k\) is a kernel, \(k = \ker(e)\); moreover, there is a unique arrow \(m: \xrightarrow{\Lambda}{A} \rightarrow B\) such that \(f = m \cdot e\). We are now going to show that this factorization \(f = m \cdot e\) satisfies the conditions (2) and (3) above.

As we have seen in the proof of the previous lemma, \(K(A) \cap \ker(f) = K(\ker(f))\) belongs to \(T\); therefore, the following sequence is exact

\[
0 \rightarrow F(I) \cong 0 \xrightarrow{F(k)} F(A) \xrightarrow{F(e)} F(\xrightarrow{\Lambda}{A}) \cong 0
\]

and then \(F(e)\) is an isomorphism. Let us also remark that this implies that the square (1) in the following diagram is a pullback:

\[
\begin{array}{ccc}
0 & \xrightarrow{k_A} & K(A) \\
\downarrow{K(e)} & & \downarrow{\eta_A} \\
0 & \xrightarrow{\xrightarrow{\Lambda}{k_A}} & F(\xrightarrow{\Lambda}{A})
\end{array}
\]

\[
\begin{array}{ccc}
& & F(e) \\
\downarrow{F(e)} & & \downarrow{\eta_A} \\
& & F(\xrightarrow{\Lambda}{A})
\end{array}
\]

In order to prove that \(m: \xrightarrow{\Lambda}{A} \rightarrow B\) is a \(\Gamma\)-covering, we shall use Lemma 4.6, namely we will prove that \(K(\xrightarrow{\Lambda}{A}) \cap \ker(m) = 0\). For this, it suffices to show that \(K(\ker(m)) = 0\), since \(K(\xrightarrow{\Lambda}{A}) \cap \ker(m) = K(\ker(m))\).

Let us consider the following commutative diagram

\[
\begin{array}{ccc}
I & \xrightarrow{\phi} & \ker(f) \\
\downarrow{F} & & \downarrow{k_{\ker(m)}} \\
K(A) & \xrightarrow{k_A} & A \\
\downarrow{K(e)} & & \downarrow{k_{\xrightarrow{\Lambda}{A}}} \\
K(\ker(m)) & \xrightarrow{k_{\ker(m)}} & \ker(m)
\end{array}
\]

where \(\tilde{e}: \ker(f) \rightarrow \ker(m)\) is the comparison arrow between the kernels induced by \(e\), and \(\phi\) is the unique arrow induced by the universal property of the front pullback. Let us observe that \(\tilde{e}\) is a regular epimorphism, since \(e\) is a
regular epi. The fact that the square (1) is a pullback easily implies that \( \phi \) is a regular epimorphism. So the (regular epi, mono)-factorization of \( e \cdot k \) is given by \( (\phi, \frac{1}{\Delta} \cdot K(ker(m))) \). Since \( e \cdot k = 0 \), it follows that \( K(ker(m)) = 0 \), as desired.

Now let \( f = m' \cdot e' \) be another factorization of \( f \) such that \( m': C \to B \) is a \( \Gamma \)-covering. Since \( ker(m') \in \mathcal{X} \), one has that \( e' \cdot k = 0 \). The universal property of \( e = coker(k) \) guarantees the existence of a unique arrow \( \psi: \frac{\Delta}{\Gamma} \to C \) such that \( e' = \psi \cdot e \); from this observation, the last statement in the theorem easily follows. \( \square \)

**Remark 4.8.** The existence of this factorization system can be also deduced from the main results in [12]. We prefer to provide a direct proof of this property in order to make the paper more self-contained. Note also that this factorization system is stable under pullbacks.

**Lemma 4.9.** Let \((T, \mathcal{X})\) be a quasi-hereditary torsion theory in \( \mathcal{C} \). Then, any \( \Gamma \)-covering is trivial if and only if the functor \( F: \mathcal{C} \to \mathcal{X} \) preserves the kernel of any \( \Gamma \)-covering.

**Proof.** Under our assumptions, one can easily check that the functor \( F: \mathcal{C} \to \mathcal{X} \) preserves any pullback of any arrow along a trivial covering. Therefore, if we suppose that any \( \Gamma \)-covering is trivial, we clearly have that \( F \) preserves the kernel of any \( \Gamma \)-covering.

Conversely, let us assume that the functor \( F: \mathcal{C} \to \mathcal{X} \) preserves the kernel of any \( \Gamma \)-covering. Given a \( \Gamma \)-covering \( f: A \to B \), one considers the following commutative diagram of exact sequences

\[
\begin{array}{ccccccc}
0 & \xrightarrow{} & ker(f) & \xrightarrow{k(f)} & A & \xrightarrow{f} & B & \xrightarrow{} & 0 \\
& \xrightarrow{\eta_{ker(f)}} & & \eta_A & & \eta_B & & \\
0 & \xrightarrow{} & F(ker(f)) & \xrightarrow{F(ker(f))} & F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{} & 0.
\end{array}
\]

Since \( \eta_{ker(f)} \) is an isomorphism, it follows from **Proposition 2.3(1)** that the right-hand square is a pullback. This means exactly that \( f: A \to B \) is a trivial covering. \( \square \)

**Proposition 4.10.** Given a quasi-hereditary torsion theory \((T, \mathcal{X})\) in a semi-abelian category \( \mathcal{C} \), let \( \Gamma = ((\mathcal{C}, \mathcal{E}), (\mathcal{X}, \mathcal{Z}), F \dashv U) \) be the admissible Galois structure induced by \((T, \mathcal{X})\). When \( \mathcal{X} \) is closed in \( \mathcal{C} \) under quotients, then every \( \Gamma \)-covering is trivial.

**Proof.** We recall that, in any semi-abelian category, the regular image of a kernel along a regular epi is always a kernel [22,9]. Since, for every morphism \( f: A \to B \), one has that \( F(ker(f)) = \eta_A(ker(f)) \), the functor \( F \) does preserve the kernel of \( f \) (under our assumptions a kernel is always the kernel of its cokernel). One concludes by the previous lemma. \( \square \)

The situation is quite different when we consider a quasi-hereditary torsion theory in a homological category. Indeed, we are going to show in the next section that there are examples of \( \Gamma \)-coverings that are not trivial in the category of topological groups.

**Remark 4.11.** Recall from the categorical Galois theory that, with any object \((E, p)\) in \( \mathcal{E}(B) \), it is possible to associate the so-called Galois pregrouploid \( Gal(p) \) [20]. One first considers the internal equivalence relation

\[
\begin{array}{ccccccc}
R[p] \times_E R[p] & \xrightarrow{m_1} & R[p] & \xrightarrow{\pi_1} & E,
\end{array}
\]

arising as the kernel pair of \( p: E \to B \) in \( \mathcal{C} \), and then \( Gal(p) \) is defined as its reflection in \( \mathcal{X} \):

\[
\begin{array}{ccccccc}
F(R[p] \times_E R[p]) & \xrightarrow{F(m_1)} & F(R[p]) & \xrightarrow{F(\pi_1)} & F(E).
\end{array}
\]

For any Galois structure arising from a torsion theory \((T, \mathcal{X})\) in a homological category \( \mathcal{C} \), we claim that \( Gal(p) \) is always an internal groupoid in \( \mathcal{X} \). We omit the proof of this fact, which essentially follows the lead of Theorem 8.3
in [17], and extends its validity from semi-abelian categories to homological categories. Furthermore, the categorical Galois theory allows one to give a detailed description of the category \( \text{Spl}(E, p) \) in the present situation. More precisely, one can establish a category equivalence between the category \( \text{Spl}(E, p) \) and a category \( \{\text{Gal}(p), \mathcal{X}\} \) of “internal actions with global support” on \( \text{Gal}(p) \):

\[
\text{Spl}(E, p) \simeq \{\text{Gal}(p), \mathcal{X}\}.
\]

For more details, we refer the interested reader to the book by Borceux and Janelidze [6], or to the recent articles [17,21].

5. Applications: coverings of topological groups

In this section we shall focus our attention on the coverings corresponding to three torsion theories in the context of the topological groups.

(1) Hausdorff and indiscrete groups

By Proposition 3.2 we know that, when \( Th \) is a semi-abelian theory, the torsion theory \( (\text{Th(Ind)}, \text{Th(Haus)}) \) in \( \text{Th(Top)} \) is quasi-hereditary. The category \( \text{Th(Haus)} \) of Hausdorff semi-abelian algebras is a reflective subcategory of \( \text{Th(Top)} \):

\[
\text{Th(Haus)} \xleftarrow{F} \text{Th(Top)},
\]

where the left adjoint \( F \) sends a topological algebra \( A \) to the quotient \( A/\langle 0 \rangle_A \) of \( A \) by the topological closure of the trivial subalgebra in \( A \) [5]. Our Theorem 4.5 then provides a characterization of the coverings relative to the Galois structure induced by \( (\text{Th(Ind)}, \text{Th(Haus)}) \):

**Proposition 5.1.** An open surjective homomorphism \( f: A \to B \in \text{Th(Top)} \) is a covering with respect to the Galois structure induced by \( (\text{Th(Ind)}, \text{Th(Haus)}) \) if and only if \( \text{ker}(f) \) is a Hausdorff algebra.

In particular, when \( Th \) is the theory \( \text{Grp} \) of groups, we have that the \( \Gamma \)-coverings of topological groups relative to the torsion theory \( (\text{Grp(Ind)}, \text{Grp(Haus)}) \) are exactly the open surjective homomorphisms with a Hausdorff kernel.

The following example, which was kindly suggested to us by D. Dikranjan, shows that there are coverings that are not trivial coverings.

**Example 5.2.** We are going to show that the left adjoint functor \( F: \text{Grp(Top)} \to \text{Grp(Haus)} \) does not preserve the kernel of a \( \Gamma \)-covering (see Lemma 4.9).

Let \( (\mathbb{T}, \tau_\mathbb{T}) \) be the one dimensional torus with its standard topology. Recall that \( \mathbb{T} \simeq \mathbb{R}/\mathbb{Z} \), and let \( \pi: \mathbb{R} \to \mathbb{R}/\mathbb{Z} \) be the canonical quotient. If we denote by \( \tau_\mathbb{R} \) the initial topology induced by \( \pi \) on \( \mathbb{R} \), one sees that \( \langle 0 \rangle_\mathbb{R} = \mathbb{Z} \), and then \( \eta_\mathbb{R} = \pi: \mathbb{R} \to \mathbb{R}/\mathbb{Z} \).

Now, let us consider the quotient homomorphism \( f: (\mathbb{R}, \tau_\mathbb{R}) \to (\mathbb{R}/\langle \sqrt{2} \rangle, \tau_q) \), where \( \tau_q \) is the quotient topology induced by \( (\mathbb{R}, \tau_\mathbb{R}) \). This open surjection \( f \) is a \( \Gamma \)-covering, since its kernel \( \text{ker}(f) = \langle \sqrt{2} \rangle \) is a Hausdorff group. We are now going to show that its kernel is not preserved by the functor \( F \).

On the one hand, one has that

\[
F(\mathbb{R}/\langle \sqrt{2} \rangle) = \frac{\mathbb{R}/\langle \sqrt{2} \rangle}{\langle 0 \rangle_{\mathbb{R}/\langle \sqrt{2} \rangle}} = \frac{\mathbb{R}/\langle \sqrt{2} \rangle}{\mathbb{R}/\langle \sqrt{2} \rangle} = 0,
\]

so that \( \text{ker}(F(f)) = F(\mathbb{R}) = \mathbb{R}/\mathbb{Z} \).

On the other hand, the fact that \( \langle \sqrt{2} \rangle \) is a Hausdorff group implies that \( F(\text{ker}(f)) = F(\langle \sqrt{2} \rangle) = \langle \sqrt{2} \rangle \), and \( F(\text{ker}(f)) \neq \text{ker}(F(f)) \), as desired.
(2) Totally disconnected and connected Hausdorff groups

Let us denote by $\text{Grp}(\text{HConn})$ the category of connected Hausdorff groups and by $\text{Grp}(\text{TotDis})$ the category of totally disconnected groups. The pair $(\text{Grp}(\text{HConn}), \text{Grp}(\text{TotDis}))$ is easily seen to be a torsion theory in the category $\text{Grp}(\text{Haus})$ of Hausdorff groups. This torsion theory, which is not quasi-hereditary, gives rise to the following reflection

$$
\text{Grp}(\text{TotDis}) \xrightarrow{\downarrow G} \text{Grp}(\text{Haus}),
$$

where the left adjoint $G$ sends a topological Hausdorff group $A$ to the quotient $A/c(0)$ of $A$ by the connected component $c(0)$ of 0 in $A$.

The following result of Arkhangel’skii [1] (see also [15]) will be useful to obtain a description of the coverings:

**Theorem 5.3** ([1]). For every Hausdorff group $B$ there exists an open surjective homomorphism $p: E \to B$ in $\text{Grp}(\text{Haus})$ with the property that $E$ is a totally disconnected group.

**Proposition 5.4.** An open surjective homomorphism $f: A \to B \in \text{Grp}(\text{Haus})$ is a covering with respect to the Galois structure induced by $(\text{Grp}(\text{HConn}), \text{Grp}(\text{TotDis}))$ if and only if $\ker(f)$ is a totally disconnected group.

**Proof.** Given a covering $f: A \to B$, we can apply the same arguments used in the implication (1) $\Rightarrow$ (2) of the **Theorem 4.5**, where the quasi-heredity assumption of the torsion theory is not used.

Conversely, let us assume that $\ker (f) \in \text{Grp}(\text{TotDis})$. By the previous theorem there exists an open surjective homomorphism $p: E \to B$ with $E \in \text{Grp}(\text{TotDis})$. Consider the following commutative diagram of exact sequences

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \ker(\pi_E) & \longrightarrow & E \times_B A & \longrightarrow & E & \longrightarrow & 0 \\
& & \overline{\pi}_A & \downarrow & \pi_E & & \pi \\
0 & \longrightarrow & \ker(f) & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

where $E \times_B A$ is the pullback of $f$ and $p$, and $\overline{\pi}_A: \ker(\pi_E) \to \ker(f)$ is the comparison arrow between the kernels. Since $\overline{\pi}_A$ is an isomorphism, one has that $\ker(\pi_E) \cong \ker(f) \in \text{Grp}(\text{TotDis})$. By **Lemma 2.6(4)**, the arrow $\pi_E$ lies in $\text{Grp}(\text{TotDis})$, and $f: A \to B$ is $(E, p)$-split. $\Box$

(3) Totally disconnected and connected groups

The third example concerns the torsion theory $(\text{Grp}(\text{Conn}), \text{Grp}(\text{TotDis}))$ in the category $\text{Grp}(\text{Top})$ of topological groups. We shall denote by $\Gamma'$ the Galois structure associated with this new torsion theory, which corresponds to the following composite reflection

$$
\text{Grp}(\text{TotDis}) \xrightarrow{\downarrow G} \text{Grp}(\text{Haus}) \xrightarrow{\downarrow F} \text{Grp}(\text{Top}).
$$

By using the previous characterizations of the coverings, we are now ready to determine which are the coverings relative to $\Gamma'$:

**Theorem 5.5.** An open surjective homomorphism $f: A \to B \in \text{Grp}(\text{Top})$ is a $\Gamma'$-covering if and only if $\ker(f)$ is a totally disconnected group.

**Proof.** Throughout the proof we shall denote by $\eta$ the unit of the adjunction $F \dashv U$.

Given a $\Gamma'$-covering $f: A \to B$, we can again apply the same arguments used in the implication (1) $\Rightarrow$ (2) of **Theorem 4.5** to conclude that $\ker(f)$ is a totally disconnected group.

Conversely, let $\ker(f) \in \text{Grp}(\text{TotDis})$. Since $F(A) \in \text{Grp}(\text{Haus})$, by **Theorem 5.3** there exists an open surjective homomorphism $p: E \to F(A)$ with $E \in \text{Grp}(\text{TotDis})$. If we denote by $\Gamma_1$ the Galois structure induced by the
torsion theory ($\text{Grp}(\text{Ind}), \text{Grp}(\text{Haus})$) in $\text{Grp}(\text{Top})$, we have that $f$ is a $\Gamma_1$-covering and then $(A, f)$ is normal by Theorem 4.5. Therefore, all the faces in the following commutative cube are pullbacks

$$
\begin{array}{c}
R[f] \\
\downarrow \phi \\
T \\
\downarrow \pi_T \\
E \\
\downarrow \pi_E \\
A \\
\downarrow \pi_A \\
F(A)
\end{array}
\begin{array}{c}
F \circ \eta_{R[f]} \\
\downarrow \pi_1 \\
F \circ \eta_A \\
\downarrow \pi_2 \\
E \\
\downarrow \pi_E \\
A \\
\downarrow \pi_A \\
F(A)
\end{array}
\begin{array}{c}
\eta_{R[f]} \\
\downarrow \\
\eta_A
\end{array}
$$

where $(R[f], \pi_1, \pi_2)$ is the kernel pair of $f$, $T$ is the pullback of $p$ and $F(\pi_1)$, and $\phi: T \times_{F(R[f])}(R[f]) \to E \times_{F(A)} A$ is the comparison arrow between the pullbacks.

Thanks to this observation and to Proposition 2.3, one then has that $\ker(f) \cong \ker(\pi_1) \cong \ker(F(\pi_1))$. But $\ker(f) \in \text{Grp}(\text{TotDis})$, then $F(\pi_1)$ is a $\Gamma_1$-covering, where $\Gamma_1$ is the Galois structure induced by the torsion theory ($\text{Grp}(\text{HConn}), \text{Grp}(\text{TotDis})$) in $\text{Grp}(\text{Haus})$. In particular, $(F(R[f]), F(\pi_1))$ is $(E, p)$-split and then, since $E \in \text{Grp}(\text{TotDis})$, $T \in \text{Grp}(\text{TotDis})$ as well. The fact that the following commutative diagram

$$
\begin{array}{c}
T \times_{F(R[f])}(R[f]) \\
\downarrow \pi_T \\
T \\
\downarrow \pi_1 \\
E \times_{F(A)} A \\
\downarrow \pi_E \\
E
\end{array}
\begin{array}{c}
\phi \\
\downarrow \\
\phi
\end{array}
$$

is a pullback, together with the previous observation, imply that $(A, f)$ is $(E \times_{F(A)} A, f \circ \pi_A)$-split with respect to the Galois structure $\Gamma$, where $\pi_A$ is the second projection of the pullback $E \times_{F(A)} A$.  

We observe that the notion of $\Gamma$-covering in the last theorem extends the notion of cover given by Berestovskii and Plaut in the category of compact Hausdorff connected groups [3].

Acknowledgements

The authors warmly thank Dikran Dikranjan for several useful discussions, and also for suggesting Example 5.2. They are also very grateful to Jiri Adamek and to the anonymous referees for helpful comments on a preliminary version of the paper.

References