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Existence of selections and disconnectedness properties for the hyperspace of an ultrametric space

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Abstract

We characterize the separable complete ultrametric spaces whose Wijsman hyperspace admits a continuous selection; such an investigation is closely connected to a similar result of V. Gutev about the Ball hyperspace. The characterization may be obtained in terms of a suitable property either of the base space (X, d) (*condition* (#)) or of the Wijsman hyperspace itself (total disconnectedness). We also give a necessary and sufficient condition for the zero-dimensionality of the Wijsman hyperspace of a (separable) ultrametric space, and we provide an example where such a hyperspace turns out to be connected. © 1998 Elsevier Science B.V. All rights reserved.

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Introduction

In [6], V. Gutev proved that for any separable complete ultrametric space (X, d), the so-called "Ball topology" $\mathbb{B}_d$ on the hyperspace $\mathrm{CL}(X)$ of X admits a continuous selection. This means that there exists a function φ from the collection $\mathrm{CL}(X)$ of all closed nonempty subsets of X to X, which is continuous with respect to the Ball topology and such that $\varphi(C) \in C$ for every $C \in \mathrm{CL}(X)$.

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In this paper, we characterize the separable complete ultrametric spaces for which the above result of Gutev's extends to the much more common and useful Wijsman topology $\mathbb{W}_d$ on $\mathrm{CL}(X)$.

It turns out that the existence of a continuous selection is both equivalent to a suitable property of the base space (X,d) (*condition* (#)), and to the total disconnectedness of $(\mathrm{CL}(X),\mathbb{W}_d)$. This establishes a first link between existence of continuous selections and disconnectedness properties of the hyperspace, that will be developed in the remaining part of the paper.

We show, in particular, that if (X,d) is a separable ultrametric space, then $(\mathrm{CL}(X),\mathbb{B}_d)$ is always zero-dimensional; and we give a characterization for $(\mathrm{CL}(X),\mathbb{W}_d)$ to be zero-dimensional (even in the case where (X,d) is not separable). Such a characterization (*condition* (Δ)) is formally very similar to that relative to the existence of a continuous selection, which is easily seen to be a weaker property. This is perfectly in accordance with an old result of Michael's [5].

The final part of the paper exhibits some examples which answer natural questions about the Wijsman hyperspace of an ultrametric space. In particular, we show that:
- it may admit no continuous selections, even if it is hereditarily disconnected;
- it may be connected;
- it may admit continuous selections, or even be zero-dimensional, and yet fail to coincide with the Ball hyperspace.

We also show that the hypotheses of separability and completeness are actually necessary for the existence of continuous selections for the Ball and Wijsman hyperspace.

1. Definitions and statement of the main results

First of all, let us recall that a metric d on a set X is said to be an *ultrametric* if it satisfies the *strong triangular inequality*:

$$\forall x,y,z\in X\colon\ d(x,z)\leqslant \max\{d(x,y),d(y,z)\}.$$

Note that, as it is well known, in an ultrametric space two open (or closed) balls either are disjoint or the one having the minimum radius is contained within the other one (see, for instance, [1, Proposition 1.8]).

We also need to recall some hyperspace topologies. Suppose (X,d) is a metric space. For every $x\in X$ and $\varepsilon>0$ we put $S_d(x,\varepsilon)=\{y\in X\mid d(x,y)<\varepsilon\}$ and $\overline{S}_d(x,\varepsilon)=\{y\in X\mid d(x,y)\leqslant\varepsilon\}$; we also put:

$$\mathcal{A}_d^+(x,\varepsilon)=\{C\in\mathrm{CL}(X)\mid d(x,C)>\varepsilon\},\quad\text{and}$$
$$\mathcal{A}_d^-(x,\varepsilon)=\{C\in\mathrm{CL}(X)\mid d(x,C)<\varepsilon\}.$$

The Wijsman topology $\mathbb{W}_d$ on $\mathrm{CL}(X)$ is that having as a subbase the collection:

$$\Delta_{\mathbb{W}_d}=\{\mathcal{A}_d^-(x,\varepsilon)\mid x\in X,\ \varepsilon>0\}\cup\{\mathcal{A}_d^+(x,\varepsilon)\mid x\in X,\ \varepsilon>0\}.$$

The Ball topology $\mathbb{B}_d$ is that having as a subbase the collection

$$\Delta_{\mathbb{B}_d}=\{S_i(x,\varepsilon)^-\mid x\in X,\ \varepsilon>0\}\cup\{(X\backslash\overline{S}_d(x,\varepsilon))^+\mid x\in X,\ \varepsilon>0\},$$

where

$$A^- = \{C \in \mathrm{CL}(X) \mid C \cap A \neq \emptyset\} \quad \text{and} \quad A^+ = \{C \in \mathrm{CL}(X) \mid C \subseteq A\}$$

for every $A \subseteq X$.

Observe that by the above definitions, every collection of kind A^- with A open in X is both $\mathbb{W}_d$-open and $\mathbb{B}_d$-open. The topology $\mathbb{V}^-$ generated by the collection

$$\Delta_{\mathbb{V}^-} = \{A^- \mid A \text{ open in } X\}$$

is called the *lower Vietoris topology*, and is hence coarser than $\mathbb{W}_d$ and $\mathbb{B}_d$ for every metric space (X, d). The *Vietoris topology* $\mathbb{V}$ on $\mathrm{CL}(X)$ is that generated by the collection

$$\Delta_{\mathbb{V}} = \{A^- \mid A \text{ open in } X\} \cup \{A^+ \mid A \text{ open in } X\},$$

and is clearly finer than $\mathbb{B}_d$. It is also well known that $\mathbb{W}_d \leqslant \mathbb{B}_d$ for every compatible metric d on X; conditions for equality are given in [7, Theorem 3.1]. Such a paper investigates as well the relationships between the Ball topology and another similar hypertopology, the "Ball proximal", which is not comparable with the previous one, in general. It turns out, however, that they coincide if the base space is ultrametric; therefore, all results in this paper about the Ball topology also apply to the Ball proximal topology.

Let's recall the already mentioned Gutev's result [6, Theorem 4.2], about the existence of a continuous selection for the Ball topology.

Theorem 1. *Let (X, d) be a separable complete ultrametric space. Then there exists a continuous selection from $(\mathrm{CL}(X), \mathbb{B}_d)$ to X.*

As we have already pointed out in the introduction, in this paper we deal with the above selection problem for the Wijsman topology. In particular, we get a partial (or the best possible) extension of Theorem 1 to the Wijsman topology. To state our main result, we need the following basic notion.

Definition 2. Let (X, d) be a metric space and x any point of X. We will say that the *condition* (#) holds at x if for every $\varepsilon > 0$ there exist $\delta, \vartheta \in \mathbb{R}$, with $0 < \delta < \vartheta \leqslant \varepsilon$, such that $S_d(x, \delta) = S_d(x, \vartheta)$.

Of course, there exist several (separable, complete) ultrametric spaces which fail to fulfill condition (#) (cf., for instance, Examples 25 and 29 in Section 4).

The main result of this paper now reads as follows.

Theorem 3. *If (X, d) is a separable complete ultrametric space, then a necessary and sufficient condition for the existence of a continuous selection from $(\mathrm{CL}(X), \mathbb{W}_d)$ to X is that (X, d) satisfies condition (#) at each $x \in X$.*

We will prove this theorem in the next section. Note that the hypotheses of separability and completeness are only needed to get necessity. In fact, we will show that the condition (#) on an ultrametric space (X, d) is equivalent—even without any hypothesis of separability and completeness—to the total disconnectedness of $(\mathrm{CL}(X), \mathbb{W}_d)$ (Theorem 14).

On the other hand, the condition (#) on an ultrametric space (X, d) does not imply the zero-dimensionality of the Wijsman hyperspace. A similar property to condition (#)—the *condition* (Δ)—will be used in Section 3 to characterize the zero-dimensionality of the Wijsman hyperspace (see Definition 18 and Theorem 19). In this way, it will also turn out that our Theorem 3 does not follow from well-known selection results, like [5, Theorem 2].

We point out that in this paper the notions of zero-dimensional and strongly zero-dimensional space are used following the generally accepted terminology of [4]. It does not correspond, however, to that of [1] and [5], where the word "zero-dimensional" is used in the habitual meaning of "strongly zero-dimensional".

2. Existence of continuous selections

Our goal in this section is to prove the previously stated Theorem 3, that characterizes the existence of continuous selections for the Wijsman topology.

First of all, it is useful to have some alternative definitions of the above introduced condition (#) (Definition 2); to this end, we need some further notation. For every x in a metric space (X, d), let we put:

$$R_d(x) = \{d(x,y) \mid y \in X\} \quad \text{and} \quad RC_d(x) = \{d(x,C) \mid C \in \mathrm{CL}(X)\}.$$

It is easily observed that for every $x \in X$ we have the equality $RC_d(x) = \mathrm{Cl}_{\rho\omega}(R_d(x))$, where $\rho\omega$ denotes the topology of the right-open (and left-closed) intervals on the real line (in the following, to avoid ambiguities, we will also denote by τ the Euclidean topology on the real line).

It turns out that condition (#) holds at a point x of X if and only if one of the following (equivalent) conditions is fulfilled.

(#′) $\forall \varepsilon > 0$: $\exists \delta,\ 0 < \delta < \varepsilon$: $\delta \notin RC_d(x)$;

(#″) $\forall \varepsilon > 0$: $\exists \delta > 0$: $\exists \vartheta > 0$: ($\delta < \vartheta < \varepsilon$ and $]\delta, \vartheta[\cap RC_d(x) = \emptyset$);

(#‴) $\forall \varepsilon > 0$: $\exists \delta > 0$: $\exists \vartheta > 0$: ($\delta < \vartheta < \varepsilon$ and $]\delta, \vartheta[\cap R_d(x) = \emptyset$).

We also point out here, for easy reference, an elementary fact about ultrametrics which will be used in the following.

Lemma 4. *Let (X, d) be an ultrametric space and $x, y, z \in X$ be such that $d(x,y) \neq d(y,z)$. Then $d(x,z) = \max\{d(x,y), d(y,z)\}$.*

Proof. Easy. □

Remark 5. It follows immediately from the above lemma that if $x, y \in X$ and $\varepsilon > 0$ are such that $0 < \varepsilon < d(x,y)$ (or $0 < \varepsilon \leqslant d(x,y)$), then $\forall z \in \overline{S}_d(y, \varepsilon)$: $d(x,z) = d(x,y)$ ($\forall z \in S_d(y, \varepsilon)$: $d(x,z) = d(x,y)$).

We prove now a series of preliminary lemmas, that will also have further applications in the next sections.

Lemma 6. *Let (X,d) be a metric space. If $\mathcal{C}$ is a $\mathbb{W}_d$-closed nonempty subset of $\mathrm{CL}(X)$ and $\mathcal{M}$ is a maximal chain in $\mathcal{C}$ (with respect to set-theoretic inclusion), then $\mathcal{M}$ has a maximum.*

Proof. Let $C^\sharp = \bigcup\mathcal{M}$ and $C = \overline{C^\sharp}$: it will suffice to show that $C \in \mathcal{M}$ or, equivalently—as $\mathcal{M}$ is a maximal chain in $\mathcal{C}$ and $F \subseteq C$ for every $F \in \mathcal{M}$, that $C \in \mathcal{C}$.

Since $\mathcal{C}$ is closed, we may simply show that every $\mathbb{W}_d$-neighbourhood of C meets $\mathcal{C}$. Then, let

$$\mathcal{V} = \left(\bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i,\delta_i)\right) \cap \left(\bigcap_{j=1}^{m} \mathcal{A}_d^+(y_j,\varepsilon_j)\right)$$

be any basic $\mathbb{W}_d$-neighbourhood of C, with $n,m \in \omega$ (the possibilities $n=0$ or $m=0$ are always allowed), $d(x_i,C) < \delta_i$ for $i \in \{1,\dots,n\}$ and $d(y_j,C) > \varepsilon_j$ for $j \in \{1,\dots,m\}$. Since $d(x,C) = d(x,C^\sharp)$ for every $x \in X$, we have on the one hand that for every $i \in \{1,\dots,n\}$ there exists $z_i \in C^\sharp$ such that $d(x_i,z_i) < \delta_i$; pick $M_i \in \mathcal{M}$ such that $z_i \in M_i$, and let $\widetilde{M} = \max\{M_1,\dots,M_n\}$: since $z_1,\dots,z_n \in \widetilde{M}$, it follows that $\widetilde{M} \in \bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i,\delta_i)$. On the other hand, for every $j \in \{1,\dots,m\}$ we have the inequality $d(y_j,\widetilde{M}) \geqslant d(y_j,C^\sharp) = d(y_j,C) > \varepsilon_j$; thus $\widetilde{M} \in \mathcal{V}$ and hence $\mathcal{V} \cap \mathcal{C} \neq \emptyset$.

Lemma 7. *Let (X,d) be an ultrametric space, $M \in \mathrm{CL}(X)$ and $\mathcal{V}$ a $\mathbb{W}_d$-neighbourhood of M. Then for every $\hat{y} \in X$ with $\hat{y} \notin M$, it is possible to find $n,m \in \omega$, $x_1,\dots,x_n$, $y_1,\dots,y_m \in X$, $\delta_1,\dots,\delta_n,\varepsilon_1,\dots,\varepsilon_m$, $\hat{\varepsilon} > 0$, such that*

$$M \in \left(\bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i,\delta_i)\right) \cap \left(\bigcap_{j=1}^{m} \mathcal{A}_d^+(y_j,\varepsilon_j)\right) \cap \mathcal{A}_d^+(\hat{y},\hat{\varepsilon}) \subseteq \mathcal{V}$$

and that $\hat{y} \notin \overline{S}_d(y_j,\varepsilon_j)$ for every $j \in \{1,\dots,m\}$.

Proof. We know that there must exist two finite subsets F, G of X, and for every $x \in F$ a $\delta'_x > 0$, and for every $y \in G$ an $\varepsilon'_y > 0$, such that

$$M \in \left(\bigcap_{x\in F} \mathcal{A}_d^-(x,\delta'_x)\right) \cap \left(\bigcap_{y\in G} \mathcal{A}_d^+(y,\varepsilon'_y)\right) \subseteq \mathcal{V}. \tag{1}$$

Let $\widehat{G} = \{y \in G \mid \hat{y} \in \overline{S}_d(y,\varepsilon'_y)\}$: then for every $y \in \widehat{G}$ it is easily proved that $\mathcal{A}_d^+(y,\varepsilon'_y) = \mathcal{A}_d^+(\hat{y},\varepsilon'_y)$ (simply use the strong triangular inequality and the fact that if, for example, $C \notin \mathcal{A}_d^+(y,\varepsilon'_y)$, then $\inf_{z\in C} d(y,z) \leqslant \varepsilon'_y$). It follows that, if $\widehat{G} \neq \emptyset$, then

$$\bigcap_{y\in\widehat{G}} \mathcal{A}_d^+(y,\varepsilon'_y) = \bigcap_{y\in\widehat{G}} \mathcal{A}_d^+(\hat{y},\varepsilon'_y) = \mathcal{A}_d^+(\hat{y},\hat{\varepsilon}), \tag{2}$$

where $\hat{\varepsilon} = \max\{\varepsilon'_y \mid y \in \widehat{G}\}$. Let $x_1, \ldots, x_n$ be the elements of F, and $y_1, \ldots, y_m$ the elements of $G \backslash \widehat{G}$; for every $i \in \{1, \ldots, n\}$ we put $\delta'_{x_i} = \delta_i$, and for every $j \in \{1, \ldots, m\}$, $\varepsilon'_{y_j} = \varepsilon_j$: if $\widehat{G} \neq \emptyset$, the thesis follows from (1), (2) and the fact that $y_j \notin \widehat{G}$ for every $j \in \{1, \ldots, m\}$. If, on the contrary, $\widehat{G} = \emptyset$, then it is sufficient to choose any $\hat{\varepsilon}$ with $0 < \hat{\varepsilon} < d(\hat{y}, M)$ to get the thesis. □

Lemma 8. *Let (X, d) an ultrametric space and $\hat{y} \in X$, $\lambda > 0$ such that $R_d(\hat{y}) \cap]0, \lambda[$ is dense in $]0, \lambda[$ (with respect to the Euclidean topology). Let us also suppose that $\hat{x} \in X$ is such that $d(\hat{y}, \hat{x}) < \lambda$. Then, every $\mathbb{W}_d$-clopen collection $\mathcal{L}$ containing $\{\hat{x}\}$ also contains $\{\hat{x}, \hat{y}\}$.*

Proof. We first prove the following fact: for every finite sequence $r_1, \ldots, r_l$ of real numbers greater than 0, there exists $D \in \mathcal{L}$ such that $\hat{x}, \hat{y} \in D$ and for every $z \in D \backslash \{\hat{x}\}$: $d(\hat{y}, z) \notin \{r_1, \ldots, r_l\}$. Let

$$\mathcal{S} = \{C \in \mathcal{L} \mid \hat{x} \in C \text{ and } \forall z \in C \backslash \{\hat{x}\}\colon\ d(\hat{y}, z) \notin \{r_1, \ldots, r_l\}\};$$

it is easily seen that $\mathcal{S}$ is $\mathbb{W}_d$-closed. Indeed, for every $h \in \{1, \ldots, l\}$, the collection $\mathcal{W}_h = \{C \in \mathrm{CL}(X) \mid \exists z \in C \backslash \{\hat{x}\}\colon\ d(\hat{y}, z) = r_h\}$ is $\mathbb{W}_d$-open since, if $C \in \mathcal{W}_h$ and $z \in C \backslash \{\hat{x}\}$ is such that $d(\hat{y}, z) = r_h$, then putting $\vartheta = \min\{r_h, d(\hat{x}, z)\}$ it is easily seen by Remark 5 that $\mathcal{A}_d^-(z, \vartheta)$ is entirely contained in $\mathcal{W}_h$; therefore,

$$\mathcal{S} = \mathcal{L} \cap \left(\bigcap_{h=1}^{l} \mathcal{W}_h^c \right) \cap \left(\bigcap_{\varepsilon > 0} \left(\mathcal{A}_d^+(\hat{x}, \varepsilon)\right)^c \right)$$

is $\mathbb{W}_d$-closed. Moreover, $\mathcal{S}$ is nonempty, as it contains at least the element $\{\hat{x}\}$. Let $\mathcal{M}$ be a maximal chain in $\mathcal{S}$: by Lemma 6, $\mathcal{M}$ has a maximum M; we claim that $\hat{y} \in M$, so that we may put $D = M$. By contradiction, suppose $\hat{y} \notin M$: by Lemma 7 it is possible to find $n, m \in \omega$, $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$, $\delta_1, \ldots, \delta_n, \varepsilon_1, \ldots, \varepsilon_m, \hat{\varepsilon} > 0$ such that

$$M \in \left(\bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i, \delta_i) \right) \cap \left(\bigcap_{j=1}^{m} \mathcal{A}_d^+(y_i, \varepsilon_i) \right) \cap \mathcal{A}_d^+(\hat{y}, \hat{\varepsilon}) \subseteq \mathcal{L}$$

and $\hat{y} \notin \overline{S}_d(y_j, \varepsilon_j)$ for every $j \in \{1, \ldots, m\}$. Pick $\gamma^\sharp \in R_d(\hat{y}) \cap]\hat{\varepsilon}, d(\hat{y}, M)[$ such that $\gamma^\sharp \neq r_h$ for $h = 1, \ldots, l$ and $\gamma^\sharp \neq d(\hat{y}, y_j)$ for $j = 1, \ldots, m$; then choose $y^\sharp \in X$ such that $d(\hat{y}, y^\sharp) = \gamma^\sharp$. It follows that $d(y^\sharp, y_j) > \varepsilon_j$ for $j = 1, \ldots, m$—if $d(y^\sharp, y_j) \leqslant \varepsilon_j$ for some j, then by Remark 5: $\gamma^\sharp = d(\hat{y}, y^\sharp) = d(\hat{y}, y_j)$. Consequently, $d(x_i, M \cup \{y^\sharp\}) \leqslant d(x_i, M) < \delta_i$ for $i = 1, \ldots, n$, $d(y_j, M \cup \{y^\sharp\}) = \min\{d(y_j, M), d(y_j, y^\sharp)\} > \varepsilon_j$ for $j = 1, \ldots, m$ and $d(\hat{y}, M \cup \{y^\sharp\}) = \min\{d(\hat{y}, M), d(\hat{y}, y^\sharp)\} > \hat{\varepsilon}$; thus $M \cup \{y^\sharp\} \in \mathcal{L}$, and by $\hat{x} \in M \subseteq M \cup \{y^\sharp\}$, $d(\hat{y}, z) \notin \{r_1, \ldots, r_l\}$ for every $z \in M \backslash \{\hat{x}\}$ (as $M \in \mathcal{S}$) and $d(\hat{y}, y^\sharp) = \gamma^\sharp \notin \{r_1, \ldots, r_l\}$, we also have that $M \cup \{y^\sharp\} \in \mathcal{S}$. This is a contradiction, because $y^\sharp \notin M$ (as $d(\hat{y}, y^\sharp) = \gamma^\sharp < d(\hat{y}, M)$), and hence $\mathcal{M} \cup \{M \cup \{y^\sharp\}\}$ is a chain in $\mathcal{S}$ which strictly extends $\mathcal{M}$.

To prove that $\{\hat{y}, \hat{x}\}$ belongs to $\mathcal{L}$, it is sufficient to show that it belongs to the closure of $\mathcal{L}$. Therefore, let

$$\mathcal{V} = \left(\bigcap_{k=1}^{p} \mathcal{A}_d^-(z_k, \alpha_k) \right) \cap \left(\bigcap_{h=1}^{l} \mathcal{A}_d^+(w_h, \beta_h) \right)$$

be any basic $\mathbb{W}_d$-neighbourhood of $\{\hat{y}, \hat{x}\}$, and let $r_h = d(\hat{y}, w_h)$ for $h = 1, \ldots, l$. Then, as we have just shown, there exists $D \in \mathcal{L}$ such that $\hat{y}, \hat{x} \in D$ and $\forall z \in D \setminus \{\hat{x}\}$: $d(\hat{y}, z) \notin \{r_1, \ldots, r_l\}$. Thus, on the one hand, the relations $\{\hat{y}, \hat{x}\} \subseteq D$ and

$$\{\hat{y}, \hat{x}\} \in \bigcap_{k=1}^{p} \mathcal{A}_d^-(z_k, \beta_k)$$

imply that

$$D \in \bigcap_{k=1}^{p} \mathcal{A}_d^-(z_k, \beta_k).$$

On the other hand, from

$$\{\hat{y}, \hat{x}\} \in \bigcap_{h=1}^{l} \mathcal{A}_d^+(w_h, \beta_h)$$

we obtain that $d(\hat{y}, w_h) > \beta_h$ for $h = 1, \ldots, l$, and this implies that $d(w_h, D) > \beta_h$ for $h = 1, \ldots, l$ (from which $D \in \mathcal{V} \cap \mathcal{L}$). Indeed, if $d(w_h, D) \leqslant \beta_h$ for some h, then there exists $\bar{z} \in D$ such that $d(w_h, \bar{z}) < \min\{r_h, d(\hat{x}, w_h)\}$ (owing to $\{\hat{y}, \hat{x}\} \in \mathcal{V}$); by Lemma 4, $d(\hat{y}, \bar{z}) = \max\{d(\hat{y}, w_h), d(w_h, \bar{z})\} = r_h$, and clearly $\bar{z}$ cannot coincide with $\hat{x}$: this contradicts the choice of D. □

Proposition 9. *Let (X, d) be an ultrametric space, and suppose that there exists $\hat{y} \in X$ at which condition (#) is not satisfied. Then there exists no continuous selection from $(\mathrm{CL}(X), \mathbb{W}_d)$ to X.*

Proof. By hypothesis, there exists $\lambda > 0$ such that $R_d(\hat{y}) \cap]0, \lambda[$ is dense in $]0, \lambda[$ (see the equivalent form (#$'''$) of (#)). Suppose by contradiction that there exists a $\mathbb{W}_d$-continuous selection φ. Choose $\vartheta \in]0, \lambda[$, let $\mathcal{W} = \varphi^{-1}(S_d(\hat{y}, \vartheta))$ and consider a suitable basic $\mathbb{W}_d$-neighbourhood $\mathcal{V}$ of $\{\hat{y}\}$ such that $\{\hat{y}\} \in \mathcal{V} \subseteq \mathcal{W}$. Thus:

$$\mathcal{V} = \left(\bigcap_{i=1}^{n} \mathcal{A}_d^-(\tilde{x}_i, \tilde{\delta}_i) \right) \cap \left(\bigcap_{h=1}^{l} \mathcal{A}_d^+(\tilde{y}_h, \tilde{\varepsilon}_h) \right),$$

with $\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_1, \ldots, \tilde{y}_l \in X$ and $\tilde{\delta}_1, \ldots, \tilde{\delta}_n, \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_l > 0$.

In particular, we have that $d(\hat{y}, \tilde{y}_h) > \tilde{\varepsilon}_h$ for every $h \in \{1, \ldots, l\}$. Since $R_d(\hat{y}) \cap]\vartheta, \lambda[$ is dense in $]\vartheta, \lambda[$, there exists $\hat{z} \in X$ such that $d(\hat{y}, \hat{z}) \in]\vartheta, \lambda[$ and $d(\hat{y}, \hat{z}) \neq d(\hat{y}, \tilde{y}_h)$ for every $h \in \{1, \ldots, l\}$. From $\tilde{\varepsilon}_h < d(\hat{y}, \tilde{y}_h)$ for every $h \in \{1, \ldots, l\}$, we have by Remark 5 that $d(z, \hat{y}) = d(\tilde{y}_h, \hat{y})$ for every $z \in \overline{S}_d(\tilde{y}_h, \tilde{\varepsilon}_h)$, and hence $\hat{z} \notin \overline{S}_d(\tilde{y}_h, \tilde{\varepsilon}_h)$. Then $\{\hat{y}, \hat{z}\} \in \mathcal{V}$.

Since $\hat{z} \notin S_d(\hat{y}, \vartheta)$, we have that $\{\hat{z}\} \in \mathcal{L} = \mathrm{CL}(X)\backslash\mathcal{W}$. However, $\mathcal{L}$ is $\mathbb{W}_d$-clopen because so is $\mathcal{V}$. Hence, by Lemma 8, $\{\hat{y}, \hat{z}\} \in \mathcal{L}$. A contradiction. □

Observe that the above result actually holds for every $\mathbb{W}_d$-continuous function $\varphi: \mathrm{CL}(X) \to X$ such that $\varphi(\{x\}) = x$ for every $x \in X$.

We turn now to show the converse implication of the preceding statement, with the additional hypotheses of separability and completeness. We first need to introduce an additional property.

Definition 10. A metric space (X, d) has the *Ball Approximation Property* for $\mathbb{W}_d$, or the $\mathbb{W}_d$-BAP, if whenever B is an open ball in X, or $B = X$, we have that for every $\varepsilon > 0$ there exists a $\mathbb{W}_d$-continuous $\varphi: B^- \to B$ such that $d(\varphi(C), C) < \varepsilon$ for every $C \in B^-$.

Lemma 11. *Let (X, d) be a complete ultrametric space having the $\mathbb{W}_d$-BAP. Then there exists a $\mathbb{W}_d$-continuous selection for* $\mathrm{CL}(X)$.

Proof. It will suffice to construct a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of $\mathbb{W}_d$-continuous maps $\varphi_n: \mathrm{CL}(X) \to X$ such that, for every $n \in \mathbb{N}$ (where $\mathbb{N} = \omega\backslash\{0\}$) and $C \in \mathrm{CL}(X)$,

(a) $d(\varphi_n(C), C) < 2^{-n}$;

(b) $d(\varphi_n(C), \varphi_{n+1}(C)) < 2^{-n}$.

Merely, by (b), $\{\varphi_n\}$ is uniformly Cauchy; so, it must converge to some $\mathbb{W}_d$-continuous φ. By (a), φ is a selection for $\mathrm{CL}(X)$.

To construct these φ_n's, proceed by induction. By the $\mathbb{W}_d$-BAP, there exists a $\mathbb{W}_d$-continuous $\varphi_1: \mathrm{CL}(X) \to X$ with $d(\varphi_1(C), C) < 1/2$ for $C \in \mathrm{CL}(X)$. Suppose $\varphi_n: \mathrm{CL}(X) \to X$ is a $\mathbb{W}_d$-continuous map for which (a) holds. Since $\mathcal{S} = \{S_d(x, 2^{-n}) \mid x \in X\}$ is a discrete open cover of X, $\varphi_n^{-1}(\mathcal{S}) = \{\varphi_n^{-1}(B) \mid B \in \mathcal{S}\}$ is a discrete $\mathbb{W}_d$-open cover of $\mathrm{CL}(X)$. By condition, for every $B \in \mathcal{S}$ there exists a $\mathbb{W}_d$-continuous $\varphi_B: B^- \to B$ with $d(\varphi_B(C), C) < 2^{-(n+1)}$ for $C \in B^-$. As d is an ultrametric, we have that $\varphi_n^{-1}(B) \subseteq B^-$ for every $B \in \mathcal{S}$; thus the $\mathbb{W}_d$-continuous map $\varphi_{n+1}: \mathrm{CL}(X) \to X$, defined by $\varphi_{n+1}|_{\varphi_n^{-1}(B)} = \varphi_B$ for $B \in \mathcal{S}$, satisfies both (a) (for $n = n + 1$) and (b). □

Lemma 12. *Let (X, d) be a separable ultrametric space such that the condition (#) holds at each point of X. Then (X, d) has the $\mathbb{W}_d$-BAP.*

Proof. Let B be an open ball in X (or $B = X$), and let $\varepsilon > 0$. By condition, for every $x \in B$ there exist $0 < \delta(x) < \vartheta(x) < \varepsilon$ such that $S_d(x, \delta(x)) = S_d(x, \vartheta(x)) \subseteq B$. Since B is separable, there is a countable subset $\{V_n \mid n \in \mathbb{N}\}$ of $\{S_d(x, \delta(x)) \mid x \in B\}$ which covers B. For every $n \in \mathbb{N}$ take a $b_n \in B$ for which $V_n = S_d(b_n, \delta(b_n))$. Also, for every $C \in B^-$ set $n(C) = \min\{n \in \mathbb{N} \mid C \cap V_n \neq \emptyset\}$. Finally, define $\varphi: B^- \to B$ by $\varphi(C) = b_{n(C)}$, $C \in B^-$. This φ is as required. Obviously, $d(\varphi(C), C) < \delta(\varphi(C))$ for every $C \in B^-$. To see that φ is $\mathbb{W}_d$-continuous, take $C \in B^-$ and then set

$$\mathcal{U} = V_{n(C)}^- \cap \left(\bigcap_{m<n(C)} \mathcal{A}_d^+(b_m, \delta(b_m)) \right)$$

(in particular, for $n(C) = 1$ we have $\mathcal{U} = V_1^-$). This $\mathcal{U}$ is a $\mathbb{W}_d$-neighbourhood of C with $\varphi(D) = \varphi(C)$ for every $D \in \mathcal{U}$. □

Proposition 13. *Let (X,d) be a separable complete ultrametric space such that condition (#) holds at each point of X. Then there exists a continuous selection from $(\mathrm{CL}(X), \mathbb{W}_d)$ to X.*

Proof. It follows immediately from the preceding two lemmas. □

This completes the proof of Theorem 3.

3. Disconnectedness properties of the hyperspace

The first result in this section consists of showing that the above defined condition (#), which characterizes the existence of Wijsman-continuous selections in the case where the ultrametric space (X,d) is separable and complete, is also equivalent to a disconnectedness property of the Wijsman hyperspace (without hypotheses of separability or completeness).

We recall that a topological space X is said to be *totally disconnected* if for any two distinct points $x, y \in X$ there exists a clopen set $M \subseteq X$ such that $x \in M$ and $y \notin M$. Equivalently, X is totally disconnected if the *quasi-component* of every point $x \in X$ (that is, the intersection of all clopen sets containing x) is $\{x\}$. Clearly, every totally disconnected space is hereditarily disconnected, but not vice versa (see [4, Exercises 6.3.23(b) and 6.3.24]).

Theorem 14. *Let (X,d) be any ultrametric space. Then condition (#) holds at all points of X if and only if $(\mathrm{CL}(X), \mathbb{W}_d)$ is totally disconnected.*

Proof. Suppose first that (X,d) satisfies condition (#), and let C, D be any two distinct elements of $\mathrm{CL}(X)$: we may suppose that there exists $\bar{x} \in C \setminus D$ (if $\bar{x} \in D \setminus C$, the proof is analogous). As condition (#) holds at $\bar{x}$, there exists r with $0 < r < d(\bar{x}, D)$ such that $r \notin RC_d(\bar{x})$. Then $(\mathcal{A}_d^-(\bar{x}, r))^c = \mathcal{A}_d^+(\bar{x}, r)$, and hence $\mathcal{A}_d^-(\bar{x}, r)$ is a clopen subset of $(\mathrm{CL}(X), \mathbb{W}_d)$ containing C but not D.

Suppose now that there exists $\hat{y} \in X$ such that condition (#) does not hold at $\hat{y}$. Then there exists $\lambda > 0$ such that $R_d(\hat{y}) \cap \,]0, \lambda[$ is dense in $]0, \lambda[$. Choose an $\hat{x} \in X$ such that $d(\hat{y}, \hat{x}) = \hat{\alpha} \in]0, \lambda[$: by Lemma 8 every $\mathbb{W}_d$-clopen collection $\mathcal{L}$ containing $\{\hat{x}\}$ also contains $\{\hat{x}, \hat{y}\}$, which implies that $(\mathrm{CL}(X), \mathbb{W}_d)$ is not totally disconnected. □

Corollary 15. *If (X,d) is a separable, complete ultrametric space, then the existence of a Wijsman-continuous selection on the hyperspace of X is equivalent to the total disconnectedness of $(\mathrm{CL}(X), \mathbb{W}_d)$.*

We deal now with the problem of establishing in which cases the hyperspace of an ultrametric space, endowed with one of the previously introduced hypertopologies, is zero-dimensional. It will turn out that there are rather strict analogies with the results about the existence of continuous selections.

Lemma 16. *Let* (X,d) *be an ultrametric separable space. Then the set*

$$R_d = \{d(x,y) \mid x,y \in X\} = \bigcup_{x\in X} R_d(x)$$

is countable.

Proof. Let us consider, on the space $Y = X \times X$, the compatible metric

$$\rho\big((x_1,x_2),(x_1',x_2')\big) = \max\{d(x_1,x_1'),d(x_2,x_2')\}$$

(which is easily seen to be in fact an ultrametric).

For every $r>0$, the set $T_r = \{(x,y)\in Y \mid d(x,y)=r\}$ is open: indeed, if $(x',y')\in S_\rho((x,y),r)$, then by Lemma 4 we have that $d(x',y) = \max\{d(x',x),d(x,y)\} = r$ and $d(x',y') = \max\{d(x',y),d(y,y')\} = r$. As Y is separable, $|\{r>0 \mid T_r \neq \emptyset\}| \leqslant \aleph_0$. □

Theorem 17. *If* (X,d) *is a separable ultrametric space, then* $(\mathrm{CL}(X),\mathbb{B}_d)$ *is zero-dimensional.*

Proof. Let $\mathbb{R}^+ = \,]0,+\infty[$: as R_d is countable, the set $\mathbb{R}^+\backslash R_d$ is dense in $\mathbb{R}^+$. Observe that for every $r\in\mathbb{R}^+\backslash R_d$ and every $x\in X$, the equality $S_d(x,r) = \overline{S}_d(x,r)$ implies that the collections $(S_d(x,r))^-$ and $((\overline{S}_d(x,r))^c)^+$ are complementary in $\mathrm{CL}(X)$; consequently, $(S_d(x,r))^-$ is clopen with respect to $\mathbb{B}_d$. On the other hand, for every $s\in\mathbb{R}^+$ and every $x\in X$ we have that

$$\Big(\big((\overline{S}_d(x,s))^c\big)^+\Big)^c = \Big(\big((\overline{S}_d(x,s))^-\big)^c\Big)^c = \big(\overline{S}_d(x,s)\big)^-,$$

which is $\mathbb{B}_d$-open since $\overline{S}_d(x,s)$ is open.

Now, let $C\in\mathrm{CL}(X)$ and $\mathcal{V}$ be any $\mathbb{B}_d$-neighbourhood of C. We may suppose

$$\mathcal{V} = \left(\bigcap_{i=1}^{n}\big(S_d(x_i,\delta_i)\big)^-\right)\cap\left(\bigcap_{j=1}^{m}\Big(\big(\overline{S}_d(y_j,s_j)\big)^c\Big)^+\right),$$

where $d(x_i,C)<\delta_i$ for $i=1,\dots,n$, $C\cap\overline{S}_d(y_j,s_j)=\emptyset$ for $j=1,\dots,m$ and n,m may possibly be equal to 0. For every $i\in\{1,\dots,n\}$, pick $r_i\in\,]d(x_i,C),\delta_i[\,\backslash R_d$ and put

$$\mathcal{U} = \left(\bigcap_{i=1}^{n}\big(S_d(x_i,r_i)\big)^-\right)\cap\left(\bigcap_{j=1}^{m}\Big(\big(\overline{S}_d(y_j,s_j)\big)^c\Big)^+\right);$$

then $C\in\mathcal{U}\subseteq\mathcal{V}$ and $\mathcal{U}$ is clopen with respect to $\mathbb{B}_d$. □

We provide now necessary and sufficient conditions for the Wijsman hyperspace of an ultrametric space to be zero-dimensional. To this end, we introduce a new notion which is strictly related to that of condition (#) (see Definition 2).

Definition 18. A point x of a metric space (X, d) satisfies *condition* (Δ) if for every $0 < \alpha < \beta$ there exist $\gamma, \delta \in \mathbb{R}$ with $\alpha < \gamma < \delta < \beta$, such that $S_d(x, \gamma) = S_d(x, \delta)$.

Again, it is useful to point out some properties which are equivalent to the fact that a point x fulfills the above condition:

(Δ') the set $R_d(x)$ is τ-nowhere dense;

(Δ'') the set $RC_d(x) = \mathrm{Cl}_{\rho\omega}(R_d(x))$ is τ-nowhere dense;

(Δ''') $\mathrm{Int}_\tau(RC_d(x)) = \emptyset$.

Theorem 19. *Let (X, d) be any ultrametric space. Then $(\mathrm{CL}(X), \mathbb{W}_d)$ is zero-dimensional if and only if condition (Δ) holds at each point of (X, d).*

Proof. Suppose first that $RC_d(x)$ is τ-nowhere dense for every $x \in X$. Observe that, for every $x \in X$ and $r \in \mathbb{R}^+ \setminus RC_d(x)$, the collections $\mathcal{A}_d^-(x, r)$ and $\mathcal{A}_d^+(x, r)$ are complementary to each other, and hence they are clopen. To prove zero-dimensionality, let $\widehat{C}$ be any element of $\mathrm{CL}(X)$ and $\mathcal{V}$ a $\mathbb{W}_d$-neighbourhood of $\widehat{C}$. Then there exist $n, m \in \omega$, $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ and $\delta_1, \ldots, \delta_n, \varepsilon_1, \ldots, \varepsilon_m > 0$ such that

$$\widehat{C} \in \left(\bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i, \delta_i)\right) \cap \left(\bigcap_{j=1}^{m} \mathcal{A}_d^+(y_j, \varepsilon_j)\right) \subseteq \mathcal{V};$$

for every $i \in \{1, \ldots, n\}$ pick $\delta_i' \in]d(x_i, \widehat{C}), \delta_i[\setminus RC_d(x_i)$, and for every $j \in \{1, \ldots, m\}$ pick $\varepsilon_j' \in]\varepsilon_j, d(y_j, \widehat{C})[\setminus RC_d(y_j)$. Then

$$\begin{aligned} \widehat{C} &\in \left(\bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i, \delta_i')\right) \cap \left(\bigcap_{j=1}^{m} \mathcal{A}_d^+(y_j, \varepsilon_j')\right) \\ &\subseteq \left(\bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i, \delta_i)\right) \cap \left(\bigcap_{j=1}^{m} \mathcal{A}_d^+(y_j, \varepsilon_j)\right) \subseteq \mathcal{V}, \end{aligned}$$

and

$$\left(\bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i, \delta_i')\right) \cap \left(\bigcap_{j=1}^{m} \mathcal{A}_d^+(y_j, \varepsilon_j')\right)$$

is $\mathbb{W}_d$-clopen.

Suppose now that there exist $r, s \in \mathbb{R}$ with $0 < r < s$, $\hat{y} \in X$ and T τ-dense in $]r, s[$, such that $\forall \gamma \in T$: $\exists y_\gamma \in X$: $d(\hat{y}, y_\gamma) = \gamma$. The $\mathbb{W}_d$-open collection

$$\mathcal{V} = \{C \in \mathrm{CL}(X) \mid r < d(\hat{y}, C) < s\} = \mathcal{A}_d^-(\hat{y}, s) \cap \mathcal{A}_d^+(\hat{y}, r)$$

is nonempty, since it contains, for example, every $\{y_\gamma\}$ with $\gamma \in T \cap]r, s[$; by contradiction, suppose that there exists a $\mathbb{W}_d$-clopen nonempty collection $\mathcal{S}$ such that $\mathcal{S} \subseteq \mathcal{V}$. Let $\mathcal{M}$ be a maximal chain in $\mathcal{S}$: by Lemma 6, $\mathcal{M}$ has a maximum M, which must satisfy the inequalities $r < d(\hat{y}, M) < s$. Since, in particular, $\hat{y} \notin M$, by Lemma 7 we have that there exist $n, m \in \omega$, $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ and $\delta_1, \ldots, \delta_n, \varepsilon_1, \ldots, \varepsilon_m, \hat{\varepsilon} > 0$ such that

$$M \in \left(\bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i, \delta_i) \right) \cap \left(\bigcap_{j=1}^{m} \mathcal{A}_d^+(y_j, \varepsilon_j) \right) \cap \mathcal{A}_d^+(\hat{y}, \hat{\varepsilon}) \subseteq \mathcal{S}$$

and $d(\hat{y}, y_j) > \varepsilon_j$ for $j = 1, \ldots, m$.

Take $\gamma^\sharp \in T \cap \,]\max\{\hat{\varepsilon}, r\}, d(\hat{y}, M)[$ such that $\gamma^\sharp \neq d(y_j, \hat{y})$ for $j \in \{1, \ldots, m\}$: then $d(\hat{y}, y_{\gamma^\sharp}) > \hat{\varepsilon}$ and $d(y_j, y_{\gamma^\sharp}) > \varepsilon_j$ for $j = 1, \ldots, m$ (apply Remark 5). Consequently, $M \cup \{y_{\gamma^\sharp}\} \in \mathcal{S}$; since $d(\hat{y}, y_{\gamma^\sharp}) = \gamma^\sharp < d(\hat{y}, M)$, we have that $y_{\gamma^\sharp} \notin M$ and this contradicts the maximality of $\mathcal{M}$. □

In the case where the ultrametric space (X, d) is separable, the above "local" characterization may be replaced by a "global" one. Recall also that if (X, d) is separable, then $(\mathrm{CL}(X), \mathbb{W}_d)$ is separable and metrizable (see [2, Theorem 2.1.5]).

Theorem 20. *For a separable ultrametric space (X, d) the following are equivalent:*

(1) $(\mathrm{CL}(X), \mathbb{W}_d)$ *is (strongly) zero-dimensional;*

(2) (X, d) *satisfies condition* (Δ) *at each* $x \in X$;

(3) $\mathrm{Int}_\tau(RC_d) = \emptyset$, *where*

$$RC_d = \{d(x, C) \mid x \in X,\ C \in \mathrm{CL}(X)\} = \bigcup_{x \in X} RC_d(x).$$

Proof. Equivalence between (1) and (2) has just been proved, and (3) ⇒ (2) is obvious; therefore, we prove that (2) ⇒ (3).

Let $RC_d(x)$ be nowhere dense for every $x \in X$, and suppose by contradiction that there exists $0 < r < s$ with $]r, s[\subseteq RC_d$. We can associate by transfinite induction, to every $\alpha \in \omega_1$, an $x_\alpha \in X$ such that

$$\left(RC_d(x_\alpha) \cap \,]r, s[\right) \setminus \left(\bigcup_{\alpha' < \alpha} RC_d(x_{\alpha'}) \right) \neq \emptyset$$

(this is possible since $]r, s[$ is a Baire space with respect to the topology induced by τ, every $RC_d(x) \cap]r, s[$ is τ-nowhere dense in $]r, s[$ and $]r, s[= \bigcup_{x \in X}(RC_d(x) \cap]r, s[)$).

Let $\alpha' < \alpha'' < \omega_1$: the choice of the points x_α ensures us that

$$RC_d(x_{\alpha''}) \cap \,]r, s[\not\subseteq RC_d(x_{\alpha'}) \cap \,]r, s[$$

and, since $RC_d(x) = \mathrm{Cl}_{\rho\omega}(R_d(x))$, we also have that

$$R_d(x_{\alpha''}) \cap \,]r, s[\not\subseteq R_d(x_{\alpha'}) \cap \,]r, s[$$

for $\alpha' < \alpha''$. We are going to infer from this that $S_d(x_{\alpha'}, r) \cap S_d(x_{\alpha''}, r) = \emptyset$ for $\alpha' < \alpha''$, which clearly contradicts the separability of X.

Indeed, if $\alpha' < \alpha''$, then pick

$$\vartheta'' \in \left(R_d(x_{\alpha''}) \cap \,]r, s[\right) \setminus \left(R_d(x_{\alpha'}) \cap \,]r, s[\right)$$

and let $y'' \in X$ be such that $d(x_{\alpha''}, y'') = \vartheta''$. By contradiction, suppose $S_d(x_{\alpha'}, r) \cap S_d(x_{\alpha''}, r) \neq \emptyset$, whence $S_d(x_{\alpha'}, r) = S_d(x_{\alpha''}, r)$ and in particular $d(x_{\alpha'}, x_{\alpha''}) < r$: since $d(x_{\alpha''}, y'') = \vartheta'' > r$, we have by Lemma 4 that

$$d(x_{\alpha'}, y'') = \max\{d(x_{\alpha'}, x_{\alpha''}), d(x_{\alpha''}, y'')\} = \vartheta''$$

and hence $\vartheta'' \in R_d(x_{\alpha'}) \cap]r, s[$, a contradiction. □

Remark 21. For a separable ultrametric space, the zero-dimensionality of the Wijsman hypertopology *is not equivalent* to the fact that RC_d (or R_d) is τ-nowhere dense. Let $X = \bigcup_{q \in \mathbb{Q} \cap]0,1[} \{0, q\} \times \{q\}$, endowed with the ultrametric:

$$d((x, q'), (y, q'')) = \begin{cases} |x - y| & \text{if } q' = q''; \\ 1 & \text{if } q' \neq q''. \end{cases}$$

Then $RC_d = R_d = \mathbb{Q} \cap [0, 1]$, while $R_d(x)$ and $RC_d(x)$ are three-element sets for every $x \in X$.

It is clear by the above characterizations and Theorem 3 that for every separable complete ultrametric space (X, d), zero-dimensionality of the Wijsman hyperspace implies the existence of a continuous selection. Such a result could also be deduced from Theorem 2 of [5]. Indeed, for (X, d) as above, the Wijsman hyperspace is certainly paracompact (in fact, metrizable) and the multifunction ψ from such a hyperspace to the closed nonempty subsets of X, defined as $\psi(C) = C$ for every $C \in \mathrm{CL}(X)$, is lower semicontinuous (because it is continuous as a function from $(\mathrm{CL}(X), \mathbb{W}_d)$ to $(\mathrm{CL}(X), \mathbb{V}^-)$—see Section 1). Therefore, by the above Michael's theorem, there exists a continuous function $\varphi : (\mathrm{CL}(X), \mathbb{W}_d) \to X$ such that $\varphi(C) \in \psi(C) = C$ for every $C \in \mathrm{CL}(X)$.

In the next section we will provide some examples giving further informations about relationships between the above properties.

4. Examples

We showed that if (X, d) is an ultrametric space, then the Ball hypertopology is always zero-dimensional (and, if X is also separable and complete, $(\mathrm{CL}(X), \mathbb{B}_d)$ admits a continuous selection); moreover, we characterized the cases where the same holds for the Wijsman hypertopology. One may wonder whether the zero-dimensionality of the Wijsman hyperspace does imply its coincidence with the Ball hyperspace; the following example gives a negative answer.

Example 22. Let $X \subseteq l^\infty$ be defined as follows:

$$X = \{\mathbf{0}\} \cup \left\{ \frac{1}{k} \boldsymbol{e}_k \,\Big|\, k \equiv 1 \bmod 3 \right\} \cup \{\boldsymbol{e}_i \mid i \equiv 2 \bmod 3\} \cup \left\{ \frac{n+1}{n} \boldsymbol{e}_n \,\Big|\, n \equiv 0 \bmod 3 \right\},$$

where $\mathbf{0}$ is the null sequence and $\boldsymbol{e}_i$ is the sequence whose ith value is 1 and all the others are 0. Then X, endowed with the metric d induced by $\|\cdot\|_\infty$, is ultrametric, separable and complete. Moreover, the Wijsman hyperspace is zero-dimensional, and $\mathbb{W}_d \neq \mathbb{B}_d$.

Proof. It is very easy to verify that (X, d) is ultrametric, separable and complete. Since

$$RC_d = \{0,1\} \cup \left\{\frac{1}{k} \,\middle|\, k \equiv 1 \bmod 3\right\} \cup \left\{\frac{n+1}{n} \,\middle|\, n \equiv 0 \bmod 3\right\},$$

the Wijsman hyperspace is zero-dimensional (Theorem 20). To see that $\mathbb{W}_d \neq \mathbb{B}_d$, apply [7, Theorem 3.1], considering $B = \overline{S}_d(0,1)$. For instance, if $\varepsilon = 1/2$, then $\{x \in X \mid d(x,B) < \varepsilon\}$ equals B, which is not strictly d-included in itself. □

If X is separable, ultrametric and complete, the condition which characterizes the Wijsman hyperspace's zero-dimensionality also implies the existence of a selection. On the other hand, the existence of a selection does not imply the hyperspace's zero-dimensionality.

Example 23. Let $X = \mathbb{Q} \cap [1, +\infty[$, endowed with the metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y; \\ \max\{x,y\} & \text{if } x \neq y. \end{cases}$$

Then X is ultrametric, separable, complete. Moreover, there exists for $\mathrm{CL}(X)$ a $\mathbb{W}_d$-continuous selection and the Wijsman hypertopology is not zero-dimensional.

Proof. Again, verifications on X are trivial. Note that $RC_d = \{0\} \cup [1, +\infty[$: this clearly implies (Theorems 3 and 20) that there exists a $\mathbb{W}_d$-continuous selection and that the Wijsman hyperspace is not zero-dimensional. □

We observe that the existence of a $\mathbb{W}_d$-continuous selection for the preceding example was guaranteed by the fact that $RC_d \cap]0,1[= \emptyset$; moreover, the same result can be obtained in a space where $RC_d = [0, +\infty[$.

Example 24. Let $X = \bigcup_{i=1}^{+\infty} X_n$, where $X_n = (\mathbb{Q} \cap [1/n, 1]) \times \{n\}$, endowed with the following ultrametric:

$$d\big((x,n),(y,m)\big) = \begin{cases} 0 & \text{if } (x,n) = (y,m); \\ \max\{x,y\} & \text{if } n = m \text{ and } x \neq y; \\ 1 & \text{if } n \neq m. \end{cases}$$

Then (X, d) is a separable, complete, ultrametric space. Moreover, $RC_d = [0,1]$ and there exists a continuous selection from $(\mathrm{CL}(X), \mathbb{W}_d)$ to X.

Proof. Easy. □

Observe that both the Wijsman hyperspaces constructed in Examples 23 and 24 are separable metrizable spaces which turn out to be totally disconnected (by Theorem 14) but not zero-dimensional. In general, the construction of this kind of spaces is not completely trivial (see, for example, [4, 6.2.19]). Taking the Wijsman hyperspace of a separable ultrametric space which satisfies condition (#) but not condition (Δ) seems to be a relatively manageable method to produce such spaces.

We show now that the hyperspace may be hereditarily disconnected, without admitting a continuous selection (hence, without being totally disconnected).

Example 25. Let $X = \mathbb{Q} \cap [0, +\infty[$, endowed with a metric defined as in Example 23. Then (X, d) is a separable, complete ultrametric space. Moreover, there is no continuous selection for the Wijsman hyperspace, and it is hereditarily disconnected.

Proof. Verifications on X are trivial. Note that $RC_d(0) = [0, +\infty[$ (if $r \in [0, +\infty[$, then $d(x, \{x \in X \mid x > r\}) = r$), and hence by Theorem 3 there is no continuous selection from $(\mathrm{CL}(X), \mathbb{W}_d)$ to X (and, by Theorem 14, $(\mathrm{CL}(X), \mathbb{W}_d)$ is not totally disconnected).

Let us show that $(\mathrm{CL}(X), \mathbb{W}_d)$ is hereditarily disconnected. Observe that for every $x \in X$ with $x \neq 0$, condition (#) is satisfied at x. Let $\mathcal{S}$ be any subcollection of $\mathrm{CL}(X)$ with at least two elements. If $\mathcal{S}$ has exactly two elements, then it is trivially disconnected. Otherwise, it is always possible to find $C, D \in \mathcal{S}$ and $x \in C \setminus D$ such that $x \neq 0$, and hence a similar argument to that used in the proof of the first implication of Theorem 14 shows that there exists a $\mathbb{W}_d$-clopen collection $\mathcal{L}$ containing C but not D; therefore, $\mathcal{S}$ is disconnected. □

The question we are going to approach now is whether the ultrametric structure of (X, d) should imply some kind of disconnectedness for the Wijsman hyperspace. Example 29 below will provide a quite negative answer.

We first define a property which turns out to imply connectedness for the Wijsman hyperspace of any metric space.

Definition 26. Let (X, d) be a metric space. We say that (X, d) satisfies condition $(*)$ if:

$$\begin{aligned} &\forall z \in X\colon\ \forall m \in \mathbb{N}\colon\ \forall y_1, \ldots, y_m \in X\colon\ \forall \varepsilon_1 \geqslant \varepsilon_2 \geqslant \cdots \geqslant \varepsilon_m > 0\colon\ \forall \vartheta \geqslant 0\colon \\ &\quad \Big(\big(\vartheta < d(z, y_1) - \varepsilon_1 \text{ and } \forall j \in \{1, \ldots, m\}\colon\ d(z, y_j) > \varepsilon_j\big) \\ &\quad\quad \Longrightarrow \exists w \in X\colon\ \big(d(w, y_1) < d(z, y_1) - \vartheta \text{ and} \\ &\quad\quad\quad\quad \forall j \in \{1, \ldots, m\}\colon\ \varepsilon_j < d(w, y_j)\big)\Big) \qquad (*) \end{aligned}$$

(in particular, $\varepsilon_1 < d(w, y_1) < d(z, y_1) - \vartheta$).

Lemma 27. *Let (X, d) be a metric space which satisfies condition $(*)$, and let $\mathcal{H} \subseteq \mathrm{CL}(X)$ be a nonempty clopen set for the Wijsman topology. Then $X \in \mathcal{H}$.*

Proof. Let $\mathcal{M}$ be a maximal chain in $\mathcal{H}$ (with respect to set-theoretic inclusion). Then by Lemma 6, $\mathcal{M}$ has a maximum M. Since $M \in \mathcal{H}$ and $\mathcal{H}$ is $\mathbb{W}_d$-open, there exist $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$, $\delta_1, \ldots, \delta_n > 0$ and $\varepsilon_1 \geqslant \cdots \geqslant \varepsilon_m > 0$ such that $d(x_i, M) < \delta_i$ for every $i = 1, \ldots, n$, $d(y_j, M) > \varepsilon_j$ for every $j = 1, \ldots, m$ and

$$\mathcal{V} = \left(\bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i, \delta_i)\right) \cap \left(\bigcap_{j=1}^{m} \mathcal{A}_d^+(y_j, \varepsilon_j)\right) \subseteq \mathcal{H}$$

(where both n and m may be equal to 0). If we can show that $m = 0$, then by the definition of $\mathcal{V}$ we will have that $S' \in \mathcal{V}$ whenever $S' \in \mathrm{CL}(X)$ and $S' \supseteq S$ for some $S \in \mathcal{V}$; and hence, in particular, $X \in \mathcal{V} \subseteq \mathcal{H}$.

Suppose $m > 0$. Since $d(y_1, M) - \varepsilon_1 > 0$, choosing any $z \in M$ we have that $d(y_1, z) - d(y_1, M) < d(y_1, z) - \varepsilon_1$. Let $\vartheta = d(y_1, z) - d(y_1, M)$: since for all $j = 1, \ldots, m$ we have that $d(z, y_j) \geqslant d(y_j, M) > \varepsilon_j$, and X satisfies condition $(*)$, there exists $w \in X$ such that $d(w, y_1) < d(z, y_1) - \vartheta = d(z, y_1) - d(z, y_1) + d(y_1, M) = d(y_1, M)$ and $\varepsilon_j < d(w, y_j)$ for every $j = 1, \ldots, m$. Therefore, once we define $M' = M \cup \{w\}$, we have that $d(y_j, M') = \min\{d(y_j, M), d(y_j, w)\} > \varepsilon_j$ for every $j = 1, \ldots, m$. Thus $M' \in \bigcap_{j=1}^{m} \mathcal{A}_d^+(y_j, \varepsilon_j)$, and since also $M' \in \bigcap_{i=1}^{n} \mathcal{A}_d^-(x_i, \delta_i)$, $M' \in \mathcal{V} \subseteq \mathcal{H}$. On the other hand, from $d(y_1, w) < d(y_1, M)$ it follows that $w \notin M$, and this contradicts the maximality of $\mathcal{M}$. □

Corollary 28. *If (X, d) satisfies condition $(*)$, then $(\mathrm{CL}(X), \mathbb{W}_d)$ is connected.*

Proof. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two nonempty subsets of $(\mathrm{CL}(X), \mathbb{W}_d)$, which are clopen. Then they both contain X, and hence cannot be disjoint (nor complementary). □

Now we construct the promised example.

Example 29. Let X be the collection of all sequences $x = (x_n)_{n\in\mathbb{N}}$ such that $x_n \in \mathbb{Q} \cap [0, +\infty[$, $x_n \geqslant x_{n+1}$ for all $n \in \mathbb{N}$, and $\lim_{n\to+\infty} x_n = 0$. Let $d: X \times X \to [0, +\infty[$ defined as follows:

$$d\big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\big) = \begin{cases} 0 & \text{if } (x_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}; \\ \max\{x_k, y_k\} & \text{if } k = \min\{n \in \mathbb{N} \mid x_n \neq y_n\}. \end{cases}$$

Then (X, d) is a separable complete ultrametric space such that $(\mathrm{CL}(X), \mathbb{W}_d)$ is connected.

Proof. (1) d is clearly an ultrametric.

(2) (X, d) is separable, since the collection of sequences which are definitively 0 is countable and dense.

(3) (X, d) is complete. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X, where $x_n = (x_{n,m})_{m\in\mathbb{N}}$, then it is easy to prove that for all $m \in \mathbb{N}$ the sequence $(x_{n,m})_{n\in\mathbb{N}}$ is definitively constant or converges to zero; hence, for all $m \in \mathbb{N}$, $\lim_{n\to+\infty} x_{n,m}$ exists and is rational: call it a_m, and let $a = (a_m)_{m\in\mathbb{N}}$. Since $(x_{n,m})_{m\in\mathbb{N}}$ is nonincreasing (with respect to m), $(a_m)_{m\in\mathbb{N}}$ is in turn nonincreasing; moreover, $\lim_{m\to+\infty} a_m = 0$. Thus $a \in X$, and $\lim_{n\to+\infty} d(x_n, a) = 0$.

(4) (X, d) verifies condition $(*)$. Let

$$z = (z_n)_{n\in\mathbb{N}},$$
$$y_1 = (y_{1,n})_{n\in\mathbb{N}}, \qquad y_2 = (y_{2,n})_{n\in\mathbb{N}}, \qquad \ldots, \qquad y_m = (y_{m,n})_{n\in\mathbb{N}}$$

be elements of X, and let $\vartheta, \varepsilon_1, \ldots, \varepsilon_m$ be real numbers such that $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \cdots \geqslant \varepsilon_m > 0$, $0 \leqslant \vartheta < d(z, y_1) - \varepsilon_1$ and $d(z, y_j) > \varepsilon_j$ for every $j \in \{1, \ldots, m\}$. Since $\lim_{n\to+\infty} y_{1,n} = 0$, there exists $k = \min\{n \in \mathbb{N} \mid y_{1,n} < d(z, y_1) - \vartheta\}$. Let us choose a rational number r such that $\varepsilon_1 < r < d(z, y_1) - \vartheta$ and $r \neq y_{j,k}$ for all $j \in \{1, \ldots, m\}$. We define $w = (w_n)_{n\in\mathbb{N}}$ as follows:

$$w_n = \begin{cases} y_{1,n} & \text{if } n < k; \\ r & \text{if } n = k; \\ 0 & \text{if } n > k. \end{cases}$$

To show that $w \in X$, we only need to verify that $y_{1,k-1} \geqslant r$ (if $k = 1$, there are no problems). Indeed, by the definition of k: $y_{1,k-1} \geqslant d(z, y_1) - \vartheta > r$.

Since $w_n = y_{1,n}$ for $n < k$, while $w_k = r \neq y_{1,k}$, we have that $d(w, y_1) = \max\{r, y_{1,k}\}$; this implies on the one hand that $d(w, y_1) \geqslant r > \varepsilon_1$, and on the other hand that $d(w, y_1) < d(z, y_1) - \vartheta$.

If we verify that $d(w, y_j) > \varepsilon_j$ for all $j \in \{2, \ldots, m\}$, the proof will be complete. Fix j and let $i = \min\{n \in \mathbb{N} \mid w_n \neq y_{j,n}\}$: then $i \leqslant k$ (since $w_k = r \neq y_{j,k}$). Hence

$$d(w, y_j) = \max\{w_i, y_{j,i}\} \geqslant w_i \geqslant w_k = r > \varepsilon_1 \geqslant \varepsilon_j. \qquad \square$$

Our proofs of the main results in Section 2 exploit the separability and completeness of the base space X. Such hypotheses cannot be dropped.

Example 30. Let X be a set with cardinality greater than $\aleph_0$, endowed with the 0-1 metric d. Then there is no continuous selection from $(\mathrm{CL}(X), \mathbb{B}_d)$ (nor from $(\mathrm{CL}(X), \mathbb{W}_d)$) to X.

Proof. Let us observe that

$$(S_d(x,\varepsilon))^- = \begin{cases} \{C \in \mathrm{CL}(X) \mid x \in C\} & \text{if } \varepsilon \leqslant 1; \\ \mathrm{CL}(X) & \text{if } \varepsilon > 1. \end{cases}$$

$$(X \backslash \overline{S}_d(x,\varepsilon))^+ = \begin{cases} \{C \in \mathrm{CL}(X) \mid x \notin C\} & \text{if } \varepsilon < 1; \\ \emptyset & \text{if } \varepsilon \geqslant 1. \end{cases}$$

Thus, if we put

$$\mathcal{B}_x = \{C \in \mathrm{CL}(X) \mid x \in C\}, \qquad \mathcal{D}_x = \{C \in \mathrm{CL}(X) \mid x \notin C\},$$

the collection $\{\mathcal{B}_x \mid x \in X\} \cup \{\mathcal{D}_x \mid x \in X\}$ is a subbase for $\mathbb{B}_d$, and for any $C \in \mathrm{CL}(X)$ a fundamental system of $\mathbb{B}_d$-neighborhoods of C is given by all collections of kind:

$$\left(\bigcap_{i=1}^{n} \mathcal{B}_{x_i}\right) \cap \left(\bigcap_{j=1}^{m} \mathcal{D}_{y_j}\right),$$

where $x_i \in C$ for $i = 1, \dots, n$, $y_j \in X \setminus C$ for $j = 1, \dots, m$ and n, m may possibly be equal to 0.

Now, suppose φ is a continuous selection from $(\mathrm{CL}(X), \mathbb{B}_d)$ to X. Then for all $x \in X$, there exists F_x finite subset of $X \setminus \{x\}$ such that

$$\forall C \in \mathcal{B}_x \cap \left(\bigcap_{y \in F_x} \mathcal{D}_y \right): \ \varphi(C) = x.$$

Put, for every $y \in X$, $G_y = \{x \in X \mid y \notin F_x\}$: we claim that there exists $\overline{z} \in X$ such that $G_{\overline{z}}$ is infinite. Otherwise, fix a subset M of X such that $|M| = \aleph_0$: as G_y is finite for every $y \in M$, we have that $G = \bigcup_{y \in M} G_y$ has cardinality $\aleph_0$, and hence there exists $\overline{x} \in X \setminus G$. For this $\overline{x}$, we have that $y \in F_{\overline{x}}$ for every $y \in G$, that is $G \subseteq F_{\overline{x}}$, which is clearly impossible.

Since $G_{\overline{z}}$ is infinite, there exists $\overline{w} \in G_{\overline{z}} \setminus (F_{\overline{z}} \cup \{\overline{z}\})$. Consider the closed set $\widetilde{C} = \{\overline{z}, \overline{w}\}$: from $\widetilde{C} \in \mathcal{B}_{\overline{w}} \cap (\bigcap_{y \in F_{\overline{w}}} \mathcal{D}_y)$, we have that $\varphi(\widetilde{C}) = \overline{w}$, and from $\widetilde{C} \in \mathcal{B}_{\overline{z}} \cap (\bigcap_{y \in F_{\overline{z}}} \mathcal{D}_y)$, we have that $\varphi(\widetilde{C}) = \overline{z}$: a contradiction. □

As for completeness, it is known (see [3, Theorem 6.1] or the much stronger Corollary 3.5 of [8]) that there exists no continuous selection from $(\mathrm{CL}(\mathbb{Q}), \mathbb{V})$ to $\mathbb{Q}$. Giving $\mathbb{Q}$ any compatible ultrametric d, we also have that there exists no continuous selection from $(\mathrm{CL}(\mathbb{Q}), \mathbb{B}_d)$ (or $(\mathrm{CL}(\mathbb{Q}), \mathbb{W}_d)$) to $\mathbb{Q}$. Observe that the ultrametric d can be chosen in such a way that condition (Δ) (hence, condition (#)) holds at each point of X.

Indeed, let $X = \{f : \mathbb{N} \to \{0, 1\} \mid \exists n \in \mathbb{N}: \ \forall n' \geqslant n: f(n') = 0\}$; for every $f, g \in X$, let $n(f, g) = \min\{n' \in \mathbb{N} \mid f(n') \neq g(n')\}$. Define an ultrametric d on X by:

$$d(f, g) = \begin{cases} 0 & \text{if } f = g; \\ \dfrac{1}{n(f, g)} & \text{if } f \neq g. \end{cases}$$

Clearly,

$$R_d = RC_d = \{0\} \cup \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\},$$

so that condition (Δ) is automatically satisfied at each point. On the other hand, X is a countable metrizable space without isolated points, and hence it is homeomorphic to $\mathbb{Q}$.

Observe that the above considerations and Example 30 also show that, in the statement of Proposition 13, we cannot replace the hypothesis of completeness of (X, d), or that of separability, by the assumption that $(\mathrm{CL}(X), \mathbb{W}_d)$ is zero-dimensional.

Finally, we observe that if (X, d) is not ultrametric, then condition (#) does not characterize any more the existence of a Wijsman-continuous selection; indeed, if we consider $X = [0, 1]$ endowed with the Euclidean metric, then condition (#) is not satisfied but there is a continuous selection from $(\mathrm{CL}(X), \mathbb{W}_d)$ to X (for instance, let $\varphi(C) = \min C$). On the other hand, the real line endowed with the Euclidean metric constitutes an example of a separable complete metric space whose Ball hyperspace (in fact, Vietoris hyperspace) admits no continuous selection; see [3, Proposition 5.1].

We do not know examples of separable complete metric spaces satisfying condition (#), whose Wijsman hyperspaces have no continuous selections.

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