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Systems Engineering Procedia 3 (2012) 312 – 318

Procedia
Systems Engineering

Performance analysis of distributed consensus on regular networks

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Abstract

This paper presents the analysis of the performance of the consensus on regular network structures in information engineering. The consensus protocols introduced are distributed in the sense that each agent only needs information from its neighbors, which reduces the complexity of links between agents significantly. By graph Laplacian spectrum and frequency domain analysis, the performance of the consensus, characterized by the convergence rate and robustness to the communication constraints, are investigated in details. We establish direct connections between the system parameters and the performance of the consensus. Specifically, the convergence rates and the maximum fixed time-delays that can be tolerated by the various regular networks are found explicitly.

Keywords: consensus; dynamical networks; regular graphs; convergence rate; communication time-delays; information engineering

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 doi:10.1016/j.sepro.2011.11.035

1. Introduction

The distributed coordination of dynamical networks has fascinated the researchers in many disciplines, from physics and statistical mechanics, to mathematical biology and information science [1-7]. In particular, the research on shared information of interest in dynamical networks has significantly facilitates the distributed coordinated behavior [8-11]. From a point of view of control science, the consensus problems are defined to be the analysis and synthesis of effective protocols and mechanisms that are able to guide the dynamic agents to converge to a consistent sense, or view, of the shared information of interest, with the constraints on information exchange. Therefore the consensus problem provides a useful unified framework for the research of cooperative control. The consensus of dynamical networks finds potential vast applications in many areas, such as cooperative control of multiple vehicle system [12-17], swarming and flocking [3, 18, 19], and sensor networks [20].

Underlying this research is the implication that the structural properties of a network must have some bearing on the dynamics taking place on it. Moreover, the study of dynamical models unfolding on complex networks has generated results of conceptual and practical relevance. Therefore, the performances of consensus protocols are expected to be strongly affected by the connectivity pattern of the underlying network substrates: certain network topology structures facilitate while others impair them. The focus of this paper is to consider the performance of linear consensus protocols when information is exchanged by regular network topologies including fully connected, star-like and $2K$ -regular networks. By graph Laplacian spectrum method, the convergence rate and robustness to the communication constraints are discussed in details.

This paper is organized as follows. Section 2 deals with basic concepts and notational details used throughout the paper and introduces the consensus protocols. The main theoretical results are given in Section 3. Finally, brief concluding remarks and some directions for future work are given in Section 4.

2. Consensus problems on graphs

2.1. Dynamics and network structure

Consider *n* dynamic agents with dynamics described by single integrator dynamics:

$$\dot{q}_i(t) = u_i(t)\,, \tag{1}$$

where $i = 1, 2, \ldots, n$, and $q_i(t) \in R$, and $u_i(t) \in R$ denote the information state and the control input of agent i , respectively. Define the information states of the whole network as $\boldsymbol{q}(t) = [q_1(t), \ldots, q_n(t)]^{\mathrm{T}}$. Information exchange between agents can be naturally modeled by the undirected graph $G = (V, E, \boldsymbol{A})$, where $V = \{v_i\}$ is the set of agents, $E \subseteq V \times V$ is the set of links between the agents and $\boldsymbol{A}$ is the corresponding weighted adjacency matrix. The adjacency matrix $\boldsymbol{A} = [a_{ij}] \in R^{n\times n}$ is defined such that $a_{ij} = 1$ if $(v_j, v_i) \in E$, while $a_{ij} = 0$ if $(v_j, v_i) \notin E$. Let matrix $\boldsymbol{L} = [l_{ij}]$ be defined as $l_{ii} = \sum_{j\neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ where $i \neq j$. Following algebraic graph theory, $\boldsymbol{L}$ is positive semi-definite and is called the graph Laplacian matrix. The set of neighbors of agent i is defined as $N_i = \{v_j : (v_j, v_i) \in E\}$. The graph *G* does not contain a loop, a link joining an agent to itself.

2.2. Consensus protocols

The distributed consensus protocol for a network of single integrator agents is presented as follows:

$$u_i(t) = \sum_{j\in N_i} a_{ij}[\beta(q_j(t) - q_i(t)], \tag{2}$$

where β is positive constant denoting the feedback gain of the consensus protocol.

The consensus problem discussed in this paper is defined exactly as follows.

Definition 1: For the network of single integrator systems, consensus is said to be reached globally asymptotically among dynamic agents if $\left|q_i(t) - q_j(t)\right| \to 0,\ \forall i \neq j$ as $t \to \infty$ for any $q_i(0)$.

3. Main results

It is noted that the graph *G* is necessarily connected, otherwise the consensus is never be achieved. Therefore the Laplacian matrix $\boldsymbol{L}$ of *G* has a simple zero eigenvalue and all the other eigenvalues are positive real numbers. Therefore we write the eigenvalues of $\boldsymbol{L}$ in the form $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_{n-1} \leq \lambda_n$. Furthermore λ_2 is called the algebraic connectivity since it characterizes some connectivity properties of a graph [21].

Guided by consensus protocol (2), the closed-loop dynamics for the single integrator agents take the following form:

$$\dot{q}_i(t) = \sum_{j\in N_i} a_{ij}[\beta(q_j(t) - q_i(t)] \tag{3}$$

with a collective dynamics

$$\dot{\boldsymbol{q}}(t) = -\beta \boldsymbol{L} \boldsymbol{q}(t). \tag{4}$$

Using the Laplacian transformation, it can be shown that the poles of the above linear dynamical systems are given by the following equations:

$$s + \lambda_i \beta = 0\,, \tag{5}$$

where s is the Laplacian variable and $s \in C$.

In this scenario, let $v_i = \min\{|s| : s + \lambda_i \beta = 0, s \in C\}$. According to linear system theory, we know $v := \min_{i \geq 2}\{v_i\}$ reflects the convergence rate of the consensus.

Now we are ready to give our main results in the following.

3.1. Convergence rate

The performance of the convergence rate of the consensus protocol for regular graphs is summarized in the following.

Theorem 1: Consider a network of *n* dynamic agents with dynamics described by Eq. (1). Assume that the network is connected and that each agent receives the information from its neighboring agents and applies the control law Eq. (2). Then the consensus is achieved with convergence rate v for the following regular network structures.

i) For a fully connected network, $v = \beta n$.

ii) For a star-like network, $v = \beta$.

iii) For a $2K$ -regular network, $v = 4\beta \sum_{l=1}^{K} \sin^2(\frac{\pi l}{n})$.

Proof: Following the multivariable control theory, the information states of all the agents achieve consensus globally asymptotically (i.e., $|q_i(t) - q_j(t)| \to 0, \ \forall i \neq j$) if and only if the poles given by Eq. (5) are located on the open left half plane (LHP) except for the isolated zero pole.

Due to the connectedness of the underlying graph topology, the above arguments are satisfied and the consensus is hence achieved.

Furthermore, since the algebraic connectivity λ_2 is the smallest nonidentically zero eigenvalue of the Laplacian matrix, we obtain

$$v = \beta \lambda_2 . \tag{6}$$

Given the Laplacian spectra for the typical regular graph topologies [21], we have:

i) For a fully connected network, we have $\lambda_2 = n$, which means $v = \beta n$.

ii) For a star-like network, we have $\lambda_2 = 1$, which means $v = \beta$.

iii) For a $2K$ -regular network, the vertices are located on a circle, and each vertex is connected to its $2K$ nearest neighbors. Then the algebraic connectivity is given by $\lambda_2 = 4\sum_{l=1}^{K} \sin^2(\frac{\pi l}{n})$, which means $v = 4\beta \sum_{l=1}^{K} \sin^2(\frac{\pi l}{n})$.

The proof of the theorem is complete. □

3.2. robustness to the communication constraints

Owing to the limited communication capacity of dynamic agents, the communication time-delays are inevitable in reality. It is well-known that unmodelled time-delay effects may deteriorate the performance of the dynamic system and even destabilize it. Therefore by the frequency domain analysis, we will prove that the network of dynamic agents can achieve consensus asymptotically for appropriate communication time-delays if the network is connected. Furthermore, we derived the theoretical result for the typical regular graph structures in terms of the corresponding tight upper bounds on communication time-delays such that the consensus can still be reached. The robustness to the communication time-delays for the consensus protocol for regular graphs is given in Theorem 2.

Theorem 2: Consider a network of n dynamic agents with dynamics described by Eq. (1). Assume that the network is connected and that each agent receives the information from its neighboring agents after a constant communication time-delay $\tau \geq 0$ and applies the control law Eq. (2). Then the consensus is achieved, if and only if $\tau \in [0, \tau^*)$ for the following regular network structures.

i) For a fully connected network, $\tau^* = \dfrac{\pi}{2\beta n}$.

ii) For a star-like network, $\tau^* = \dfrac{\pi}{2\beta n}$.

iii) For a $2K$ -regular network, $\tau^* = \pi \Big/ 8\beta \sum_{l=1}^{K} \sin^2 (\dfrac{\pi l(n-1)}{n})$.

Proof: For a network with communication time-delays, the closed-loop dynamics for the single integrator agents take the following form:

$$\dot{q}_i(t) = \sum_{j \in N_i} a_{ij} [\beta(q_j(t-\tau) - q_i(t-\tau)] \tag{7}$$

with a collective dynamics

$$\dot{\boldsymbol{q}}(t) = -\beta \boldsymbol{L} \boldsymbol{q}(t-\tau). \tag{8}$$

Following the similar line of reasoning in Theorem 1, the poles of the above linear delay dynamical systems are given by the following equations:

$$s + \mathrm{e}^{-\tau s} \lambda_i \beta = 0. \tag{9}$$

When $\tau = 0$, Eq. (9) degenerates to Eq. (5) and the poles of the ith subsystem are $s = -\lambda_i \beta$. Consequently, the consensus is achieved. When $\tau > 0$, the proposed protocol can not guarantee the consensus unless the communication time-delay τ is smaller than some given upper bound. Note that the pole associated with λ_1 is not dependent on time-delay τ. Hence it is sufficient to consider the case of λ_i $(i \geq 2)$. A tight upper bound on the time-delay τ can be determined as follows.

Let us find the smallest value of the time-delay $\tau > 0$ such that Eq. (9) has some roots on the imaginary axis. Setting $s = \mathrm{j}\omega$ in Eq. (9), we have

$$\mathrm{j}\omega + \mathrm{e}^{-\mathrm{j}\omega\tau} \lambda_i \beta = 0. \tag{10}$$

Assuming $\omega > 0$ and using Eq. (10) give

$$\tau = \frac{2k\pi + \dfrac{\pi}{2}}{\beta \lambda_i}, \tag{11}$$

where $k = 0, \pm 1, \pm 2, \ldots$.

The smallest $\tau > 0$ occurs at $k = 0$ and is thus given by

$$\tau^* = \min_{i \geq 2}\{\frac{\pi}{2\beta\lambda_i}\} = \frac{\pi}{2\beta\lambda_2}. \quad (12)$$

One can repeat a very similar argument for the case that $\omega < 0$ and get the same conclusion.

Owing to the fact that for $\tau = 0$ all the roots of Eq. (9) other than *s*=0 are located on the open LHP and the continuous dependence of the roots of Eq. (9) on time-delay τ, for all $\tau \in [0, \tau^*)$, the roots of Eq. (9) with $i \geq 2$ are on the open LHP. Hence the first subsystem (for $i = 1$) is marginally stable, while all the other subsystems (for $i \geq 2$) are asymptotically stable [22]. Then the poles of the consensus system other than *s*=0 are all on the open LHP under the given condition and therefore are all stable. Consequently, the consensus is achieved. Moreover, $\boldsymbol{q}(t)$ has a globally asymptotically stable oscillatory solution when $\tau = \tau^*$.

Now we consider the typical regular graph topologies:

i) For a fully connected network, we have $\lambda_n = n$, which means $\tau^* = \frac{\pi}{2\beta n}$.

ii) For a star-like network, we have $\lambda_n = n$, which means $\tau^* = \frac{\pi}{2\beta n}$.

iii) For a $2K$-regular network, λ_n is given by $4\sum_{l=1}^{K}\sin^2(\frac{\pi l(n-1)}{n})$, which means $\tau^* = \pi \Big/ 8\beta\sum_{l=1}^{K}\sin^2(\frac{\pi l(n-1)}{n})$.

The proof of the theorem is complete. □

4. Conclusion and future work

This paper has presented the analytical results for the performance of the consensus on regular network structures. In our framework, the performance of the consensus are characterized by the convergence rates and the maximum tolerable communication time-delays, and the tuneable parameters of the consensus system are the Laplacian spectrum of the network topology and the feedback gain of the protocol. Using several tools from algebraic graph theory and control theory, we developed direct explicit links between the tuneable system parameters and the performance of the consensus. In addition, the performance of the consensus on complex networks will be a topic of future research.

Acknowledgements

This work was supported by the Fundamental Research Funds for the Central Universities under Grant 2010-IV-014.

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