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Distance edge-colourings and matchings☆

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A B S T R A C T

We consider a distance generalisation of the strong chromatic index and the maximum induced matching number. We study graphs of bounded maximum degree and Erdős–Rényi random graphs. We work in three settings. The first is that of a distance generalisation of an Erdős–Nešetřil problem. The second is that of an upper bound on the size of a largest distance matching in a random graph. The third is that of an upper bound on the distance chromatic index for sparse random graphs. One of our results gives a counterexample to a conjecture of Skupień.

1. Introduction

Given a graph $G = (V, E)$, a strong edge-colouring of G is a proper edge-colouring such that no edge is adjacent to two edges of the same colour. Equivalently, a strong edge-colouring of G is a colouring of the edges such that no two edges within distance 2 are given the same colour. (The distance between two edges is defined as the number of vertices in a shortest path between them. Adjacent edges have distance 1.) The strong chromatic index of G, which we denote as $\chi'_2(G)$ in this paper, is the least integer k such that there exists a strong edge-colouring of G using k colours. The strong chromatic index has a rich history, going back to problems posed by Erdős and Nešetřil in 1985 (cf. [5]).

We shall study a distance-based generalisation of the strong chromatic index. Given a positive integer t, a distance-t edge-colouring of G is a colouring of the edges such that no two edges within distance t are given the same colour. Note that a distance-1 edge-colouring is a proper edge-colouring. A distance-2 edge-colouring is a strong edge-colouring. The distance-t chromatic index of G, denoted as $\chi'_t(G)$, is the least integer k such that there exists a distance-t edge-colouring of G using k colours. The distance-t chromatic index was first considered by Skupień [11] in the early 1990's. Recently, Ito et al. [7] developed two polynomial-time algorithms for finding distance-t edge-colourings of partial k-trees and planar graphs.

The distance-t edge-colouring problem is related to the colouring of powers of graphs. Observe that $\chi'_t(G) = \chi((L(G))^t)$, where $\chi(\cdot)$ denotes the chromatic number, $L(\cdot)$ denotes the line graph, and the tth power of a graph is the graph obtained by adding the edges between pairs of vertices at distance at most t.

We will also study another parameter that is related to maximum induced matchings. Given a graph $G = (V, E)$, a distance-t matching of G is a set of edges no two of which are within distance t. This particular generalisation of matchings was first studied by Stockmeyer and Vazirani [12]. (A distance-1 matching is a matching; a distance-2 matching is an induced

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matching.) Note that a distance-t edge-colouring is an edge-colouring such that each colour class induces a distance-t matching. The distance-t matching number of G, here denoted as $\mu_t(G)$, is the largest integer k such that there exists a distance-t matching in G with k edges. Observe that $\chi'_t(G) \geq |E|/\mu_t(G)$, since in any distance-$t$ edge-colouring, each colour class has at most $\mu_t(G)$ edges.

In this paper, we will consider three relatively self-contained problems concerning the distance-t chromatic index. In Section 2, we consider graphs of bounded maximum degree and propose a distance-t version of the Erdős–Nešetřil problem. In Propositions 1 and 2, we describe examples that give good lower bounds on the distance-t chromatic index. In Sections 3 and 4, we study Erdős–Rényi random graphs: in the former section we upper bound the distance-t matching number (Theorem 4) which also implies a lower bound for the distance-t chromatic index, and in the latter we upper bound the distance-t chromatic index (Theorem 5) for sparse enough random graphs.

2. A distance-t Erdős–Nešetřil problem

If G has maximum degree Δ, then

$$\chi'_t(G) \leq 2\sum_{j=1}^{t}(\Delta-1)^j + 1$$

is a trivial upper bound, as the maximum degree of $(L(G))^t$ is at most $2\sum_{j=1}^{t}(\Delta-1)^j$. Note that $2\sum_{j=1}^{t}(\Delta-1)^j \sim 2\Delta^t$ as $\Delta \to \infty$. By Vizing's theorem, if G has maximum degree Δ, then $\chi'_1(G) \in \{\Delta, \Delta+1\}$. Erdős and Nešetřil in 1985 asked what is the optimal upper bound on $\chi'_2(G)$ in terms of the maximum degree (cf. [5]). They mentioned a natural lower bound example, which consists of a 5-cycle with its vertices multiplied; this example demonstrates that there exists a graph G with arbitrarily large maximum degree Δ such that $\chi'_2(G) \geq 5\Delta^2/4$. Over a decade later, Molloy and Reed [9] showed using the probabilistic method that, if G has maximum degree Δ for Δ sufficiently large, then $\chi'_2(G) \leq 1.998\Delta^2$. The example in the following proposition shows that any general upper bound on χ'_t for graphs of maximum degree Δ is $\Omega(\Delta^t)$.

Proposition 1. *Fix an integer $t \geq 2$. For arbitrarily large Δ, there exists a regular graph with degree Δ such that $\chi'_t(G) > \Delta^t/\left(2(t-1)^{t-1}\right)$.*

Proof. Fix an integer $x > 1$. Let the vertex set consist of $(t-1)$-tuples with coordinates chosen from $\{1, \ldots, x\}$. The number of vertices is x^{t-1}. Two vertices $(x_1, \ldots, x_{t-1})$ and $(x'_1, \ldots, x'_{t-1})$ are adjacent if and only if there exists $i \in \{1, \ldots, t-1\}$ such that $x_j = x'_j$ for all $j \neq i$, i.e. if and only if they differ in exactly one coordinate. Every vertex has degree exactly $(t-1)(x-1)$, and thus the number of edges in G is $(t-1)(x-1)x^{t-1}/2$. Between any pair of vertices there is a path with at most t vertices, so between any pair of edges there is a path with at most t vertices. Therefore, $(L(G))^t$ is a clique, and so

$$\chi'_t(G) = (t-1)(x-1)\frac{x^{t-1}}{2} > \frac{((t-1)(x-1))^t}{2(t-1)^{t-1}}.$$

Since x was an arbitrary integer greater than 1, the result follows. □

This example is tight neither for $t = 2$ (because of the multiplied 5-cycle) nor for $t = 3$ (as we shall soon see from the next proposition); however, we have been unable to find any better alternative constructions for when $t \geq 4$. For a graph G of maximum degree Δ, we believe that it is possible to obtain an upper bound of $(2-\varepsilon)\Delta^t$ for some fixed $\varepsilon > 0$, by bounding the number of edges that span the neighbourhood sets of $(L(G))^t$, as Molloy and Reed did in the case $t = 2$, but the details seem more complicated for $t \geq 3$.

It should be noted that the above example is a Δ-regular graph with $\Omega(\Delta^t)$ edges that has distance-t matching number 1. Skupień [11] suggested such a graph could be found by multiplying the cycle C_{2t+1}; however, this results in a Δ-regular graph of distance-t matching number 1 with only $O(\Delta^2)$ edges.

Faudree et al. [5] asked about the optimal upper bound for the strong chromatic index in terms of the maximum degree when restricted to bipartite graphs. Note that the complete bipartite graph $K_{\Delta,\Delta}$ shows that, for graphs of maximum degree Δ, any such bound is at least Δ^2. The following example demonstrates that any upper bound on χ'_3 for bipartite graphs of maximum degree Δ is at least $\Delta^3 - o(\Delta^3)$. Furthermore, this example disproves Conjecture 2 in [11].

Proposition 2. *For arbitrarily large Δ, there exists a bipartite, regular graph with degree Δ such that $\chi'_3(G) = \Delta^3 - \Delta^2 + \Delta$.*

Proof. Let q be a prime power and let P be the projective plane with q^2+q+1 points and q^2+q+1 lines. Consider the point–line incidence graph for P, i.e. the bipartite graph G with parts A and B, where A is the set of points, B is the set of lines, and there is an edge between two vertices $a \in A$ and $b \in B$ if the point a lies in the line b. Note that G is a regular graph with degree $q+1$. Consider two edges a_1b_1 and a_2b_2 in G. If $a_1 \neq a_2$, then there is a unique line b which contains both points a_1 and a_2. The path a_1ba_2 verifies that a_1b_1 and a_2b_2 are at distance at most 3 in G. As a_1b_1 and a_2b_2 were arbitrary, this shows that $\mu_3(G) = 1$; thus, $\chi'_3(G) = |E(G)| = (q+1)(q^2+q+1) = (q+1)^3 - (q+1)^2 + (q+1)$. □

3. The distance-t matching number of random graphs

As is standard, we let $G_{n,p}$ denote a random graph on vertex set $[n] = \{1, \dots, n\}$ where each possible pair of vertices is included as an edge independently at random with probability p. The induced matching number of $G_{n,p}$ was first considered by El Maftouhi and Márquez Gordones [4] and they obtained a fairly tight estimate. In this section, we shall bound $\mu_t(G_{n,p})$ for $p = d/n$, with d sufficiently large, by estimating the expected number of distance-t matchings with k edges. Our approach is inspired by the method used by Atkinson and Frieze [2] to study what they called b-independent sets, which are vertex subsets in which no two vertices are within distance b. We use the following lemma in this estimation.

Lemma 3. *Suppose K is a matching with k edges in the complete graph on $[n]$ and*

$$p_d = \frac{d(d^{t-1}-1)}{(d-1)n}$$

where $p = d/n$ satisfies $p < 1/2$. Then, in $G_{n,p}$, letting

$$q^K = \mathbb{P}\left(K \text{ is a distance-}t \text{ matching}\right),$$

we have

$$p^k (1-p_d)^{4\binom{k}{2}} \le q^K \le p^k (1-p_d)^{4\binom{k}{2}} \exp\left(O\left(k^3 n^{2t-3} p^{2t-1}\right)\right),$$

as $n \to \infty$.

Proof. We first count the edge sets of paths on at most t vertices (including endpoints) which could connect a pair of distinct edges in K. Given two edges in K, the endpoints of such a potential path can be chosen in four ways. We may enumerate these path edge sets as follows: P_i, $i \in [N]$, where

$$N = 4\binom{k}{2}\sum_{r=2}^{t}(n-4)\cdots(n-r-1).$$

Now let $\mathscr{P}_i$, $i \in [N]$, be the event that the edges of P_i are all present in $G_{n,p}$. Note that since $p < 1/2$ and K is a matching, we have that $\mathbb{P}(\mathscr{P}_i) < 1/2$ for all i. Note also that, given that the edges of K are present, then K is a distance-t matching if and only if none of the $\mathscr{P}_i$ occur. We use Janson's inequality (cf. [1, Theorem 8.1.1]) to bound this probability. Set

$$\Delta = \sum_{P_i \cap P_j \neq 0 | i \neq j} \mathbb{P}\left(\mathscr{P}_i \cap \mathscr{P}_j\right) \quad \text{and} \quad M = \prod_{i=1}^{N}\left(1 - \mathbb{P}\left(\mathscr{P}_i\right)\right).$$

Then, by Janson's inequality (also using $\mathbb{P}(\mathscr{P}_i) < 1/2$ for all i),

$$M \le q^K = \mathbb{P}\left(\bigcap_{i=1}^{N} \bar{\mathscr{P}}_i\right) \le e^{\Delta} M.$$

We have, by splitting the product according to path lengths,

$$M = \left(\prod_{r=2}^{t}\left(1-p^{r-1}\right)^{(n-4)\cdots(n-r-1)}\right)^{4\binom{k}{2}}$$
$$= \left(1 - \sum_{r=2}^{t} p^{r-1} n^{r-2}\right)^{4\binom{k}{2}} e^{O(1)} = \left(1 - \frac{d(d^{t-1}-1)}{(d-1)n}\right)^{4\binom{k}{2}} e^{O(1)}.$$

Observe that the $e^{O(1)}$ term is at least 1, so this implies the lower bound on P.

For the upper bound, we bound Δ as follows. Subject to a penalty factor of 2 in the final count, let us assume that P_i is the shorter of P_i and P_j. If we fix a quantity ℓ_i for the path length of P_i, then the probability weight on the choices for P_i is at most $4\binom{k}{2} n^{\ell_i - 1} p^{\ell_i}$. Next we fix a quantity $\ell_j \ge \ell_i$ for the path length of P_j and condition on $s \ge 1$ being the size of the common edge set between P_i and P_j. With this conditioning, we see that the probability weight on the choices for P_j is at most $kn^{\ell_j - s - 1} p^{\ell_j - s}$. We thus have

$$\Delta \le 2\sum_{\ell_i=3}^{t} 4\binom{k}{2} n^{\ell_i-1} p^{\ell_i} \sum_{\ell_j=\ell_i}^{t} \sum_{s=1}^{\ell_i-1} \binom{\ell_i}{s} kn^{\ell_j-s-1} p^{\ell_j-s}$$
$$= O\left(k^3 n^{2t-3} p^{2t-1}\right).$$

Since the probability that the edges of K are present is p^k, we obtain the desired upper bound. □

With this lemma we can prove an upper bound on $\mu_t(G_{n,p})$ with the first moment method. Clearly, this also implies a lower bound for $\chi'_t(G_{n,p})$ (a.a.s.).

Theorem 4. *Let $\varepsilon > 0$ and suppose $p = d/n$ and $p < 1/2$. Furthermore, let*

$$k_t = \frac{n}{2d^{t-1}} \left(t \log d - \log\log d - \log et + \varepsilon\right).$$

There exists d_0 such that, if $d \geq d_0$, then a.a.s. $\mu_t\left(G_{n,p}\right) \leq k_t$.

Proof. Let the random variable X_k denote the number of distance-t matchings with k edges in $G_{n,p}$. Set $k = k_t$ and compute the expectation as follows using Lemma 3:

$$\begin{aligned}
\mathbb{E}\left(X_k\right) &= \sum_{\text{matching } K, |K|=k} \mathbb{P}\left(K \text{ is a distance-}t \text{ matching}\right) \\
&\leq \binom{n}{2k}\binom{2k}{2,\ldots,2}\frac{p^k}{k!}(1-p_d)^{4\binom{k}{2}} \exp\left(O\left(k^3 n^{2t-3} p^{2t-1}\right)\right) \\
&\leq \frac{n!}{k!(n-2k)!}\left(\frac{p}{2}\right)^k \exp\left(-4\binom{k}{2}p_d\right)\exp\left(O\left(k^3 n^{2t-3} p^{2t-1}\right)\right) \\
&\leq n^{2k}\left(\frac{p}{2ek}\right)^k \exp\left(-4\binom{k}{2}p_d\right)\exp\left(O\left(\frac{k(\log d)^2}{d^t}\right)\right) \\
&= \exp\left(k\left(\log\left(\frac{n^2 p}{2ek}\right) - 2\left(k-1\right)p_d + o(1)\right)\right)
\end{aligned}$$

as $d \to \infty$. Now, we have

$$\begin{aligned}
&\log\left(\frac{n^2 p}{2ek}\right) = t \log d - \log\log d - \log et + o(1) \quad \text{and} \\
&2(k-1)p_d = t \log d - \log\log d - \log et + \varepsilon + o(1)
\end{aligned}$$

as $d \to \infty$. Combining these, $\mathbb{E}\left(X_k\right) \leq \exp(-\varepsilon k/2)$ for d large enough. Then, by Markov's inequality, $\mathbb{P}\left(\mu_t(G_{n,p}) > k\right) \leq \mathbb{P}\left(X_k > 0\right) \leq \mathbb{E}\left(X_k\right) \to 0$ as $n \to \infty$. This establishes the required upper bound for $\mu_t(G_{n,p})$. □

If we consider together the precise a.a.s. formulas obtained by El Maftouhi and Márquez Gordones for $\mu_2(G_{n,p})$ and by Atkinson and Frieze for the b-independence number of $G_{n,p}$, as well as the lower bound on q^K given in Lemma 3, then it is natural to speculate that the expression k_t in the above theorem is close to the correct formula for the asymptotic value of $\mu_t(G_{n,p})$. Proving the required lower bound on $\mu_t(G_{n,p})$ is likely to demand significant technical overhead, which we defer to future work.

4. Distance-t edge-colouring of sparse random graphs

The strong chromatic index of random graphs was first considered by El Maftouhi and Márquez Gordones [4]. It was then considered in a succession of papers by Palka [10], Vu [13], Czygrinow and Nagle [3] and Frieze et al. [6]. Frieze et al. gave a precise description of the strong chromatic index of $G_{n,p}$ for p satisfying $np \leq (\log n/\log\log n)^{1/2}/100$. In this section, we show that their method can be extended to show an analogous result for the distance-t chromatic index. Before stating the result, we need to give a few definitions.

Let $G = (V, E)$ be a graph and s be a positive integer. If $v \in V$, then $\deg_s(v)$ is defined to be the number of edges within distance s of v (so $\deg_1(v)$ is just the usual degree of v); if $e \in E$, then $\deg_s(e)$ is defined to be the number of edges within distance s of e (including e itself). We let

$$\Delta_t(G) = \begin{cases} \max\left\{\deg_{t/2}(e) : e \in E\right\} & \text{if } t \text{ is even} \\ \max\left\{\deg_{(t+1)/2}(v) : v \in V\right\} & \text{if } t \text{ is odd} \end{cases}$$

(deviating slightly from the notation of Frieze et al.) and

$$\lambda_t = \left(\frac{\log n}{\log\log n}\right)^{1/t}.$$

We remark here that $\Delta_t(G)$ is the size of a largest clique in $(L(G))^t$.

Theorem 5. *If p satisfies $np \le \lambda_t/100$, then $\chi'_t(G_{n,p}) = \Delta_t(G_{n,p})$ a.a.s.*

Proof. Let $G = (V, E)$ be a graph with maximum degree Δ. Consider the subgraph $\beta(G)$ induced by the set of vertices of G that are at distance at most t from the set of vertices of degree at least $(\Delta/4)^{1/t}$. By an induction argument that is similar to Lemma 3 of Frieze et al., we can show that if $\beta(G)$ is acyclic then $\chi'_t(G) = \Delta_t(G)$. Then it suffices to show that if $np \le \lambda_t/100$, then $\beta(G_{n,p})$ is acyclic a.a.s. For this, consider the subgraph ξ of $G_{n,p}$ induced by the set of vertices that are at distance at most t from the set of vertices of degree at least $\lambda_t/3$.

Let C be a shortest cycle in ξ and let ℓ be the length of C. We claim that there are at least $\ell/(4t+2)$ vertex-disjoint paths of length at most t connecting vertices of the cycle to vertices of degree at least $\lambda_t/3$. Since there is always at least one such path, assume that $\ell \ge 4t+2$. Let $v_1, \ldots, v_s$ be a largest set of vertices of C such that the distance along the cycle between any pair of them is at least $2t+1$. Clearly $s = \lfloor \ell/(2t+1) \rfloor \ge \ell/(4t+2)$. Since C is the shortest cycle in ξ, the distance in ξ between any pair $v_i \ne v_j$ is also at least $2t+1$. By the definition of ξ, for each $i \in \{1, \ldots, s\}$ there is a path P_i of length at most t from v_i to a vertex w_i of degree at least $\lambda_t/3$. All vertices of P_i belong to ξ and two paths $P_i, P_j, i \ne j$ are vertex disjoint since the distance in ξ between v_i and v_j is at least $2t+1$. The subgraph induced by the intersection of P_i and C is connected; otherwise C would not be a shortest cycle in ξ; thus, there is a unique vertex u_i in the intersection that has a neighbour in $P_i \setminus C$. We let P'_i denote the part of P_i which is edge-disjoint from C. (P'_i contains w_i and u_i.) Let H be the union of all paths P'_i and C. We now estimate the probability that $G_{n,p}$ contains such a subgraph H.

For this estimation, we use Lemma 4 of Frieze et al. which holds with λ replaced by λ_t. That is, the following holds.

Lemma 6. *Let p be such that $np \le \lambda_t/100$. Let A be a set of vertices of size x and let B be a set of edges of size at most cx for some constant c. Conditioning on the event that all edges in B are present in $G_{n,p}$, the probability that all vertices in A have degree at least $\lambda_t/3$ is at most $2e^{-\lambda_t x/10}$.*

Now, the number of ways to choose C is at most n^ℓ and the probability that it appears in $G_{n,p}$ is p^ℓ. We can choose the set of vertices u_i in at most $\binom{\ell}{s} \le 2^\ell$ ways. The path between u_i and w_i has length 0 up to t and there are at most $(t+1)^{\ell/(2t+1)}$ ways to choose the lengths for all the paths P'_i. The number of paths of length r is at most n^r and the probability that such a path appears is p^r. After we choose the paths P'_i, the vertices w_i are determined and we expose a set B of at most $\ell + t(\ell/(2t+1)) \le 2\ell \le 2(4t+2)s$ edges of $G_{n,p}$. Therefore, by the above lemma, the probability that all of the vertices w_i have degree at least $\lambda_t/3$ is bounded by $2e^{-\lambda_t s/10} \le 2e^{-\lambda_t \ell/(10(4t+2))}$.

In summary, the probability that there is a cycle in $\beta(G_{n,p})$ is at most

$$\begin{aligned}
&\sum_{\ell \ge 3} n^\ell p^\ell 2^\ell (t+1)^{\ell/(2t+1)} \left(\sum_{r=0}^{t} (np)^r \right)^{\ell/(2t+1)} 2e^{-\lambda_t \ell/(10(4t+2))} \\
&\quad \le \sum_{\ell \ge 3} (2(t+1)np)^\ell \left((t+1)(np)^t \right)^{\ell/(2t+1)} 2e^{-\lambda_t \ell/(10(4t+2))} \\
&\quad \le \sum_{\ell \ge 3} ((t+1)\lambda_t/50)^{2\ell} 2e^{-\lambda_t \ell/(10(4t+2))} = o(1).
\end{aligned}$$

This completes the proof. □

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