Families Parametrized by Coalgebras

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I. Introduction

Since Hopf [H] first introduced the coalgebra structure on the homology ring of a grouplike manifold, making it into what is now called a Hopf algebra, coalgebras have been appearing more and more frequently in many branches of mathematics, in particular in algebraic topology, homological algebra and in algebraic geometry. This prompted fundamental work on the structure of coalgebras themselves, a good part of which is outlined in the influential article of Milnor and Moore [M-M], in Sweedler's book [S] and in articles mentioned later in this text.

At first glance, coalgebras are strange objects. Although they are defined in algebraic terms, our algebraic intuition breaks down when trying to understand them. They really are more geometric than algebraic. This is somewhat explained by the partial duality that exists between algebras and coalgebras (with scalars from a field). The dual of a coalgebra is an algebra and although the dualizing functor is not an equivalence of categories, it does have an adjoint on the right ( )^0 [S, p. 109]. These two functors restrict to a contravariant equivalence between finite dimensional coalgebras and finite dimensional algebras. The category of cocommutative coalgebras is similar in many respects to the opposite of the category of commutative k-algebras, and this is the same as the category of affine schemes over k. So, it is not surprising that cocommutative coalgebras are geometric entities. In fact, they correspond to formal schemes [C, T].

Due to this partial duality between coalgebras and algebras, many definitions in the theory of coalgebras were suggested by the corresponding concepts for algebras (such as the cotensor $\square_C$ of [M-M]), and the statements of many theorems are inspired by the corresponding results for algebras. But in many respects, the category of cocommutative coalgebras is much better than the category of algebras. It is cartesian closed, i.e., there is a *coalgebra* of morphisms from one coalgebra to another, with the appropriate universal property (see [ML, p. 95] for the definition of "cartesian closed"). This makes Coalg into a monoidal category [K, p. 21] and

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as Beck has pointed out (unpublished), many categories occurring in algebra are enriched (i.e., are V-categories [K, p. 23]) over Coalg. This has been studied by Barr [Ba2] and Fox [Fo1, Fo2]. Cartesian closedness says that Coalg is like the category of sets to some extent. Another property which is set-like (or space-like) is that coproducts are disjoint and universal, i.e., they are like disjoint unions (see [Gro2, p. 243] and our Theorem V.2.3). Many results which hold for algebras with finiteness conditions have "duals" which hold in general for coalgebras. For example, a module over a finite product of rings is the same as a finite sequence of modules, one over each ring. This does not hold for infinite products, but the corresponding result for comodules holds for arbitrary coproducts of coalgebras.

Thus one might expect that results could be proved directly for coalgebras and then dualized to give results about algebras. But, because of the unintuitive nature of coalgebras, they are used in this way very little. What is needed are new techniques and especially a new conceptual framework, independent of the theory of algebras. It is the purpose of this paper to propose such a framework.

The main idea is to treat the category of cocommutative coalgebras as if it were the category of sets for some generalized theory of sets. We have already indicated that Coalg has a number of properties in common with Set. However, the main property of sets that we wish to exploit is that they are used to index families of objects or morphisms in a category. This is a fundamental concept (e.g., it is necessary to know what such families are in order to define infinite products, etc.) and we will show that cocommutative coalgebras can be used to parametrize families of objects arising in algebra.

The idea that families can be parametrized by objects other than sets goes back to Riemann (see, e.g., [M-F, p. 96]) who wanted to make the set of all curves of a given genus into a space of some kind (a moduli space). This idea is used considerably today in differential and algebraic geometry. In all cases, the parametrizing objects are some kind of spaces and the category of such has a number of properties in common with Set.

But Set is used in another way too. It is the *basic category* out of which other categories are built (the objects are sets with structure). Here, these two roles of Set are split between two categories: Coalg is the category of parametrizing objects, and Vect, the category of vector spaces, is the basic category out of which others are built.

Our whole theory is developed from the following three basic definitions. The main one is that if C is a cocommutative coalgebra, then a C-*indexed family of vector spaces* is a C-comodule. The notation Vect^C for the category of C-comodules is intended to reflect this. The second basic definition is that the memberwise tensor product of two families of vector

spaces is what was called the cotensor product of two comodules in [M-M, p. 219]. This definition is justified by Theorem IV.2.3. We introduce the notation $\otimes^C$ for the cotensor because it is important for our theory that we think of it as a memberwise $\otimes$ of families.

According to the theory of indexed categories [P-S], a C-indexed family of cocommutative coalgebras should be taken to be a cocommutative coalgebra over C, i.e., a morphism $D \to C$ in Coalg. We define the underlying family of vector spaces of such a family of coalgebras to be D with the C-comodule structure given by "corestriction of scalars". These are the three basic definitions.

A first result suggested by these concepts is that a family of vector spaces indexed by a direct sum of coalgebras, $\oplus C_v$ (which we want to think of as a disjoint union), should be an ordinary family whose members are C_v-comodules, one for each v. This is the content of Theorem III.3.2.

We can consider cocommutative comonoids in Vect^C made into a monoidal category by $\otimes^C$. The family interpretations of Vect^C and $\otimes^C$ suggest that these should be the same as C-indexed families of coalgebras. Theorem V.2.2 says they are.

These two results together imply that a family of coalgebras indexed by a coproduct $\oplus C_v$ is the same as an ordinary family of C_v-indexed families, one for each v (Theorem V.2.3). This means that coproducts in Coalg are disjoint and universal, so that we are justified in thinking in terms of disjoint unions.

Not necessarily cocommutative comonoids in Vect^C correspond to general coalgebras over C with a property dual to the commuting of scalars with the elements of an algebra (Theorem V.2.7).

The Beck condition for the indexed coproduct Σ plays an important role in the theory of indexed categories. Its proof for coalgebras is given in V.2.5.3.

In Section VII, we introduce families of algebras as monoids in Vect^C with respect to the tensor $\otimes^C$. A number of examples are given to justify the introduction of this new concept. It explains what Sweedler's measurings are in terms of families and suggests a number of constructions which can be performed on them (VII.3.1.3–3.2.2). The tensor algebra functor and, in characteristic 0, the symmetric algebra functor are shown to be indexed, i.e., they extend to families in a compatible way (Theorems VII.2.1 and VII.2.5.2). This explains some results of Sweedler's on measurings induced on tensor and symmetric algebras. Another unifying aspect of this point of view is that 2-cocycles give rise to certain families of algebras (VII.3.2.3) and the usual construction of an algebra from a cocycle is interpreted as the indexed product of these families.

The existence of indexed products and coproducts of C-indexed families is discussed in chapters VI and VII.

Another aspect of sets is that "collections of things" can be considered as sets, e.g., all algebra homomorphisms from one algebra to another form a set. We can also discuss "coalgebras of things" in our framework. We have already mentioned that Coalg is cartesian closed, i.e., there is a coalgebra of homomorphisms from one coalgebra to another. In [S, p. 143], Sweedler showed that there existed coalgebras of homomorphisms between two algebras. The universal properties of these coalgebras were somewhat ad hoc. Using families, we can say what these should be in general, and not only for single objects but for families too.

If A and B are two C-indexed families of vector spaces, we give the universal property which the C-indexed family of coalgebras, $\mathrm{Hom}^C(A, B)$, should have in order to be the memberwise hom of A and B. $\mathrm{Hom}^C(A, B)$ exists for all B if and only if A is *coflat*, and Theorem V.4.2. characterizes these. We construct a number of other such examples under some coflatness conditions.

At the end of the paper, we construct the coalgebra of all finite dimensional algebras. Even to make sense of this requires a knowledge of families of algebras.

Throughout the paper, a number of counterexamples are given. For example, V.2.6 shows that epimorphisms in Coalg are not stable under pullback, so Coalg/C cannot be cartesian closed. We also show that for coalgebras with scalars from $\mathbf{Z}$, the cotensor $\otimes^C$ is not associative (IV.2.5), nor is the forgetful functor $\mathbf{Z}$-Coalg $\to$ Ab indexed. It is for such reasons that we were obliged to restrict our theory to coalgebras over a field.

II. Preliminaries

1. *Outline of Indexed Categories*

The formal setting in which to study families of objects parametrized by objects other than sets is the theory of indexed categories. In this theory, sketched below, the families are given axiomatically as extra structure on a category. For more details see [P-S, ML-P].

1.1. Let **S** be a category with finite limits (pullbacks and terminal object). It is by the objects of **S** that we want to index families, i.e., the objects of **S** are to be thought of as the "spaces of parameters."

1.1.1. DEFINITION. An **S**-*indexed category* $\mathscr{A}$ consists of categories $\mathbf{A}^I$, one for each object **I** of **S**, and of functors $\alpha^*\colon \mathbf{A}^I \to \mathbf{A}^J$, one for each morphism $\alpha\colon \mathbf{J} \to \mathbf{I}$ of **S**, such that $1_I^* \cong 1_{\mathbf{A}^I}$ and $\beta^*\alpha^* \cong (\alpha\beta)^*$ for all objects I and composable pairs of morphisms α and β. These natural isomorphisms

are required to satisfy certain coherence conditions which can be found in [ML-P].

The objects of $\mathbf{A}^I$ are called the *I-indexed families* of $\mathscr{A}$ and they should be thought of in this way. The morphisms of $\mathbf{A}^I$ are thought of as *I*-indexed families of morphisms of $\mathscr{A}$ between corresponding members of two *I*-indexed families of objects. The functor α^* is the substitution of α into the family. If we write an *I*-indexed family suggestively as $\langle A_i \rangle_{i \in I}$, then $\alpha^*(\langle A_i \rangle_{i \in I})$ is meant to be $\langle A_{\alpha(j)} \rangle_{j \in J}$. If 1 is the terminal object of **S**, we write $\mathbf{A}$ for $\mathbf{A}^1$. If $\tau: I \to 1$ is the unique morphism, τ^* is written $\varDelta_I: \mathbf{A} \to \mathbf{A}^I$ and can be thought of as a diagonal functor. We also often write A_i for i^*A when $i: 1 \to I$.

1.1.2. DEFINITION.   An **S**-*indexed functor* $F: \mathscr{A} \to \mathscr{B}$ consists of functors $F^I: \mathbf{A}^I \to \mathbf{B}^I$, one for each object I of **S**, such that for each $\alpha: J \to I$, $\alpha^*F^I \cong F^J\alpha^*$. Again, these natural isomorphisms must satisfy coherence conditions [ML-P].

We also write F for F^1. When α in the above definition is $i: 1 \to I$, the condition becomes $(F^I(A))_i = F(A_i)$. Thus the idea is that $F^I(\langle A_i \rangle_{i \in I}) = \langle F(A_i) \rangle_{i \in I}$, although, of course, a family is not usually determined by its members at such i.

We can also define *indexed natural transformations* (and so indexed adjoints). We leave it to the reader to formulate the correct definition, which can be checked in [ML-P].

1.2.   Coproducts of *I*-indexed families are defined by requiring that the diagonal functor $\varDelta_I: \mathbf{A} \to \mathbf{A}^I$ have a left adjoint Σ_I. In practice we want more; we want families of coproducts.

1.2.1. DEFINITION.   We say that $\mathscr{A}$ has **S**-*indexed coproducts* if for each $\alpha: J \to I$ in **S**, the functor $\alpha^*: \mathbf{A}^I \to \mathbf{A}^J$ has a left adjoint Σ_α, and the *Beck condition* is satisfied, i.e., for every pullback diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\ \delta\ } & K \\
\scriptstyle\gamma\big\downarrow & & \big\downarrow\scriptstyle\beta \\
J & \xrightarrow[\ \alpha\ ]{} & I
\end{array}
$$

the canonical map $\Sigma_\delta \gamma^* \to \beta^* \Sigma_\alpha$ is an isomorphism.

The Beck condition insures that these coproducts are well behaved. In particular it gives us the following interpretation:

$$\Sigma_\alpha(\langle B_j \rangle_{j \in J}) = \langle \Sigma_{\alpha(j)=i} B_j \rangle_{i \in I}.$$

We also say that $\mathscr{A}$ has a certain type of (ordinary) colimits (e.g., coequalizers) if each $\mathbf{A}^I$ has them and each α^* preserves them. This *stability* is important and much like the Beck condition: without it the colimits are only of limited use. S-indexed products Π_α and limits are treated dually.
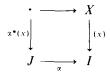
1.3. If A and B are objects of an indexed category, we can say what it means for the morphisms from A to B to form, not just a set, but an object of S. If there were such an object $\text{Hom}(A, B)$ in S, then for any I in S, a morphism $I \to \text{Hom}(A, B)$ would be an I-indexed family of morphisms from A to B, i.e., a morphism from $\varDelta_I A$ to $\varDelta_I B$ in $\mathbf{A}^I$. Thus we say that there is a small class of morphisms from A to B if there is an object $\text{Hom}(A, B)$ in S and a bijection

$$\cong \frac{I \to \text{Hom}(A, B) \text{ in } \mathbf{S}}{\varDelta_I A \to \varDelta_I B \text{ in } \mathbf{A}}$$

which is natural in I. This is called a *smallness* condition because we are considering S as a replacement for the category of small sets.

Notation. The term *Hom* denotes the hom object in S (i.e., in Coalg in this paper), whereas *hom* denotes the vector space hom. The set of morphisms in $\mathbf{A}$ is denoted $\mathbf{A}(\ ,\ )$.

1.4. The base category S itself can be indexed by letting $\mathbf{S}^I = \mathbf{S}/I$, the category of objects over I (i.e., the category of morphisms to I). We are thinking of $x: X \to I$ as the family $\langle x^{-1}(i) \rangle_{i \in I}$. Substitution $\alpha^*: \mathbf{S}/I \to \mathbf{S}/J$ along α is given by the pullback

$$
\begin{array}{ccc}
\cdot & \longrightarrow & X \\
{\scriptstyle \alpha^*(x)}\downarrow & & \downarrow{\scriptstyle (x)} \\
J & \xrightarrow{\ \alpha\ } & I
\end{array}
$$

in S.

If $y: Y \to J$ is an object of $\mathbf{S}^J$ and we define $\Sigma_\alpha(y) = (\alpha y)$ in $\mathbf{S}^I$, then we easily see that Σ_α is left adjoint to α^* and the Beck condition is satisfied. Thus $\mathscr{S}$ has S-indexed coproducts. But $\mathscr{S}$ has S-indexed products (if and) only if each category $\mathbf{S}/I$ is cartesian closed. Also, $\text{Hom}(A, B)$ exists for all pairs of objects A and B of S if and only if S is cartesian closed.

1.5. In working with indexed categories, a general principle is that all concepts should be defined not merely for single objects but for families

of objects as we did, for example, with S-indexed coproducts. There should also be a condition of stability under change of indexing object, such as the Beck condition.

If A and B are I-families of objects of $\mathscr{A}$, $\mathrm{Hom}^I(A, B)$ is defined to be an object of $\mathbf{S}^I$ with the property that there is a bijection

$$\cong \frac{(\alpha) \to \mathrm{Hom}^I(A, B) \text{ in } \mathbf{S}^I}{\alpha^* A \to \alpha^* B \text{ in } \mathbf{A}^J}$$

which is natural in $\alpha: J \to I$. It is easily seen that if $\mathrm{Hom}^I(A, B)$ exists it is *stable* in the sense that

$$\alpha^* \mathrm{Hom}^I(A, B) \cong \mathrm{Hom}^J(\alpha^* A, \alpha^* B).$$

Specializing this to the case where α is $i: 1 \to I$, we see that $\mathrm{Hom}^I(A, B)$ is to be interpreted as the family $\langle \mathrm{Hom}(A_i, B_i) \rangle_{i \in I}$.

1.6. Although the basic theory of indexed categories works for any category $\mathbf{S}$ with finite limits, we only get relevant results when the objects of $\mathbf{S}$ are sufficiently "set-like" or "space-like." For example, $\mathbf{Ab}$, the category of abelian groups is not a good base category for indexing, although it has finite limits (in fact all limits and colimits) and internal homs. The hom objects in $\mathbf{Ab}$ do not satisfy the bijection of 1.3, i.e., $\mathbf{Ab}$ is not cartesian closed. The cartesian product acts more like a sum and it is the tensor product which acts like a product.

These considerations lead us to cocommutative coalgebras over $\mathbf{Z}$ where the tensor product becomes the cartesian product. Barr [Ba2] has shown that this category is cartesian closed, and Beck pointed out (unpublished) that many categories (occurring in homological algebra) are enriched over it (see [Fo2]). This looks promising but our theory as it stands now does not go too far over $\mathbf{Z}$, since the tensor product in $\mathbf{Ab}$ is not exact. So we consider mainly cocommutative coalgebras over a field (we could take a commutative von Neumann regular ring).

2. *Basic Properties of Coalgebras*

2.1. We shall be considering the category Coalg of *cocommutative coalgebras* over a *field* k. The basic reference on coalgebras is [S]. Below, we give a few results which will be used later.

2.2. The finite dimensional coalgebras are a generating set for Coalg. The forgetful functor U: Coalg $\to$ Vect creates all colimits and Coalg

is co-well-powered since epimorphisms are onto. By the special adjoint functor theorem, U has a right adjoint R. U also creates all limits which are preserved by the tensor power functors $(\ )^{(0)}$, $(\ )^{(2)}$, $(\ )^{(3)}$. $(\ )^{(0)}$ is the constant functor k and it preserves a limit if and only if the diagram in question is nonempty and connected.

2.3. The terminal object in Coalg is k with its unique coalgebra structure. For any coalgebra C, the unique morphism to k is given by the counit $\varepsilon: C \to k$. The binary product of two coalgebras C and D is $C \otimes D$ together with the projections $C \otimes D \to^{C \otimes \varepsilon} C \otimes k \cong C$ and $C \otimes D \to^{\varepsilon \otimes D} k \otimes D \cong D$. Given two morphisms of coalgebras $\phi: E \to C$ and $\psi: E \to D$, then the corresponding morphism into the product is the composite $E \to^{\delta} E \otimes E \to^{\phi \otimes \psi} C \otimes D$. (The comultiplication of a coalgebra will usually be denoted by δ.) For any coalgebra C, the functor $C \otimes -: \text{Coalg} \to \text{Coalg}$ preserves colimits and so, by the special adjoint functor theorem, has a right adjoint $\text{Hom}(C, -)$. Thus Coalg is cartesian closed.

2.4. Every morphism of coalgebras can be factored into an onto map followed by a one-to-one map in Coalg. This is because it has such a factorization in Vect and $\otimes$ preserves monomorphisms in Vect. Since U has a right adjoint, onto maps are the same as epimorphisms in Coalg. We shall see presently that one-one maps are the regular monomorphisms. So we have a regular image factorization in Coalg.

2.4.1. PROPOSITION. *Intersections in* Vect *are absolute, i.e., are preserved by all functors.*

Proof. Let $V_1 \subsetneqq V$ and $V_2 \subsetneqq V$ be two subobjects in Vect and let $V_0 \subsetneqq V$ be their intersection. Choose a complement W_1 of V_0 in V_1 and a complement W of $V_2 + W_1$ in V. Then $V_1 = V_0 + W_1$ and $V = V_2 + W_1 + W$. In the diagram

$$
\begin{array}{ccccc}
V_0 & \longrightarrow & V_1 & \xrightarrow{\ s_1\ } & V_0 \\
\downarrow & & \downarrow & & \downarrow \\
V_2 & \longrightarrow & V & \xrightarrow{\ s\ } & V_2
\end{array}
$$

where s_1 and s are the obvious projections, the top and bottom composites are identities, both squares commute and $V_1 \to V$ is a split monomorphism. Since the large square is an absolute pullback and $V_1 \to V$ is an absolute monomorphism the left hand square is an absolute pullback. ∎

2.4.2. A pair of morphisms $f, g: A \rightrightarrows B$ in any category is called

coreflexive if there is a morphism $s: B \to A$ such that $sf = 1_A = sg$. The equalizer of a coreflexive pair is the same as the pullback of the two morphisms since $fx = gy$ if and only if $x = y$ and $fx = gx$. Since f and g must be monic, the pullback is merely the intersection. Almost all equalizers which we use will be coreflexive. See [L], where the ubiquity of reflexive coequalizers is manifest.

2.4.3. COROLLARY. $U: \text{Coalg} \to \text{Vect}$ *creates   intersections   and   also equalizers of U-coreflexive pairs.*

Proof.   The intersection of two subcoalgebras (one-one maps) can be taken in Vect, since by 2.4.1 the tensor $\otimes$ preserves intersections. A pair $\phi, \psi: D \rightrightarrows C$ is U-coreflexive if $U\phi, U\psi: UD \rightrightarrows UC$ is coreflexive. The result now follows from the above.   ∎

COROLLARY.   *The regular monomorphisms in* Coalg *are precisely the one to one maps.*

Proof.   The equalizer of an arbitrary pair $\phi, \psi: D \rightrightarrows C$ is the same as the equalizer of the coreflexive pair

$$D \xrightarrow{\ \delta\ } D \otimes D \underset{D \otimes \psi}{\overset{D \otimes \phi}{\rightrightarrows}} D \otimes C$$

(split by the projection) which can be calculated in Vect. Thus equalizers in Coalg are one-one. Conversely, for any one-one map, take its cokernel pair in Coalg, which is the same as in Vect. This is a U-coreflexive pair, so its equalizer is the same as in Vect and we get the same subobject back.   ∎

2.4.4. Of course, pullbacks can also be calculated using coreflexive equalizers. The pullback of $\phi: D_1 \to C$ and $\psi: D_2 \to C$ is given by the equalizer of the coreflexive pair

$$D_1 \otimes D_2 \underset{D_1 \otimes \psi \otimes D_2 \cdot D_1 \otimes \delta}{\overset{D_1 \otimes \phi \otimes D_2 \cdot \delta \otimes D_2}{\rightrightarrows}} D_1 \otimes C \otimes D_2.$$

Pulling back along projections is easier. Indeed, the diagram

$$
\begin{array}{ccc}
E \otimes D & \xrightarrow{\ p_1\ } & E \\
{\scriptstyle \phi \otimes D}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\
C \otimes D & \xrightarrow[\ p_1\ ]{} & C
\end{array}
$$

is a pullback.

2.4.5. If V is a subspace of a coalgebra C, then there exists a largest sub-coalgebra contained in V. Consider the family of all subcoalgebras $C_\alpha \subseteq C$ such that $C_\alpha \subseteq V$. The image of $\bigoplus C_\alpha \to C$ is clearly the largest sub-coalgebra contained in V.

It is now easy to see that the equalizer of ϕ, $\psi: D \rightrightarrows C$ is the largest sub-coalgebra of D contained in the vector space equalizer of ϕ and ψ.

Similarly, if $\phi: D_1 \to C$ is one-to-one and $\psi: D_2 \to C$ is arbitrary then the pullback of ϕ along ψ is the largest subcoalgebra of D_2 contained in $\psi^{-1}(D_1)$. Another way to calculate this pullback is to take the intersection of

$$
\begin{array}{ccc}
& & D_2 \otimes D_1 \\
& & \downarrow {\scriptstyle D_2 \otimes \phi} \\
D_2 \xrightarrow{\ \delta\ } D_2 \otimes D_2 \xrightarrow[\ D_2 \otimes \psi\ ]{} D_2 \otimes C.
\end{array}
$$

Thus the pullback is $\{x \in D_2 \mid (D_2 \otimes \psi)\, \delta(x) \in D_2 \otimes D_1\}$.

III. THE INDEXED CATEGORY *Vect*

1. *Definition and Basic Properties*

The indexing of the category of vector spaces Vect by the category of coalgebras is of fundamental importance in our considerations. Many other categories which are indexed by Coalg are constructed from *Vect*.

1.1. Let C be a coalgebra. A C-comodule (A, α) is a vector space A together with a linear map $\alpha: A \to C \otimes A$ such that

$$
\begin{array}{ccccccc}
A \xrightarrow{\ \alpha\ } C \otimes A & & & & A \xrightarrow{\ \alpha\ } C \otimes A \\
{\scriptstyle \cong}\searrow \quad \downarrow{\scriptstyle \varepsilon \otimes A} & \text{and} & & \alpha\downarrow \qquad \qquad \downarrow{\scriptstyle \delta \otimes A} \\
\qquad k \otimes A & & & & C \otimes A \xrightarrow[\ C \otimes \alpha\ ]{} C \otimes C \otimes A
\end{array}
$$

commute.

We define a *C-indexed family of vector spaces* to be a C-comodule, i.e., we let Vect^C be the category of C-comodules. The forgetful functor $U: \text{Vect}^C \to \text{Vect}$ has a right adjoint R given by $R(V) = (C \otimes V, \delta \otimes V)$, the cofree comodule on V. U creates all colimits and any limits preserved by $C \otimes -$ and $C \otimes C \otimes -$. In particular, finite products and equalizers in Vect^C are the same as in Vect. Thus we conclude that Vect^C is an abelian category.

A morphism of coalgebras $\phi: D \to C$ gives rise to an adjoint pair of functors between Vect^D and Vect^C. There is first of all a functor Σ_ϕ commuting with the forgetful functors

$$\mathrm{Vect}^D \xrightarrow{\ \Sigma_\phi\ } \mathrm{Vect}^C$$

$$U \searrow \qquad \swarrow U$$

$$\mathrm{Vect}$$

which is defined by $\Sigma_\phi(B, \beta) = (B, (\phi \otimes B)\beta)$. This functor has a right adjoint

$$\phi^*: \quad \mathrm{Vect}^C \to \mathrm{Vect}^D$$

given by the following equalizer in Vect^D (which can be calculated in Vect),

$$\phi^*(A, \alpha) \to D \otimes A \xrightarrow{\ \delta \otimes A\ } D \otimes D \otimes A$$

$$D \otimes \alpha \searrow \qquad \downarrow D \otimes \phi \otimes A$$

$$D \otimes C \otimes A$$

where $D \otimes A$, $D \otimes D \otimes A$ and $D \otimes C \otimes A$ are given the cofree D-structures. If $\psi: E \to D$ is another morphism of coalgebras, then clearly $\Sigma_\phi \Sigma_\psi = \Sigma_{\phi\psi}$. By uniqueness of adjoints we get a canonical isomorphism $(\phi\psi)^* \cong \psi^* \phi^*$. Moreover, $1_C^* \cong 1_{\mathrm{Vect}^C}$ since $\Sigma_{1_C} = 1_{\mathrm{Vect}^C}$. In this way we define a Coalg-indexed category *Vect*.

1.2. Since $U\Sigma_\phi = U$, passing to right adjoints, we see that $\phi^*(C \otimes A, \delta \otimes A) \cong (D \otimes A, \delta \otimes A)$. In fact, $\mathrm{Vect}^k \simeq \mathrm{Vect}$, $\Sigma_\varepsilon = \Sigma_C = U$ and $\varepsilon^* = \Delta_C = R$. The above isomorphism is nothing but $\phi^* \Delta_C \cong \Delta_D$. Using this, we can see that ϕ^* applied to the C-morphism $\delta \otimes V: C \otimes V \to C \otimes C \otimes V$ is given by $D \otimes V \to^{\delta \otimes V} D \otimes D \otimes V \to^{D \otimes \phi \otimes V} D \otimes C \otimes V$.

1.3. Later, we shall see that Σ satisfies the Beck condition, but a direct proof of this seems difficult. However, the following explicit description of the substitution functor along a projection makes it easy to show that Σ satisfies the Beck condition at 1.

1.3.1. LEMMA. *If* $p: D \otimes C \to C$ *is the projection then* $p^*(A, \alpha) = (D \otimes A, \sigma_{23}(\delta \otimes \alpha))$, *where* $\sigma_{23}: D \otimes D \otimes C \otimes A \to D \otimes C \otimes D \otimes A$ *is the canonical isomorphism which switches the second and third factors.*

Proof. Since the comultiplication of $D \otimes C$ is $\sigma_{23}(\delta \otimes \delta)$ we see from the definition that $p^*(A, \alpha)$ is the equalizer of the pair

$$D \otimes C \otimes A \underset{D \otimes \delta \otimes A}{\overset{D \otimes C \otimes \alpha}{\rightrightarrows}} D \otimes C \otimes C \otimes A.$$

This pair is $D \otimes -$ applied to the pair

$$C \otimes A \underset{\delta \otimes A}{\overset{C \otimes \alpha}{\rightrightarrows}} C \otimes C \otimes A$$

which has equalizer A. Thus the equalizer of the original pair is $D \otimes A$ and $\sigma_{23}(\delta \otimes \alpha)$ is the unique morphism making

$$
\begin{array}{ccc}
D \otimes A & \xrightarrow{\ D \otimes \alpha\ } & D \otimes C \otimes A \\
{\scriptstyle \sigma_{23}(\delta \otimes \alpha)} \downarrow & & \downarrow {\scriptstyle \sigma_{23}(\delta \otimes \delta \otimes A)} \\
D \otimes C \otimes D \otimes A & \xrightarrow[D \otimes C \otimes D \otimes \alpha]{} & D \otimes C \otimes D \otimes C \otimes A
\end{array}
$$

commute. ∎

1.3.2. THEOREM. *Σ satisfies the Beck condition at 1, i.e., for any two coalgebras C and D,*

$$
\begin{array}{ccc}
\mathbf{Vect}^C & \xrightarrow{\ \Sigma_C\ } & \mathbf{Vect} \\
{\scriptstyle p_2^*} \uparrow & & \downarrow {\scriptstyle \Delta_D} \\
\mathbf{Vect}^{D \otimes C} & \xrightarrow[\Sigma_{p_1}]{} & \mathbf{Vect}^D
\end{array}
$$

commutes

Proof. Let (A, α) be a C-comodule. Then $\Delta_D \Sigma_C(A, \alpha) = \Delta_D A = (D \otimes A, \delta \otimes A)$. By 1.3.1, $\Sigma_{p_1} p_2^*(A, \alpha) = \Sigma_{p_1}(D \otimes A, \sigma_{23}(\delta \otimes \alpha))$, which is $D \otimes A$ with the structure map

$$D \otimes A \xrightarrow{\ \delta \otimes \alpha\ } D \otimes D \otimes C \otimes A \xrightarrow{\ \sigma_{23}\ } D \otimes C \otimes D \otimes A \xrightarrow{\ D \otimes \varepsilon \otimes D \otimes A\ } D \otimes D \otimes A$$

which is equal to $\delta \otimes A$. Thus $\Sigma_{p_1} p_2^*(A, \alpha) = (D \otimes A, \delta \otimes A) = \Delta_D \Sigma_C(A, \alpha)$. ∎

If we think of the family interpretation, this seems obvious. That it is true supports the idea that we can think of these things as families.

1.4. There is also a global description of *Vect* as a fibration (see

[Gro1, pp. 145–194, Bé2] for more about fibrations). Let Comod be the category whose objects are pairs (C, A) where C is a coalgebra and A is a C-comodule. A morphism is a pair

$$(C, A) \xrightarrow{(\phi, f)} (D, B)$$

such that $\phi: C \to D$ is a morphism of coalgebras and $f: \Sigma_\phi A \to B$ is a D-morphism. The condition on f is simply that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\
C \otimes A & \xrightarrow[\phi \otimes f]{} & D \otimes B
\end{array}
$$

commutes. There is a canonical forgetful functor

$$P: \quad \text{Comod} \to \text{Coalg}$$

which is a fibration (the lifting property is given by the ϕ^*). The fibre over C is precisely Vect^C and so Comod can be viewed as the category of all families of vector spaces.

2. Stability

Since the ϕ^* have left adjoints Σ_ϕ, all limits are stable (i.e., preserved by the ϕ^*). As we shall see, coproducts are stable too but in general coequalizers are not.

2.1. PROPOSITION. *Coproducts in Vect are stable.*

Proof. Let $\phi: D \to C$ be a morphism of coalgebras and let (A_i, α_i) be a family of C-comodules. Then $\phi^*(\bigoplus (A_i, \alpha_i))$ is given by the equalizer

$$
\phi^*(\bigoplus (A_i, \alpha_i)) \to D \otimes (\bigoplus A_i) \xrightarrow{\delta \otimes (\bigoplus A_i)} D \otimes D \otimes (\bigoplus A_i)
$$

$$
\begin{array}{c}
{\scriptstyle D \otimes (\bigoplus \alpha_i)} \searrow \qquad \downarrow {\scriptstyle D \otimes \phi \otimes (\bigoplus A_i)} \\
D \otimes C \otimes (\bigoplus A_i)
\end{array}
$$

which is isomorphic to the equalizer

$$
\bigoplus \phi^*(A_i, \alpha_i) \to \bigoplus (D \otimes A_i) \xrightarrow{\bigoplus (\delta \otimes A_i)} \bigoplus (D \otimes D \otimes A_i)
$$

$$
\begin{array}{c}
{\scriptstyle \bigoplus (D \otimes \alpha_i)} \searrow \qquad \downarrow {\scriptstyle \bigoplus (D \otimes \phi \otimes A_i)} \\
\bigoplus (D \otimes C \otimes A_i). \quad \blacksquare
\end{array}
$$

Note that this proposition also applies to the empty coproduct 0, i.e., 0 is stable.

2.2. COUNTEREXAMPLE. Coequalizers in *Vect* are not always stable. Let $C = k \oplus kd$ with $\delta(1) = 1 \otimes 1$, $\delta(d) = d \otimes 1 + 1 \otimes d$, $\varepsilon(1) = 1$, $\varepsilon(d) = 0$ and let $D = k$ with the unique coalgebra structure. $\phi: D \to C$ is taken to be the inclusion. A D-comodule is simply a vector space. A C-comodule is easily seen to be a vector space A with an endomorphism $\alpha: A \to A$ such that $\alpha^2 = 0$ (the costructure of such an A is given by the function $f: A \to C \otimes A$ defined by $f(a) = 1 \otimes a + d \otimes \alpha(a)$). A morphism $g: (A, \alpha) \to (B, \beta)$ is a linear map such that $\beta g = g\alpha$. $\phi^*(A, \alpha)$ is then nothing but the kernel of α.

In *Vect* we have the coequalizer

$$(k, 0) \underset{i_2}{\overset{0}{\rightrightarrows}} \left(k \oplus k, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \xrightarrow{p_1} (k, 0).$$

ϕ^* of this is

$$k \underset{i_2}{\overset{0}{\rightrightarrows}} 0 \oplus k \xrightarrow{0} k$$

which is not a coequalizer.

Thus the functors ϕ^* are not generally exact. Thus as an indexed category *Vect* does not have coequalizers, and so it is not an indexed abelian category.

However, we do have the following.

2.3. PROPOSITION. *If $p: D \otimes C \to C$ is the projection, then p^* preserves coequalizers.*

Proof. By Lemma 1.3.1, $p^*(A, \alpha) = (D \otimes A, \sigma_{23}(\delta \otimes \alpha))$. Since $D \otimes (\ )$ preserves coequalizers and $U: \mathrm{Vect}^{D \otimes C} \to \mathrm{Vect}$ creates them, p^* preserves them. ∎

3. *The Equivalence* $\mathrm{Vect}^{\oplus C_v} \simeq \pi \, \mathrm{Vect}^{C_v}$

If $C = \bigoplus C_v$, then a C-comodule should be the same as a family of C_v-comodules. We shall use the following proposition to prove this.

3.1. PROPOSITION. *Let $\phi: D \to C$ be a one-to-one coalgebra map and let (A, α) be a C-comodule. Then $\phi^*(A, \alpha)$ is given by the pullback in* Vect
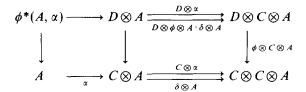
$$
\begin{array}{ccc}
\phi^*(A, \alpha) & \longrightarrow & A \\
\downarrow & & \downarrow{\alpha} \\
D \otimes A & \underset{\phi \otimes A}{\longrightarrow} & C \otimes A
\end{array}
$$

and its D-comodule structure is the restriction of the cofree structure on
$D \otimes A$.

Proof.   In the diagram

$$
\begin{array}{ccc}
\phi^*(A, \alpha) \longrightarrow D \otimes A \begin{array}{c} \xrightarrow{\; D \otimes \alpha \;} \\[-4pt] \xrightarrow[\; D \otimes \phi \otimes A \cdot \delta \otimes A \;]{} \end{array} D \otimes C \otimes A \\[6pt]
\Big\downarrow \qquad\qquad \Big\downarrow \qquad\qquad \Big\downarrow {\scriptstyle \phi \otimes C \otimes A} \\[6pt]
A \xrightarrow{\;\;\alpha\;\;} C \otimes A \begin{array}{c} \xrightarrow{\; C \otimes \alpha \;} \\[-4pt] \xrightarrow[\; \delta \otimes A \;]{} \end{array} C \otimes C \otimes A
\end{array}
$$

the top part is an equalizer by definition of $\phi^*(A, \alpha)$ and the bottom is an equalizer by basic cotriple theory. A simple diagram chase, using the fact that the diagram commutes (in the obvious sense) and that $\phi \otimes A$ and $\phi \otimes C \otimes A$ are monomorphisms, shows that the left hand square is a pullback. By definition, the D-comodule structure on $\phi^*(A, \alpha)$ comes from the cofree structure on $D \otimes A$.   ∎

3.2. THEOREM.   *Let C_v be a family of coalgebra indexed by a set I and let $i_v : C_v \to \bigoplus C_v$ be the injection. Then the functor*

$$
(i_v^*) : \quad \mathrm{Vect}^{\oplus C_v} \to \Pi \, \mathrm{Vect}^{C_v}
$$

is an equivalence of categories.

Proof.   Let $C = \bigoplus C_v$. If (A_v, α_v) is C_v-comodule for each v, then $\bigoplus A_v$ has a natural C-comodule structure. This C-comodule is $\bigoplus_v \Sigma_{i_v}(A_v, \alpha_v)$. The functor $\bigoplus_v \Sigma_{i_v}$ is left adjoint to (i_v^*). By Proposition 3.1, for every μ and v,

$$
i_\mu^* \left(\bigoplus_v \Sigma_{i_v}(A_v, \alpha_v) \right) \cong \bigoplus_v i_\mu^* \Sigma_{i_v}(A_v, \alpha_v).
$$

We compute $i_\mu^* \Sigma_{i_v}(A_v, \alpha_v)$ as the equalizer of

$$
C_\mu \otimes A_v \begin{array}{c} \xrightarrow{\; C_\mu \otimes i_v \otimes A_v \cdot C_\mu \otimes \alpha_v \;} \\[-4pt] \xrightarrow[\; C_v \otimes i_\mu \otimes A_v \cdot \delta_\mu \otimes A_v \;]{} \end{array} C_\mu \otimes \left(\bigoplus C_v \right) \otimes A_v.
$$

If $\mu = v$, then since i_v is monic the equalizer is (A_v, α_v). If $\mu \neq v$, then i_v and i_μ are disjoint so the equalizer is 0. Thus the composite $(i_v^*) \bigoplus_v \Sigma_{i_v}$ is isomorphic to the identity.

Conversely, let (A, α) be a C-comodule. There is a canonical morphism

$$
\bigoplus \Sigma_{i_v}(i_v^*)(A, \alpha) \to (A, \alpha),
$$

and it is sufficient to show that it is an isomorphism at the level of vector spaces. At this level Σ_{i_v} does nothing, so the domain of the map is the coproduct of the equalizers

$$\bigoplus_v i_v^* A \to \bigoplus_v (C_v \otimes A) \xrightarrow{\oplus_v(\delta_v \otimes A)} \bigoplus_v (C_v \otimes C_v \otimes A)$$

$$\searrow{\scriptstyle \oplus_v(C_v \otimes \alpha)} \qquad \qquad \Big\downarrow{\scriptstyle \oplus_v(C_v \otimes i_v \otimes A)}$$

$$\bigoplus_v (C_v \otimes C \otimes A)$$

which is again an equalizer. This is isomorphic to

$$\bigoplus_v i_v^* A \to \left(\bigoplus_v C_v\right) \otimes A \xrightarrow{(\oplus_v \delta_v) \otimes A} \bigoplus_v (C_v \otimes C_v) \otimes A$$

$$\searrow{\scriptstyle (\oplus_v C_v) \otimes \alpha} \qquad \qquad \Big\downarrow{\scriptstyle (i_v \otimes i_v) \otimes A}$$

$$\left(\bigoplus_v C_v\right) \otimes C \otimes A$$

which is the same as

$$\bigoplus_v i_v^* A \to C \otimes A \underset{\delta \otimes A}{\overset{C \otimes \alpha}{\rightrightarrows}} C \otimes C \otimes A.$$

Thus $\bigoplus_v i_v^* A$ is canonically isomorphic to A. ∎

Remark. The theorem is also valid when I is empty, i.e., $\mathrm{Vect}^0 \simeq 1$. It follows easily from the counit law that a 0-comodule is itself 0.

If $C = \bigoplus_I k$ is the I-fold coproduct of copies of k, then $\mathrm{Vect}^C \simeq \Pi \mathrm{Vect}^k \simeq \mathrm{Vect}^I$. Thus a C-indexed family of vector spaces is an ordinary family of vector spaces indexed by the set I. A coalgebra homomorphism $\phi: \bigoplus_I k \to \bigoplus_J k$ is induced by a function $f: I \to J$ and then $\phi^*: \mathrm{Vect}^{\oplus_J k} \to \mathrm{Vect}^{\oplus_I k}$ corresponds to substitution of f into a J-family of vector spaces. In particular $\Delta_{\oplus_I k}: \mathrm{Vect} \to \mathrm{Vect}^{\oplus_I k}$ corresponds to the ordinary diagonal functor. Thus ordinary coproducts (and products) of vector spaces are special cases of the indexed ones.

IV. THE INDEXED FUNCTOR $\otimes$

We index the functor $\otimes: \mathrm{Vect} \times \mathrm{Vect} \to \mathrm{Vect}$ by defining a functor $\otimes^C: \mathrm{Vect}^C \times \mathrm{Vect}^C \to \mathrm{Vect}^C$ for each coalgebra C, such that $\otimes^k = \otimes$ and such that

$$\begin{array}{ccc}
\mathrm{Vect}^C \times \mathrm{Vect}^C & \xrightarrow{\;\otimes^C\;} & \mathrm{Vect}^C \\
{\scriptstyle \phi^* \times \phi^*}\downarrow & & \downarrow{\scriptstyle \phi^*} \\
\mathrm{Vect}^D \times \mathrm{Vect}^D & \xrightarrow[\;\otimes^D\;]{} & \mathrm{Vect}^D
\end{array}$$

commutes up to coherent isomorphism for every morphism $\phi: D \to C$.

1. The Definition and Elementary Properties of $\otimes^C$

1.1. Let C be a coalgebra. If (A, α) and (B, β) are C-comodules, let $(A, \alpha) \otimes^C (B, \beta)$ be the equalizer

$$(A, \alpha) \otimes^C (B, \beta) \to A \otimes B \xrightarrow[\alpha' \otimes B]{\overset{A \otimes \beta}{\Longrightarrow}} A \otimes C \otimes B,$$

where $\alpha' = (A \xrightarrow{\alpha} C \otimes A \xrightarrow{\sigma} A \otimes C)$ is the twisted C-structure on A. The equalizer may be considered to be in Vect^C if $A \otimes B$ and $A \otimes C \otimes B$ are given the C-comodule structures $\alpha \otimes B$ and $\alpha \otimes C \otimes B$ respectively. This gives $(A, \alpha) \otimes^C (B, \beta)$ its comodule structure, i.e., the unique map making

$$\begin{array}{ccc}
(A, \alpha) \otimes^C (B, \beta) & \to & A \otimes B \\
\downarrow & & \downarrow{\scriptstyle \alpha \otimes B} \\
C \otimes ((A, \alpha) \otimes^C (B, \beta)) & \to & C \otimes A \otimes B
\end{array}$$

commute.

Remark. $\otimes^C$ was introduced in [M-M] (see also [Gru1]), where it is denoted $\square_C$ and called the cotensor. We prefer to write $\otimes^C$ rather than $\square_C$ because we believe that it is best thought of as an extension of the usual tensor product of vector spaces, and not a simple dualization of the usual notion of tensor product over an algebra. We may write $A \otimes^C B$ for $(A, \alpha) \otimes^C (B, \beta)$ when no confusion is possible.

1.2. As evidence that we are dealing with a tensor product of some sort, we first show that $\otimes^C$ makes Vect^C into a symmetric monoidal category [K, pp. 28, 29]. We give the proof in detail, even though it is straightforward, because the result does not hold for coalgebras over a ring, as 2.5 shows.

1.2.1. PROPOSITION. *Let* (X, ξ), (Y, θ), (Z, ζ) *be C-comodules. Then*

(i) *the switching map* $\sigma: X \otimes Y \to Y \otimes X$ *restricts to an isomorphism*

$$\sigma: \quad (X, \xi) \otimes^C (Y, \theta) \to (Y, \theta) \otimes^C (X, \xi),$$

(ii) *the associativity isomorphism* $\alpha: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ *restricts to an isomorphism*

$$\alpha: ((X, \xi) \otimes^C (Y, \theta)) \otimes^C (Z, \zeta) \to (X, \xi) \otimes^C ((Y, \theta) \otimes^C (Z, \zeta)).$$

Proof. The diagram

$$
\begin{array}{ccc}
X \otimes^C Y \to X \otimes Y & \overset{X \otimes \theta}{\underset{\xi' \otimes Y}{\rightrightarrows}} & X \otimes C \otimes Y \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma_{13}} \\
Y \otimes^C X \to Y \otimes X & \overset{\theta' \otimes X}{\underset{Y \otimes \xi}{\rightrightarrows}} & Y \otimes C \otimes X
\end{array}
$$

commutes, which establishes (i). Now consider

$$
\begin{array}{ccc}
(X \otimes^C Y) \otimes Z & \overset{(X \otimes^C Y) \otimes \xi}{\underset{\psi' \otimes Z}{\rightrightarrows}} & (X \otimes^C Y) \otimes C \otimes Z \\
\downarrow & & \downarrow \\
X \otimes (Y \otimes^C Z) \quad \to \quad X \otimes Y \otimes Z & \overset{X \otimes Y \otimes \zeta}{\underset{X \otimes \theta' \otimes Z}{\rightrightarrows}} & X \otimes Y \otimes C \otimes Z \\
{\scriptstyle \zeta' \otimes (Y \otimes^C Z)} \Big\Downarrow {\scriptstyle X \otimes \phi} \quad {\scriptstyle \zeta' \otimes Y \otimes Z} \Big\Downarrow {\scriptstyle X \otimes \theta \otimes Z} & & {\scriptstyle \zeta' \otimes Y \otimes C \otimes Z} \Big\Downarrow {\scriptstyle X \otimes \theta \otimes C \otimes Z} \\
X \otimes C \otimes (Y \otimes^C Z) \to X \otimes C \otimes Y \otimes Z & \overset{X \otimes C \otimes Y \otimes \zeta}{\underset{X \otimes C \otimes \theta' \otimes Z}{\rightrightarrows}} & X \otimes C \otimes Y \otimes C \otimes Z
\end{array}
$$

where ϕ and ψ are the structure maps of $Y \otimes^C Z$ and $X \otimes^C Y$, respectively. The diagram commutes in the obvious sense. The rows and columns are equalizers since $V \otimes (\ )$ preserves equalizers for every vector space V. The 3×3 lemma says that the equalizer of the top row, which is $(X \otimes^C Y) \otimes^C Z$, is isomorphic to the equalizer of the left hand column, which is $X \otimes^C (Y \otimes^C Z)$. The isomorphism makes

$$
\begin{array}{ccc}
(X \otimes^C Y) \otimes^C Z & \to & (X \otimes Y) \otimes Z \\
\downarrow & & \downarrow{\scriptstyle \alpha} \\
X \otimes^C (Y \otimes^C Z) & \to & X \otimes (Y \otimes Z)
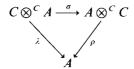\end{array}
$$

commute. ∎

1.2.2. PROPOSITION. *For any C-comodule (A, α) and vector space V, there is a natural isomorphism $(A, \alpha) \otimes^C (C \otimes V, \delta \otimes V) \cong (A \otimes V, \alpha \otimes V)$. Moreover,*

$$A \otimes V \xrightarrow{\ \cong\ } A \otimes^{C} (C \otimes V)$$

$$\alpha' \otimes V \downarrow \qquad\qquad \downarrow A \otimes^{C} (\delta \otimes V)$$

$$A \otimes C \otimes V \xrightarrow[\cong]{} A \otimes^{C} (C \otimes C \otimes V)$$

commutes.

1.2.3. COROLLARY. (C, δ) *is a unit for* $\otimes^{C}$.

1.2.4. COROLLARY. $(C \otimes V, \delta \otimes V) \otimes^{C} (C \otimes W, \delta \otimes W) \cong (C \otimes V \otimes W, \delta \otimes V \otimes W)$, *i.e., the tensor of two cofrees is cofree.*

Since the associativity and symmetry isomorphisms for $\otimes^{C}$ are the restrictions of those for $\otimes$, they satisfy all the required coherence conditions. To verify that $\lambda: C \otimes^{C} A \to_{\cong} A$ and $\rho: A \otimes^{C} C \to_{\cong} A$ satisfy

$$C \otimes^{C} A \xrightarrow{\ \sigma\ } A \otimes^{C} C$$

$$\lambda \searrow \quad \swarrow \rho$$

$$A$$

and

$$(A \otimes^{C} C) \otimes^{C} B \xrightarrow{\ \alpha\ } A \otimes^{C} (C \otimes^{C} B)$$

$$\rho \otimes^{C} B \searrow \qquad \swarrow A \otimes^{C} \lambda$$

$$A \otimes^{C} B$$

is an easy calculation.

Thus we have established the following:

1.2.5. THEOREM. $(\mathrm{Vect}^{C}, \otimes^{C}, C, \alpha, \lambda, \rho, \sigma)$ *is a symmetric monoidal category and the forgetful functor* $\Sigma_{C}: \mathrm{Vect}^{C} \to \mathrm{Vect}$ *is comonoidal (a "comorphisme de c.m." in the notation of* [Bé1]*) via the maps* $A \otimes^{C} B \to A \otimes B$ *and* $\varepsilon: C \to k$.

2. *The Indexedness of* $\otimes$

2.1. The indexedness of $\otimes$ is a consequence of the easily established fact that for each coalgebra C the bifunctor $\otimes^{C}$ preserves equalizers in each variable.

2.1.1. PROPOSITION. $A \otimes^{C} (\ )$ *preserves finite limits.*

2.1.2. PROPOSITION. $(\ ) \otimes^{C} (\ )$ *preserves equalizers of coreflexive pairs.*

Proof. This is true in general for any bifunctor $T: \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ which preserves equalizers in each variable, as an easy diagram chase shows. ▮

This last result gives us a new formula for $A \otimes^C B$ which is more symmetric in A and B.

2.2. PROPOSITION. *If (A, α) and (B, β) are C-comodules, then $A \otimes^C B \cong \delta^*(A \otimes B)$, where $\delta: C \to C \otimes C$ is the diagonal and $A \otimes B$ has the $C \otimes C$- comodule structure $\gamma = (A \otimes B \to^{\alpha \otimes \beta} C \otimes A \otimes C \otimes B \to^{\sigma_{23}} C \otimes C \otimes A \otimes B)$, i.e., $A \otimes^C B$ is the equalizer of the diagram of cofree comodules*

$$C \otimes A \otimes B \xrightarrow[\substack{\sigma_{34} \cdot C \otimes \alpha \otimes \beta \\ C \otimes \delta \otimes A \otimes B \cdot \delta \otimes A \otimes B}]{} C \otimes C \otimes C \otimes A \otimes B$$

where σ_{34} is the map which switches the third and fourth factors in $C \otimes C \otimes A \otimes C \otimes B$.

Proof. For any C-comodule (X, ξ)

$$X \xrightarrow{\xi} C \otimes X \xrightarrow[\delta \otimes X]{C \otimes \xi} C \otimes C \otimes X$$

is a coreflexive equalizer in Vect^C with coreflexivity morphism $C \otimes \varepsilon \otimes X$. Thus by the previous theorem

$$A \otimes^C B \to (C \otimes A) \otimes^C (C \otimes B) \xrightarrow[\substack{(C \otimes \alpha) \otimes^C (C \otimes \beta) \\ (\delta \otimes A) \otimes^C (\delta \otimes B)}]{} (C \otimes C \otimes A) \otimes^C (C \otimes C \otimes B)$$

is an equalizer diagram which by 1.2.2 and 1.2.4 is the same as

$$A \otimes^C B \to C \otimes A \otimes B \xrightarrow[x]{C \otimes \alpha \otimes \beta} C \otimes C \otimes A \otimes C \otimes B.$$

The morphism x corresponding to

$$(\delta \otimes A) \otimes^C (\delta \otimes B) = ((C \otimes C \otimes A) \otimes^C (\delta \otimes B)) \cdot ((\delta \otimes A) \otimes^C (C \otimes B))$$

is, again by 1.2.2, the same as

$$((\delta \otimes C \otimes A)' \otimes B)(\delta \otimes A \otimes B) = \sigma_{34}(C \otimes \delta \otimes A \otimes B)(\delta \otimes A \otimes B). ▮$$

2.3. THEOREM. *For any C-comodules (A, α) and (B, β) and any coalgebra map $\phi: D \to C$,*

$$\phi^*(A \otimes^C B) \cong \phi^*(A) \otimes^D \phi^*(B),$$

i.e., $\otimes: \text{Vect} \times \text{Vect} \to \text{Vect}$ is indexed.

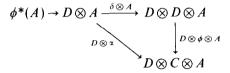*Proof.* Apply ϕ^* to the equalizer of the previous proposition to get the equalizer

$$\phi^*(A \otimes^C B) \to D \otimes A \otimes B \xrightarrow{\phi^*(\delta \otimes A \otimes B)} D \otimes C \otimes A \otimes B$$

with $\sigma_{34} \cdot D \otimes \alpha \otimes \beta$ and $D \otimes \delta \otimes A \otimes B$ to $D \otimes C \otimes C \otimes A \otimes B$

where by III.1.2, $\phi^*(\delta \otimes A \otimes B) = (D \otimes \phi \otimes A \otimes B)(\delta \otimes A \otimes B)$. We also have the coreflexive equalizer

$$\phi^*(A) \to D \otimes A \xrightarrow{\delta \otimes A} D \otimes D \otimes A$$

with $D \otimes \alpha$ and $D \otimes \phi \otimes A$ to $D \otimes C \otimes A$

and a similar one involving (B, β). Applying $\otimes^D$ to these we get, by Proposition 2.1.2, an equalizer

$$\phi^*(A) \otimes^D \phi^*(B) \to D \otimes A \otimes B \xrightarrow{(\delta \otimes A) \otimes^C (\delta \otimes B)} D \otimes D \otimes A \otimes D \otimes B$$

with $D \otimes \alpha \otimes \beta$ and $D \otimes \phi \otimes A \otimes \phi \otimes B$ to $D \otimes C \otimes A \otimes C \otimes B$

Using the value x of $(\delta \otimes A) \otimes^C (\delta \otimes B)$ as computed in the previous proposition, we easily see that the bottom map is the same as $\sigma_{34}(D \otimes \delta \otimes A \otimes B)(D \otimes \phi \otimes A \otimes B)(\delta \otimes A \otimes B)$, and so $\phi^*(A \otimes^C B) \cong \phi^*(A) \otimes^D \phi^*(B)$. ∎

Thus we can say that $(Vect, \otimes, k)$ is an *indexed monoidal category*.

If $p: k \to C$ is a point of C and if we denote $p^*(A)$ by A_p (the pth member of the family), then Theorem 2.3 says

$$(A \otimes^C B)_p \cong A_p \otimes B_p.$$

This indicates how $\otimes^C$ should be interpreted. It is the memberwise tensor product of two families of vector spaces. This is another reason why we have changed the $\square_C$ notation of [M-M] to $\otimes^C$ and call it the tensor product rather than the cotensor product.

If $C = \bigoplus_I k$ is an I-fold coproduct of copies of k, the equivalence

$$Vect^C \simeq \Pi_I \, Vect$$

is given by taking j^* for all injections $j: k \to \oplus k$, and so 2.3 tells us that $\otimes^C$ corresponds exactly to the memberwise $\otimes$ of two I-families of vector spaces in this case.

2.4. Let (A, α) be a *finite dimensional* C-comodule. The natural map $\mu: A \otimes V \to \hom(A^*, V)$ in Vect defined by $\mu(a \otimes v)(f) = f(a) v$ is an isomorphism in this case and the unique map h, for which

$$
\begin{array}{ccc}
A \otimes V & \xrightarrow{\ \alpha \otimes V\ } & C \otimes A \otimes V \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle C \otimes \mu} \\
\hom(A^*, V) & \xrightarrow[\ h\ ]{} & C \otimes \hom(A^*, V)
\end{array}
$$

commutes, makes $\hom(A^*, V)$ into a C-comodule.

Moreover, $A^* = \hom(A, k)$ gets a C-comodule structure via the sequence of natural bijections in Vect

$$\underline{A \xrightarrow{\ \alpha\ } A \otimes C}$$

$$\underline{A \xrightarrow{\ \rho\ } \hom(A^*, C)}$$

$$\underline{A^* \otimes A \xrightarrow{\ \theta\ } C}$$

$$\underline{A^* \xrightarrow{\ \rho\ } \hom(A, C)}$$

$$A^* \xrightarrow{\ \bar{\alpha}\ } C \otimes A^*$$

where $\rho = \mu \alpha'$, $\bar{\rho} = \mu \bar{\alpha}$ and θ is either leg of the following commutative square:

$$
\begin{array}{ccc}
A^* \otimes A & \xrightarrow{\ A^* \otimes \alpha'\ } & A^* \otimes A \otimes C \\
{\scriptstyle \bar{\alpha} \otimes A}\downarrow & & \downarrow{\scriptstyle ev \otimes C} \\
C \otimes A^* \otimes A & \xrightarrow[\ C \otimes ev\ ]{} & C.
\end{array}
$$

It is now easy to check that the diagram of equalizers

$$
\begin{array}{ccc}
A \otimes^C B & \to & A \otimes B \rightrightarrows A \otimes C \otimes B \\
& & \downarrow \qquad\qquad\qquad\qquad \downarrow \\
\mathrm{Vect}^C(A^*, B) & \to \hom(A^*, B) \rightrightarrows \hom(A^*, C \otimes B),
\end{array}
$$

with arrows labelled $A \otimes \beta$, $\alpha' \otimes B$ and $\hom(A^*, \beta)$.

where x is

$$\text{hom}(A^*, B) \xrightarrow{C \otimes -} \text{hom}(C \otimes A^*, C \otimes B) \xrightarrow{\text{hom}(\bar{a}, C \otimes B)} \text{hom}(A^*, C \otimes B),$$

commutes in the obvious sense for any C-comodule (B, β). Thus, we have a natural isomorphism

$$A \otimes^C B \cong \text{Vect}^C(A^*, B)$$

if A is finite dimensional.

2.5. COUNTEREXAMPLE (which shows that for a coalgebra C over $\mathbf{Z}$, $\otimes^C$ is not associative):

Let C be the $\mathbf{Z}$-coalgebra which is the linear dual of the $\mathbf{Z}$-algebra $\mathbf{Z}[x]/(x^2)$, i.e., $C = \mathbf{Z} \oplus \mathbf{Z}d$ with $\varepsilon(1) = 1$, $\varepsilon(d) = 0$, $\delta(1) = 1 \otimes 1$, $\delta(d) = d \otimes 1 + 1 \otimes d$.

As in III.2.2, we can see that a C-comodule is the same as an abelian group A with an endomorphism $\alpha: A \to A$ such that $\alpha^2 = 0$ (i.e., A is a $\mathbf{Z}[x]/(x^2)$-module). The comodule structure $f: A \to C \otimes A$ is given by $f(a) = 1 \otimes a + d \otimes \alpha(a)$. A C-homomorphism $g: (A, \alpha) \to (B, \beta)$ is a homomorphism of groups such that $g\alpha = \beta g$. Equalizers in this category are taken in **Ab**.

Given (A, α) and (B, β), then $(A, \alpha) \otimes^C (B, \beta)$ is the equalizer of

$$A \otimes B \underset{\phi_B}{\overset{\phi_A}{\rightrightarrows}} A \otimes C \otimes B$$

where $\phi_A(a \otimes b) = a \otimes 1 \otimes b + \alpha(a) \otimes d \otimes b$ and $\phi_B(a \otimes b) = a \otimes 1 \otimes b + a \otimes d \otimes \beta(b)$. Thus, $A \otimes^C B$ is

$$\{\Sigma a_i \otimes b_i \in A \otimes B \mid \Sigma \alpha(a_i) \otimes b_i = \Sigma a_i \otimes \beta(b_i)\}$$

with the endomorphism γ defined by $\gamma(\Sigma a_i \otimes b_i) = \Sigma \alpha(a_i) \otimes b_i$. In particular

$$(A, 0) \otimes^C (B, 0) = (A \otimes B, 0)$$

$$(A, 0) \otimes^C (\mathbf{Z}/n^2\mathbf{Z}, n) = (\{[\alpha] \in A/n^2 A \mid n[\alpha] = 0\}, 0).$$

Now consider the map $(\mathbf{Z}/n\mathbf{Z}, 0) \to^n (\mathbf{Z}/n^2\mathbf{Z}, 0)$. Then the induced homomorphism

$$((\mathbf{Z}, 0) \otimes^C (\mathbf{Z}/n^2\mathbf{Z}, n)) \otimes^C (\mathbf{Z}/n\mathbf{Z}, 0)$$

$$\to ((\mathbf{Z}, 0) \otimes^C (\mathbf{Z}/n^2\mathbf{Z}, n)) \otimes^C (\mathbf{Z}/n^2\mathbf{Z}, 0)$$

is

$$nZ/n^2Z \otimes Z/nZ \xrightarrow{\ nZ/n^2Z\,\otimes\, n\ } nZ/n^2Z \otimes Z/n^2Z$$

$$\cong \Big\downarrow \qquad\qquad\qquad\qquad\qquad \Big\downarrow \cong$$

$$Z/nZ \xrightarrow{\qquad\quad 0 \quad\qquad} Z/nZ$$

while

$$(Z, 0) \otimes^C ((Z/n^2Z, n) \otimes^C (Z/nZ, 0))$$

$$\to (Z, 0) \otimes^C ((Z/n^2Z, n) \otimes^C (Z/n^2Z, 0))$$

is the same as

$$Z/nZ \xrightarrow{\ n\ } nZ/n^2Z$$

$$\cong \Big\downarrow \qquad\qquad\qquad \Big\downarrow \cong$$

$$Z/nZ \xrightarrow{\quad 1 \quad} Z/nZ.$$

3. *The Global* $\otimes$

3.1. In the global description III.1.4 of *Vect*, there is a global $\otimes$

$$\text{Comod} \times \text{Comod} \xrightarrow{\ \otimes\ } \text{Comod}$$

$$P \times P \Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow P$$

$$\text{Coalg} \times \text{Coalg} \xrightarrow{\quad \otimes \quad} \text{Coalg}$$

defined by $(C, A) \otimes (D, B) = (C \otimes D, A \otimes B)$ where the $C \otimes D$-comodule structure on $A \otimes B$ is given by
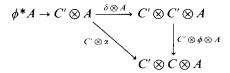
$$A \otimes B \xrightarrow{\ \alpha \otimes \beta\ } C \otimes A \otimes D \otimes B \xrightarrow{\ \sigma_{23}\ } C \otimes D \otimes A \otimes B.$$

This tensor $\otimes$ is clearly functorial, unitary, associative and symmetric. In terms of families, we think of tensoring each member of A with each member of B to get a family indexed by the product coalgebra. This is made more precise by the theorem below which, when specialized to points $p: k \to C$ and $q: k \to D$, says that $(A \otimes B)_{p \otimes q} \cong A_p \otimes B_q$.

3.2. THEOREM. *If (C, A) and (D, B) are objects of* Comod, *then for any coalgebra maps* $\phi: C' \to C$ *and* $\psi: D' \to D$

$$(\phi \otimes \psi)^* (A \otimes B) \cong \phi^* A \otimes \psi^* B.$$

Proof. $\phi^* A$ is defined by the coreflexive equalizer

$$\phi^* A \to C' \otimes A \xrightarrow{\delta \otimes A} C' \otimes C' \otimes A$$

with maps $C' \otimes \alpha$ and $C' \otimes \phi \otimes A$ to $C' \otimes C \otimes A$

and a similar coreflexive equalizer defines $\psi^* B$. The tensor product of these two equalizers is again an equalizer (Proposition 2.1.2) and the pair of maps to be equalized is isomorphic to the pair whose equalizer defines $(\phi \otimes \psi)^* (A \otimes B)$. ∎

This theorem says that the global $\otimes$ is a morphism of fibrations over the $\otimes$ for coalgebras.

The relationship between the global $\otimes$ and the local $\otimes^C$ can be inferred by their interpretations in terms of families. To get $\otimes^C$ from $\otimes$, note that for a point p, $(A \otimes^C A')_p = A_p \otimes A'_p = (A \otimes A')_{p \otimes p} = (A \otimes A')_{\delta(p)} = (\delta^*(A \otimes A'))_p$. So $A \otimes^C A'$ should be $\delta^*(A \otimes A')$. To get $\otimes$ from $\otimes^C$, we observe that for points p and q, $(A \otimes B)_{p \otimes q} = A_p \otimes B_q = A_{\pi_1(p \otimes q)} \otimes B_{\pi_2(p \otimes q)} = (\pi_1^* A_{p \otimes q} \otimes \pi_2^* B_{p \otimes q} = (\pi_1^* A \otimes^{C \otimes D} \pi_2^* B)_{p \otimes q}$. Thus $A \otimes B$ should be $(\pi_1^* A) \otimes^{C \otimes D} (\pi_2^* B)$. That these considerations do give the correct answers is the content of the following.

3.3. PROPOSITION. (1) *If (A, α) and (A', α') are two C-comodules, then*

$$(A, \alpha) \otimes^C (A', \alpha') \cong \delta^*((A, \alpha) \otimes (A', \alpha'))$$

where $\delta: C \to C \otimes C$ *is the diagonal.*

(2) *If (A, α) is a C-comodule and (B, β) is a D-comodule, then*

$$(A, \alpha) \otimes (B, \beta) = \pi_1^*(A, \alpha) \otimes^{C \otimes D} \pi_2^*(B, \beta)$$

where $\pi_1: C \otimes D \to C$ *and* $\pi_2: C \otimes D \to D$ *are the projections.*

Proof. (1) is a restatement of Proposition 2.2, and (2) is an easy calculation. ∎

Some of the results previously obtained now follow easily from these results.

3.3.1. COROLLARY. $\otimes^C$: $\mathrm{Vect}^C \times \mathrm{Vect}^C \to \mathrm{Vect}^C$ *is   symmetric   and associative.*

Proof. $A \otimes^C B \cong \delta^*(A \otimes B) \cong \delta^*\sigma^*(B \otimes A) \cong (\sigma\delta)^*(B \otimes A) \cong$ $\delta^*(B \otimes A) \cong B \otimes^C A$, and $(X \otimes^C Y) \otimes^C Z \cong \delta^*(\delta^*(X \otimes Y) \otimes Z) \cong$ $\delta^*(\delta \otimes C)^* (X \otimes Y \otimes Z) \cong ((\delta \otimes C)\delta)^* (X \otimes Y \otimes Z) \cong ((C \otimes \delta)\delta)^*$ $(X \otimes Y \otimes Z) \cong \delta^*(C \otimes \delta)^* (X \otimes Y \otimes Z) \cong X \otimes^C (Y \otimes^C Z)$. ∎

3.3.2. COROLLARY. $\otimes$: *Vect* × *Vect* → *Vect* *is indexed.*

Proof.   Let (A, α) and (B, β) be C-comodules. If $\phi: D \to C$ is a coalgebra map, then $\phi^*(A \otimes^C B) \cong \phi^*\delta^*(A \otimes B) \cong (\delta\phi)^* (A \otimes B) \cong ((\phi \otimes \phi)\delta)^*$ $(A \otimes B) \cong \delta^*(\phi \otimes \phi)^* (A \otimes B) \cong \delta^*(\phi^*A \otimes \phi^*B) \cong \phi^*A \otimes^D \phi^*B$. ∎

V. THE INDEXED CATEGORY *Coalg*

Since Coalg has finite limits, it indexes itself in the usual way (II.1.4),

$$\mathrm{Coalg}^C = \mathrm{Coalg}/C,$$

i.e., we are defining a C-indexed family of cocommutative coalgebras to be a cocommutative coalgebra D together with a homomorphism $\phi: D \to C$. In particular, a k-indexed family is the same as a single coalgebra.

For a homomorphism $\psi: E \to C$, ψ^*: $\mathrm{Coalg}^C \to \mathrm{Coalg}^E$ is given by pulling back along ψ, i.e., $\psi^*(\phi)$ is given by the pullback diagram of coalgebras

$$
\begin{array}{ccc}
P & \longrightarrow & D \\
\downarrow {\scriptstyle \psi^*(\phi)} & & \downarrow {\scriptstyle \phi} \\
E & \underset{\psi}{\longrightarrow} & C.
\end{array}
$$

As usual, ψ^* has the left adjoint Σ_ψ given by composition with ψ. As always in this situation, Σ satisfies the Beck condition.

1. *The Indexed Functor U: Coalg → Vect*

1.1.   For any coalgebra C we define a functor

$$U^C: \mathrm{Coalg}^C \to \mathrm{Vect}^C$$

as follows. If $\phi: D \to C$ is an object of Coalg^C, then $U^C(\phi) = (D, \phi \otimes D \cdot \delta)$. If $f: (\psi) \to (\phi)$ is a morphism in Coalg^C, i.e.,

$$
\begin{array}{ccc}
E & \xrightarrow{\quad f \quad} & D \\
& \psi \searrow \quad \swarrow \phi & \\
& C &
\end{array}
$$

commutes, then

$$
\begin{array}{ccccc}
E & \xrightarrow{\quad \delta \quad} & E \otimes E & \xrightarrow{\quad \psi \otimes E \quad} & C \otimes E \\
f \downarrow & & \downarrow f \otimes f & & \downarrow C \otimes f \\
D & \xrightarrow{\quad \delta \quad} & D \otimes D & \xrightarrow{\quad \phi \otimes E \quad} & C \otimes D
\end{array}
$$

also commutes and we can define $U^C(f) = f$. When $C = k$, then $\mathrm{Coalg}^C \simeq \mathrm{Coalg}$, $\mathrm{Vect}^C \simeq \mathrm{Vect}$ and U^C is simply the usual forgetful functor $U: \mathrm{Coalg} \to \mathrm{Vect}$. Thus the following theorem says that the functors U^C are the components of the indexed version of the forgetful functor $\mathrm{Coalg} \to \mathrm{Vect}$.

1.2. THEOREM. *U: Coalg → Vect is indexed, i.e., for any coalgebra map* $\psi: E \to C$

$$
\begin{array}{ccc}
\mathrm{Coalg}^C & \xrightarrow{\quad \psi^* \quad} & \mathrm{Coalg}^E \\
U^C \downarrow & & \downarrow U^E \\
\mathrm{Vect}^C & \xrightarrow{\quad \psi^* \quad} & \mathrm{Vect}^E
\end{array}
$$

commutes.

Proof. Let $\phi: D \to E$ be any object of Coalg^C. Then $U^C(\phi) = (D, \phi \otimes D \cdot \delta)$ and so $\psi^* U^C(\phi)$ is given by the equalizer

$$
X \xrightarrow{\quad x \quad} E \otimes D \xrightarrow[\;E \otimes \psi \otimes D \cdot \delta \otimes D\;]{\;E \otimes \phi \otimes D \cdot E \otimes \delta\;} E \otimes C \otimes D.
$$
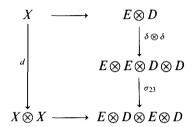
The equalizer is taken in Vect and then made into an E-comodule by the unique ξ making

$$
\begin{array}{ccc}
X & \xrightarrow{\quad\quad} & E \otimes D \\
\xi \downarrow & & \downarrow \delta \otimes D \\
E \otimes X & \xrightarrow{\quad\quad} & E \otimes E \otimes D
\end{array}
$$

commute. In the other direction, $\psi^*(\phi)$ is given by the pullback in Coalg

$$
\begin{array}{ccc}
\cdot & \longrightarrow & D \\
{\scriptstyle \psi^*(\phi)}\downarrow & & \downarrow{\scriptstyle \phi} \\
E & \underset{\psi}{\longrightarrow} & C
\end{array}
$$

which, by II.2.4.4, can also be calculated as the above equalizer in Vect. The comultiplication on X is the unique map d making

$$
\begin{array}{ccc}
X & \longrightarrow & E \otimes D \\
& & \downarrow{\scriptstyle \delta \otimes \delta} \\
{\scriptstyle d}\downarrow & & E \otimes E \otimes D \otimes D \\
& & \downarrow{\scriptstyle \sigma_{23}} \\
X \otimes X & \longrightarrow & E \otimes D \otimes E \otimes D
\end{array}
$$

commute, and $\psi^*(\phi) = (X \xrightarrow{x} E \otimes D \xrightarrow{E \otimes \varepsilon} E \otimes k \cong E)$. Thus $U^E \psi^*(\phi)$ is X with the comodule structure

$$
X \xrightarrow{d} X \otimes X \xrightarrow{x \otimes X} E \otimes D \otimes X \xrightarrow{E \otimes \varepsilon \otimes X} E \otimes k \otimes X \cong E \otimes X,
$$

which is easily seen to satisfy the defining property of ξ. $\blacksquare$

1.3. THEOREM. *The indexed functor U preserves indexed coproducts, i.e., for any homomorphism $\phi: D \to C$,*

$$
\begin{array}{ccc}
\mathrm{Coalg}^D & \xrightarrow{\Sigma_\phi} & \mathrm{Coalg}^C \\
{\scriptstyle U^D}\downarrow & & \downarrow{\scriptstyle U^C} \\
\mathrm{Vect}^D & \xrightarrow[\Sigma_\phi]{} & \mathrm{Vect}^C
\end{array}
$$

commutes.

Proof. Let $\psi: E \to D$ be any object in Coalg^D. Then $U^C \Sigma_\phi(\psi)$ and $\Sigma_\phi U^D(\psi)$ are both E with the structure map

$$
E \xrightarrow{\delta} E \otimes E \xrightarrow{\psi \otimes E} D \otimes E \xrightarrow{\phi \otimes E} C \otimes E. \quad \blacksquare
$$

COROLLARY. *U preserves all Coalg-indexed and ordinary colimits.*

Proof. It is easily seen that, for every coalgebra C, U^C preserves all ordinary colimits, in particular coequalizers. ∎

1.4. THEOREM. *For any two objects (ψ) and (ϕ) of Coalg^C,*

$$U^C((\psi) \times (\phi)) \cong U^C(\psi) \otimes^C U^C(\phi).$$

Proof. With the notation of Theorem 1.2, $E \otimes^C D$ is X with the C-comodule structure given by that of E, namely $(\psi \otimes E)\,\delta$. Thus, $E \otimes^C D = \Sigma_\psi(X, \xi)$. But by the same theorem $(X, \xi) \cong U^E \psi^*(\phi)$. So by Theorem 1.3

$$E \otimes^C D = \Sigma_\psi(X, \xi) \cong \Sigma_\psi U^E \psi^*(\phi) \cong U^C \Sigma_\psi \psi^*(\phi) \cong U^C((\psi) \otimes (\phi)). \quad ∎$$

We could now use the special adjoint functor theorem to show that each U^C has a right adjoint R^C. It follows from 1.3 that the resulting functor $R: Vect \to Coalg$ is indexed. In the following theorem, R^C is constructed explicitly from the cofree functor R.

1.5. THEOREM. *U has an indexed right adjoint R.*

Proof. For any vector space V, let RV denote the cofree coalgebra on V [S, p. 125]. Let C be any coalgebra and (A, α) a C-comodule. For any $\phi: D \to C$ in Coalg^C the natural transformations

$$
\begin{array}{c}
(\phi) \to C \otimes RA \text{ in Coalg}^C \\
\cong \ \rule{4cm}{0.4pt} \\
D \to RA \text{ in Coalg} \\
\cong \ \rule{4cm}{0.4pt} \\
D \xrightarrow{\ u\ } A \text{ in Vect} \\
\downarrow \ \rule{4cm}{0.4pt} \\
D \xrightarrow{\ u\ } A \xrightarrow{\ \alpha\ } C \otimes A \\
\\
D \xrightarrow{\ \delta\ } D \otimes D \xrightarrow{\ \phi \otimes u\ } C \otimes A \text{ in Vect} \\
\downarrow \ \rule{5cm}{0.4pt} \\
D \rightrightarrows R(C \otimes A) \text{ in Coalg} \\
\cong \ \rule{4cm}{0.4pt} \\
(\phi) \rightrightarrows C \otimes R(C \otimes A) \text{ in Coalg}^C
\end{array}
$$

give two morphisms

$$C \otimes RA \underset{\ }{\overset{r}{\rightrightarrows}} C \otimes R(C \otimes A)$$

in Coalg^C. Define $R^C(A, \alpha)$ to be the equalizer of r and s in Coalg^C. Then for any (ϕ) in Coalg^C we have the natural bijections

$$
\cfrac{(\phi) \to R^C(A, \alpha) \text{ in } \text{Coalg}^C}
{\cfrac{(\phi) \xrightarrow{\ t\ } C \otimes RA \text{ such that } rt = st}
{\cfrac{D \xrightarrow{\ u\ } A \text{ in Vect such that } \alpha u = C \otimes u \cdot \phi \otimes D \cdot \delta}
{U^C(\phi) \to (A, \alpha) \text{ in } \text{Vect}^C.}}}
$$

Thus $U^C \dashv R^C$.

Finally, the functors R^C make R into an indexed functor because U preserves Σ. Indeed, the left adjoints of all functors in the diagram

$$
\begin{array}{ccc}
\text{Vect}^C & \xrightarrow{\ \phi^*\ } & \text{Vect}^D \\
{\scriptstyle R^C}\downarrow & & \downarrow{\scriptstyle R^D} \\
\text{Coalg}^C & \xrightarrow[\phi^*]{} & \text{Coalg}^D
\end{array}
$$

give the commutative diagram of Theorem 1.3. ∎

1.6. COUNTEREXAMPLE (to the indexedness of $U: \mathbf{Z}\text{-Coalg} \to \mathbf{Ab}$):   Let $C = \mathbf{Z} \oplus (\mathbf{Z}/2\mathbf{Z})$ with $\varepsilon(a, b) = a$ and $\delta(a, b) = a(1, 0) \otimes (1, 0) + b((1, 0) \otimes (0, 1) + (0, 1) \otimes (1, 0) + (0, 1) \otimes (0, 1))$. It is straightforward to check that (C, ε, δ) is a cocommutative $\mathbf{Z}$-coalgebra with the two points $(1, 0)$ and $(1, 1)$.

Let the points $(1, 0)$ and $(1, 1)$ be given by $p: \mathbf{Z} \to C$ and $q: \mathbf{Z} \to C$ respectively. We shall see that

$$
\begin{array}{ccccccc}
\text{Coalg}/C & = & \text{Coalg}^C & \xrightarrow{\ p^*\ } & \text{Coalg}^{\mathbf{Z}} & \cong & \text{Coalg} \\
{\scriptstyle U^C}\downarrow & & {\scriptstyle U^C}\downarrow & & \downarrow{\scriptstyle U^{\mathbf{Z}}} & & \downarrow{\scriptstyle U} \\
C\text{-Comod} & = & \mathbf{Ab}^C & \xrightarrow[p_*]{} & \mathbf{Ab}^{\mathbf{Z}} & \cong & \mathbf{Ab}
\end{array}
$$

does not commute up to isomorphism. Since

$$
\begin{array}{ccc}
p^*(q) & \longrightarrow & \mathbf{Z} \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle q} \\
\mathbf{Z} & \xrightarrow[p]{} & C
\end{array}
$$

commutes, it follows that ϕ maps into $2\mathbf{Z}$. However, if $\psi: D \to \mathbf{Z}$ is any coalgebra map then factoring the top and bottom maps in

$$
\begin{array}{ccc}
D & \xrightarrow{\ \psi\ } & \mathbf{Z} \\
{\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\cong} \\
D \otimes D & \xrightarrow[\ \psi \otimes \psi\ ]{} & \mathbf{Z} \otimes \mathbf{Z}
\end{array}
$$

we see that $\operatorname{im}(\psi) \subseteq \operatorname{im}(\psi) \cdot \operatorname{im}(\psi)$. Thus $\operatorname{im}(\psi) = 0$ or $\mathbf{Z}$. We conclude that $\phi = 0$ and $p^*(q) = 0$. Hence, $Up^*(q) = 0$ in **Ab**. On the other hand, $U^C(q) = \mathbf{Z}$ made into a C-comodule via q and $p^* U^C(q) = \operatorname{eq}(\mathbf{Z} \rightrightarrows^q_p C) = 2\mathbf{Z}$.
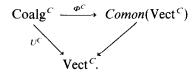
2. Comonoids

2.1. Vect^C is a symmetric monoidal category with $\otimes^C$ as multiplication, and so we can consider the category of cocommutative comonoids in Vect^C, which we shall denote $Comon(\operatorname{Vect}^C)$.

An object in Vect^C is meant to be thought of as a family of vector spaces and $\otimes^C$ as the memberwise $\otimes$ of families of vector spaces. Thus an object of $Comon(\operatorname{Vect}^C)$ should be a C-indexed family of coalgebras. We already have a definition of a C-indexed family of coalgebras and the following theorem says that the two concepts agree.

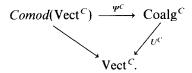**2.2. THEOREM.** *There is an equivalence of categories* $Comon(\operatorname{Vect}^C) \cong \operatorname{Coalg}^C$.

Proof. The functor $U^C: \operatorname{Coalg}^C \to \operatorname{Vect}^C$ takes the cartesian product in Coalg^C into $\otimes^C$ in Vect^C, by Theorem 1.4. In a cartesian category, every object has a unique structure of cocommutative comonoid given by the diagonal and the unique map to the terminal object. Thus, for any $(\phi) \in \operatorname{Coalg}^C$, $U^C(\phi)$ has a canonical cocommutative comonoid structure. Also, every morphism gets sent to a homomorphism. So U^C extends to a functor

$$
\begin{array}{ccc}
\operatorname{Coalg}^C & \xrightarrow{\ \varPhi^C\ } & Comon(\operatorname{Vect}^C) \\[2pt]
{\scriptstyle U^C}\searrow & & \swarrow \\[2pt]
& \operatorname{Vect}^C.
\end{array}
$$

The definition of $\otimes^C$ as an equalizer gives morphisms $m: A \otimes^C B \to A \otimes B$ which, together with $\varepsilon: C \to k$, make $\Sigma_C: \operatorname{Vect}^C \to \operatorname{Vect}$ into a symmetric comonoidal functor. Thus Σ_C takes comonoids in Vect^C into comonoids in Vect. If (A, d, e) is a comonoid in Vect^C, then there is a homomorphism

$\phi: (A, d, e) \to (C, 1_C, 1_C)$ into the terminal comonoid $(C, 1_C, 1_C)$. If we apply Σ_C to ϕ we get an object of $Coalg^C$, namely $\phi: (A, md, \varepsilon e) \to (C, \delta, \varepsilon)$. Thus we have a functor

$$Comod(Vect^C) \xrightarrow{\Psi^C} Coalg^C$$
$$\searrow \qquad \swarrow U^C$$
$$Vect^C.$$

A simple calculation shows that Φ^C and Ψ^C are inverse equivalences. ∎

2.2.1. In fact, we see from the proof that everything is indexed. So we get an indexed equivalence

$$Coalg \simeq Comon(Vect),$$

i.e., for any $\phi: D \to C$ we have a commutative (up to isomorphism) diagram

$$
\begin{array}{ccccc}
Coalg^C & \xrightarrow{\Phi^C} & Comon(Vect^C) & \xrightarrow{\Psi^C} & Coalg^C \\
\phi^* \downarrow & & Comon(\phi^*) \downarrow & & \downarrow \phi^* \\
Coalg^D & \xrightarrow[\Phi^D]{} & Comon(Vect^D) & \xrightarrow[\Psi^D]{} & Coalg^D.
\end{array}
$$

We are now in a position to prove the analogue of Theorem III.3.2 for the indexed category $Coalg$.

2.3. THEOREM. *If $\langle C_\nu \rangle$ is a family of coalgebras indexed by a set I, then*

$$Coalg^{\oplus C_\nu} \simeq \Pi\, Coalg^{C_\nu}.$$

Proof. Let $i_\nu: C_\nu \to \oplus C_\nu$ be the νth injection. Then Theorem III.3.2 says that the functor

$$(i_\nu^*): \quad Vect^{\oplus C_\nu} \to \Pi\, Vect^{C_\nu}$$

is an equivalence of categories. $\Pi\, Vect^{C_\nu}$ can be made into a monoidal category by defining $\langle A_\nu \rangle \otimes \langle B_\nu \rangle$ to be $\langle A_\nu \otimes^{C_\nu} B_\nu \rangle$. Since $\otimes$ is indexed, (i_ν^*) preserves $\otimes$ and so extends to an equivalence
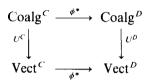
$$Comon(Vect^{\oplus C_\nu}) \simeq Comon(\Pi\, Vect^{C_\nu})$$

which, together with Theorem 2.2, establishes the result. ∎

This result can be understood from the point of view of families. A family indexed by a coproduct should be the same as an ordinary family of

families indexed by each summand. A special case of Theorem 2.3 says that a family of coalgebras indexed by $\bigoplus_I k$ is the same as an ordinary family of coalgebras indexed by the set I.

Theorem 2.3 can be viewed as saying that coproducts in Coalg are disjoint and universal [Gro2, p. 243]. This is seen more directly by considering the commutative diagram

$$
\begin{array}{ccc}
\text{Coalg}^C & \xrightarrow{\;\phi^*\;} & \text{Coalg}^D \\
{\scriptstyle U^C}\downarrow & & \downarrow{\scriptstyle U^D} \\
\text{Vect}^C & \xrightarrow[\phi^*]{} & \text{Vect}^D
\end{array}
$$

arising from a homomorphism $\phi: D \to C$. The bottom ϕ^* preserves coproducts (Proposition I.3.1), U^C preserves colimits and U^D reflects them. Thus the top ϕ^* preserves coproducts. This shows that coproducts are universal. That they are disjoint follows from the fact that they are the same as for vector spaces.

2.4. As an application of 2.3 we get the following well-known result [S, p. 163, Gru2].

PROPOSITION. *Every coalgebra can be expressed uniquely as a coproduct of irreducible subcoalgebras (i.e., subcoalgebras which are not themselves coproducts of nontrivial subcoalgebras).*

Proof. If $C \neq 0$, let $X(C)$ be the set of minimal subcoalgebras of C. Since a finite dimensional algebra is uniquely the product of local algebras, the result certainly holds for finite dimensional coalgebras. In this case $C = \bigoplus_{S \in X(C)} C_S$, where C_S is the irreducible subcoalgebra containing the minimal subcoalgebra S, and we can define a coalgebra map $\chi: C \to \bigoplus_{X(C)} k$ by letting $\chi = \bigoplus_{X(C)} \varepsilon$. For a general coalgebra C, let $kX(C) = \bigoplus_{X(C)} k$. If $\{\kappa_v: C_v \to C\}$ is a covering of C by finite dimensional subcoalgebras, then $X(C) = \bigcup X(C_v)$ and $kX(C) = \varinjlim kX(C_v)$. Let $\chi = \varinjlim \chi_v: C \to kX(C)$. By 2.3, we have $C \cong \bigoplus_{S \in X(C)} C_S$ and the C_S are clearly irreducible. ∎

2.5. By 2.2, an object $\phi: D \to C$ of Coalg^C can be considered as a comonoid in Vect^C. A coassociative counitary coaction of (ϕ) on an object of Vect^C will be called a (ϕ)-comodule and the category of such will be denoted (ϕ)-comod. According to our interpretation, (ϕ) is a C-indexed family of coalgebras and such a coaction is a C-indexed family of com-

odules, one over each member of (ϕ). The analogue of Theorem III.3.2 for C-indexed coproducts would say that a (ϕ)-comodule is the same as a comodule over $\Sigma_C(\phi) = D$. That this is the case is the content of the following proposition.

2.5.1. PROPOSITION. *There is an equivalence of categories*

$$(\phi)\text{-comod} \simeq \mathrm{Vect}^D.$$

Proof. A (ϕ)-comodule in Vect^C consists of a C-comodule structure $\alpha: A \to C \otimes A$ together with a C-homomorphism $\beta: A \to D \otimes^C A$. A straightforward calculation shows that

$$A \xrightarrow{\ \beta\ } D \otimes^C A \to D \otimes A$$

makes A into a D-comodule. This gives the equivalence in one direction.

In the other direction, if $\alpha': A \to D \otimes A$ is a D-comodule structure, then $(A, \phi \otimes A \cdot \alpha')$ is a C-comodule and α' factors through $D \otimes^C A \to D \otimes A$. This gives a (ϕ)-comodule structure on A. It is easily checked that the above constructions are inverse to each other. ∎

2.3.2. *Remark.* If $\gamma: (\psi) \to (\phi)$ is a morphism of Coalg^C, we can define a functor

$$\Sigma_\gamma: \quad (\psi)\text{-comod} \to (\phi)\text{-comod}$$

by

$$\Sigma_\gamma(A, \alpha) = (A, \gamma \otimes^C A \cdot \alpha).$$

This functor has a left adjoint γ^* given by the same equalizer formula as in III.1.1 with $\otimes$ replaced by $\otimes^C$. It is clear from the above proof that the equivalences constructed there commute with Σ_γ and, by adjointness, also with γ^*.

2.5.3. THEOREM. *The Σ for Vect satisfies the Beck condition, i.e., if*

$$
\begin{array}{ccc}
D' & \xrightarrow{\ \phi'\ } & C' \\
\theta \downarrow & & \downarrow \psi \\
D & \xrightarrow[\ \phi\ ]{} & C
\end{array}
$$

is a pullback diagram of coalgebras, then the following diagram commutes up to canonical isomorphism

$$\begin{array}{ccc} \mathrm{Vect}^D & \xrightarrow{\Sigma_\phi} & \mathrm{Vect}^C \\ {\scriptstyle \theta^*}\downarrow & & \downarrow{\scriptstyle \psi^*} \\ \mathrm{Vect}^{D'} & \xrightarrow[\Sigma_{\phi'}]{} & \mathrm{Vect}^{C'}. \end{array}$$

Proof. Replace Vect, C, D in Theorem III.1.3.2 by Vect^C, (ϕ), (ψ), respectively. Reinterpreting in the light of 2.5.1 and 2.5.2, we get our result. ∎

2.6. COUNTEREXAMPLE (to the stability of epimorphisms under pullbacks): Let $C_{mn} = kx_0 \oplus (\bigoplus_{i=m}^{n} kx_i)$ with $\varepsilon(x_j) = \delta_{0j}$ and $\delta(x_j) = \Sigma_{r+s=j} x_r \otimes x_s$. Consider the pullback

$$\begin{array}{ccc} P & \longrightarrow & C_{13} \\ {\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \psi} \\ C_{33} & \xrightarrow[\phi]{} & C_{23} \end{array}$$

where $\phi(x_i) = x_i$ for $i = 0, 3$ and $\psi(x_1) = 0$, $\psi(x_i) = x_i$ for $i = 0, 2, 3$. By I.1.4.5, $P = \{x \in C_{13} \mid (C_{13} \otimes \psi)\, \delta(x) \in C_{13} \otimes C_{33}\}$. If $x = ax_0 + bx_1 + cx_2 + dx_3$, then $\delta(x) = ax_0 \otimes x_0 + b(x_0 \otimes x_1 + x_1 \otimes x_0) + c(x_0 \otimes x_2 + x_1 \otimes x_1 + x_2 \otimes x_0) + d(x_0 \otimes x_3 + x_1 \otimes x_2 + x_2 \otimes x_1 + x_3 \otimes x_0)$ and $(C_{13} \otimes \psi)\, \delta(x) = ax_0 \otimes x_0 + bx_1 \otimes x_0 + c(x_0 \otimes x_2 + x_2 \otimes x_0) + d(x_0 \otimes x_3 + x_1 \otimes x_2 + x_3 \otimes x_0)$. For this result to be in $C_{13} \otimes C_{33}$, all terms with x_2 in the second factor must be 0, i.e., $c = 0 = d$. Thus $P = kx_0 \oplus kx_1 = C_{11}$ which is a subcoalgebra of C_{13} and the restriction ψ' of ψ is given by $\psi'(x_0) = x_0$, $\psi'(x_1) = 0$. This map $\psi' : P \to C_{33}$ is not an epimorphism.

2.7. Until now we have considered only cocommutative coalgebras. We can also index the category coalg^C (with a lower case "c") of arbitrary coalgebras by defining coalg^C to be the category of comonoids in Vect^C. Since $\phi^* : \mathrm{Vect}^C \to \mathrm{Vect}^D$ preserves $\otimes$ (Theorem IV.2.3), comonoids are also preserved, thus giving

$$\phi^* : \quad \mathrm{coalg}^C \to \mathrm{coalg}^D$$

and making the category of coalgebras into an indexed category *coalg*.

Although $\Sigma_\phi \colon \mathrm{Vect}^D \to \mathrm{Vect}^C$ does not preserve $\otimes$, there are comparison maps

$$\Sigma_\phi(A \otimes^D B) \to (\Sigma_\phi A) \otimes^C (\Sigma_\phi B)$$

$$\Sigma_\phi D \to C$$

which make Σ_ϕ into a comonoidal functor so it preserves comonoids. This gives

$$\Sigma_\phi \colon \quad \mathrm{coalg}^D \to \mathrm{coalg}^C$$

which is left adjoint to ϕ^*. These Σ_ϕ satisfy the Beck condition as they are the same as for *Vect*. Thus *coalg* has Coalg-indexed coproducts.

As cocommutative comonoids in Vect^C are the same as cocommutative coalgebras over C (Theorem 2.2), it is not surprising that (not necessarily cocommutative) comonoids in Vect^C correspond to certain coalgebras over C. The following proposition says that they do and the corresponding coalgebras are those satisfying a condition dual to the commuting of scalars with elements of an algebra.

PROPOSITION. *The category* coalg^C *is equivalent to the full subcategory of* coalg/C *determined by those* $\phi \colon D \to C$ *such that*

$$
\begin{array}{ccc}
D & \xrightarrow{\;\;\delta\;\;} & D \otimes D \\
{\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle D \otimes \phi} \\
D \otimes D \xrightarrow[\phi \otimes D]{} C \otimes D \xrightarrow[\cong]{} & & D \otimes C.
\end{array}
$$

Proof. If $d \colon D \to D \otimes^C D$, $\phi \colon D \to C$ is a comonoid in Vect^C, then an easy calculation shows that $\delta = (D \to^d D \otimes^C D \to D \otimes D)$ and $\varepsilon = (D \to^\phi C \to^\varepsilon k)$ make D into a coalgebra for which $\phi \colon D \to C$ is a homomorphism. This gives a full and faithful functor

$$\mathrm{coalg}^C \to \mathrm{coalg}/C$$

whose image is precisely the full subcategory of those coalgebras over C satisfying the condition of the proposition. ∎

VI. Σ, Hom, Π, COFLATNESS

We have seen that the substitution functors ϕ^* always have left adjoints Σ_ϕ. By III.2.2 and V.2.6, the ϕ^* and thus also the functors $(\;)\otimes^C A$ do not

preserve coequalizers in general. The question of when right adjoints, Π_ϕ and $\hom^C(A, \ )$, for these functors exist will be dealt with in the present chapter.

1. *On the Stability of Colimits*

It is easy to see from the definition that, for every coalgebra C, the bifunctor $\otimes^C$ preserves coproducts in each variable (it is true for $\otimes$ in Vect and coproducts commute with equalizers). This result has the following generalization to Coalg-indexed coproducts.

1.1. THEOREM. *Let* $\phi: D \to C$ *be a coalgebra map. If* (A, α) *is a* D-*comodule and* (B, β) *a* C-*comodule, then*

$$\Sigma_\phi (A \otimes^D \phi^* B) \cong (\Sigma_\phi A) \otimes^C B.$$

In particular

$$\Sigma_\phi \phi^* B \cong D \otimes^C B.$$

Proof. $\phi^* B$ is defined by the equalizer in Vect^D

$$\phi^* B \to D \otimes B \xrightarrow{\ \delta \otimes B\ } D \otimes D \otimes B$$

with maps $D \otimes \beta$ and $D \otimes \phi \otimes B$ to $D \otimes C \otimes B$.

Since $A \otimes^D (\ )$ and Σ_ϕ preserve equalizers we get an equalizer diagram

$$A \otimes^D \phi^* B \to A \otimes^D (D \otimes B) \xrightarrow{\ A \otimes^D (\delta \otimes B)\ } A \otimes^D (D \otimes D \otimes B)$$

with maps $A \otimes^D (D \otimes \beta)$ and $A \otimes^D (D \otimes \phi \otimes B)$ to $A \otimes^D (D \otimes C \otimes B)$.

which, by IV.1.2.2 and the definition of Σ_ϕ and $\otimes^C$, is isomorphic to the equalizer diagram

$$(\Sigma_\phi A) \otimes^C B \to A \otimes B \xrightarrow{\ \alpha' \otimes B\ } A \otimes D \otimes B$$

with maps $A \otimes \beta$ and $A \otimes \phi \otimes B$ to $A \otimes C \otimes B$.

The resulting isomorphism $\Sigma_\phi (A \otimes^D \phi^* B) \cong (\Sigma_\phi A) \otimes^C B$ is in Vect^C since, for every vector space V, the isomorphism $\Sigma_\phi (A \otimes^D (D \otimes V)) \cong$

$(A \otimes V, (\phi \otimes A \otimes V)(\alpha \otimes V))$ is an isomorphism of C-comodules. The particular case is obtained by letting $(A, \alpha) = (D, \delta)$. ∎

1.2. The corresponding result in *Coalg* is of course a simple consequence of the fact that the pullback along a composite is the same as pasting the individual pullbacks together.
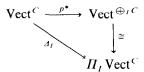
PROPOSITION. *For any coalgebra map $\phi: D \to C$, any object (ξ) of* Coalg^D *and any object (ζ) of* Coalg^C,

$$\Sigma_\phi((\xi) \times \phi^*(\zeta)) \cong (\Sigma_\phi(\xi)) \times (\zeta).$$

In particular

$$\Sigma_\phi \phi^*(\zeta) \cong (\phi) \times (\zeta).$$

If $D = \bigoplus_I C$ and ϕ is the projection $p: \bigoplus_I C \to C$, then we get the original result for ordinary coproducts since, by III.3.2, a D-comodule is nothing but an ordinary family of C-comodules indexed by the set I, and

commutes. Thus, the left adjoint Σ_p of p^* is $\bigoplus_I$ and $(\bigoplus_I A_i) \otimes^C B = \bigoplus_I (A_i \otimes^C B)$.

1.3. Given a coalgebra map $\phi: D \to C$, then $\phi^*: \text{Coalg}^C \to \text{Coalg}^D$ preserves coequalizers (colimits) if $\phi^*: \text{Vect}^C \to \text{Vect}^D$ does. This is because for every algebra E the forgetful functor $U^E: \text{Coalg}^E \to \text{Vect}^E$ preserves and reflects colimits. The converse of the above statement is also true. To prove this, it suffices to find an indexed functor $\Phi: \text{Vect} \to \text{Coalg}$ such that, for every coalgebra E, Φ^E preserves and reflects coequalizers. For any vector space V we can define a coalgebra structure on $k \oplus V$ by making each $v \in V$ primitive relative to the point 1 of k. The functor Φ is the indexed version of this.

1.3.1. Let C be a coalgebra and (A, α) a C-comodule. Define a coalgebra $(C \oplus A, \bar{\varepsilon}, \bar{\delta})$ by the formulas $\bar{\varepsilon}(c) = \varepsilon(c)$, $\bar{\varepsilon}(a) = 0$, $\bar{\delta}(c) = \delta(c)$ and $\bar{\delta}(a) = \alpha(a) + \alpha'(a)$ where $a \in A$, $c \in C$ and $\alpha' = (A \to^\alpha C \otimes A \to^\sigma A \otimes C)$. It is easily seen that all equations for a coalgebra are satisfied and that the projection $p: C \oplus A \to C$ is a coalgebra homomorphism. Define a functor

$$\Phi^C: \quad \text{Vect}^C \to \text{Coalg}^C$$

by $\Phi^C(A, \alpha) = (p: C \oplus A \to C)$ and $\Phi^C(f) = C \oplus f$. Φ^C is characterized by the natural bijection

$$\frac{(\phi) \to \Phi^C(A, \alpha) \text{ in } \operatorname{Coalg}^C}{U^C(\phi) \xrightarrow{\theta} (A, \alpha) \text{ in } \operatorname{Vect}^C \text{ such that } (\theta \otimes \theta)\, \delta = 0.}$$

1.3.2. PROPOSITION. $\Phi: Vect \to Coalg$ is an indexed functor.

Proof.   Let $\phi: D \to C$ be a coalgebra map. For any object $\psi: E \to D$ of Coalg^D and any C-comodule A, we have the natural bijections

$$\frac{(\psi) \to \Phi^D \phi^* A \text{ in } \operatorname{Coalg}^D}{U^D(\psi) \xrightarrow{\lambda} \phi^* A \text{ in } \operatorname{Vect}^D \quad \text{such that } (\lambda \otimes \lambda)\, \delta = 0}$$

$$\frac{}{\Sigma_\phi U^D(\psi) \xrightarrow{\mu} A \text{ in } \operatorname{Vect}^C \quad \text{such that } (\mu \otimes \mu)\, \delta = 0}$$

$$\frac{}{U^C \Sigma_\phi(\psi) \xrightarrow{\theta} A \text{ in } \operatorname{Vect}^C \quad \text{such that } (\theta \otimes \theta)\, \delta = 0}$$

$$\frac{}{\Sigma_\phi(\psi) \to \Phi^C A \text{ in } \operatorname{Coalg}^C}$$

$$(\psi) \to \phi^* \Phi^C A \text{ in } \operatorname{Coalg}^D. \quad \blacksquare$$

Since for every coalgebra C the functors U^C and Φ^C preserve and reflect epimorphisms, we have:

1.3.3. COROLLARY.   *Let* $\phi: D \to C$ *be   a   coalgebra   map.   Then* $\phi^*: \operatorname{Coalg}^C \to \operatorname{Coalg}^D$ *preserves   coequalizers   (colimits)   if   and   only   if* $\phi^*: \operatorname{Vect}^C \to \operatorname{Vect}^D$ *does.* $\blacksquare$

2. *Coflatness*, Hom *and* Π

2.1. DEFINITION.   A C-comodule B is said to be *coflat* if $(\ ) \otimes^C B$: $\operatorname{Vect}^C \to \operatorname{Vect}^C$ preserves epimorphisms.

2.1.1. PROPOSITION.   *If* $\phi: D \to C$ *is a coalgebra map and B is a coflat C-comodule, then $\phi^* B$ is a coflat D-comodule, i.e., coflatness is stable.*

Proof.   Let e be   an   epimorphism   in Vect^D. Since Σ_ϕ preserves epimorphisms and B is coflat, both $\Sigma_\phi e$ and $(\Sigma_\phi e) \otimes^C B$ are epimorphisms in Vect^C. By Theorem 1.1, we have $(\Sigma_\phi e) \otimes^C B \cong \Sigma_\phi(e \otimes^D \phi^* B)$ and, since Σ_ϕ reflects epimorphisms, $e \otimes^D \phi^* B$ is an epimorphism. $\blacksquare$

2.1.2.   Every vector space V is a coflat k-comodule. Thus every

cofree C-comodule $(C \otimes V, \delta \otimes V)$ is coflat. Since $\otimes^C$ preserves coproducts in each variable, we see that a coproduct $\bigoplus B_v$ is coflat in Vect^C if and only if each factor B_v is.

2.2. If C is a coalgebra and A and B are C-comodules, then the Coalg^C-valued hom of A and B, if it exists, will be denoted $\mathrm{Hom}^C(A\ B)$ (with a capital "H" to distinguish it from hom^C defined in the next theorem). It is characterized by the bijection

$$\frac{(\phi) \to \mathrm{Hom}^C(A, B) \text{ in } \mathrm{Coalg}^C}{\phi^* A \to \phi^* B \text{ in } \mathrm{Vect}^C}$$

natural in $\phi: D \to C$. (See II.1.5.)

2.3. THEOREM. *The following are equivalent for a C-comodule (A, α):*

(1)   *A is coflat.*

(2) $(\ ) \otimes^C A: \mathrm{Vect}^C \to \mathrm{Vect}^C$ *has a right adjoint* $\mathrm{hom}^C(A,)$.

(3) $\mathrm{Hom}^C(A, B)$ *exists for every B in* Vect^C.

Proof. $(1) \Rightarrow (2)$. Vect^C is an abelian category and the finite dimensional C-comodules form a set of generators for Vect^C. $(\ ) \otimes^C A$ preserves coproducts by 1.1 and epimorphisms since A is coflat. Thus it preserves all colimits. The existence of a right adjoint now follows from the special adjoint functor theorem.

$(2) \Rightarrow (3)$.   For any B in Vect^C and any $\phi: D \to C$, we have the natural bijections

$$\frac{(\phi) \to R^C \mathrm{hom}^C(A, B) \text{ in } \mathrm{Coalg}^C}{U^C(\phi) \to \mathrm{hom}^C(A, B) \text{ in } \mathrm{Vect}^C}$$

$$\frac{U^C(\phi) \otimes^C A \to B \text{ in } \mathrm{Vect}^C}{\Sigma_\phi \phi^* A \to B \text{ in } \mathrm{Vect}^C}$$

$$\phi^* A \to \phi^* B \text{ in } \mathrm{Vect}^D$$

where $R: \mathrm{Vect} \to \mathrm{Coalg}$ is the indexed "cofree coalgebra functor" (V.1.5) and $U^C(\phi) \otimes^C (\ ) \simeq \Sigma_\phi \phi^*$ by 1.1. Thus $\mathrm{Hom}^C(A, B) = R^C \mathrm{hom}^C(A, B)$ is the Coalg^C valued hom of A and B.

$(3) \Rightarrow (1)$.   The bottom part of the above sequence of bijections and the characterization of $\mathrm{Hom}^C(A, B)$ show that $U^C(\ ) \otimes^C A$ has a right adjoint,

namely $\mathrm{Hom}^C(A,)$. Thus $U^C( )\otimes^C A$ preserves epimorphisms. We shall
now use the functor $\Phi\colon Vect \to Coalg$ to show that A is coflat. If $e\colon X \to Y$ is
an epimorphism in $Vect^C$, then so is $\Phi^C(E)$ in $Coalg^C$ and so
$U^C\Phi^C(e)\otimes^C A$ is an epimorphism in $Vect^C$. This epimorphism is the
bottom map in

$$
\begin{array}{ccc}
A \oplus (X\otimes^C A) & \xrightarrow{A \oplus (e\otimes^C A)} & A \oplus (Y\otimes^C A) \\
\cong \downarrow & & \downarrow \cong \\
(C \oplus X)\otimes^C A & \xrightarrow{(C\oplus e)\otimes^C A} & (C \oplus Y)\otimes^C A
\end{array}
$$

and so $e\otimes^C A\colon X\otimes^C A \to Y\otimes^C A$ is an epimorphism in $Vect^C$. Thus A is
coflat. ∎

If $C = k$ then hom^k is the ordinary vector space hom and we see that
$Vect$ has small homs at 1. Although homs do not exist in general, those
that do are stable.

2.4. THEOREM.   Let $\phi\colon D \to C$ be a coalgebra map. If A and B are
C-comodules and A is coflat, then $\phi^* \mathrm{hom}^C(A, B) \cong \mathrm{hom}^D(\phi^*A, \phi^*B)$ and
$\phi^* \mathrm{Hom}^C(A, B) \cong \mathrm{Hom}^D(\phi^*A, \phi^*B)$.

Proof.   Let X be an arbitrary D-comodule and consider the natural
bijections

$$X \to \mathrm{hom}^D(\phi^*A, \phi^*B) \text{ in } Vect^D$$
_____

$$X\otimes^D \phi^*A \to \phi^*B \text{ in } Vect^D$$
_____

$$\Sigma_\phi(X\otimes^D \phi^*A) \to B \text{ in } Vect^C$$
_____

$$(\Sigma_\phi X)\otimes^C A \to B \text{ in } Vect^C$$
_____

$$\Sigma_\phi X \to \mathrm{hom}^C(A, B) \text{ in } Vect^C$$
_____

$$X \to \phi^* \mathrm{hom}^C(A, B) \text{ in } Vect^D$$

where Theorem 1.1 is used for the third bijection. The indexedness of
$R\colon Vect \to Coalg$ completes the argument. ∎

3. Coflat Morphisms and Π

3.1. DEFINITION.   An object $\phi\colon D \to C$ of $Coalg^C$ is said to be coflat if D
is a coflat C-comodule.

If $\psi: E \to C$ is a coalgebra map, then $U^E\psi^*(\phi) \cong \psi^*U^C(\phi)$, thus the stability of coflatness in *Coalg* follows from its stability in *Vect* and the indexedness of $U: Coalg \to Vect$.

Every projection $p: C \otimes D \to C$ is coflat as $U^C(p)$ is cofree as C-comodule and finite products of coflats are coflat since $U^C((\phi) \times (\psi)) \cong U^C(\phi) \otimes^C U^C(\psi) \cong D \otimes^C E$.

3.2. THEOREM. *The following are equivalent for a coalgebra map* $\phi: D \to C$:

(1) ϕ *is coflat.*

(2) $\phi^*: \text{Vect}^C \to \text{Vect}^D$ *has a right adjoint* Π_ϕ.

(3) $\phi^*: \text{Coalg}^C \to \text{Coalg}^D$ *has a right adjoint* Π_ϕ.

(4) $(\phi) \times (\ ): \text{Coalg}^C \to \text{Coalg}^C$ *has a right adjoint* $\text{Hom}^C((\phi),)$.

Proof. First note that Coalg^C and Vect^C satisfy the hypotheses of the special adjoint functor theorem.

$(1) \Leftrightarrow (2)$. $\phi^*: \text{Vect}^C \to \text{Vect}^D$ preserves coproducts by III.2.1. So, ϕ has a right adjoint if and only if it preserves coequalizers. $U^C(\phi) \otimes^C (\ ) \simeq \Sigma_\phi \phi^*(\ )$ by 1.1, and Σ_ϕ preserves and reflects epimorphisms. Thus, ϕ^* has a right adjoint if and only if ϕ is coflat.

$(2) \Leftrightarrow (3)$. $\phi^*: \text{Coalg}^C \to \text{Coalg}^D$ preserves coproducts by V.2.3 and, by 1.3.3, it preserves coequalizers if and only if $\phi^*: \text{Vect}^C \to \text{Vect}^D$ does.

$(3) \Leftrightarrow (4)$. $(\phi) \times (\ ) \simeq \Sigma_\phi \phi^*(\ )$ by 1.2 and Σ_ϕ preserves and reflects colimits. Thus, $(\phi) \times (\ )$ has a right adjoint if and only if ϕ^* has. ∎

3.2.1. COROLLARY. $\phi = \bigoplus \phi_v: \bigoplus D_v \to \bigoplus C_v$ *is coflat if and only if each* $\phi_v: C_v \to D_v$ *is.*

Proof. The diagram

$$
\begin{array}{ccc}
\text{Vect}^{\oplus\, C_v} & \xrightarrow{\ \phi^*\ } & \text{Vect}^{\oplus\, D_v} \\
{\scriptstyle (i_v^*)} \downarrow & & \downarrow {\scriptstyle (i_v^*)} \\
\Pi\, \text{Vect}^{C_v} & \xrightarrow[\ \Pi\, \phi_v^*\ ]{} & \Pi\, \text{Vect}^{D_v}
\end{array}
$$

in which the vertical functors are equivalences, commutes up to isomorphism. $\Pi\, \phi_v^*$ has a right adjoint if and only if each ϕ_v^* has. The "if" part is clear. The "only if" part follows from the fact that the projection functor $P_v: \Pi\, \text{Vect}^{C_v} \to \text{Vect}^{C_v}$ has the functor Q_v which sends a C_v-comodule (A, α) to the family with vth member (A, α) and all other

members 0 as left and right adjoints. The result follows from the commutativity of

$$\begin{array}{ccc}
\Pi \, \mathrm{Vect}^{C_v} & \xrightarrow{\;\Pi\,\phi_v^*\;} & \Pi \, \mathrm{Vect}^{D_v} \\
{\scriptstyle Q_v}\big\uparrow & & \big\downarrow{\scriptstyle P_v} \\
\mathrm{Vect}^{C_v} & \xrightarrow[\;\phi_v^*\;]{} & \mathrm{Vect}^{D_v}. \quad \blacksquare
\end{array}$$

3.2.2. COROLLARY.   *If ϕ is a sum of projections then ϕ is coflat. In particular, coproduct injections are coflat.*

3.2.3. COROLLARY.   *Let $E \xrightarrow{\psi} D \xrightarrow{\phi} C$ be coalgebra maps.*

   (i)   *If ϕ and ψ are coflat then so is $\phi \circ \psi$.*

   (ii)   *If $\phi \circ \psi$ is coflat and ϕ^* is faitful then ψ is coflat.*

   3.3.   Since Σ satisfies the Beck condition (VI.2.1), it follows by adjointness that Π does whenever it exists, i.e., if

$$\begin{array}{ccc}
D' & \xrightarrow{\;\phi'\;} & C' \\
{\scriptstyle \theta}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\
D & \xrightarrow[\;\phi\;]{} & C
\end{array}$$

is a pullback with ψ coflat, then θ is coflat and

$$\begin{array}{ccc}
\mathrm{Vect}^{C'} & \xrightarrow{\;\phi'^*\;} & \mathrm{Vect}^{D'} \\
{\scriptstyle \Pi_\psi}\big\downarrow & & \big\downarrow{\scriptstyle \Pi_\theta} \\
\mathrm{Vect}^{C} & \xrightarrow[\;\phi^*\;]{} & \mathrm{Vect}^{D}
\end{array}$$

commutes up to isomorphism.

   In particular, for any coalgebra E and any coalgebra map $\phi: D \to C$, the square

$$\begin{array}{ccc}
E \otimes D & \xrightarrow{\;E \otimes \phi\;} & E \otimes C \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
D & \xrightarrow[\;\phi\;]{} & C
\end{array}$$

is a pullback and the projections p are coflat. Thus Π satisfies the Beck condition at 1, which means that for every coalgebra E the Coalg-*indexed product functor*

$$\Pi_E: \quad \text{Vect}^E \to \text{Vect}$$

is indexed. This is the indexed version of $\Pi_E: \text{Vect}^E \to \text{Vect}$, which can be computed by the formula $\Pi_E (A, \alpha) = \text{Vect}^E((E, \delta), (A, \alpha))$, and should be interpreted as the product of families of vector spaces indexed by E.

If $E = \bigoplus_I k$ is the I-fold coproduct of k, then $\Pi_E = \Pi_I$, the ordinary product functor.

4. Characterization of Coflat Comodules

Let us prove first that over an irreducible (colocal) coalgebra, coflatness is the same as cofreeness. This is the "dual" of the well-known fact that a finitely generated projective module over a local ring is free.
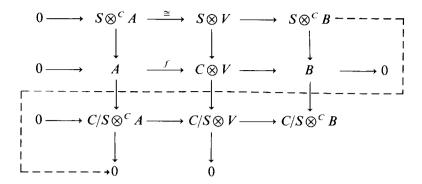
4.1. PROPOSITION. *Let* C *be an irreducible* (*colocal*) *coalgebra with minimal* (*simple*) *subcoalgebra* $\kappa: S \hookrightarrow C$ *and let* A *be a* C-*comodule. Then*

(i) $\kappa^* A = 0$ *if and only if* $A = 0$.

(ii) A *is coflat if and only if* A *is cofree.*

Proof. (i) If $A \neq 0$ then A contains a minimal (simple) subcomodule. This is because every finite dimensional $A \neq 0$ does and the finite dimensional C-comodules form a set of generators for Vect^C. Thus it suffices to show that $\kappa^*(A) \neq 0$ if $A \neq 0$ and simple. Since $\text{Vect}^C(A, C) \cong \text{Vect}(A, k)$, there is a nonzero C-homomorphism $f: A \to C$ which is one-to-one since A is simple and factors through κ since C is colocal. Thus, $A \cong S$ and $\kappa^*(A) \cong \kappa^*(S) = S \neq 0$.

(ii) $\kappa^*(A)$ is cofree as an S-comodule, since S is simple (i.e., S^* is a finite field extension of k). Thus, $\kappa^*(A) \cong S \otimes V$ for some vector space V. This isomorphism extends to a C-homomorphism $f: A \to C \otimes V$ such that

$$
\begin{array}{ccc}
\Sigma_\kappa \kappa^*(A) & \overset{\cong}{\longrightarrow} & \Sigma_k(S \otimes V) \\
\downarrow & & \downarrow \\
A & \underset{f}{\longrightarrow} & C \otimes V
\end{array}
$$

commutes. f is injective by (i), since $\kappa^*(\ker f) = 0$. If $B = \text{coker} f$, we get the following diagram with exact rows and columns in Vect^C

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S \otimes^C A & \overset{\cong}{\longrightarrow} & S \otimes V & \longrightarrow & S \otimes^C B & \dashrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & A & \overset{f}{\longrightarrow} & C \otimes V & \longrightarrow & B & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & C/S \otimes^C A & \longrightarrow & C/S \otimes V & \longrightarrow & C/S \otimes^C B & \\
 & & \downarrow & & \downarrow & & & \\
 & & 0 & & 0 & & &
\end{array}
$$

The first column is exact since A is coflat. By the snake lemma, it follows that $\Sigma_\kappa \kappa^* B = S \otimes^C B = 0$. Thus, $B = 0$ by (i) and f is an isomorphism. ∎

4.2. THEOREM. *A C-comodule B is coflat if and only if $C \cong \bigoplus C_v$ with $i_v^* B$ cofree, i.e., $i_v^* B \simeq C_v \otimes V_v$, for vector spaces V_v.*

Proof. First assume that $C \cong \bigoplus C_v$ and that $i_v^* B \cong C_v \otimes V_v$ for every v. By III.3.2

$$(i_v^*): \quad \mathrm{Vect}^C \xrightarrow{\cong} \Pi_v \, \mathrm{Vect}^{C_v}$$

is an equivalence of categories. $C_v \otimes V_v$ is coflat in Vect^{C_v} since every vector space is coflat and since coflatness is stable. Since (i_v^*) preserves $\otimes$, we see that B must be coflat. Conversely, suppose that B is a coflat C-comodule. Express C as the coproduct of its irreducible components $C \cong \bigoplus_{X(C)} C_S$ as in V.2.4. The C_S-comodule $i_S^*(B)$ is coflat by 2.1.1 and thus cofree by the previous proposition. ∎
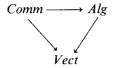
VII. FAMILIES OF ALGEBRAS

The two previous sections have illustrated that $\otimes^C$ can usefully be thought of as the memberwise tensor product of two C-indexed families of vector spaces. Cocommutative comonoids with respect to $\otimes^C$ turned out to be cocommutative coalgebras over C, i.e., C-indexed families of such coalgebras. In this section, we introduce C-indexed families of (commutative) algebras as (commutative) monoids with respect to $\otimes^C$. We justify this new concept with numerous examples. In particular, we give an interpretation of measurings [S, p. 139] in terms of families, which suggests a way of composing measurings and explains how actions of coalgebras on vector spaces induce measurings on the tensor and symmetric algebras. Also, 2-cocycles on algebras appear as families of algebras and the usual

construction of an algebra from a 2-cocycle is a special case of Π_C. We also construct a coalgebra of all finite dimensional algebras. The concept of a family of algebras is necessary even to say what this means.

1. *Definitions and Elementary Properties*

1.1. Let Alg^C and Comm^C be the category of monoids and the category of commutative monoids in the monoidal category $(\text{Vect}^C, \otimes^C)$. For any coalgebra map $\phi: D \to C$, $\phi^*: \text{Vect}^C \to \text{Vect}^D$ preserves the tensor and so lifts to $\phi^*: \text{Alg}^C \to \text{Alg}^D$ and $\phi^*: \text{Comm}^C \to \text{Comm}^D$, thus giving indexed categories *Alg* and *Comm*. We have the obvius forgetful functors

$$Comm \longrightarrow Alg$$
$$\searrow \qquad \swarrow$$
$$Vect$$

all of which are obviously indexed. An object of $\text{Alg}^C(\text{Comm}^C)$ is called a *C-algebra* (resp. a *commutative C-algebra*).

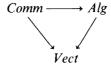1.2. PROPOSITION. *Let C_v be a family of coalgebras indexed by a set I. Then we have the equivalences*

$$\text{Alg}^{\oplus C_v} \simeq \Pi \, \text{Alg}^{C_v} \qquad \text{and} \qquad \text{Comm}^{\oplus C_v} \simeq \Pi \, \text{Comm}^{C_v}.$$

In particular, $\text{Alg}^0 \simeq 1 \simeq \text{Comm}^0$.

Proof. This is an immediate consequence of Theorem III.3.2 and IV.2.3. ∎

Since $\text{Alg}^k \simeq \text{Alg}$ and $\text{Comm}^k \simeq \text{Comm}$, we see that ordinary families of algebras indexed by the set I are the same as families of algebras indexed by the coalgebra $\oplus_I k$.

1.3. PROPOSITION. *The forgetful functors*

$$Comm \longrightarrow Alg$$
$$\searrow \qquad \swarrow$$
$$Vect$$

create all $\varprojlim$ and any Π_ϕ which exists in Vect. The $\varprojlim$ are stable. If ϕ is coflat then Π_ϕ exists and satisfies the Beck condition.

Proof. We shall prove the statements concerning Π_ϕ, the case of $\varprojlim$ being similar. Let $\phi: D \to C$ be a coalgebra map for which Π_ϕ exists in *Vect*

(by Theorem VI.3.2, it exists exactly if ϕ is coflat). Let A be a D-algebra with multiplication $m: A \otimes^D A \to A$ and unit $u: D \to A$. By adjointness of ϕ^* and Π_ϕ, there exist unique $m': (\Pi_\phi A) \otimes^C (\Pi_\phi A) \to \Pi_\phi A$ and $u': C \to \Pi_\phi A$ such that the following diagrams commute

$$\phi^*(\Pi_\phi A \otimes^C \Pi_\phi A) \cong (\phi^* \Pi_\phi A) \otimes^D (\phi^* \Pi_\phi A) \xrightarrow{\varepsilon \otimes^D \varepsilon} A \otimes^D A$$

$$\phi^*(m') \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow m$$

$$\phi^* \Pi_\phi A \xrightarrow{\qquad\qquad\qquad \varepsilon \qquad\qquad\qquad} A$$

$$\phi^* C \xrightarrow{\;\cong\;} D$$

$$\phi^*(u') \downarrow \qquad\qquad \downarrow u$$

$$\phi^* \Pi_\phi A \xrightarrow{\;\varepsilon\;} A.$$

The uniqueness of m' and u' permits the transfer of the associative (commutative) and unit laws from A to $\Pi_\phi A$, giving a unique (commutative) monoid structure on $\Pi_\phi A$. If B is a C-algebra then the natural bijection

$$\frac{B \xrightarrow{\;f\;} \Pi_\phi A}{\phi B \xrightarrow{\;g\;} A}$$

given by $g = \varepsilon \phi^*(f)$, restricts to monoid homomorphisms, so that Π_ϕ lifts to a right adjoint for $\phi^*: \mathrm{Alg}^C \to \mathrm{Alg}^D$ and for $\phi^*: \mathrm{Comm}^C \to \mathrm{Comm}^D$. Since the forgetful functors reflect isomorphisms and the Beck condition holds for Π_ϕ in *Vect*, it also holds in *Alg* and *Comm*. $\blacksquare$

1.4. If A and B are in Alg^C, then the Coalg^C-valued hom of A and B is characterized by the bijection

$$\frac{(\phi) \to \mathrm{Hom}^C_{Alg}(A, B) \text{ in } \mathrm{Coalg}^C}{\phi^* A \to \phi^* B \text{ in } \mathrm{Alg}^C}$$

for any $\phi: D \to C$ in Coalg^C.

1.4.1. THEOREM. *Let A and B be in Alg^C (or in Comm^C). If A is coflat as a C-comodule then $\mathrm{Hom}^C_{Alg}(A, B)$ exists and*

$$\phi^* \mathrm{Hom}^C_{Alg}(A, B) \cong \mathrm{Hom}^D_{Alg}(\phi^* A, \phi^* B)$$

for any coalgebra map $\phi: D \to C$.

Proof. Since A is coflat, so is $A \otimes^C A$. Of course, C itself is coflat. Thus, by Theorem VI.2.3, $\operatorname{Hom}^C_{Vect}(A, B)$, $\operatorname{Hom}^C_{Vect}(A \otimes^C A, B)$ and $\operatorname{Hom}^C_{Vect}(C, B)$ all exist. Now define $\operatorname{Hom}^C_{Alg}(A, B)$ by the limit of the following diagram in Coalg^C

$$\begin{array}{ccc}
\operatorname{Hom}^C_{Vect}(A \otimes^C A, B \otimes^C B) & \xrightarrow{\operatorname{Hom}^C_{Vect}(A \otimes^C A, m)} & \operatorname{Hom}^C_{Vect}(A \otimes^C A, B) \\
\Big\uparrow s_1 & & \\
\operatorname{Hom}^C_{Vect}(A, B) & & \\
\Big\downarrow s_2 & & \\
\operatorname{Hom}^C_{Vect}(C, C) & \xrightarrow{\operatorname{Hom}^C_{Vect}(C, u)} & \operatorname{Hom}^C_{Vect}(C, B)
\end{array}$$

with diagonal arrows $\operatorname{Hom}^C_{Vect}(m, B)$ and $\operatorname{Hom}^C_{Vect}(u, B)$ from $\operatorname{Hom}^C_{Vect}(A, B)$.

where s_1 and s_2 are "strength morphisms" given as follows. For a coalgebra map $\phi: D \to C$, a morphism $f: (\phi) \to \operatorname{Hom}^C_{Vect}(A, B)$ corresponds to a D-comodule map $\bar{f}: \phi^* A \to \phi^* B$. Then $s_1 f$ corresponds to
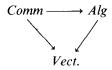
$$\phi^*(A \otimes^C A) \cong \phi^* A \otimes^D \phi^* A \xrightarrow{\bar{f} \otimes^D \bar{f}} \phi^* B \otimes^D \phi^* B \cong \phi^*(B \otimes^C B),$$

and $s_2 f$ corresponds to 1_D. The required bijection is easily checked.

Clearly, $\operatorname{Hom}^C_{Comm}(A, B)$ exists and is isomorphic to $\operatorname{Hom}^C_{Alg}(A, B)$ if A and B are in Comm^C. The last assertion holds in *Vect* and, since ϕ^* preserves $\varprojlim$, it also holds in *Alg*. ∎

2. Tensor Algebra and Symmetric Algebra

This section is concerned with the extension of the tensor algebra and the symmetric algebra constructions to families of vector spaces indexed by coalgebras, i.e., with the question of existence and indexedness of left adjoints to the forgetful functors

$$\begin{array}{ccc}
Comm & \longrightarrow & Alg \\
& \searrow \quad \swarrow & \\
& Vect. &
\end{array}$$

In the process, it will be necessary to construct equalizers in Alg^C and Comm^C.

2.1. THEOREM. *The forgetful functor $U: Alg \to Vect$ has an indexed left adjoint.*

Proof. Since Vect^C has coproducts and $A \otimes^C -$ preserves them,

U^C: $\text{Alg}^C \to \text{Vect}^C$ has a left adjoint T^C [ML2, p. 168, Theorem 2]. $T^C A$ is given by $\bigoplus_{n \in N} A^{(n)}$, where $A^{(n)} = A \otimes^C (A \otimes^C (\cdots \otimes^C A))$ with n factors. T is indexed since coproducts and tensor products in *Vect* are stable (Proposition III.2.1 and Theorem IV.2.3). ∎

2.2. COROLLARY. U^C: $\text{Alg}^C \to \text{Vect}^C$ *is tripleable (monadic) for every coalgebra C.*

Proof. Easy by standard arguments (see, e.g., [ML2, p. 152]). ∎

2.3. PROPOSITION. U: *Alg* $\to$ *Vect and* U: *Comm* $\to$ *Vect create filtered colimits.*

Proof. If X is a C-comodule, then $X \otimes^C -$: $\text{Vect}^C \to \text{Vect}^C$ preserves filtered colimits. This is because in Vect the tensor $\otimes$ preserves colimits and filtered colimits are exact. Now

$$(\varinjlim_v A_v) \otimes^C (\varinjlim_v A_v) \cong \varinjlim_v \varinjlim_\mu (A_v \otimes^C A_\mu) \cong \varinjlim_v A_v \otimes^C A_v$$

since the diagonal $\mathbf{I} \to^\Delta \mathbf{I} \times \mathbf{I}$ is cofinal for filtered diagrams. Thus, the tensor power $A^{(2)} = A \otimes^C A$ commutes with filtered colimits. Since objects of Alg^C and of Comm^C are defined in terms of the functors $(\ )^{(2)}$, $(\ )^{(3)}$, $(\ )^{(0)} = C$ and equations, any colimits preserved by these functors will be created by U^C. Thus U^C creates filtered colimits. They are stable in *Alg* and *Comm*, since they are so in *Vect*. ∎

2.4. Let B be a subcomodule of the C-algebra (A, μ, η) in Alg^C. We want to find a bound on the cardinality of the subalgebra $\bar{B}$ of A generated by B. Define a sequence of subcomodules B_n of A by

$$B_1 = B + \text{im}(\eta: C \to A),$$

$$B_{n+1} = B_n + \text{im}(B_n \otimes^C B_n \to A \otimes^C A \xrightarrow{\mu} A).$$

The directed union $\bar{B} = \bigcup B_n$ is a subcomodule of A and $\eta: C \to A$ clearly factors through $\bar{B}$. If $\Sigma a_i \otimes a_i' \in \bar{B} \otimes^C \bar{B}$, then each a_i and each a_i' is contained in some B_n. Thus there is an N such that all a_i and all a_i' are in B_N and $\mu(\Sigma a_i \otimes a_i') \in B_{N+1}$. Hence, $\mu: A \otimes^C A \to A$ restricts to $\mu: \bar{B} \otimes^C \bar{B} \to \bar{B}$. $\bar{B}$ is clearly the smallest subalgebra of A containing B. Moreover, $\text{Card}(B_1) \leqslant \max(\aleph_0, \text{Card } B, \text{Card } C)$ and $\text{Card}(B_n \otimes^C B_n) \leqslant \text{Card}(B_n \otimes B_n) \leqslant \max(\aleph_0, \text{Card } B_n)$ so that

$$\text{Card } B_{n+1} \leqslant \max(\aleph_0, \text{Card } B_n) \leqslant \max(\aleph_0, \text{Card } B, \text{Card } C).$$

Thus, $\text{Card}(\bar{B}) \leqslant \max(\aleph_0, \text{Card } B, \text{Card } C)$. We are now in a position to show that coequalizers exist in Alg^C.

2.4.1. PROPOSITION. Alg^C *and* Comm^C *have coequalizers.*

Proof. We use the adjoint functor theorem as in [Ba1, p. 310]. Since Vect^C is complete and $U^C\colon \text{Alg}^C \to \text{Vect}^C$ creates limits (Proposition 1.3), Alg^C and thus the diagram category $(\text{Alg}^C)^{\rightrightarrows}$ are complete. The diagonal functor $\Delta\colon \text{Alg}^C \to (\text{Alg}^C)^{\rightrightarrows}$, for which we seek a left adjoint, preserves limits. So it suffices to verify the solution set condition. Given a diagram $A_2 \rightrightarrows A_1$ in Alg^C, we take one representative vector space of each cardinality $\leqslant \max(\aleph_0, \text{Card } A_1, \text{Card } C) = a$. There are $\leqslant a$ of these. Then consider all comodule structures on each of these vector spaces. There are $\leqslant a^{a \times a} = 2^a$ such. On each of these comodules, consider all algebra structures: there are again $\leqslant 2^a$ of these. For any coequalizing map $f\colon A_1 \to A$ in Alg^C, let $\bar{A}$ be the subalgebra of A generated by $\text{im}(f)$. Then f factors through $\bar{A}$ and $\text{Card } \bar{A} \leqslant a$ by the previous argument. Thus the above set of algebras forms a solution set for $A_2 \rightrightarrows A_1$. The same arguments work for Comm^C. ∎

2.4.2. COROLLARY. Alg^C *is cocomplete.*

Proof. In view of 2.2, Linton's theorem [L, p. 81] applies. ∎

2.4.3 *Remark.* The coequalizers in *Alg* and *Comm* cannot be expected to be stable since coequalizers in *Vect* are not (Counterexample II.3.2), and the tensor algebra functor $T\colon Vect \to Alg$ is indexed, reflects isomorphisms, and preserves coequalizers.

2.4.4. PROPOSITION. *The inclusion* $\text{Comm}^C \to \text{Alg}^C$ *has a left adjoint.*

Proof. This is a straightforward application of the adjoint functor theorem. The above bound on $\bar{B}$ shows that a representative set (one from each isomorphism class) of the commutative algebras of cardinality $\leqslant \max(\aleph_0, \text{Card } A, \text{Card } C)$ is a solution set for A in Alg^C. ∎

2.4.5. COROLLARY. Comm^C *is cocomplete and* $U\colon \text{Comm}^C \to \text{Vect}^C$ *has a left adjoint and is tripleable.*

2.5. QUESTION: Do the left adjoints of 2.4.4 and 2.4.5. actually give indexed left adjoints to *Comm* → *Alg* and $U\colon$ *Comm* → *Vect*?

A partial answer is given in Theorem 2.5.2.

For this, we need the following result about free commutative monoids in a symmetric monoidal category **V**. For B in **V**, let $B^{(n)}$ denote the n-fold tensor product of B, i.e., $B^{(n)} = ((B \otimes B) \otimes B) \cdots \otimes B$. For any permutation $\sigma \in S_n$, there is a morphism $\bar{\sigma}\colon B^{(n)} \to B^{(n)}$ in **V** obtained by permuting the factors according to σ (see [ML1, sect. 4]). Let $q_n\colon B^{(n)} \to B_n$ be the joint

coequalizer of all $\bar{\sigma}$, i.e., q_n is the universal morphism such that $q_n\bar{\sigma} = q_n\bar{\tau}$ for all $\sigma, \tau \in S_n$.

2.5.1. PROPOSITION. *If* V *has countable coproducts and all the coequalizers defining the* B_n, *and if for every* $X \in V$, $X \otimes -$ *preserves all these colimits, then* $\sum_{n=0}^{\infty} B_n$ *is the free commutative monoid generated by* B.

Proof. The proof is a long but straightforward calculation along the lines of [ML2, p. 168]. ∎

2.5.2. THEOREM. *If* k *is of characteristic* 0, *then* U: Comm → Vect *has an indexed left adjoint, the symmetric algebra functor* S.

Proof. Let B be a C-comodule and define $\psi: B^{(n)} \to B^{(n)}$ by

$$\psi = \frac{1}{n!} \sum_{\sigma \in S_n} \bar{\sigma}.$$

For $\theta: B^{(n)} \to A$, $\theta\bar{\sigma} = \theta\bar{\tau}$ for all $\sigma, \tau \in S_n$ if and only if $\theta\psi = \theta$, so the joint coequalizer of all $\bar{\sigma}$ is the same as the coequalizer of $\psi, 1: B^{(n)} \rightrightarrows B^{(n)}$. It is easily seen that ψ is idempotent and so this coequalizer is absolute (see [Fr, p. 61] and [P1]), i.e., preserved by any functor. Thus the joint coequalizer of all $\bar{\sigma}: B^{(n)} \to B^{(n)}$ is preserved by any additive functor, in particular by all $X \otimes^C -$ and ϕ^*. Since $X \otimes^C -$ preserves all coproducts, the left adjoint S^C of U^C: Comm^C → Vect^C is computed as in Proposition 2.5.1 and therefore is stable under ϕ^*. ∎

2.5.3. *Remark.* If the characteristic of k is not 0 but for some C every C-comodule is coflat, then we will get S^C by the formula of Proposition 2.5.1 and it is stable. This holds, for example, if $C = \bigoplus_I k$.

2.5.4. *Remark.* The existence and indexedness (in characteristic 0) of the exterior algebra functor can be proved in a way analogous to the symmetric algebra. For any permutation $\sigma \in S_n$ and any C-comodule B, define $\hat{\sigma}: B^{(n)} \to B^{(n)}$ to be $\text{sgn}(\sigma) \bar{\sigma}$ where $\bar{\sigma}$ is as above and $\text{sgn}(\sigma)$ is the sign of the permutation σ. Also define $\tilde{\psi}: B^{(n)} \to B^{(n)}$ by $\tilde{\psi} = (1/n!) \sum_{\sigma \in S_n} \hat{\sigma}$. Then $\tilde{\psi}$ is idempotent and everything works exactly as in the symmetric case. In particular, we let $B^{(n)} \to E_n^C(B)$ be the (absolute) coequalizer of $\tilde{\psi}$ and $1_{B^{(n)}}$, and $E^C(B) = \bigoplus E_n^C(B)$. The universal property of $E^C(B)$ is as follows: If A is in Alg^C and $f: B \to A$ in Vect^C is such that the diagram

$$
\begin{array}{ccc}
B \otimes^C B & \xrightarrow{\;\;f \otimes^C f\;\;} & A \otimes^C A \\
{\scriptstyle f \otimes^C f}\downarrow & & \downarrow{\scriptstyle \mu} \\
A \otimes^C A & \xrightarrow[-s]{} A \otimes^C A \xrightarrow[\mu]{} & A
\end{array}
$$

commutes, then there exists a unique $g: E^C(B) \to A$ such that

$$B \longrightarrow E^C(B)$$

(diagram: $B \longrightarrow E^C(B)$, with f going down-right from B to A, and g going down from $E^C(B)$ to A)
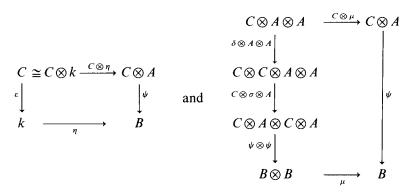
commutes.

E^C has, for example, the following properties:

(1) If $B = C \otimes V$ with $\dim V = n$, then $E_n^C(B) \cong C \otimes E_n(V) \cong C$ and $E_m^C(B) = 0$ for $m > n$.

(2) If $B \hookrightarrow C \otimes V$ with $\dim V = n$ (i.e. B is finitely cogenerated), then $E_m^C(B) = 0$ for all $m > n$.

3. Measurings

3.1. For algebras A and B, Sweedler [S, p. 139] defines a *measuring* from A to B as a pair (ψ, C) where C is a coalgebra and $\psi: C \otimes A \to B$ is such that

(left diagram)
$$C \cong C \otimes k \xrightarrow{C \otimes \eta} C \otimes A$$
with ε down on left to k, ψ down on right to B, and $k \xrightarrow{\eta} B$

and

(right diagram)
$$C \otimes A \otimes A \xrightarrow{C \otimes \mu} C \otimes A$$
$$\downarrow{\delta \otimes A \otimes A} \qquad\qquad \downarrow{\psi}$$
$$C \otimes C \otimes A \otimes A$$
$$\downarrow{C \otimes \sigma \otimes A}$$
$$C \otimes A \otimes C \otimes A$$
$$\downarrow{\psi \otimes \psi}$$
$$B \otimes B \xrightarrow{\mu} B$$

commute. As we can see from the following proposition, such a measuring is the same as a C-indexed family of homomorphisms between constant families of algebras.

3.1.1. PROPOSITION. *For any coalgebra C, there is a natural bijection between measurings $\psi: C \otimes A \to B$ and morphisms $\phi: \Delta_C A \to \Delta_C B$ in Alg^C.*

Proof. A morphism $\phi: \Delta_C A \to \Delta_C B$ in Alg^C is a map $\Delta_C A \to \Delta_C B$ in Vect^C which preserves unit and multiplication. We have a natural bijection

$$\phi: \quad \Delta_C A \to \Delta_C B \text{ in } \mathrm{Vect}^C$$
$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$
$$\psi: \quad C \otimes A \to B \text{ in } \mathrm{Vect}$$

and an easy calculation shows that preservation of unit and multiplication corresponds exactly to the conditions for ψ to be a measuring. ∎

Thus, in our context, a measuring $C \otimes A \to B$ can be thought of as a C-indexed family of algebra homomorphisms. This is a suggestive way of looking at measurings. For example, the identity $1: \varDelta_C A \to \varDelta_C A$ corresponds to the measuring $C \otimes A \xrightarrow{\varepsilon \otimes A} k \otimes A \cong A$. If $f: \varDelta_C A_1 \to \varDelta_C A_2$ and $g: \varDelta_C A_2 \to \varDelta_C A_3$ are C-algebra homomorphisms, we can compose them to get $gf: \varDelta_C A_1 \to \varDelta_C A_3$. There must be a corresponding "composition" of measurings: given $\phi: C \otimes A_1 \to A_2$ and $\psi: C \otimes A_2 \to A_3$, we can define

$$\psi * \phi = (C \otimes A_1 \xrightarrow{\delta \otimes A_1} C \otimes C \otimes A_1 \xrightarrow{C \otimes \phi} C \otimes A_2 \xrightarrow{\psi} A_3).$$

This composition is associative and unitary. Thus the category of algebras and C-measurings forms a category (which is equivalent to a full subcategory of Alg^C). Of course, this can be done without families of algebra homomorphisms, but it is more intuitive in terms of families. An example, which is expanded below, is that isomorphisms and monomorphisms are easily understood in Alg^C but look somewhat artificial in terms of measurings.

3.1.2. EXAMPLE. Let $k\langle d \rangle$ be the infinitesimal coalgebra, i.e., the coalgebra with basis $\{1, d\}$ where 1 is a point and d is a primitive with respect to 1. As shown in [S, p. 139], a measuring $\psi: k\langle d \rangle \otimes A \to B$ consists of a pair of linear maps ψ_0, $\psi_1: A \to B$, where ψ_0 is an algebra homomorphism and ψ_1 is a ψ_0-derivation. The underlying space of $\varDelta_{k\langle d \rangle} A$ is $k\langle d \rangle \otimes A \cong A \oplus A$. With this identification, the $k\langle d \rangle$-algebra homomorphism corresponding to ψ is represented by the matrix

$$\begin{pmatrix} \psi_0 & \psi_1 \\ 0 & \psi_0 \end{pmatrix}: A \oplus A \to B \oplus B.$$

Thus the "identity derivation" is $(1_A, 0)$ and composition of derivations is given by $(\psi_0', \psi_1')(\psi_0, \psi_1) = (\psi_0'\psi_0, \psi_0'\psi_1 + \psi_1'\psi_0)$. Similarly, (ψ_0, ψ_1) corresponds to an isomorphism (monomorphism) if and only if ψ_0 is an isomorphism (resp. monomorphism) with no condition on ψ_1.

3.1.3. EXAMPLE. Let $F: \mathrm{Vect} \to \mathrm{Alg}$ be any of the indexed functors T, S, E of section 2. If V and W are vector spaces, then any C-indexed family of linear maps $f: \varDelta_C V \to \varDelta_C W$ induces a C-indexed family of algebra homomorphisms

$$g = (\varDelta_C F(V) \cong F^C(\varDelta_C V) \xrightarrow{F^C(f)} F^C(\varDelta_C W) \cong \varDelta_C F(W)).$$

Thus, any linear map $\bar{f}: C \otimes V \to W$ induces a measuring $\bar{g}: C \otimes FV \to FW$. This explains the meaning of Theorems 7.1.1 and 7.1.3 of [S, p. 147] in terms of families (for cocommutative C): a linear map $\bar{f}: C \otimes V \to W$ is a C-indexed family of linear maps $V \to W$ and $\bar{g}: C \otimes FV \to FW$ is the C-indexed family of algebra homomorphisms obtained by applying F to each member of $\bar{f}$.

3.1.4. EXAMPLE: Another example is that of universal measurings. For two algebras A and B, Theorem 1.4.1 says that there exists a coalgebra $\mathrm{Hom}(A, B)$ with the property that there is a natural bijection

$$
\begin{array}{l}
\phi: \quad C \to \mathrm{Hom}(A, B) \text{ in Coalg} \\
\hline
f: \quad \Delta_C A \to \Delta_C B \text{ in Alg}^C
\end{array}
$$

If C is taken to be $\mathrm{Hom}(A, B)$, then corresponding to $\phi = 1_{\mathrm{Hom}(A, B)}$ there is a generic family of algebra homomorphisms

$$
g: \quad \Delta_{\mathrm{Hom}(A, B)} A \to \Delta_{\mathrm{Hom}(A, B)} B.
$$

Then the above bijection can be rephrased by saying that for every C-algebra homomorphism $f: \Delta_C A \to \Delta_C B$, there exists a unique coalgebra homomorphism $\phi: C \to \mathrm{Hom}(A, B)$ such that $\phi^*(g) = f$. We interpret $\mathrm{Hom}(A, B)$ as the coalgebra of all algebra homomorphisms from A to B (although $\mathrm{Hom}(A, B)$ contains much more than homomorphisms; there are homomorphisms corresponding to points, derivations corresponding to primitives, etc.) and g is the *family* of all such homomorphisms parametrized by $\mathrm{Hom}(A, B)$. This g corresponds to the universal measuring $\theta: \mathrm{Hom}(A, B) \otimes A \to B$ with the universal property given in [S, p. 143].

Because we can compose families of homomorphisms, $\mathrm{Hom}(A, A)$ becomes a monoid in Coalg, i.e., a bialgebra. Indeed, for a coalgebra C, we have the following natural transformations

$$
\begin{array}{l}
C \to \mathrm{Hom}(A, A) \otimes \mathrm{Hom}(A, A) \\
\cong \overline{} \\
C \rightrightarrows \mathrm{Hom}(A, A) \\
\cong \overline{} \\
f_1, f_2: \quad \Delta_C A \rightrightarrows \Delta_C A \\
\downarrow \overline{} \\
f_2 f_1: \quad \Delta_C A \to \Delta_C A \\
\cong \overline{} \\
C \to \mathrm{Hom}(A, A)
\end{array}
$$

which, by the Yoneda lemma, give a morphism

$$\mathrm{Hom}(A, A) \otimes \mathrm{Hom}(A, A) \to \mathrm{Hom}(A, A)$$

which is associative and unitary (the unit is given by $1_A \colon A \to A$).

3.1.5 EXAMPLE. Let $f \colon A \to B$ be a morphism in Alg^C, i.e., a C-indexed family of algebra homomorphisms. If A and B are coflat, we can extract from f the subfamily of all isomorphisms, i.e., there is a subcoalgebra $C_0 \hookrightarrow C$ such that for any $\phi \colon D \to C$, $\phi^*(f)$ is an isomorphism if and only if ϕ factors through C_0 (see [P2, Lemma 20]). By Theorem 1.4.1, $\mathrm{Hom}^C(A, A)$, $\mathrm{Hom}^C(B, A)$, and $\mathrm{Hom}^C(B, B)$ exist and it is easily seen that $C_0 \hookrightarrow C$ is the inverse limit in Coalg^C of the diagram

$$
\begin{array}{c}
\mathrm{Hom}^C(B, A) \xrightarrow[\mathrm{Hom}^C(f, A)]{\ulcorner 1_A \urcorner} \mathrm{Hom}^C(A, A) \\
{\scriptstyle \mathrm{Hom}^C(B, f)} \Big\downarrow\Big\downarrow {\ulcorner 1_B \urcorner} \\
\mathrm{Hom}^C(B, B)
\end{array}
$$

If we apply this to the generic family of homomorphisms

$$g \colon \quad \Delta_{\mathrm{Hom}(A,A)} A \to \Delta_{\mathrm{Hom}(A,A)} A,$$

we get a subobject $\mathrm{Iso}(A, A) \hookrightarrow \mathrm{Hom}(A, A)$. In the same way as 3.1.4, we can see that $\mathrm{Iso}(A, A)$ is a Hopf algebra, with the antipode given by taking inverses.

3.2.   Up until now, although we have considered families of algebra homomorphisms, they have always involved only constant families of algebras. As mentioned after Proposition 3.1.1, the C-measurings form a category, but it is not a very good one. It does not have limits or colimits, for example. From a categorical point of view, Alg^C and Comm^C are the appropriate categories, i.e., we should also consider non-constant families of algebras.

3.2.1. EXAMPLE.   If F is the indexed functor T, or in characteristic 0, S or E, then for any nonconstant C-indexed family of vector spaces A (i.e., a C-comodule which is not cofree), $F^C(A)$ is a nonconstant family of algebras. Every C-indexed family of algebras is a coequalizer of maps between families of the form $T^C(A)$, so if we want the functors T^C and coequalizers, we are forced to take all of Alg^C.

3.2.2. EXAMPLE. Let $\phi, \psi: C \otimes A \to A'$ be two C-measurings. These correspond to homomorphisms $\bar\phi$, $\bar\psi: \varDelta_C A \to \varDelta_C A'$ whose equalizer $A_0 \to \varDelta_C A$ will be in general a non-constant family of algebras.

By 1.3, $\varPi_C$ (i.e., $\varPi_\varepsilon$ for $\varepsilon: C \to k$) exists, so we get an object $\varPi_C A_0$ in Alg^k, i.e., we get an "honest" algebra. $\varPi_C A_0$ has the following universal property: for any algebra B and coalgebra D, measurings $D \otimes B \to \varPi_C A_0$ are in natural bijection with measurings $\theta: C \otimes D \otimes B \to A$ such that $\phi(C \otimes \theta)(\delta \otimes D \otimes B) = \psi(C \otimes \theta)(\delta \otimes D \otimes B)$. This follows from the natural bijections

$D \otimes B \to \varPi_C A_0$ measuring
_____

$\varDelta_D B \to \varDelta_D \varPi_C A_0$ in Alg^D
_____

$\varDelta_D B \to \varPi_{p_2} p_1^* A_0$ in Alg^D
_____

$p_2^* \varDelta_D B \to p_1^* A_0$ in $\mathrm{Alg}^{C \otimes D}$
_____

$\bar\theta: p_2^* \varDelta_D B \to p_1^* \varDelta_C A$ in $\mathrm{Alg}^{C \otimes D}$ s.t.
_____

$\bar\theta: \varDelta_{C \otimes D} B \to \varDelta_{C \otimes D} A$ in $\mathrm{Alg}^{C \otimes D}$ s.t. $\bar\phi\bar\theta = \bar\psi\bar\theta$
_____

$\theta: C \otimes D \otimes B \to A$ measuring
     such that $\phi(C \otimes \theta)(\delta \otimes D \otimes B) = \psi(C \otimes \theta)(\delta \otimes D \otimes B)$

where p_1 and p_2 are the projections from $C \otimes D$ to C and D, respectively. The first and last bijections follows from 3.1.1, the second from the Beck condition, the third by the adjointness $p_2^* \dashv \varPi_{p_2}$, and the fourth because p_1^* preserves equalizers.

Another way that measurings give rise to families of algebras is by pullback. Let $\phi: C \otimes A' \to A$ and $\psi: D \otimes A'' \to A$ be measurings. These give algebra homomorphisms $\bar\phi: \varDelta_C A' \to \varDelta_C A$ and $\bar\psi: \varDelta_D A'' \to \varDelta_D A$. If $p_1: C \otimes D \to C$ and $p_2: C \otimes D \to D$ are the projections, then we can construct the following pullback in $\mathrm{Alg}^{C \otimes D}$:

$$
\begin{array}{ccc}
P & \longrightarrow & \varDelta_{C \otimes D} A' \\
\downarrow & & \downarrow{\scriptstyle p_1^* \bar\phi} \\
\varDelta_{C \otimes D} A'' & \xrightarrow[p_2^* \bar\psi]{} & \varDelta_{C \otimes D} A.
\end{array}
$$

This P will be in general a nonconstant family of algebras. Again, $\varPi_{C \otimes D} P$ gives us an honest algebra with a corresponding universal property.

3.2.3. EXAMPLE. Let $C = k \oplus kd$ be the "infinitesimal" coalgebra of III.2.2. A C-comodule is a vector space A with an endomorphism $\alpha: A \to A$ such that $\alpha^2 = 0$. If $A_0 = \ker(\alpha)$ and $A_1 = \mathrm{im}(\alpha)$ and $i: A_1 \hookrightarrow A_0$ is the inclusion, then $A \cong A_0 \oplus A_1$ and α corresponds to the matrix

$$\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}: \quad A_0 \oplus A_1 \to A_0 \oplus A_1.$$

If $i: B_1 \hookrightarrow B_0$ comes from another C-comodule (B, β), a morphism $(A, \alpha) \to (B, \beta)$ corresponds to a matrix

$$\begin{pmatrix} f_0 & g \\ 0 & f_1 \end{pmatrix}: \quad A_0 \oplus A_1 \to B_0 \oplus B_1$$

where $f_0: A_0 \to B_0$ and $g: A_1 \to B_0$ are linear maps and $f_1: A_1 \to B_1$ is the restriction of f_0 to A_1. An easy calculation shows that $(A, \alpha) \otimes^C (B, \beta)$ corresponds to the inclusion $i \otimes i: A_1 \otimes B_1 \hookrightarrow A_0 \otimes B_0$ and C itself corresponds to $1: k \to k$. Thus a C-family of algebras is given by an inclusion of vector spaces $A_1 \hookrightarrow A_0$ and two matrices

$$\begin{pmatrix} u_0 & v \\ 0 & u_1 \end{pmatrix}: \quad k \oplus k \to A_0 \oplus A_1$$

and

$$\begin{pmatrix} \mu_0 & \phi \\ 0 & \mu_1 \end{pmatrix}: \quad (A_0 \otimes A_0) \oplus (A_1 \otimes A_1) \to A_0 \oplus A_1.$$

The unit and associativity laws say that A_0 is an algebra (with multiplication μ_0) and A_1 is a subalgebra, and that $z = v(1) \in A_0$ and $\phi: A_1 \otimes A_1 \to A_0$ satisfy the 2-cocycle conditions

$$za + \phi(1, a) = 0 = az + \phi(a, 1),$$

$$a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)\, c = 0$$

for all $a, b, c \in A_1$.

If A is such a C-family of algebras, then $\Pi_C A$ is the ordinary algebra built out of a cocycle in the usual way, i.e., $A_0 \times A_1$ with multiplication given by $(a_0, a_1) \cdot (b_0, b_1) = (a_0 b_1 + a_1 b_0 + \phi(a_1, b_1), a_1 b_1)$ and unit $(z, 1)$.

3.3. We finish by giving a construction of a coalgebra of all finite dimensional algebras. For this, we must know what a C-indexed family of finite dimensional algebras is.
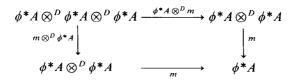
3.3.1. DEFINITION. A C-comodule A is called *finite dimensional* if there is a decomposition of C, $C = \bigoplus C_v$, and finite integers n_v such that, for each injection $i_v \colon C_v \to \bigoplus C_v$, $i_v^* A \cong C_v^{n_v}$. A C-algebra A is finite dimensional if its underlying comodule is.
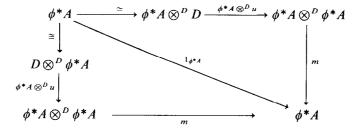
The following theorem says that there is a coalgebra C_0 of all finite dimensional algebras in the sense that, for any coalgebra D, homomorphisms $D \to C_0$ are the same as D-indexed families of finite dimensional algebras.

3.3.2. THEOREM. *There exists a coalgebra C_0 and a C_0-indexed family of finite dimensional algebras A_0 with the property that for any coalgebra D and any D-indexed family of finite dimensional algebras B, there exists a homomorphism $\phi \colon D \to C_0$ such that $B \cong \phi^* A_0$.*

Proof. Let $C = \bigoplus_{\mathbf{N}} k$, the direct sum of countably many copies of k. Since $\mathrm{Vect}^C \simeq \Pi_{\mathbf{N}} \mathrm{Vect}$, we can let A be the C-comodule corresponding to the sequence of vector spaces $(0, k, k^2, k^3, \ldots)$. Thus $A \cong \bigoplus_{n \in \mathbf{N}} k^n$. So C is the coalgebra of all finite dimensional vector spaces and A is their C-indexed family. Since A is coflat, so are $A \otimes^C A$ and $A \otimes^C A \otimes^C A$ and therefore, by Theorem VI.2.3, all the Hom^C used below exist.

For any $\phi \colon D \to C$, morphisms $(\phi) \to \mathrm{Hom}^C(A \otimes^C A, A) \times_C \mathrm{Hom}^C(C, A)$ in Coalg^C are in bijection with pairs of maps $m \colon \phi^* A \otimes^D \phi^* A \to \phi^* A$ and $u \colon D \to \phi^* A$ in Vect^D. From such a pair, we construct the following not necessarily commutative diagrams

$$
\begin{array}{ccc}
\phi^* A \otimes^D \phi^* A \otimes^D \phi^* A & \xrightarrow{\phi^* A \otimes^D m} & \phi^* A \otimes^D \phi^* A \\
{\scriptstyle m \otimes^D \phi^* A}\big\downarrow & & \big\downarrow{\scriptstyle m} \\
\phi^* A \otimes^D \phi^* A & \xrightarrow{\quad m \quad} & \phi^* A
\end{array}
$$

$$
\begin{array}{ccccc}
\phi^* A & \xrightarrow{\ \simeq\ } & \phi^* A \otimes^D D & \xrightarrow{\phi^* A \otimes^D u} & \phi^* A \otimes^D \phi^* A \\
{\scriptstyle \cong}\big\downarrow & & & & \big\downarrow{\scriptstyle m} \\
D \otimes^D \phi^* A & & {\scriptstyle 1_{\phi^* A}} & & \\
{\scriptstyle \phi^* A \otimes^D u}\big\downarrow & & & & \\
\phi^* A \otimes^D \phi^* A & & \xrightarrow{\quad m \quad} & & \phi^* A
\end{array}
$$

which give two morphisms $(\phi) \to \mathrm{Hom}^C(A \otimes^C A \otimes^C A, A)$ and three

morphisms $(\phi) \to \operatorname{Hom}^C(A, A)$. These associations are natural in (ϕ), so they induce (by the Yoneda lemma) the five morphisms in the diagram

$$\operatorname{Hom}^C(A \otimes^C A, A) \times {}_C \operatorname{Hom}^C(C, A) \rightrightarrows \operatorname{Hom}^C(A \otimes^C A \otimes^C A, A)$$

$$\Big\downarrow\Big\downarrow\Big\downarrow$$

$$\operatorname{Hom}^C(A, A)$$

Let $(\gamma: C_0 \to C)$ be the limit in Coalg^C of this diagram. For any coalgebra D, morphisms $D \to C_0$ are in bijection with pairs $(\phi): D \to C$ and $(\phi) \to (\gamma)$ in Coalg^C. A morphism $(\phi) \to (\gamma)$ is the same as a morphism

$$(\phi) \to \operatorname{Hom}^C(A \otimes^C A, A) \times {}_C \operatorname{Hom}^C(C, A)$$

equalizing the five maps above, and this in turn is the same as $m: \phi^*A \otimes^D \phi^*A \to \phi^*A$ and $u: D \to \phi^*A$ such that (ϕ^*A, m, u) is a D-algebra. But since $C \cong \bigoplus_{\mathbf{N}} k$, $\phi: D \to C$ is the same as a decomposition of D into $\bigoplus_{n \in \mathbf{N}} D_n$ and ϕ into $\bigoplus \phi_n: \bigoplus D_n \to \bigoplus k$. Thus, for each injection $i_n: D_n \to \bigoplus D_n$, $i_n^* \phi^*A \cong \phi_n^* i_n^* A \cong \phi_n^* k^n \cong D_n^n$. Thus ϕ^*A is finite dimensional.

So, homomorphisms $D \to C_0$ are in natural one-one correspondence with partitions of D into $\bigoplus D_n$ and algebra structures on the finite dimensional D_n-comodules D_n^n. The algebra A_0 corresponds to $1_{C_0}: C_0 \to C_0$. ∎

The coalgebra C_0 is non-trivial: its points "are" finite dimensional algebras, its primitives "are" cocycles as above, etc. Further, A_0 is a non-constant family of algebras.

Using exactly the same method, we can construct a coalgebra of all finite dimensional commutative algebras, finite dimensional coalgebras, finite dimensional commutative coalgebras, or finite dimensional Hopf algebras. Each such coalgebra comes equipped with a generic family of the objects in question.

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