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Integral closedness of MI and the formula of Hoskin and Deligne for finitely supported complete ideals

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Abstract

We obtain necessary and sufficient conditions for a finitely supported monomial ideal I in a polynomial ring of dimension at least three for MI to be integrally closed. This is obtained via the higher-dimensional analogue of the formula of Hoskin and Deligne for the length of a finitely supported ideal in a regular local ring.

Keywords: Complete ideals; Regular local ring; Quadratic transform

1. Introduction

It is well known that the theory of complete ideals in a two-dimensional regular local ring was founded by Zariski [33]. His work was inspired by the birational theory of linear systems on smooth surfaces. Due to the complexity in higher dimension, an higher-dimensional analogue was not easy to obtain, as observed by Zariski and evident from the work of Lipman, Cutkosky, Piltant, Lejeune, etc. For example see [1,3,20,21]. Their work also reveals that a trivial generalization of Zariski's work is not possible to obtain.

Zariski proved that in a two-dimensional regular local ring product of complete ideals is complete. Moreover, every complete ideal can be uniquely factorized as a product of

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simple complete ideals. Both these results do not hold true in higher dimension. A substantial amount of work on the unique factorization has been done by Cutkosky, Lipman and Piltant.

Of recent interest is the following question. If I is an ideal in a Noetherian local ring (R, M), when is MI integrally closed (see [2,12])? This is closely related to the Cohen–Macaulayness of the fiber cone of I, $F(I) := R[It] \otimes_R R/M = \bigoplus_{n \geqslant 0} I^n/MI^n$. Hence two topics of interest arise: the integral closedness of MI and the Cohen–Macaulayness of $F(I)$. By an example we show that if the dimension of the ring is at least three and I is a finitely supported complete ideal in a regular local ring, then the fiber cone of I need not be Cohen–Macaulay, in contrast to the case of dimension two where the fiber cone of an M-primary complete ideal is always Cohen–Macaulay [6, Corollary 2.5].

Let (R, M) be a regular local ring. In this paper we give necessary and sufficient conditions in terms of the number of generators of a finitely supported monomial complete ideal I for MI to be integrally closed. In a two-dimensional regular local ring, if I is an M-primary ideal, then $\mu(I) \leqslant 1 + o(I)$, where $o(I)$ is the M-adic order of I and $\mu(I)$ is the minimal number of generators of I. If I is complete, then equality holds [28, Lemma 3.1]. More generally, if I is an M-primary complete ideal in a regular local ring R of dimension $d \geqslant 2$, then $\mu(I) \geqslant \binom{o(I)+d-1}{d-1}$ [5]. If I is finitely supported, then, $\mu(I) \leqslant o(I)^{d-1} + d - 1$ (see Theorem 5.3). There exists examples of finitely supported complete ideals where the upper bound is attained (see Example 7.1).

In this paper we prove:

Theorem 1.1. *Let I be a finitely supported monomial complete ideal in a regular local ring (R, M) of dimension at least three. Assume that $k = R/M$ is algebraically closed field. Then MI is integrally closed if and only if*

$$\mu(I) = \binom{o(I) + d - 1}{d - 1}.$$

An essential ingredient in the proof is the formula of Hoskin and Deligne which expresses the length of a finitely supported complete ideal in terms of the order of the strict transform of I (see Theorem 1.4). Note that the condition "finitely supported" is necessary (see [21, Lemma 1.21.1]). This formula was known only for complete ideals of height two in a two-dimensional regular local ring and was proved independently by Hoskin [11] and Deligne [8]. Due to several hurdles the higher-dimensional analogue was not easy to obtain. A formula in dimension three was obtained by Lejeune-Jalabert [20]. One of the main obstacles is that the theory of complete ideals in a two-dimensional regular local ring which was founded by Zariski [33] does not directly generalize to higher dimension even though the definitions do. In this paper we obtain the higher-dimensional analogue of this length formula.

Let $\overline{I}$ denote the integral closure of I. A formula for $\ell(R/\overline{I^n})$ for all $n \geqslant 1$ was obtained by Morales in [23]. In fact, his formula holds true for a finitely generated normal k-algebra over an algebraically closed field. He also gave a geometrical interpretation for the coefficients of the Hilbert–Samuel polynomial of I.

For the sake of completeness we state the formula of Hoskin and Deligne:

Theorem 1.2. ([11, Theorem 5.2], [8, Theorem 2.13]) *Let (R, M) be a two-dimensional regular local ring with infinite residue field. Let I be an M-primary complete ideal in R. Let $\mathcal{C}_I = \{Q_0, Q_1, \ldots, Q_t\}$ be the base points of I and R_i the local ring at Q_i $(R_0 = R)$. Then*

$$\ell\left(\frac{R}{I}\right) = \sum_{i=0}^{t} \binom{o(I^{R_i}) + 1}{2} [R_i/M_i : R/M]$$

where M_i is the maximal ideal of R_i $(M_0 = M)$ and $[R_i/M_i : R/M]$ denotes the degree of the field extension $R_i/M_i \supseteq R/M$.

In higher dimension one can verify that:

Theorem 1.3. *Let (R, M) be a regular local ring of dimension d with infinite residue field. Let I be a finitely supported complete ideal. Let $\mathcal{C}_I = \{Q_0, Q_1, \ldots, Q_t\}$ be the base points of I and R_i the local ring at Q_i $(R_0 = R)$. Then*

$$\ell\left(\frac{R}{I}\right) \leqslant \sum_{i=0}^{t} \binom{o(I^{R_i}) + d - 1}{d} [R_i/M_i : R/M]$$

where M_i is the maximal ideal of R_i $(M_0 = M)$ and $[R_i/M_i : R/M]$ denotes the degree of the field extension $R_i/M_i \supseteq R/M$.

If the dimension of the ring is at least three then the inequality in Theorem 1.3 may be strict (see Examples 7.2, 7.3). The gap between the terms on the left and the right can be estimated in terms of the length of the right derived functors of the direct image sheaf.

In this paper we prove:

Theorem 1.4. *Let (R, M) be a regular local ring. Assume that $k = R/M$ is an algebraically closed field. Let I be a finitely supported complete ideal. Let $\mathcal{C}_I = \{Q_0, Q_1, \ldots, Q_t\}$ be the base points of I and let $\sigma : X(\mathcal{C}_I) \to \operatorname{Spec} R$ be the birational map obtained by a sequence of blowing up of points of $\mathcal{C}_I$. Then*

$$\ell\left(\frac{R}{I}\right) + \sum_{i=1}^{d-2} (-1)^{i+1} \ell\big(R^i \sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})\big) = \sum_{i=0}^{t} \binom{o(I^{R_i}) + d - 1}{d}$$

where R_i the local ring at the point Q_i $(R_0 = R)$ and M_i is the maximal ideal of R_i $(M_0 = M)$.

As a consequence of Theorem 1.4 we are able to recover the formula for the multiplicity and the mixed-multiplicities of finitely supported complete ideals [17,19,26].

It is evident, from Theorems 1.3 and 1.4 that $\sum_{i=1}^{d-2} (-1)^{i+1} \ell(R^i \sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})) \geqslant 0$. This inequality can be strict if I is not a monomial ideal (see Example 7.2). If I is a monomial ideal, then $\sum_{i=1}^{d-2}(-1)^{i+1}\ell(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)}))=0$. This is a consequence of the length formula obtained by Morales [24, Lemma 3]. We independently obtain a closed formula for finitely supported monomial ideals (see Theorem 4.3) which implies the same result.

We also list a few applications. Let (R,M) be a regular local ring of dimension $d\geqslant 2$. Then the graded ring associated to the filtration $\mathcal{F}=\{\overline{I^n}\}_{n\geqslant 0}$, $G(\mathcal{F}):=\bigoplus_{n\geqslant 0}\overline{I^n}/\overline{I^{n+1}}$ is of interest. The normalization of the Rees ring $R[It]$ denoted by $\overline{R[It]}$ is the graded ring $\bigoplus_{n\geqslant 0}\overline{I^n}$. Since the depth of $G(\mathcal{F})\geqslant 1$, $\operatorname{depth}(G(\mathcal{F}))=\operatorname{depth}(\overline{R[It]}\otimes_R R/I)$.

In a two-dimensional regular local ring, it is well known that if I is a complete ideal then $G(\mathcal{F})=G(I):=\bigoplus_{n\geqslant 0}I^n/I^{n+1}$ and $G(I)$ is Cohen–Macaulay [18,22]. Even in the three-dimensional case $G(I)$ need not be Cohen–Macaulay. The first example was given by Cutkosky in [4]. In this paper, under certain conditions we are able to deduce the Cohen–Macaulayness of $G(\mathcal{F})$ for a finitely supported complete ideal in a regular local ring of dimension three.

We end this paper with some examples which probably will put more light on the results of this paper.

2. Preliminaries

Let (R,M) be a regular local ring. For all the definitions in this section we refer to [1,21,33].

Let I be an ideal in R. The *integral closure* of I denoted by $\bar{I}$ is the set

$$\left\{x\in R \mid x^n+a_1x^{n-1}+\cdots+a_n=0;\ a_j\in I^j,\ 1\leqslant j\leqslant n\right\}.$$

The *completion* of I is the ideal $I'=\bigcap_{v\in S}IR_v$ where S denotes the set of all non-trivial valuations which are non-negative on R. Zariski proved that $I'=\bar{I}$. Hence the integral closure of I is an ideal of R. An ideal I is *integrally closed or complete* if $I=\bar{I}$.

Let X be a non-singular variety and let O be a point on X. Put $R=\mathcal{O}_{X,O}$ and let M be the maximal ideal of R. Let $f:X_1\to \operatorname{Spec} R$ denote the blowing up of M. The (first) *quadratic transforms* of R are the local rings $\mathcal{O}_{X_1,P}$, where $P\in f^{-1}\{M\}$ is a point on X_1.

A point Q on a variety Y is *infinitely near to* a point O on X, $Q\succcurlyeq O$ in symbol, if

(1) there exists a sequence of blowing ups

$$\sigma: Y=X_n\xrightarrow{f_n}X_{n-1}\xrightarrow{f_{n-1}}\cdots\longrightarrow X_1\xrightarrow{f_1}X_0=\operatorname{Spec} R$$

where each f_{i+1}, $0\leqslant i\leqslant n-1$, is obtained by blowing up M_i, the maximal ideal of $\mathcal{O}_{X_i,P_i}$, $P_i\in f_i^{-1}\{M_{i-1}\}\subseteq X_i$ $(M_0=M,\ P_0=O)$;

(2) $Q\in f_{n-1}^{-1}\{M_{n-1}\}$.

Put $R_i:=\mathcal{O}_{X_i,P_i}$ and $S:=R_n$. The sequence $R=R_0\subset R_1\subset\cdots\subset R_i\subset\cdots\subset R_n=S$ is called the *quadratic sequence* from R to S. We say that S is *infinitely near* to R and denote it by $S \succcurlyeq R$. Given any ideal $I \subset R$, the *transform* of I in $S = R_n = \mathcal{O}_{X_n,P_n}$ denoted by I^S is defined inductively as follows:

(1) $I^{R_0} = I$;
(2) $I^{R_i} = M_{i-1}^{-m_{i-1}} I^{R_{i-1}}$, where $m_{i-1} = o(I^{R_{i-1}}) = \max\{n \mid I_{i-1} \subseteq M_{i-1}^n\}$.

The *point basis* of a non-zero ideal I is a family of non-negative integers $\mathcal{B}(I) = \{o(I^S) \mid S \succcurlyeq R\}$. We say that a point $Q \in Y$ is a *base point* of I if $Q \succcurlyeq O$ and $o(I^S) < \infty$, where $S = \mathcal{O}_{Y,Q}$. We say that an ideal I is *finitely supported* if $I \neq 0$ and if I has at most finitely many base points [21]. For a finitely supported ideal I we will denote the set base points of I by $\mathcal{C}_I = \{Q_0 = O, Q_1, \dots, Q_t\}$.

3. Hoskin–Deligne formula for finitely supported complete ideals

Let k be an algebraically closed field and let X be a non-singular variety of dimension at least two. Let $O \in X$ be a point. Put $R = \mathcal{O}_{X,O}$. Let M be the maximal ideal at O.

The notion of $*$-product was introduced in [21]. Let I and J be ideals in R. The $*$-product of I and J denoted by $I * J$ is the ideal $\overline{IJ}$. An ideal I is $*$-simple of it cannot be decomposed as a $*$-product of proper complete ideals [21]. Notice that $I * J = \bar{I} * \bar{J}$.

To prove Theorem 1.4 we first need to consider the first blow up. The following lemma is basically a consequence of [21, Lemma 2.3]. We prove it since it is the crucial result in proving Theorem 1.4.

Lemma 3.1. *Let $f : X_1 \to \operatorname{Spec} R$ denote the blowing up of M. Let $\mathcal{I}$ be the coherent $\mathcal{O}_{X_1}$-ideal whose stalk at any point $P \in f^{-1}\{M\}$ is a complete ideal and $\mathcal{I}_P = \mathcal{O}_{X_1,P}$ if $P \notin f^{-1}\{M\}$. Then there exists a complete ideal $I \subseteq R$ such that $\mathcal{I}_P = \overline{I^{\mathcal{O}_{X_1,P}}}$, where $P \in f^{-1}\{M\}$. Moreover, there exists a positive integer N such that for all $n \geqslant N$*

(1) $R^j f_*(M^{o(I)+n}\mathcal{I}) = 0$ *for all* $j > 0$;
(2) $\ell_R\left(\dfrac{R}{M^n * I}\right) = \dbinom{o(I)+n+d-1}{d} + \displaystyle\sum_{P \in f^{-1}\{M\}} \ell_R\left(\dfrac{\mathcal{O}_{X_1,P}}{\mathcal{I}_P}\right)$.

Proof. The existence of a complete ideal $I \subseteq R$ satisfying the assumptions in the lemma was proved in [21, Lemma 2.3].

From [10, Proposition 8.5, p. 251]

$$R^j f_*\big(M^{o(I)+n}\mathcal{I}\big) = H^j\big(X_1, M^{o(I)+n}\mathcal{I}\big)$$

and for $j \geqslant 0$ and all n large, $H^j(X_1, M^{o(I)+n}\mathcal{I}) = 0$.

We now prove (2). The exact sequence

$$0 \to M^{o(I)+n}\mathcal{I}\mathcal{O}_{X_1} \to M^{o(I)+n}\mathcal{O}_{X_1} \to M^n\mathcal{O}_{X_1}/\mathcal{I}\mathcal{O}_{X_1} \to 0$$

gives the long exact sequence

$$0 \to H^0\big(X_1, M^{o(I)+n}\mathcal{I}\mathcal{O}_{X_1}\big) \to H^0\big(X_1, M^{o(I)+n}\mathcal{O}_{X_1}\big) \to H^0\big(X_1, M^n\mathcal{O}_{X_1}/\mathcal{I}\mathcal{O}_{X_1}\big)$$
$$\to H^1\big(X_1, M^{o(I)+n}\mathcal{I}\mathcal{O}_{X_1}\big) \to H^1\big(X_1, M^{o(I)+n}\mathcal{O}_{X_1}\big) \to \cdots. \tag{1}$$

Now $H^0(X_1, M^{o(I)+n}\mathcal{I}\mathcal{O}_{X_1}) = \overline{M^n I}$ and $H^0(X_1, M^{o(I)+n}\mathcal{O}_{X_1}) = \overline{M^{o(I)+n}}$ (see [21, proof of Lemma 2.3]). Since $H^0(X_1, M^n\mathcal{O}_{X_1}/\mathcal{I}\mathcal{O}_{X_1}) = \sum_{P\in f^{-1}\{M\}} \ell(\mathcal{O}_{X_1,P}/\mathcal{I}_P)$ and for all $n \gg 0$, $H^1(X_1, M^{o(I)+n}\mathcal{I}\mathcal{O}_{X_1}) = 0$ (see [10, Theorem 5.2, p. 228]), plugging these in the exact sequence (1) gives the result. $\square$

Proposition 3.2. [21, Proposition 2.1, Corollary 2.2] *Let $S \succcurlyeq R$. Then there exists a unique $*$-simple complete ideal $p_{RS} \subseteq R$ satisfying the following properties*:

(1) *R/p_{RS} is artinian*;
(2) *p_{RR} is the maximal ideal of R*;
(3) *For all regular local rings T with $S \succcurlyeq T \succcurlyeq R$, $p_{TS} = \overline{(p_{RS})^T}$.*

We now apply Lemma 3.1 recursively to a sequence of point blowing ups.

Lemma 3.3. *Let I be a finitely supported ideal and $\mathcal{C}_I = \{Q_0 = O, Q_1, \ldots, Q_t\}$ the base points of I. Let R_i denote the local ring at $Q_i \in \mathcal{C}_I$. Put $m_i = o(I^{R_i})$. There exists integers $N_0, \ldots, N_t$ such that for all $n_i \geqslant N_i$,*

(1) $R^j f_*\left(\prod_{i=0}^{*t}\big(p_{RR_i}^{n_i} * I\big)\mathcal{O}_{X_1(\mathcal{C}_I)}\right) = 0$ *for all* $j \geqslant 0$;

(2) $\ell\left(\dfrac{R}{\prod_{i=0}^{*t} p_{RR_i}^{n_i} * I}\right) = \sum_{i=0}^{t}\binom{m_i + \sum_{R_j \succcurlyeq R_i} n_j + d - 1}{d}$.

Here $\prod^*$ denotes $*$-product.

Proof. The first part follows from [10, Proposition 8.5, p. 251].

We prove (2) by induction on $t = |\mathcal{C}_I| - 1$. If $t = 0$, then $\overline{I} = M^n$ for some $n > 0$ and (2) follows trivially.

Let $t > 0$. Let $f : X_1 \to \operatorname{Spec} R$ denote the blowing up of $\operatorname{Spec} R$ at M. Let $\mathcal{I}$ be the $\mathcal{O}_{X_1}$-ideal sheaf whose stalk at every point $Q_i \in \mathcal{C}_I \cap X_1$ is

$$\mathcal{I}_{Q_i} = \overline{I^{R_i}} = \overline{M^{-o(I)} I \mathcal{O}_{X_1,Q_i}}$$

and $\mathcal{I}_Q = \mathcal{O}_{X_1,Q}$ for $Q \in X_1 \setminus (X_1 \cap \mathcal{C}_{\mathcal{I}})$.

Let $\mathcal{C}_{\mathcal{I}_{Q_i}} = \{Q_{i(0)} = Q_i, \ldots, Q_{i(s_i)}\}$. For all $Q_i \in X_1 \cap \mathcal{C}_I$, $\mathcal{C}_{\mathcal{I}_{Q_i}} \subset \mathcal{C}_{\mathcal{I}}$ and hence $|\mathcal{C}_{\mathcal{I}_{Q_i}}| < |\mathcal{C}_{\mathcal{I}}|$.

By induction hypothesis, for each $Q_i \in \mathcal{C}_I \cap X_1$, there exist integers $N_{i(0)}, \dots, N_{i(s_i)}$ such that for all $n_{i(j)} \geqslant N_{i(j)}$, $0 \leqslant j \leqslant s_i$,

$$\ell\left(\frac{\mathcal{O}_{X_1,Q_i}}{\prod_{j=0}^{*s_i} p_{R_iR_{i(j)}}^{n_{i(j)}} * \mathcal{I}_{Q_i}}\right) = \sum_{j=0}^{s_i} \binom{m_{i(j)} + \sum_{R_k \succcurlyeq R_{i(j)}} n_k + d - 1}{d}. \tag{2}$$

Fix $n_i \geqslant N_i$, $0 \leqslant i \leqslant t$. Let $\mathcal{J}(n_1, \dots, n_t; I)$ be the $\mathcal{O}_{X_1}$-ideal sheaf whose stalk at every point $Q_i \in \mathcal{C}_I \cap X_1$ is

$$\mathcal{J}(n_1, \dots, n_t; I)_{Q_i} := \left(\prod_{j=0}^{*s_i} p_{R_iR_{i(j)}}^{n_{i(j)}} * \mathcal{I}_{Q_i}\right)$$

and $\mathcal{J}(n_1, \dots, n_t; I)_Q = \mathcal{O}_{X_1,Q}$ for $Q \in X_1 \setminus (X_1 \cap \mathcal{C}_{\mathcal{I}})$. Let $n_0 \geqslant 0$ and for each n_0 let

$$J(n_0, n_1, \dots, n_t; I) = p_{RR}^{n_0} * \prod_{j=1}^{*t} p_{RR_j}^{n_j} * I.$$

Here $p_{RR} = M$, the maximal ideal of R. Then

$$\overline{M^{-o(J(n_0,n_1,\dots,n_t;I))} J(n_0, n_1, \dots, n_t; I)\mathcal{O}_{X_1,Q_i}} = \mathcal{J}(n_1, \dots, n_t; I)_{Q_i}$$

at all points $Q_i \in X_1 \cap \mathcal{C}_I$.

Now by Lemma 3.1 there exists an integer N_0 such that for each $n_0 \geqslant N_0$,

$$\begin{aligned}
&\ell_R\left(\frac{R}{J(n_0, n_1, \dots, n_t; I)}\right) \\
&\quad = \ell_R\left(\frac{R}{p_{RR}^{o(J(n_0,n_1,\dots,n_t;I))}}\right) + \sum_{Q_i \in \mathcal{C}_I \cap X_1} \ell_{\mathcal{O}_{X_1,Q_i}}\left(\frac{\mathcal{O}_{X_1,Q_i}}{\mathcal{J}(n_1, \dots, n_t; I)_{Q_i}}\right) \\
&\quad = \binom{o(I) + n_0 + \dots + n_t + d - 1}{d} + \sum_{Q_i \in \mathcal{C}_I \cap X_1} \ell_{\mathcal{O}_{X_1,Q_i}}\left(\frac{\mathcal{O}_{X_1,Q_i}}{\mathcal{J}(n_1, \dots, n_t; I)_{Q_i}}\right).
\end{aligned} \tag{3}$$

Substituting (2) in (3) proves the lemma. □

Let $\mathcal{C}_I = \{Q_0 = O, Q_1, \dots, Q_t\}$ denote the base points of a finitely supported ideal I. Let $X(\mathcal{C}_I)$ denote the variety obtained by blowing up X_t at Q_t. Let E_i be the exceptional divisor obtained by blowing up Q_i and let E_i^* denote the exceptional divisor in X_h, $i \leqslant h \leqslant n+1$. Let $\mathcal{A}_I = \{\mathcal{C}_I, B(I)\}$ where $\mathcal{B}(I) = \{m_0, \dots, m_t\}$ is the point basis of I. Then $\mathcal{D}(\mathcal{A}_I) = \sum_{i=0}^t m_i E_i^*$ is the divisor associated to the ideal sheaf $I\mathcal{O}_{X(\mathcal{C}_I)}$. Let R_i be the regular local ring at Q_i. Then the exceptional divisor corresponding to $(\prod_{i=0}^t p_{RR_i}^{n_i} * I)\mathcal{O}_{X(\mathcal{C}_I)}$ is $\sum_{i=0}^t h_i E_i^*$ where $h_i = m_i + \sum_{R_j \succcurlyeq R_i} n_j$.

Proof of Theorem 1.4. Denote $\mathcal{L}_i = \mathcal{O}_{X(\mathcal{C}_I)}(-E_i^*)$ and let $h_i = m_i + \sum_{R_j \succcurlyeq R_i} n_j$. We have

$$\begin{aligned}
&\chi\big(\mathcal{L}_0^{\otimes h_0} \otimes \cdots \otimes \mathcal{L}_t^{\otimes h_t}\big) \\
&\quad = \ell\left(\frac{R}{\prod_{i=0}^t p_{RR_i}^{n_i} * I}\right) + \sum_{i=1}^{d-1} (-1)^{i+1} \ell\left(R^i \sigma_*\left(\prod_{i=0}^t \big(p_{RR_i}^{n_i} * I\big) \mathcal{O}_{X(\mathcal{C}_I)}\right)\right) \\
&\qquad \big(\text{by [22, Theorem 1.4]}\big) \\
&\quad = \sum_{i=0}^t \binom{m_i + \sum_{R_j \succcurlyeq R_i} n_j + d - 1}{d} + \sum_{i=1}^{d-1} (-1)^{i+1} \ell\left(R^i \sigma_*\left(\prod_{i=0}^t \big(p_{RR_i}^{n_i} * I\big)\right) \mathcal{O}_{X(\mathcal{C}_I)}\right) \\
&\qquad (\text{by Lemma 3.3}) \qquad (4)
\end{aligned}$$

for all $n_0, \ldots, n_t \gg 0$.

On the other hand, for all non-negative integers $r_0, \ldots, r_t$ there exist rational numbers $a_{i_0,\ldots,i_t}$ such that [29, Theorem 9.1]

$$\chi\big(\mathcal{L}_0^{\otimes r_0} \otimes \cdots \otimes \mathcal{L}_t^{\otimes r_t}\big) = \sum_{i_0+\cdots+i_t \leqslant d} a_{i_0,\ldots,i_t} \binom{r_0 + i_0}{i_0} \cdots \binom{r_t + i_t}{i_t}. \qquad (5)$$

If we put $r_i = h_i$ in (5), then for $n_0, \ldots, n_t \gg 0$ large the polynomials in (4) and (5) agree. This gives

$$a_{i_0,\ldots,i_t} = \begin{cases} 1, & \text{if } (i_0, \ldots, i_t) = (0, \ldots, d, \ldots, 0), \\ -1, & \text{if } (i_0, \ldots, i_t) = (0, \ldots, d-1, \ldots, 0), \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\chi\big(\mathcal{L}_0^{\otimes h_0} \otimes \cdots \otimes \mathcal{L}_t^{\otimes h_t}\big) = \sum_{i=0}^t \left[\binom{h_i + d}{d} - \binom{h_i + d - 1}{d - 1}\right]$$

for all values of $h_0, \ldots, h_t \geqslant 0$. Hence (5) is true for all values of $n_0, n_1, \ldots, n_t$. If we put $n_0 = n_1 = \cdots = n_t = 0$ in (4) and (5) we get

$$\ell\left(\frac{R}{I}\right) = \sum_{i=0}^t \binom{m_i + d - 1}{d} - \sum_{i=1}^{d-1} (-1)^{i+1} R^i \sigma_*\big(I\mathcal{O}_{X(\mathcal{C}_I)}\big).$$

It remains to show that $R^{d-1}\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)}) = 0$. Consider the exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_{X(\mathcal{C}_I)} \to I\mathcal{O}_{X(\mathcal{C}_I)} \to 0$$

where $\mathcal{F}$ is a coherent $\mathcal{O}_{X(\mathcal{C}_I)}$-module. This gives the exact sequence

$$R^{d-1}\sigma_*(\mathcal{O}_{X(\mathcal{C}_I)}) \to R^{d-1}\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)}) \to R^d\sigma_*\mathcal{F} \to R^d\sigma_*(\mathcal{O}_{X(\mathcal{C}_I)}).$$

Since σ is the composition of sequence of blowing ups and X is non-singular, $R^i\sigma_*(\mathcal{O}_{X(\mathcal{C}_I)}) = 0$ for all $i \geqslant 1$. And $R^d\sigma_*\mathcal{F} = 0$ since $\sigma^{-1}\{M\}$ has dimension $d-1$. Hence, $R^{d-1}\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)}) = 0$. $\square$

4. The length formula for finitely supported monomial ideals

The following result was proved by Morales:

Theorem 4.1. [24, Lemma 6] *Let $I_1, \dots, I_t$ be monomial ideals of height d in $R = k[x_1, \dots, x_d]$. Then for all non-negative integers $r_1, \dots, r_t$, $\ell(R/\overline{I_1^{r_1} \dots I_t^{r_t}})$ is a polynomial of degree d in $r_1, \dots, r_t$.*

Remark 4.2. Let $d \geqslant 2$ and let $R = k[x_1, \dots, x_d]$ be a polynomial ring in d variables over a field k and let $I \subseteq R$ be a complete ideal of height d.

(1) Since $M = (x_1, \dots, x_d)$ is the only maximal ideal which contains I, $\ell(R/I) = \ell(R_M/I_M)$.
(2) The first quadratic transform of R_M are the local rings $S_i = R[M/x_i]_{M/x_i}$. Each $T_i = R[M/x_i]$ is a polynomial ring in d variables over the field k.
(3) Let I be a monomial ideal in R and $o(I) = \max\{n \mid I \subseteq M^n\}$. Then $I^{T_i} := x_i^{-o(I)} I$ is a monomial ideal and $I^{S_i} = I^{T_i}_{M_i}$, where M_i is the maximal ideal of S_i.
(4) If I is a finitely supported ideal, then $\ell(T_i/I^{T_i})$ is finite and $\ell(T_i/I^{T_i}) = \ell(S_i/I^{S_i})$.

Hence it is possible to use the theory of length of ideals in local rings to obtain our main result.

Theorem 4.3. *Let I be a finitely supported monomial ideal in a polynomial ring $R = k[x_1, \dots, x_d]$. Let $M = (x_1, \dots, x_d)$. Then*

$$\ell(R/\bar{I}) = \sum_{S \succcurlyeq R} \binom{o(I^S) + d - 1}{d} [S/M_S : R/M]$$

where $[S/M_S : R/M]$ denotes the degree of the field extension $S/M_S \supseteq R/M$. Here M_S is the maximal ideal of S and $M_R = M$.

Proof. We prove the theorem by induction on $\ell_R(R/\bar{I})$. If $\ell(R/\bar{I}) = 1$, then $\bar{I} = M$ and the result is trivially true.

Let $\ell(R/\bar{I}) > 1$. Let S_i denote the first quadratic transform of R and M_i the maximal ideal of S_i. By Theorem 4.1, $\ell(S_i/\overline{(I^{S_i})^s})$ is a polynomial in s for all $s \geqslant 0$. Fix $s > 0$. Then by Lemma 3.1, there exists an integer r_s so that for all $r \geqslant r_s$,

$$\begin{aligned}\ell\big(R/\overline{M^r I^s}\big) &= \binom{r + so(I) + d - 1}{d} + \sum_{S_i} \ell_R\big(S_i/\overline{\big(I^{S_i}\big)^s}\big)\\ &= \binom{r + so(I) + d - 1}{d} + \sum_{S_i} \ell_{S_i}\big(S_i/\overline{\big(I^{S_i}\big)^s}\big)[S_i/M_i : R/M].\end{aligned} \tag{6}$$

But $\ell(R/\overline{M^r I^s})$, is a polynomial in r, s for all $r, s \geqslant 0$. Put $r = 0$ and $s = 1$ in (6). Then we have

$$\ell(R/\bar{I}) = \binom{o(I) + d - 1}{d} + \sum_{S_i} \ell_{S_i}\big(S_i/\overline{I^{S_i}}\big)[S_i/M_i : R/M].$$

Since $\ell_{S_i}(S_i/\overline{I^{S_i}}) < \ell_R(R/\bar{I})$, the result follows by induction hypothesis. □

5. Mixed-multiplicities and the integral closedness of MI

Let (R, M) be a normal local ring of dimension d with infinite residue field. It is well known that if $I_1, \ldots, I_g$ are M-primary ideals in R, then for all $n_1, \ldots, n_g \gg 0$,

$$\ell\big(R/\overline{I_1^{n_1} \ldots I_g^{n_g}}\big) = \sum_{i_1+\cdots+i_g \leqslant d} e_{i_1,\ldots,i_g}(I_1, \ldots, I_g) \binom{n_1 + i_1}{i_1} \ldots \binom{n_1 + i_g}{i_g}$$

where $e_{i_1,\ldots,i_g}(I_1, \ldots, I_g)$ are integers. For $i_1 + \cdots + i_g = d$, $e_{i_1,\ldots,i_g}$ are the mixed-multiplicities of the ideals $I_1, \ldots, I_g$ (see [28,30]). We let $\underline{i}$ denote the multi-index $\{i_1, \ldots, i_g\}$, such that $\sum i_j = d$.

For the rest of this section we will assume that k is an algebraically closed field. There is a more precise formula for the mixed multiplicities of finitely supported monomial ideals.

Theorem 5.1. ([17, Corollary 3.14], [26, Proposition 2.2]) *Let $I_1, \ldots, I_g$ be finitely supported ideals in a regular local ring (R, M) of dimension d. Then*

$$e_{\underline{i}}(I_1, \ldots, I_g) = \sum_{S \succcurlyeq R} \big(o\big(I_1^S\big)\big)^{i_1} \ldots \big(o\big(I_g^S\big)\big)^{i_g}.$$

Proof. Imitating the proof of Lemma 3.1 for several ideals, we get that for fixed $n_1, \ldots, n_g$ there exists an $N_0 \geqslant 0$ depending on $n_1, \ldots, n_g$ such that for all $n_0 \geqslant N_0$

$$\begin{aligned}&\ell\big(R/\overline{M^{n_0}I_1^{n_1}\dots I_g^{n_g}}\big)\\&\quad=\binom{n_0+n_1o(I_1)+\cdots+n_go(I_g)+d-1}{d}+\sum_{S_i}\ell_{S_i}\Big(S/\overline{\big(I_1^{S_i}\big)^{n_1}\dots\big(I_g^{S_i}\big)^{n_g}}\Big)\end{aligned}$$

where S_i are the first quadratic transform of R. Choose $n_0,n_1,\dots,n_g\gg 0$ so that both $\ell(R/\overline{M^{n_0}I_1^{n_1}\dots I_g^{n_g}})$ and $\ell_{S_i}(S/\overline{(I_1^{S_i})^{n_1}\dots(I_g^{S_i})^{n_g}})$ are polynomials. Now use the fact that $e_{0,i_1,\dots,i_g}(M,I_1,\dots,I_g)=e_{i_1,\dots,i_g}(I_1,\dots,I_g)$. $\square$

It is well known that $e_{0,\dots,d,\dots,0}(I_1,\dots,I_g)=e(I_i)$ where $(0,\dots,d,\dots,0)$ denotes the tuple where d is at the ith spot [27]. When we deal with two ideals, we will use the notation $e_i(I\mid J):=e_{d-i,i}(IJ)$. Note that $e_i(I\mid J)=e_i(\bar{I}\mid\bar{J})$.

In a regular local ring of dimension at least two, for any M-primary complete ideal we have $e_1(M\mid I)=o(I)$ ([31, Theorem 4.1], [32, Lemma 1.1]). We have an analogue of this result for finitely supported complete ideals in regular local rings of dimension at least two.

As an immediate consequence we have:

Corollary 5.2. *Let I be finitely supported ideal in a regular local ring (R,M) of dimension $d\geqslant 2$. Then*

(1) $e(I)=\sum_{S\succcurlyeq R}o(I^S)$,
(2) $e_i(M\mid I)=o(I)^i$ *for* $1\leqslant i\leqslant d-1$.

Proof. Both (1) and (2) follow directly from Theorem 5.1. $\square$

It is well known that for every M-primary complete ideal in a two-dimensional regular local ring, $\mu(I)=1+o(I)$. We have a generalization of this result.

Theorem 5.3. *Let I be a finitely supported ideal in a regular local ring (R,M) of dimension $d\geqslant 2$. Assume that R/M is an algebraically closed field. Then*

(1) $\ell(\bar{I}/\overline{MI})\geqslant\binom{o(I)+d-1}{d-1}$ *and equality holds if and only if*

$$\sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})\big)=0.$$

(2) *Assume that I integrally closed.*
  (a) *MI is integrally closed of and only if*

$$\begin{aligned}\mu(I)=\binom{o(I)+d-1}{d-1}+\sum_{i=1}^{d-2}(-1)^{i+1}\big[&\ell\big(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})\big)\\&-\ell\big(R^i\sigma_*(MI\mathcal{O}_{X(\mathcal{C}_I)})\big)\big].\end{aligned}$$

(b) $\mu(I) \geqslant \binom{o(I)+d-1}{d-1}$ *and equality holds if and only if MI is integrally closed and* $\sum_{i=1}^{d-2}(-1)^{i+1}\ell(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})) = 0$.

(3) *For $i = 1, \ldots, d-1$,*

$$\binom{o(I)+d-1}{d-1} \leqslant \mu(I) \leqslant o(I)^{d-i} + d - i + (i-1)\,\ell(R/I).$$

In particular when $i = 1$, $\mu(I) \leqslant o(I)^{d-1} + d - 1$.

Proof. We can choose an element $x \in M \setminus M^2$ such that $\overline{MI} : xR = I$ [5, Lemma 3.1]. The exact sequence

$$0 \to \frac{\bar{I}}{\overline{MI}} \to \frac{M^{o(I)}}{\overline{MI}} \xrightarrow{.x} \frac{M^{o(I)}}{\overline{MI}} \to \frac{M^{o(I)}}{\overline{MI} + xM^{o(I)}} \to 0$$

gives

$$\ell\left(\frac{\bar{I}}{\overline{MI}}\right) = \ell\left(\frac{M^{o(I)}}{\overline{MI} + xM^{o(I)}}\right) \geqslant \ell\left(\frac{M^{o(I)}}{M^{o(I)+1}}\right) = \binom{o(I)+d-1}{d-1}.$$

This proves (1).

From Theorem 1.4 it follows that

$$\begin{aligned}
\ell(R/\overline{MI}) &= \ell(R/\bar{I}) - \binom{o(I)+d-1}{d} + \binom{o(I)+d}{d-1} \\
&\quad + \sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})\big) - \sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(MI\mathcal{O}_{X(\mathcal{C}_I)})\big) \\
&= \ell(R/\bar{I}) + \binom{o(I)+d-1}{d-1} + \sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})\big) \\
&\quad - \sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(MI\mathcal{O}_{X(\mathcal{C}_I)})\big).
\end{aligned}$$

Hence

$$\begin{aligned}
\ell\left(\frac{I}{\overline{MI}}\right) = \binom{o(I)+d-1}{d-1} &+ \sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})\big) \\
&- \sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(MI\mathcal{O}_{X(\mathcal{C}_I)})\big).
\end{aligned}$$

Applying (1) we get

$$\sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})\big) \geqslant \sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(MI\mathcal{O}_{X(\mathcal{C}_I)})\big).$$

Recursively we can prove:

$$\begin{aligned}\sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})\big) &\geqslant \sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*(MI\mathcal{O}_{X(\mathcal{C}_I)})\big)\\ &\geqslant\\ &\vdots\\ &\geqslant\\ &\vdots\\ &\geqslant \sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*\big(M^nI\mathcal{O}_{X(\mathcal{C}_I)}\big)\big).\end{aligned}$$

But

$$\sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*\big(M^nI\mathcal{O}_{X(\mathcal{C}_I)}\big)\big)=0$$

for $n \gg 0$. Now, equality holds if and only if

$$\sum_{i=1}^{d-2}(-1)^{i+1}\ell\big(R^i\sigma_*\big(M^nI\mathcal{O}_{X(\mathcal{C}_I)}\big)\big)=0$$

for all $n \geqslant 0$.

If MI is integrally closed if and only if $MI=\overline{MI}$. Now apply (1). This proves (2)(a). (2)(b) follows from that fact that $\mu(I) \geqslant \ell(I/\overline{MI})$.

From [7, Theorem 2.2] it follows that, for all $i = 1, \dots, d-1$,

$$\begin{aligned}\mu(I) &\leqslant d-i+(i-1)\ell\left(\frac{R}{I}\right)+e_{d-i}(M \mid I)\\ &= d-i+(i-1)\ell\left(\frac{R}{I}\right)+o(I)^i \quad \text{(by Corollary 5.2).}\end{aligned}$$

This proves (3). □

As an immediate consequence we have:

Theorem 5.4. *If in addition to the conditions in Theorem* 5.3*, I is a monomial ideal and M is the maximal homogeneous ideal in $k[x_1, \ldots, x_d]$, then*

(1) $\ell(\bar{I}/\overline{MI}) = \binom{o(I)+d-1}{d-1}$.
(2) *Let $d \geqslant 3$. If I is integrally closed, then MI is integrally closed of and only if*

$$\mu(I) = \binom{o(I)+d-1}{d-1}.$$

Proof. If I is a monomial ideal then MI is also a monomial ideal. Now, for any monomial ideal I, $\sum_{i=1}^{d-2}(-1)^{i+1}\ell(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})) = 0$. □

Remark 5.5. If I is a complete M-primary ideal in a ring of dimension at least three and if I is not finitely supported, then Corollary 5.2 and Theorem 5.3(3) may not hold true.

6. The associated graded ring and the Rees ring

Let (R, M) be a Noetherian local ring of positive dimension d. Let I be an M-primary ideal in R. Here $R(I)$ and $G(I)$ will denote the ordinary Rees ring and the associated graded ring, respectively. The filtration $\mathcal{F} = \{\overline{I^n}\}_{n\geqslant 0}$ is a Hilbert filtration. The Rees ring of $\mathcal{F}$, $R(\mathcal{F}) := \bigoplus \overline{I^n}$ (respectively the associated graded ring of $G(\mathcal{F}) := \bigoplus \overline{I^n}/\overline{I^{n+1}}$), is a graded ring which is Noetherian and $R(\mathcal{F})$ (respectively $G(\mathcal{F})$) is a finite $R(I)$ (respectively $G(I)$) module.

An ideal $J \subseteq \bar{I}$ is a reduction of $\mathcal{F}$ if $J\overline{I^n} = \overline{I^{n+1}}$ for all $n \gg 0$ [25]. A minimal reduction of $\mathcal{F}$ is a reduction of $\mathcal{F}$ which is minimal with respect to containment.

Since $R(\mathcal{F})$ is a finite $R(I)$-module, any minimal reduction of I is also a minimal reduction of $\mathcal{F}$. By [25], minimal reductions always exist and if the residue field R/M is infinite, then any minimal reduction of I is generated by d elements. For any minimal reduction J of $\mathcal{F}$ we set $r_J(\mathcal{F}) = \sup\{n \in \mathbb{Z} \mid J\overline{I^{n-1}} \neq \overline{I^n}\}$.

The reduction number of $\mathcal{F}$, denoted by $r(\mathcal{F})$ is defined to be the least $r_J(\mathcal{F})$ over all possible minimal reductions of J of $\mathcal{F}$. For any M-primary ideal I in a local ring (R, M), let $J = (x_1, \ldots, x_d)$ be a minimal reduction of I and let $C.(n) := C.(J, \mathcal{F}, n)$ denote the complex

$$0 \to \frac{R}{\overline{I^{n-d}}} \to \cdots \to \frac{R}{\overline{I^n}} \to 0$$

where the maps are those of the Koszul complex of R with respect to $x_1, \ldots, x_d$. For details see [14]. Let $H_i(C.(n))$ denote the ith-homology. Let

$$h_i(J, \mathcal{F}) = \sum_{n\geqslant 1} \ell\big(H_i\big(C.(n)\big)\big) = \sum_{n\geqslant i+1} \ell\big(H_i\big(C.(n)\big)\big)$$

since $H_i(C.(n)) = 0$ for $n \leqslant i$.

If I is an M-primary ideal in a regular local ring of dimension two, then $r(I) \leqslant 1$ [15] and hence $G(I)$ is Cohen–Macaulay [18] and hence the Rees ring is Cohen–Macaulay by [9].

In higher dimension, if I is not a finitely supported complete ideal then it is easy to see that both the Rees ring $R(\mathcal{F})$ and the associated graded ring $G(\mathcal{F})$ need not be Cohen–Macaulay.

Lemma 6.1. *Let I be a finitely supported M-primary ideal in a regular local ring (R, M) of dimension at least three. Assume that for all $n \geqslant 1$*

$$\ell\left(\frac{R}{\overline{I^n}}\right) = \sum_{S \succcurlyeq R} \binom{no(I^S)+2}{3} [S/M_S : R/M]$$

where M_S is the maximal ideal of S, $M_R = M$. Then

(1) $\operatorname{depth} G(\mathcal{F}) \geqslant 2$ *if and only if* $r(\mathcal{F}) \leqslant 2$.
(2) *$G(\mathcal{F})$ is Cohen–Macaulay if and only if*

$$\ell\left(\frac{J+\overline{I^2}}{J}\right) = \sum_{S \succcurlyeq R} \binom{o(I_S)}{3} [S/M_S : R/M]$$

where J is a minimal reduction of I.
(3) *If $o(I) \leqslant 2$, then $G(\mathcal{F})$ is Cohen–Macaulay.*
(4) *If $o(I) \geqslant 3$, then $r_J(\mathcal{F}) \geqslant 2$ for any minimal reduction J of I.*

Proof. First note that for all $n \geqslant 3$,

$$\ell\left(\frac{R}{\overline{I^n}}\right) - 3\ell\left(\frac{R}{\overline{I^{n-1}}}\right) + 3\ell\left(\frac{R}{\overline{I^{n-2}}}\right) - \ell\left(\frac{R}{\overline{I^{n-3}}}\right) = e(I).$$

Since $\sum_{i \geqslant 2} h_i(J, \mathcal{F}) \geqslant 0$ [14, Theorem 3.7],

$$\begin{aligned}
&\sum_{i \geqslant 2} h_i(J, \mathcal{F}) \\
&\quad = h_2(J, \mathcal{F}) - h_3(J, \mathcal{F}) \\
&\quad = \sum_{n \geqslant 3} \ell\big(H_2\big(C.(n)\big)\big) - \ell\big(H_3\big(C.(n)\big)\big) \\
&\quad = \sum_{n \geqslant 3} \left[\ell\big(H_1\big(C.(n)\big)\big) - \ell\big(H_0\big(C.(n)\big)\big) + \ell\left(\frac{R}{\overline{I^n}}\right) - 3\ell\left(\frac{R}{\overline{I^{n-1}}}\right) + 3\ell\left(\frac{R}{\overline{I^{n-2}}}\right) \right. \\
&\quad\quad \left. - \ell\left(\frac{R}{\overline{I^{n-3}}}\right) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geqslant 3}\left[\ell\left(\frac{J\cap\overline{I^n}}{J\overline{I^{n-1}}}\right) - \ell\left(\frac{R}{J+\overline{I^n}}\right) + e(J)\right] \\
&= \sum_{n\geqslant 3}\ell\left(\frac{\overline{I^n}}{J\overline{I^{n-1}}}\right) \\
&\geqslant 0.
\end{aligned}
$$

Equality holds if and only if $r_J(\mathcal{F}) \leqslant 2$. And when equality holds then depth $G(\mathcal{F}) \geqslant 3-2+1=2$ [14, Theorem 3.7].

Similarly,

$$
\begin{aligned}
\sum_{i\geqslant 1} h_i(J,\mathcal{F}) &= h_1(J,\mathcal{F}) - h_2(J,\mathcal{F}) + h_3(J,\mathcal{F}) \\
&= \sum_{n\geqslant 2}\big[\ell\big(H_1\big(C.(n)\big)\big) - \ell\big(H_2\big(C.(n)\big)\big) + \ell\big(H_3\big(C.(n)\big)\big)\big] \\
&= \sum_{n\geqslant 2}\left[\ell\big(H_0\big(C.(n)\big)\big) - \ell\left(\frac{R}{\overline{I^n}}\right) + 3\ell\left(\frac{R}{\overline{I^{n-1}}}\right) - 3\ell\left(\frac{R}{\overline{I^{n-2}}}\right) + \ell\left(\frac{R}{\overline{I^{n-3}}}\right)\right] \\
&= \sum_{n\geqslant 2}\ell\left(\frac{R}{J+\overline{I^n}}\right) - \sum_{n\geqslant 3} e(J) - \ell\left(\frac{R}{\overline{I^2}}\right) + 3\ell\left(\frac{R}{\overline{I}}\right) \\
&= -\sum_{n\geqslant 2}\ell\left(\frac{J+\overline{I^n}}{J}\right) + \sum_{S\succcurlyeq R}\binom{o(I^S)}{3}[S/M_S : R/M] \\
&\geqslant 0.
\end{aligned}
$$

If the above inequality is an equality, then depth $G(\mathcal{F}) \geqslant 3$. Conversely, if depth $G(\mathcal{F}) \geqslant 3$ then there exists a minimal reduction J such that above inequality is an equality. But if depth $G(\mathcal{F}) \geqslant 3$, then $J\overline{I^n} = \overline{I^{n+1}}$ for all $n \geqslant 2$, i.e., $J + \overline{I^n} = J$ for all $n \geqslant 3$. Hence

$$
\sum_{S\succcurlyeq R}\binom{o(I^S)}{3}[S/M_S : R/M] = \ell\left(\frac{J+\overline{I^2}}{J}\right).
$$

Suppose $o(I) \leqslant 2$, then $o(I^S) \leqslant 2$ for all $S \succcurlyeq R$. Hence

$$
\sum_{i\geqslant 1} h_i(J,\mathcal{F}) = -\ell\left(\frac{J+\overline{I^2}}{J}\right) \geqslant 0.
$$

Since the length of the module appearing above is positive, it is equal to zero. Hence $\sum_{i\geqslant 1} h_i(J,\mathcal{F}) = 0$ which implies that depth $G(\mathcal{F}) \geqslant 3$, i.e., $G(\mathcal{F})$ is Cohen–Macaulay. This proves (3).

If $r_J(\mathcal{F}) = 1$ for some minimal reduction of J of I, then $G(\mathcal{F})$ is Cohen–Macaulay. Hence we have

$$0 = \ell\left(\frac{J + \overline{I^2}}{J}\right) = \sum_{S \succcurlyeq R} \binom{o(I_S)}{3} [S/M_S : R/M].$$

This implies that $o(I) \leqslant 2$. □

7. Examples

We end this paper with a few examples which will clarify our results and the assumptions we use.

If $\mu(I) = d - 1 + o(I)^{d-1}$, then $F(I)$ is Cohen–Macaulay if and only if there exists an ideal $J \subseteq I$ generated by d elements such that $JI = I^2$ [6]. Here we demonstrate an example of a monomial ideal I whose fiber cone is not Cohen–Macaulay. This example also shows that MI is not integrally closed. It is easy to see that $\mu(I) = 11 > 10$.

Example 7.1. Let $I = (x^4, x^3y, x^2z, x^2y^2, xy^2z, xyz^2, xz^3, y^3, y^2z^2, yz^3, z^5)$ be an ideal in the polynomial ring $k[x, y, z]$. Assume that k is an algebraically closed field. Then

(1) I is a finitely supported ideal.
(2) $J = (x^4 + yz^3, x^2z, y^3 + z^5)$ is a minimal reduction of I and $r_J(I) = 2$.
(3) Since $\mu(I) = 11 = o(I)^2 + 2$. Since $r(\mathcal{F}) = r(I) = 2$, $F(I)$ is not Cohen–Macaulay [6, Corollary 2.5]. The Hilbert series of $F(I)$ is

$$H\bigl(F(I), t\bigr) = \frac{1 + 8t}{(1 - t)^3}.$$

Notice that $MI \neq \overline{MI}$ since $xy^2z \in \overline{MI} \setminus MI$.
(4) $I^n = \overline{I^n}$ for all $n \geqslant 1$ and $J \cap I^n = I^{n+1}$ for all $n \geqslant 2$.
(5) $\mathcal{B}(I) = \{o(I^S) \mid S \succcurlyeq R\} = \{3, 2, 1, 1, 1, 1, 1\}$. The Hilbert function $H_I(n) = \ell(R/I^n)$ is equal to the Hilbert polynomial $P_I(n)$ for all $n \geqslant 0$. In particular,

$$\begin{aligned}\ell\left(\frac{R}{I^n}\right) &= \ell\left(\frac{R}{\overline{I^n}}\right) \\ &= \binom{3n+2}{3} + 5\binom{n+2}{3} + \binom{2n+2}{3} \\ &= 40\binom{n+2}{3} - 22\binom{n+1}{2} + \binom{n}{1}.\end{aligned}$$

(6) By [16, Theorem 17], $G(\mathcal{F}) = G(I)$ is Cohen–Macaulay.
(7) Since $\ell(J + I^2/J) = \sum_{S \succcurlyeq R} \binom{o(I^S)}{3} = 1$, by Theorem 6.1, $G(\mathcal{F}) = G(I)$ is Cohen–Macaulay.

We now show that Theorem 1.4 does not hold true if I is not a monomial ideal.

Example 7.2. (The author is very grateful to Oliver Piltant for bringing this example into light.) Consider the following example. Let k be an algebraically closed field. Let $R = k[x,y,z]$ and let $I = (z^3, y^3 - x^2z, y^2z^2, xyz^2, x^2z^2, xy^2z, x^2yz, x^3z, x^2y^2, x^3y, x^4)$ be an ideal of R. Then it is easy to verify that

(1) I is complete and $\ell(R/I) = 18$.
(2) The strict transform of I in $S = R[y/x, z/x]$ is $I^S = (x, (y/x)^3 - (z/x), (z/x)^3)$ and $\ell(S/I^S) = 9$. But $\binom{o(I)+2}{3} = 10$. Hence $\ell(R^1\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})) = 1$.
(3) Note that if $J = (z^3, y^3 - x^2z, x^4)$, then J is generated by a system of parameters in I and $JI = I^2$. Hence $F(I)$ is Cohen–Macaulay [6, Corollary 2.5]. Also $G(I)$ is Cohen–Macaulay.

We now present an infinite class of ideals where $\ell(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})) > 0$. This is a generalization of the example in [13].

Example 7.3. Let $R = \mathbb{C}[x,y,z]$ where x, y, z are variables and let

$$I = \left(x^{r+1}, (x,y,z)\left(y^r + z^r\right), (x,y,z)^{r+2}\right).$$

Then I is a finitely supported complete ideal. Put $S = R[x/y, z/y]$. Then

$$\begin{aligned}
\ell\left(\frac{R}{I}\right) &= \binom{r+3}{3} + \binom{r+3}{2} - 4,\\
\ell\left(\frac{S}{I^S}\right) + \binom{o(I)+2}{3} &= 2\binom{r+1}{2} + \binom{r+3}{3},\\
\ell\left(R^i\sigma_*(I\mathcal{O}_{X(\mathcal{C}_I)})\right) &= \ell\left(\frac{R}{I}\right) - \ell\left(\frac{S}{I^S}\right)\\
&= \binom{r-1}{2}.
\end{aligned}$$

Using the argument on the lines in [13] one can show that $G(I)$ is not Cohen–Macaulay for all $r \geqslant 3$. In particular, $x^3(y^r + z^r)^3 \in J \cap I^3 \setminus JI^2$ where $J = (x^{r+1}, z(y^r + z^r), y(y^r + z^r) + yz^{r+1})$ is a minimal reduction of I.

We end this paper with the following example.

Example 7.4. Let k be an algebraically closed field. Let $R = k[x,y,z]$. Let $M = (x,y,z)$, $I_1 = (x, y^2, yz, z^2)$ and $I_2 = (x^2, y, z)$ and put $I = I_1I_2$. Then I is not a finitely supported ideal. We demonstrate the fact that Corollary 5.2(2) and Theorem 5.3(2), (3) does not hold true if the ideal is not finitely supported.

By [30] and [28]

$$e_2(M \mid I_1) = e\big(x, y, z^2\big) = 2,$$
$$e_2(M \mid I_2) = e(x, y, z) = 1,$$
$$e(M \mid I_1 \mid I_2) = e_{1,1,1}(M, I_1, I_2) = e(x, y, z) = 1.$$

Hence

$$e_2(M \mid I_1 I_2) = e_2(M \mid I_1) + e_2(M \mid I_2) + 2e(M|I_1|I_2) = 5 > 4 = o(I),$$
$$\mu(I) = 7 > 6 = o(I)^2 + (3-1).$$

It is also easy to verify that MI_1 is integrally closed, but

$$\mu(I_1) = 4 > 3 = \binom{o(I_1)+2}{2}.$$

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