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Hybrid shrinking projection method for a generalized equilibrium problem, a maximal monotone operator and a countable family of relatively nonexpansive mappings

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ABSTRACT

The purpose of this paper is to introduce and consider a hybrid shrinking projection method for finding a common element of the set EP of solutions of a generalized equilibrium problem, the set $\bigcap_{n=0}^{\infty} F(S_n)$ of common fixed points of a countable family of relatively nonexpansive mappings $\{S_n\}_{n=0}^{\infty}$ and the set $T^{-1}0$ of zeros of a maximal monotone operator T in a uniformly smooth and uniformly convex Banach space. It is proven that under appropriate conditions, the sequence generated by the hybrid shrinking projection method, converges strongly to some point in $EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n))$. This new result represents the improvement, complement and development of the previously known ones in the literature.

1. Introduction

Let E be a real Banach space with the dual E^* and C be a nonempty closed convex subset of E. We denote by $\mathcal{N}$ and $\mathcal{R}$ the sets of nonnegative integers and real numbers, respectively. Also, we denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Recall that if E is smooth then J is single-valued and norm-to-weak* continuous, and that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We shall still denote by J the single-valued duality mapping. Let $A : C \to E^*$ be a nonlinear mapping and $f : C \times C \to \mathcal{R}$ be a bifunction. Then, consider the following generalized equilibrium problem:

$$\text{Find } u \in C \text{ such that } f(u, y) + \langle Au, y - u \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by EP, i.e.,

$$EP = \{u \in C : f(u, y) + \langle Au, y - u \rangle \geq 0, \ \forall y \in C\}.$$

Whenever $E = H$ a Hilbert space, problem (1.1) was introduced and studied by Takahashi and Takahashi [1].

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Whenever $A \equiv 0$, problem (1.1) is equivalent to finding $u \in C$ such that

$$f(u, y) \geq 0, \quad \forall y \in C,$$

which is called the equilibrium problem. The set of its solutions is denoted by $EP(f)$.

Whenever $f \equiv 0$, problem (1.1) is equivalent to finding $u \in C$ such that

$$\langle Au, y - u\rangle \geq 0, \quad \forall y \in C,$$

which is called the variational inequality of Browder type. The set of its solutions is denoted by $VI(C, A)$.

Problem (1.1) is very general in the sense that it includes, as spacial cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, e.g., [2,3]. A mapping $S : C \to E$ is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in C : Sx = x\}$. A mapping $A : C \to E^*$ is called α-inverse-strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is easy to see that if $A : C \to E^*$ is an α-inverse-strongly monotone mapping, then it is $1/\alpha$-Lipschitzian.

Very recently, motivated by Takahashi and Zembayashi [4], Chang [5] has been proved the following strong convergence theorem for finding a common element of the set of solutions to the generalized equilibrium problem (1.1) and the set of common fixed points of a pair of relatively nonexpansive mappings in a Banach space.

Theorem SSC (*See [5, Theorem 3.1]*). *Let E be a uniformly smooth and uniformly convex Banach space, and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an α-inverse-strongly monotone mapping and $f : C \times C \to \mathcal{R}$ be a bifunction satisfying the following conditions* (A1)–(A4)*:*

(A1) *$f(x, x) = 0$ for all $x \in C$,*
(A2) *f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$,*
(A3) *for all $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$,*
(A4) *for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.*

Let $S, \widetilde{S} : C \to C$ be two relatively nonexpansive mappings such that $F(S) \cap F(\widetilde{S}) \cap EP \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_0 \in C, \qquad C_0 = C; \\ z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)J\widetilde{S}z_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n\phi(v, x_n) + (1 - \beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, $\forall x, y \in E$, $\Pi_C : E \to C$ is the generalized projection operator, $J : E \to E^*$ is the single-valued normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$,
(ii) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$,

then $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap F(\widetilde{S})\cap EP} x_0$, where $\Pi_{F(S)\cap F(\widetilde{S})\cap EP}$ is the generalized projection of E onto $F(S) \cap F(\widetilde{S}) \cap EP$.

Let E be a real Banach space with the dual E^*. A multivalued operator $T : E \to 2^{E^*}$ with domain $D(T) = \{z \in E : Tz \neq \emptyset\}$ is called monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(T)$ and $y_i \in Tx_i$, $i = 1, 2$. A monotone operator T is called maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. A method for solving the inclusion $0 \in Tx$ is the proximal point algorithm. Denote by I the identity operator on $E = H$ a Hilbert space. The proximal point algorithm generates, for any initial point $x_0 = x \in H$, a sequence $\{x_n\}$ in H, by the iterative scheme

$$x_{n+1} = (I + r_nT)^{-1}x_n, \quad n = 0, 1, 2, \ldots,$$

where $\{r_n\}$ is a sequence in the interval $(0, \infty)$. Note that this iteration is equivalent to

$$0 \in Tx_{n+1} + \frac{1}{r_n}(x_{n+1} - x_n), \quad n = 0, 1, 2, \ldots.$$

This algorithm was first introduced by Martinet [2] and generally studied by Rockafellar [6] in the framework of a Hilbert space. Later many authors studied its convergence in a Hilbert space or a Banach space. See for instance, [7–12] and the references therein. Kamimura and Takahashi [13] have been recently introduced and studied the proximal-type algorithm for finding an element of $T^{-1}0$ in a uniformly smooth and uniformly convex Banach space E, which is an extension of Solodov and Svaiter's proximal-type algorithm. They derived a strong convergence theorem which extends and improves Solodov and Svaiter's result [14].

Recently, utilizing Nakajo and Takahashi's idea [15], Qin and Su [16] have been introduced one iterative algorithm (i.e., modified Ishikawa iteration) for a relatively nonexpansive mapping $S : C \to C$, with C a closed convex subset of a uniformly smooth and uniformly convex Banach space E

$$\begin{cases} x_0 \in C \quad \text{arbitrarily chosen}, \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JSx_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSz_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{1.3}$$

They proved that under appropriate conditions the sequence $\{x_n\}$ generated by algorithm (1.3), converges strongly to $\Pi_{F(S)}x_0$.

Let E be a real Banach space with the dual E^*. Assume that $T : E \to 2^{E^*}$ is a maximal monotone operator and $S : E \to E$ is a relatively nonexpansive mapping. Very recently, by combining Kamimura and Takahashi's idea [13] with Qin and Su [16], Ceng et al. [17] have been introduced a hybrid proximal-type algorithm for finding an element of $F(S) \cap T^{-1}0$ in a uniformly smooth and uniformly convex Banach space E. The authors proved that under appropriate conditions the sequence $\{x_n\}$ generated by the algorithm, converges strongly to $\Pi_{F(S)\cap T^{-1}0}x_0$.

Let E be a reflexive, strictly convex, and smooth Banach space with the dual E^* and C be a nonempty closed convex subset of E. Let $T : E \to 2^{E^*}$ be a maximal monotone operator, and $\{S_n\}_{n=0}^{\infty}$ be a countable family of relatively nonexpansive self-mappings on C. Let $A : C \to E^*$ be an α-inverse-strongly monotone mapping and $f : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)–(A4). The purpose of this paper is to introduce and investigate a hybrid shrinking projection method for finding an element of $EP \cap T^{-1} \cap (\bigcap_{n=0}^{\infty} F(S_n))$, i.e., the following iterative algorithm

$$\begin{cases} x_0 \in C_0 \quad \text{arbitrarily chosen}, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n} z_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \langle Au_n, y - u_n \rangle + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.4}$$

where $C_0 = C, J_{r_n} = (J + r_n T)^{-1}J$, $\forall n \ge 0$, $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$.

In this paper, a strong convergence result for our hybrid shrinking projection method is established in a uniformly smooth and uniformly convex Banach space; that is, under appropriate conditions, the sequence $\{x_n\}$ generated by algorithm (1.4), converges strongly to $\Pi_{EP\cap T^{-1}0\cap(\bigcap_{n=0}^{\infty} F(S_n))}x_0$. Our result improves and extends some well-known results in [5,13,14,16,17].

Throughout this paper, the symbol $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.

2. Preliminaries

Let E be a real Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. A Banach space E is called strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. It is said to be uniformly convex if $x_n - y_n \to 0$ for any two sequences $\{x_n\}, \{y_n\} \subset E$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be a unit sphere of E. Then the Banach space E is called smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. If E is smooth then J is single-valued. We shall still denote the single-valued duality mapping by J.

It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. Recall also that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. A Banach space E is said to have the Kadec–Klee property if for any sequence $\{x_n\} \subset E$, whenever $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, we have $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec–Klee property; see [18,19] for more details.

Let C be a nonempty closed convex subset of a real Hilbert space H and $P_C : H \to C$ be the metric projection of H onto C. Then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and hence, it is not available in more general Banach spaces. Nevertheless, Alber [20] has been recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined as in [20,21] by

$$\phi(x,y)=\|x\|^2-2\langle x,Jy\rangle+\|y\|^2,\quad \forall x,y\in E. \tag{2.1}$$

It is clear that in a Hilbert space H, (2.1) reduces to $\phi(x,y)=\|x-y\|^2,\ \forall x,y\in H$.

The generalized projection $\Pi_C: E\to C$ is a mapping that assigns to an arbitrary point $x\in E$ the minimum point of the functional $\phi(y,x)$; that is, $\Pi_C x=\bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x},x)=\min_{y\in C}\phi(y,x). \tag{2.2}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, e.g., [22]). In a Hilbert space H, $\Pi_C=P_C$. From [2], in uniformly smooth and uniformly convex Banach spaces, we have

$$(\|x\|-\|y\|)^2\le\phi(x,y)\le(\|x\|+\|y\|)^2,\quad\forall x,y\in E. \tag{2.3}$$

Let C be a nonempty closed convex subset of E, and let S be a mapping from C into itself. A point $p\in C$ is called an asymptotically fixed point of S [23] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $Sx_n-x_n\to 0$. The set of asymptotical fixed points of S will be denoted by $\widehat{F}(S)$. A mapping S from C into itself is called relatively nonexpansive [24–26] if $\widehat{F}(S)=F(S)$ and $\phi(p,Sx)\le\phi(p,x)$ for all $x\in C$ and $p\in F(S)$.

We remark that if E is a reflexive, strictly convex and smooth Banach space, then for any $x,y\in E$, $\phi(x,y)=0$ if and only if $x=y$. It is sufficient to show that if $\phi(x,y)=0$ then $x=y$. From (2.3), we have $\|x\|=\|y\|$. This implies that $\langle x,Jy\rangle=\|x\|^2=\|y\|^2$. From the definition of J, we have $Jx=Jy$. Therefore, we have $x=y$; see [18,19] for more details.

We need the following lemmas for the proof of our main results.

Lemma 2.1 (*See [13]*). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n,y_n)\to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n-y_n\to 0$.*

Lemma 2.2 (*See [13,20]*). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let $x\in E$ and let $z\in C$. Then*

$$z=\Pi_C x\Leftrightarrow\langle y-z,Jx-Jz\rangle\le 0,\quad\forall y\in C.$$

Lemma 2.3 (*See [13,20]*). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x,\Pi_C y)+\phi(\Pi_C y,y)\le\phi(x,y),\quad\forall x\in C \text{ and } y\in E.$$

Lemma 2.4 (*See [27]*). *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E, and let $S: C\to C$ be a relatively nonexpansive mapping. Then $F(S)$ is closed and convex.*

The following result is due to Blum and Oettli [28].

Lemma 2.5 (*See [28]*). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C\times C$ to $\mathcal{R}$ satisfying* (A1)–(A4). *Then*

$$f(z,y)+\frac{1}{r}\langle y-z,Jz-Jx\rangle\ge 0,\quad\text{for all } y\in C.$$

Motivated by Combettes and Hirstoaga [29] in a Hilbert space, Takahashi and Zembayashi [11] established the following lemma.

Lemma 2.6 (*See [4]*). *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let f be a bifunction from $C\times C$ to $\mathcal{R}$ satisfying* (A1)–(A4). *For $r>0$ and $x\in E$, define a mapping $T_r: E\to C$ as follows:*

$$T_r(x)=\left\{z\in C: f(z,y)+\frac{1}{r}\langle y-z,Jz-Jx\rangle\ge 0,\ \text{for all } y\in C\right\}$$

for all $x\in E$. Then, the following statements hold.

(i) *T_r is single-valued.*
(ii) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x,y\in E$,*

$$\langle T_r x-T_r y,JT_r x-JT_r y\rangle\le\langle T_r x-T_r y,Jx-Jy\rangle.$$

(iii) *$F(T_r)=\widehat{F}(T_r)=EP(f)$.*
(iv) *$EP(f)$ is closed and convex.*

Using Lemma 2.6, one has the following result.

Lemma 2.7 (*See [4]*)**.** *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying* (A1)–(A4)*, and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \le \phi(q, x).$$

Utilizing Lemmas 2.5–2.7 as above, Chang [5] derived the following result.

Proposition 2.1 (*See [5, Lemma 2.5]*)**.** *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an α-inverse-strongly monotone mapping, let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying* (A1)–(A4)*, and let $r > 0$. Then the following statements hold.*

(I) *for $x \in E$, there exists $u \in C$ such that*

$$f(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \ge 0, \quad \forall y \in C;$$

(II) *if E is additionally uniformly smooth and $K_r : E \to C$ is defined as*

$$K_r(x) = \left\{u \in C : f(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \ge 0,\ \forall y \in C\right\}, \quad \forall x \in E, \tag{2.4}$$

then the mapping K_r has the following properties.

(i) *K_r is single-valued,*
(ii) *K_r is a firmly nonexpansive-type mapping, i.e.,*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \le \langle K_r x - K_r y, Jx - Jy\rangle, \quad \forall x, y \in E,$$

(iii) *$F(K_r) = \widehat{F}(K_r) = EP$,*
(iv) *EP is a closed convex subset of C,*
(v) *$\phi(p, K_r x) + \phi(K_r x, x) \le \phi(p, x),\ \forall p \in F(K_r)$.*

Proof. Define a bifunction $F : C \times C \to \mathcal{R}$ as follows:

$$F(x, y) = f(x, y) + \langle Ax, y - x\rangle, \quad \forall x, y \in C.$$

Then it is easy to verify that F satisfies conditions (A1)–(A4). Therefore, statements (I) and (II) of Proposition 2.1 follow immediately from Lemmas 2.5–2.7. □

Let $T : E \to 2^{E^*}$ be a maximal monotone operator in a smooth Banach space E. We denote the resolvent of T by $J_r := (J + rT)^{-1}J$ for each $r > 0$. Then $J_r : E \to D(T)$ is a single-valued mapping. Also, $T^{-1}0 = F(J_r)$ for each $r > 0$, where $F(J_r)$ is the set of fixed points of J_r. For each $r > 0$, the Yosida approximation of T is defined by $A_r = (J - JJ_r)/r$. It is known that

$$A_r x \in T(J_r x), \quad \text{for each } r > 0 \text{ and } x \in E. \tag{2.5}$$

Lemma 2.8 (*Rockafellar [30]*)**.** *Let E be a reflexive, strictly convex, and smooth Banach space and let $T : E \to 2^{E^*}$ be a multivalued operator. Then the following statements hold.*

(i) *$T^{-1}0$ is closed and convex if T is maximal monotone such that $T^{-1}0 \ne \emptyset$.*
(ii) *T is maximal monotone if and only if T is monotone with $R(J + rT) = E^*$ for all $r > 0$.*

Lemma 2.9. *Let E be a reflexive, strictly convex, and smooth Banach space, and let $T : E \to 2^{E^*}$ be a maximal monotone operator with $T^{-1}0 \ne \emptyset$. Then the following statements hold.*

(i) *(see [3]) $\phi(z, J_r x) + \phi(J_r x, x) \le \phi(z, x)$ for all $r > 0$, $z \in T^{-1}0$ and $x \in E$.*
(ii) *(see [5]) $J_r : E \to D(T)$ is a relatively nonexpansive mapping.*

3. Main results

Throughout this section, unless otherwise stated, we assume that $\{S_n\}_{n=0}^{\infty}$ is a countable family of relatively nonexpansive self-mappings on C, $T : E \to 2^{E^*}$ is a maximal monotone operator, $A : C \to E^*$ is an α-inverse-strongly monotone mapping and $f : C \times C \to \mathcal{R}$ is a bifunction satisfying (A1)–(A4), where C is a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E. Let $EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. In this section, we study the following algorithm for finding an element of $EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n))$.

$$\begin{cases} x_0 \in C_0 \quad \text{arbitrarily chosen,} \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n} z_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \langle Au_n, y-u_n\rangle + \dfrac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{3.1}$$

where $C_0 = C$, $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$.

First we investigate the condition under which algorithm (3.1) is well defined.

Lemma 3.1. *Let E be a reflexive, strictly convex, and smooth Banach space. If $EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$, then the sequence $\{x_n\}$ generated by algorithm (3.1) is well defined.*

Proof. First, let us show that C_n is a closed and convex subset of C for all $n \geq 0$. Indeed, observe that

$$\begin{aligned} &\phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \\ &\Leftrightarrow 2\langle v, (1-\beta_n)Jz_n + \beta_n Jx_n - Ju_n\rangle \leq (1-\beta_n)\|z_n\|^2 - \|u_n\|^2 + \beta_n\|x_n\|^2 \end{aligned}$$

and

$$\begin{aligned} &\beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n) \\ &\Leftrightarrow \phi(v, z_n) \leq \phi(v, x_n) \\ &\Leftrightarrow 2\langle v, Jx_n - Jz_n\rangle \leq \|x_n\|^2 - \|z_n\|^2. \end{aligned}$$

Hence C_n is closed and convex for each $n \geq 0$.

Second, let us show that $EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \subset C_n$ for each $n \geq 0$.

Indeed, it is clear that $EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \subset C_0 = C$. Suppose that $EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \subset C_n$ for some $n \in \mathcal{N}$. Let $w \in EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n))$ be arbitrarily chosen. Then $w \in EP$, $w \in T^{-1}0$ and $w \in \bigcap_{n=0}^{\infty} F(S_n)$. Since $u_n = K_{r_n} y_n$, utilizing (3.1) and Proposition 2.1 we have

$$\begin{aligned} \phi(w, u_n) &= \phi(w, K_{r_n} y_n) \leq \phi(w, y_n) \\ &= \phi(w, J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n} z_n)) \\ &= \|w\|^2 - 2\langle w, \beta_n Jx_n + (1-\beta_n)JJ_{r_n} z_n\rangle + \|\beta_n Jx_n + (1-\beta_n)JJ_{r_n} z_n\|^2 \\ &\leq \|w\|^2 - 2\beta_n\langle w, Jx_n\rangle - 2(1-\beta_n)\langle w, JJ_{r_n} z_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|J_{r_n} z_n\|^2 \\ &= \beta_n \phi(w, x_n) + (1-\beta_n)\phi(w, J_{r_n} z_n) \\ &\leq \beta_n \phi(w, x_n) + (1-\beta_n)\phi(w, z_n) \quad \text{(using Lemma 2.9)} \\ &= \beta_n \phi(w, x_n) + (1-\beta_n)\phi(w, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_n x_n)) \\ &= \beta_n \phi(w, x_n) + (1-\beta_n)[\|w\|^2 - 2\langle w, \alpha_n Jx_n + (1-\alpha_n)JS_n x_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JS_n x_n\|^2] \\ &\leq \beta_n \phi(w, x_n) + (1-\beta_n)[\|w\|^2 - 2\alpha_n\langle w, Jx_n\rangle - 2(1-\alpha_n)\langle w, JS_n x_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|S_n x_n\|^2] \\ &= \beta_n \phi(w, x_n) + (1-\beta_n)[\alpha_n \phi(w, x_n) + (1-\alpha_n)\phi(w, S_n x_n)] \\ &\leq \beta_n \phi(w, x_n) + (1-\beta_n)[\alpha_n \phi(w, x_n) + (1-\alpha_n)\phi(w, x_n)] \\ &= \phi(w, x_n). \end{aligned}$$

This implies that $w \in C_{n+1}$. This shows that $EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \subset C_n$ for all $n \geq 0$. Therefore $x_{n+1} = \Pi_{C_{n+1}} x_0$ is well defined. Then, by induction, the sequence $\{x_n\}$ generated by (3.1) is well defined for each integer $n \geq 0$. □

Remark 3.1. From the above proof, we obtain that

$$EP \cap T^{-1}0 \cap \left(\bigcap_{n=0}^{\infty} F(S_n)\right) \subset C_n$$

for each integer $n \geq 0$.

We are now in a position to prove the main theorem.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space. Let $\{r_n\}_{n=0}^{\infty}$ be a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that*

$$\liminf_{n\to\infty} r_n > 0, \qquad \limsup_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad \limsup_{n\to\infty} \beta_n < 1. \tag{3.2}$$

Let $EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. *If for each integer* $m \geq 0$,

$$\lim_{n\to\infty} \|S_n x_n - S_m x_n\| = 0, \tag{UARC}$$

then the sequence $\{x_n\}$ *generated by algorithm* (3.1) *converges strongly to* $\Pi_{EP\cap T^{-1}0\cap(\bigcap_{n=0}^{\infty} F(S_n))}x_0$.

Proof. We divide the proof into several steps.

Step 1. We claim that $\{x_n\}$ is bounded, and $\phi(x_{n+1}, x_n) \to 0$.

Indeed, by the definition of C_n, we have $x_n = \Pi_{C_n}x_0$, $\forall n \geq 0$. Hence from Lemma 2.3 it follows that for each $u \in EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n))$ and each $n \geq 0$,

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{C_n}x_0) \leq \phi(u, x_0).$$

This implies that $\{\phi(x_n, x_0)\}$ is bounded, and so $\{x_n\}$, $\{S_n x_n\}$, $\{J_{r_n}x_n\}$ all are bounded. Furthermore, noticing that $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0.$$

Thus, $\{\phi(x_n, x_0)\}$ is nondecreasing, and so the limit $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. From Lemma 2.3 we have

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n}x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n}x_0, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0), \quad \forall n \geq 0,\end{aligned}$$

which leads to $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. So from Lemma 2.1 it follows that $\|x_{n+1} - x_n\| \to 0$.

Step 2. We claim that $\|z_n - J_{r_n}z_n\| \to 0$ and $\|x_n - S_m x_n\| \to 0$ for each integer $m \geq 0$.

Indeed, since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, from the definition of C_{n+1} we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \geq 0,$$

and

$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \geq 0.$$

Hence from $\phi(x_{n+1}, x_n) \to 0$ it follows that $\phi(x_{n+1}, u_n) \to 0$ and $\phi(x_{n+1}, z_n) \to 0$. Utilizing Lemma 2.1, we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\| = 0, \tag{3.3}$$

and so

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.4}$$

Again since $u_n = K_{r_n}y_n$, as in the proof of Lemma 3.1 we can deduce that

$$\phi(w, u_n) \leq \phi(w, y_n) \leq \phi(w, x_n), \quad \forall w \in EP \cap T^{-1}0 \cap \left(\bigcap_{n=0}^{\infty} F(S_n)\right). \tag{3.5}$$

Now observe that

$$\begin{aligned}\phi(u_n, y_n) &= \phi(K_{r_n}y_n, y_n) \leq \phi(w, y_n) - \phi(w, K_{r_n}y_n) \quad (\text{using Proposition 2.1})\\ &\leq \phi(w, x_n) - \phi(w, K_{r_n}y_n)\\ &= \phi(w, x_n) - \phi(w, u_n)\\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle w, Jx_n - Ju_n\rangle\\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|w\|\,\|Jx_n - Ju_n\|.\end{aligned}$$

Since $\|x_n - u_n\| \to 0$ and J is uniformly norm-to-norm continuous on bounded subsets of E, it follows that $\|Jx_n - Ju_n\| \to 0$ and so $\phi(u_n, y_n) \to 0$. Since E is smooth and uniformly convex, from Lemma 2.1 and (3.4), we have

$$\|u_n - y_n\| \to 0, \quad \text{and so } \|x_n - y_n\| \to 0. \tag{3.6}$$

Note that E is uniformly smooth and uniformly convex. Thus J and J^{-1} are uniformly norm-to-norm continuous on bounded subsets of E and E^*, respectively. Hence from (3.1) and (3.6) we have

$$(1 - \beta_n)\|JJ_{r_n}z_n - Jx_n\| = \|Jy_n - Jx_n\| \to 0,$$

and so $\|J_{r_n}z_n - x_n\| \to 0$. This together with $\|x_n - z_n\| \to 0$ yields

$$\lim_{n\to\infty} \|z_n - J_{r_n}z_n\| = \lim_{n\to\infty} \|Jz_n - JJ_{r_n}z_n\| = 0. \tag{3.7}$$

Again from (3.1) and (3.4) we have

$$(1-\alpha_n)\|JS_nx_n - Jx_n\| = \|Jz_n - Jx_n\| \to 0.$$

This implies that $\|JS_nx_n - Jx_n\| \to 0$, and so

$$\lim_{n\to\infty} \|x_n - S_nx_n\| = 0. \tag{3.8}$$

Note that for each integer $m \geq 0$,

$$\|x_n - S_mx_n\| \leq \|x_n - S_nx_n\| + \|S_nx_n - S_mx_n\|. \tag{3.9}$$

Thus, from (3.8) and condition (UARC) we infer that for each integer $m \geq 0$,

$$\lim_{n\to\infty} \|x_n - S_mx_n\| = 0. \tag{3.10}$$

Step 3. We claim that $\omega_w(\{x_n\}) \subset EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n))$, where

$$\omega_w(\{x_n\}) := \{\hat{x} \in C : x_{n_k} \rightharpoonup \hat{x} \text{ for some subsequence } \{n_k\} \subset \{n\} \text{ with } n_k \uparrow \infty\}.$$

Indeed, for any $\hat{x} \in \omega_w(\{x_n\})$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. Since S_m is relatively nonexpansive for each integer $m \geq 0$, from (3.10) and $x_{n_k} \rightharpoonup \hat{x}$ we have

$$\hat{x} \in \widehat{F}(S_m) = F(S_m).$$

Now let us show that $\hat{x} \in T^{-1}0$. Since $x_{n_k} \rightharpoonup \hat{x}$, from (3.4) and (3.7) it follows that $z_{n_k} \rightharpoonup \hat{x}$ and $J_{r_{n_k}}z_{n_k} \rightharpoonup \hat{x}$. Also, from (3.7) and $\liminf_{n\to\infty} r_n > 0$ we derive

$$\lim_{n\to\infty} \|A_{r_n}z_n\| = \lim_{n\to\infty} \frac{1}{r_n}\|Jz_n - JJ_{r_n}z_n\| = 0.$$

If $z^* \in Tz$, then it follows from (2.5) and the monotonicity of the operator T that for all integers $k \geq 0$

$$\langle z - J_{r_{n_k}}z_{n_k}, z^* - A_{r_{n_k}}z_{n_k}\rangle \geq 0.$$

Letting $k \to \infty$, we obtain $\langle z - \hat{x}, z^*\rangle \geq 0$. Then the maximality of the operator T yields $\hat{x} \in T^{-1}0$.

Next, let us show that $\hat{x} \in EP$. Since $x_{n_k} \rightharpoonup \hat{x}$, from (3.4) and (3.6) it follows that $u_{n_k} \rightharpoonup \hat{x}$ and $y_{n_k} \rightharpoonup \hat{x}$. Since J is uniformly norm-to-norm continuous on bounded subsets of E, from (3.6) we have $\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0$. From $\liminf_{n\to\infty} r_n > 0$, it follows that

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{3.11}$$

By the definition of $u_n := K_{r_n}y_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C,$$

where

$$F(u_n, y) = f(u_n, y) + \langle Au_n, y - u_n\rangle.$$

Replacing n by n_k, we have from (A2) that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k}\rangle \geq -F(u_{n_k}, y) \geq F(y, u_{n_k}), \quad \forall y \in C.$$

Since $y \mapsto f(x, y) + \langle Ax, y - x\rangle$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in the last inequality, from (3.11) and (A4) we have

$$F(y, \hat{x}) \leq 0, \quad \forall y \in C.$$

For t, with $0 < t \leq 1$, and $y \in C$, let $y_t = ty + (1-t)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, then $y_t \in C$ and hence $F(y_t, \hat{x}) \leq 0$. So, from (A1) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \hat{x}) \leq tF(y_t, y).$$

Dividing by t, we have

$$F(y_t, y) \geq 0, \quad \forall y \in C.$$

Letting $t \downarrow 0$, from (A3) it follows that

$$F(\hat{x}, y) \geq 0, \quad \forall y \in C.$$

So, $\hat{x} \in EP$. Therefore, we obtain that $\omega_w(\{x_n\}) \subset EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n))$ by the arbitrariness of $\hat{x}$.

Step 4. We claim that $\omega_w(\{x_n\}) = \{\Pi_{EP\cap T^{-1}0\cap(\bigcap_{n=0}^{\infty} F(S_n))}x_0\}$ and $x_n \to \Pi_{EP\cap T^{-1}0\cap(\bigcap_{n=0}^{\infty} F(S_n))}$.

Indeed, put $\bar{x} = \Pi_{EP\cap T^{-1}0\cap(\bigcap_{n=0}^{\infty} F(S_n))}x_0$. From $x_{n+1} = \Pi_{C_{n+1}}x_0$ and $\bar{x} \in EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \subset C_{n+1}$, we have $\phi(x_{n+1}, x_0) \leq \phi(\bar{x}, x_0)$. Now from weakly lower semicontinuity of the norm, we derive for each $\hat{x} \in \omega_w(\{x_n\})$

$$\begin{aligned}
\phi(\hat{x}, x_0) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{k\to\infty}(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2) \\
&= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \\
&\leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \\
&\leq \phi(\bar{x}, x_0).
\end{aligned}$$

It follows from the definition of $\Pi_{EP\cap T^{-1}0\cap(\bigcap_{n=0}^{\infty} F(S_n))}x_0$ that $\hat{x} = \bar{x}$ and hence

$$\lim_{k\to\infty} \phi(x_{n_k}, x_0) = \phi(\bar{x}, x_0).$$

So we have $\lim_{k\to\infty} \|x_{n_k}\| = \|\bar{x}\|$. Utilizing the Kadec–Klee property of E, we conclude that $\{x_{n_k}\}$ converges strongly to $\Pi_{EP\cap T^{-1}0\cap(\bigcap_{n=0}^{\infty} F(S_n))}x_0$. Since $\{x_{n_k}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we know that $\{x_n\}$ converges strongly to $\Pi_{EP\cap T^{-1}0\cap(\bigcap_{n=0}^{\infty} F(S_n))}x_0$. This completes the proof. □

The following corollaries can be obtained from Theorem 3.1 immediately.

Corollary 3.1. *Let E and C be the same as in Theorem 3.1. Let $T : E \to 2^{E^*}$ be a maximal monotone operator, $f : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)–(A4), and $\{S_n\}_{n=0}^{\infty}$ be a countable family of relatively nonexpansive self-mappings on C. Let $EP(f) \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases}
x_0 \in C, \qquad C_0 = C, \\
z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_nx_n), \\
y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n}z_n), \\
u_n \in C \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n\phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}}x_0, \quad n = 0, 1, 2, \ldots,
\end{cases} \tag{3.12}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy (3.2). If the condition (UARC) is satisfied, then $\{x_n\}$ converges strongly to $\Pi_{EP(f)\cap T^{-1}0\cap(\bigcap_{n=0}^{\infty} F(S_n))}x_0$.

Proof. Put $A \equiv 0$ in Theorem 3.1. Then $EP = EP(f)$. Hence from Theorem 3.1 we immediately obtain the desired conclusion. □

Corollary 3.2. *Let E and C be the same as in Theorem 3.1. Let $T : E \to 2^{E^*}$ be a maximal monotone operator, $A : C \to E^*$ be an α-inverse-strongly monotone mapping, and $\{S_n\}_{n=0}^{\infty}$ be a countable family of relatively nonexpansive self-mappings on C. Let $VI(C, A) \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases}
x_0 \in C, \qquad C_0 = C, \\
z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_nx_n), \\
y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n}z_n), \\
u_n \in C \quad \text{such that } \langle Au_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n\phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}}x_0, \quad n = 0, 1, 2, \ldots,
\end{cases} \tag{3.13}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy (3.2). If the condition (UARC) is satisfied, then $\{x_n\}$ converges strongly to $\Pi_{VI(C,A)\cap T^{-1}0\cap(\bigcap_{n=0}^{\infty} F(S_n))}x_0$.

Proof. Put $f \equiv 0$ in Theorem 3.1. Then $EP = VI(C, A)$. Hence from Theorem 3.1 we immediately obtain the desired conclusion. □

Corollary 3.3. *Let E and C be the same as in Theorem 3.1. Let $A : C \to E^*$ be an α-inverse-strongly monotone mapping, $f : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)–(A4), and $\{S_n\}_{n=0}^{\infty}$ be a countable family of relatively nonexpansive self-mappings on C. Let $EP \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jz_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{3.14}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy (3.2). If the condition (UARC) is satisfied, then $\{x_n\}$ converges strongly to $\Pi_{EP \cap (\bigcap_{n=0}^{\infty} F(S_n))} x_0$.

Proof. Put $T \equiv 0$ in Theorem 3.1. Then $EP \cap T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) = EP \cap (\bigcap_{n=0}^{\infty} F(S_n))$ and $J_r = (J + rT)^{-1}J = I$. Hence from Theorem 3.1 we immediately obtain the desired conclusion. □

Corollary 3.4. *Let E and C be the same as in Theorem 3.1. Let $T : E \to 2^{E^*}$ be a maximal monotone operator, and $\{S_n\}_{n=0}^{\infty}$ be a countable family of relatively nonexpansive self-mappings on C such that $T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n} z_n), \\ u_n = \Pi_C y_n, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{3.15}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy (3.2). If the condition (UARC) is satisfied, then $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0 \cap (\bigcap_{n=0}^{\infty} F(S_n))} x_0$.

Proof. Put $A \equiv 0$ and $f \equiv 0$ in Theorem 3.1. Then $u_n = \Pi_C y_n$, $\forall n \geq 0$. Hence from Theorem 3.1 we immediately obtain the desired conclusion. □

4. Applications

Let E be a reflexive, strictly convex, and smooth Banach space. Let $T, \widetilde{T} : E \to 2^{E^*}$ be two maximal monotone operators. For $r > 0$, define the resolvent of T and $\widetilde{T}$ by $J_r = (J + rT)^{-1}J$ and $\widetilde{J}_r = (J + r\widetilde{T})^{-1}J$, respectively. Then, J_r (resp. $\widetilde{J}_r$) is a single-valued mapping from E to $D(T)$ (resp. from E to $D(\widetilde{T})$). Also, for $r > 0$,

$$T^{-1}0 = F(J_r) \quad (\text{resp. } \widetilde{T}^{-1}0 = F(\widetilde{J}_r)), \tag{4.1}$$

where $F(J_r)$ (resp. $F(\widetilde{J}_r)$) is the set of fixed points of J_r (resp. $\widetilde{J}_r$). We can define, for $r > 0$, the Yosida approximation of T (resp. $\widetilde{T}$) by $A_r = (J - JJ_r)/r$ (resp. $\widetilde{A}_r = (J - J\widetilde{J}_r)/r$). For $r > 0$ and $x \in E$, we know that $A_r x \in TJ_r x$ and $\widetilde{A}_r x \in \widetilde{T}\widetilde{J}_r x$.

We are now in a position to apply Theorem 3.1 to prove the following result.

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space, $r > 0$ be a positive constant, $A : E \to E^*$ be an α-inverse-strongly monotone mapping, and $f : E \times E \to \mathcal{R}$ be a bifunction satisfying (A1)–(A4). Let $T, \widetilde{T} : E \to 2^{E^*}$ be two maximal monotone operators such that $EP \cap T^{-1}0 \cap \widetilde{T}^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in E, \quad C_0 = E, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J\widetilde{J}_r x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n} z_n), \\ u_n \in E \quad \text{such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in E, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4.2}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy (3.2). Then $\{x_n\}$ converges strongly to $\Pi_{EP \cap T^{-1}0 \cap \widetilde{T}^{-1}0} x_0$.

Proof. From (4.1) and Lemma 2.9 it follows that $\widetilde{J}_r : E \to D(\widetilde{T})$ is a relatively nonexpansive mapping and $\widetilde{T}^{-1}0 = F(\widetilde{J}_r)$. Now, in Theorem 3.1, put $S_n = \widetilde{J}_r$ for each integer $n \geq 0$. Then it is easy to see that for all $m \geq 0$,

$$\lim_{n\to\infty} \|S_n x_n - S_m x_n\| = \lim_{n\to\infty} \|\widetilde{J}_r x_n - \widetilde{J}_r x_n\| = 0,$$

that is, the condition (UARC) is satisfied. Then from Theorem 3.1 we immediately obtain the desired conclusion. □

From Theorem 4.1, we can derive the following corollaries.

Corollary 4.1. *Let E and $r > 0$ be the same as in Theorem 4.1. Let $A : E \to E^*$ be an α-inverse-strongly monotone mapping and $T, \widetilde{T} : E \to 2^{E^*}$ be two maximal monotone operators such that $VI(E, A) \cap T^{-1}0 \cap \widetilde{T}^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in E, \qquad C_0 = E, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J\widetilde{J}_r x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n} z_n), \\ u_n \in E \quad \text{such that } \langle Au_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in E, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4.3}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy (3.2). Then $\{x_n\}$ converges strongly to $\Pi_{VI(E,A)\cap T^{-1}0\cap\widetilde{T}^{-1}0} x_0$.

Proof. Put $f \equiv 0$ in Theorem 4.1. Then from Theorem 4.1 we immediately obtain the desired conclusion. □

Corollary 4.2. *Let E and $r > 0$ be the same as in Theorem 4.1. Let $f : E \times E \to \mathcal{R}$ be a bifunction satisfying (A1)–(A4) and $T, \widetilde{T} : E \to 2^{E^*}$ be two maximal monotone operators such that $EP(f) \cap T^{-1}0 \cap \widetilde{T}^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in E, \qquad C_0 = E, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J\widetilde{J}_r x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n} z_n), \\ u_n \in E \quad \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in E, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4.4}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy (3.2). Then $\{x_n\}$ converges strongly to $\Pi_{EP(f)\cap T^{-1}0\cap\widetilde{T}^{-1}0} x_0$.

Proof. Put $A \equiv 0$ in Theorem 4.1. Then from Theorem 4.1 we immediately obtain the desired conclusion. □

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