JOURNAL OF ALGEBRA 133, 211–231 (1990)

On the Number of Maximal Sincere Modules over Sincere Directed Algebras

LUISE UNGER

Fachbereich 17, Universität Gesamthoschule Paderborn, D-4790 Paderborn, West Germany

Communicated by Walter Feit

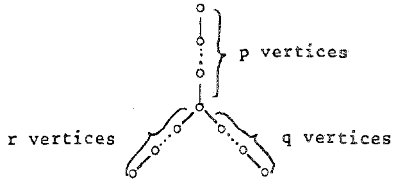
Received April 21, 1988

Let B be a finite dimensional, basic, connected algebra over an algebraically closed field k. Following [G], we write $B = k\Lambda/\mathscr{I}$, where $k\Lambda$ is the path algebra of a finite quiver without oriented cycles, and $\mathscr{I}$ is an admissible ideal. The category of finite dimensional left B-modules will be denoted by B-mod. A B-module X is called sincere if X is indecomposable and if $\operatorname{Hom}_B(X, I) \neq 0$ for all injective B-modules I.

If B is representation-finite with a sincere module, and if the Auslander–Reiten quiver $\Gamma(B)$ of B does not contain an oriented cycle, then B is called sincere directed. Equivalently, B is a representation-finite tilted algebra with a sincere module [HR]. Sincere modules play an important role for calculating sincere directed algebras [R].

Let B be sincere directed with n pairwise nonisomorphic simple modules $S(a)$, $1 \leqslant a \leqslant n$, and let X be a B-module. By $\underline{\dim}\, X \in \mathbb{Z}^n$ we denote the dimension vector of X, that is, the vector whose ath entry is the k-dimension of $\operatorname{Hom}_B(X, I(a))$, where $I(a)$ is the indecomposable injective B-module, whose socle is the simple module $S(a)$. We call X maximal sincere if X is sincere, and $\underline{\dim}\, X$ is maximal with respect to the componentwise order on all dimension vectors of indecomposable B-modules.

In a recent article [P], de la Peña proved that if B-mod admits more than one maximal sincere module (note that we consider modules only up to isomorphism), then B is a tilted algebra of $A = k\Delta^*$, where the underlying graph Δ of Δ is of type T_{pqr}, that is, Δ is of the form

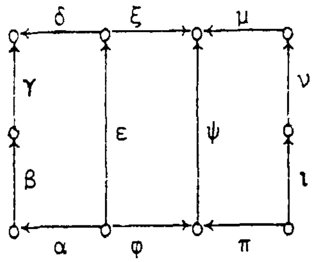


0021-8693/90 $3.00

481/133/1-15

Then all maximal sincere modules have three neighbours in the orbit graph $\mathcal{O}(\Gamma(B))$ of $\Gamma(B)$. By Δ^* we denote the opposite quiver of Δ. In this article we investigate these algebras and give optimal upper bounds for the number of maximal sincere modules. Our main purpose is to prove:

THEOREM. *Let B be a sincere directed algebra which is tilted from a wild algebra. Then B-*mod *admits more than one maximal sincere module, if and only if $B = k\Lambda/\mathcal{I}$, where Λ is the quiver*



and $\mathcal{I}$ is generated by $\alpha\beta\gamma - \varepsilon\delta$, $\varepsilon\xi - \varphi\psi$, and $\pi\psi - \iota\nu\mu$.

This result is known if B has at least 13 pairwise nonisomorphic simple modules. It follows from Bongartz' list of the large sincere directed algebras [B1, R].

The first section will be introductory. We will fix some notation, recall definitions and results connected to maximal sincere modules, and deduce some general properties of A and the A tilting module T, whose endomorphismring $\operatorname{End}_A T$ is the sincere directed algebra B.

In the second and third section we give optimal upper bounds for the number of maximal sincere B-modules, if B is tilted from a representation-finite or a tame algebra. The last chapter is devoted to the proof of the above theorem.

1. PRELIMINARIES

1.1. We briefly want to summarize the definition and some properties of maximal sincere B-modules. Basic definitions, more detailed information, and proofs can be found in [R], and will be used here frequently.

If n is the number of pairwise nonisomorphic simple B-modules $S(a)$, with $1 \leqslant a \leqslant n$, we consider the nonsymmetric bilinear form $\langle z, z' \rangle = zC^{-T}z'^{T}$ for $z, z' \in \mathbb{Z}^n$, where C denotes the Cartan matrix of B, and we denote the corresponding symmetric form by $(-, -)$.

It has been shown [B2, HR] that the quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$ with $q(z) = \langle z, z \rangle$ is weakly positive, and that there is a bijection $X \mapsto \underline{\dim}\, X$

between the indecomposable B-modules and the positive roots of q. A sincere B-module M is called maximal if its dimension vector is a maximal root of q.

A sincere root z is maximal if and only if $(z, e(a)) \geqslant 0$ for all $e(a)$, where $e(a) = \underline{\dim}\, S(a)$. It satisfies the equation

$$2 = (z, z) = \sum_{a=1}^{n} (z)_a (z, e(a)),$$

where by $(z)_a$ we denote the ath entry of z.

Since $-1 \leqslant (z, e(a)) \leqslant 1$, there are at most two a satisfying $(z, e(a)) > 0$, and these are called the exceptional vertices of z. If there is only one such a, then $(z)_a = 2$, and if there are two, say a and b, then $(z)_a = (z)_b = 1$.

Let M be a maximal sincere B-module with $\underline{\dim}\, M = z$. M is dominated by the projective module $P = \bigoplus_{a=1}^{n} P(a)^{(z, e(a))}$, where $P(a)$ denotes the indecomposable B-module whose top is $S(a)$. Then we have for all B-modules X that $(z, \underline{\dim}\, X) = \langle \underline{\dim}\, P, \underline{\dim}\, X \rangle$. Recall that for dimension vectors $\underline{\dim}\, X$ and $\underline{\dim}\, Y$ of B-modules we have

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y \rangle = \sum_{i=1}^{t} (-1)^i \dim_k \operatorname{Ext}^i_B (X, Y),$$

and since the global dimension of B is at most two [HR], $\operatorname{Ext}^i_B(X, Y) = 0$ for $i > 2$.

1.2. Being interested in the number of maximal sincere B-modules, we have according to the results of de la Peña only to consider the cases where B is tilted from $A = k\mathbf{\Delta}$, Δ of type T_{pqr}, and B-mod has a maximal sincere module with three neighbours in the orbit graph of $\Gamma(B)$.

Since $\mathcal{O}(\Gamma(B)) = \Delta$, we will enumerate the vertices of $\mathcal{O}(\Gamma(B))$ according to the vertices of Δ. Since B is sincere, $\Gamma(B)$ contains each possible orientation of Δ as a complete slice, implying that for each quiver $\mathbf{\Delta}$ there is an $A = k\mathbf{\Delta}^*$ tilting module T with $\operatorname{End}_A T = B$ [R].

We will assume in the following that $\mathbf{\Delta}$ is given by factorspace orientation, that is, that the branching point b of Δ, the unique vertex with three neighbours, is the only source of $\mathbf{\Delta}$.

The indecomposable injective (projective) A-modules will be denoted by $I_A(i)$ $(P_A(i))$, where $1 \leqslant i \leqslant n$ are the vertices of $\mathbf{\Delta}$, and the indecomposable injective (projective) B-modules will be denoted by $I_B(i)$ $(P_B(i))$, where $1 \leqslant i \leqslant n$ are the vertices of $\mathbf{\Lambda}$.

If M and M' are two distinct maximal sincere B-modules, then $M^\tau = M'^\tau$, where M^τ denotes the τ-orbit of M in $\mathcal{O}(\Gamma(B))$ [P]. This implies that there is a unique last (in the order given by the paths in $\Gamma(B)$) maximal sincere B-module, which will be denoted by $M(a)$.

$\mathscr{S}(M(a)\to)$, the slice in $\Gamma(B)$ having $M(a)$ as the only source, is a complete slice, and we may assume that it is the image of the indecomposable injective A-modules under the functor $\operatorname{Hom}_A(T,-)$. In particular, $M(a)=\operatorname{Hom}_A(T, I_A(b))$.

If $M(c)=\tau_B^r M(a)$ for some $r>0$ is maximal sincere as well, then $M(c)=\tau_B^r \operatorname{Hom}_A(T, I_A(b))=\operatorname{Hom}_A(T, \tau_A^r I_A(b))$ [R]. In fact all preinjective direct summands of T are predecessors of $\tau_A^r I_A(b)$.

Since all maximal sincere modules have three neighbours in the orbit graph, they have one exceptional vertex [R], in particular, they are dominated by an indecomposable projective module.

Let $P_B(a)$ be the projective dominating $M(a)$, and let $T(a)$ be the indecomposable summand of T with $P_B(a)=\operatorname{Hom}_A(T, T(a))$. Then $\underline{\dim}\, T(a)=(\dim_k \operatorname{Hom}_B(P_B(a), \mathscr{S}(M(a)\to)))$.

Let X be an indecomposable module in $\mathscr{S}(M(a)\to)$. Since $M(a)$ is dominated by $P_B(a)$ we obtain

$$\begin{aligned}\dim_k \operatorname{Hom}_B(P_B(a), X) &= \langle \underline{\dim}\, P_B(a), \underline{\dim}\, X\rangle \\ &= (\underline{\dim}\, M(a), \underline{\dim}\, X) \\ &= \sum_{i=0}^{2} (-1)^i \dim_k \operatorname{Ext}_B^i(M(a), X) \\ &\quad + \sum_{i=0}^{2} (-1)^i \dim_k \operatorname{Ext}_B^i(X, M(a)).\end{aligned}$$

Since the Ext^i terms vanish for $i\geqslant 1$, we get that $\dim_k \operatorname{Hom}_B(P_B(a), X)=2$ if $X=M(a)$ and $\dim_k \operatorname{Hom}_B(P_B(a), X)=1$ otherwise.

Hence the A-module $T(a)$ has

$$\underline{\dim}\, T(a) = \begin{matrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 2 & 1\cdots 1 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix}$$

Let $M(c)=\tau_B^r M(a)$ be another maximal sincere B-module, $M(c)$ dominated by $P_B(c)$, and let $T(c)$ be the corresponding tilting summand. Obviously, $(\dim_k \operatorname{Hom}_B(P_B(c), \mathscr{S}(\tau_B^r M(a)\to)))=\underline{\dim}\, T(a)$.

LEMMA. *If* $(\dim_k \operatorname{Hom}_B(P_B(a), \mathscr{S}(M(a)\to))) = (\dim_k \operatorname{Hom}_B(P_B(c), \mathscr{S}(\tau_B^r M(a)\to)))$, *then* $T(a)=\tau_A^{-r}T(c)$.

Proof. The proof involves some tilting theory for which we refer to

[R]. Let G be the functor ${}_AT_B \otimes_B -$ and F the functor $\mathrm{Hom}_A(T, -)$. Let $r > 0$, since otherwise the assertion of the lemma is trivial. If $X(i)$ is a module in $\mathscr{S}(M(a)\to)$, say $X(i) = F(I_A(i))$, then $X(i)$ and $\tau_B X(i)$ are in $\mathscr{Y}(T)$, where $(\mathscr{Y}(T), \mathscr{X}(T))$ is the torsion pair on B-mod induced by T. Hence we have that $G(X(i))$ and $G(\tau_B X(i))$ belong to $\mathscr{T}$, where $(\mathscr{F}, \mathscr{T})$ is the torsion pair on A-mod. Then $F(\tau_A G(X(i))) = \tau_B(FG(X(i))) = \tau_B(X(i))$ [R], implying that $\tau_A G(X(i)) = GF(\tau_A G(X(i))) = G(\tau_B(X(i)))$.

Assume that $\mathrm{Hom}_B(P_B(a), X(i)) = \mathrm{Hom}_B(P_B(c), \tau_B^r X(i))$. Then

$$\mathrm{Hom}_A(G(P_B(a)), G(X(i))) = \mathrm{Hom}_A(G(P_B(c)), G(\tau_B^r X(i))),$$

hence

$$\begin{aligned}\mathrm{Hom}_A(T(a), I_A(i)) &= \mathrm{Hom}_A(T(c), G(\tau_B^r X(i)))\\ &= \mathrm{Hom}_A(T(c), \tau_A^r(G(X(i)))),\end{aligned}$$

applying the above consideration inductively. But then $\mathrm{Hom}_A(T(a), I_A(i)) = \mathrm{Hom}_A(T(c), \tau_A^r I_A(i)) = \mathrm{Hom}_A(\tau_A^{-r} T(c), I_A(i))$, implying that $\underline{\dim}\, T(a) = \underline{\dim}\, \tau_A^{-r} T(c)$, hence $T(a) = \tau_A^{-r} T(c)$.

As an immediate consequence of this lemma we obtain:

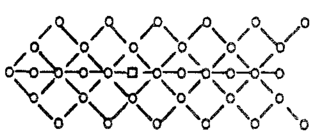
PROPOSITION. *Let B be a sincere directed algebra, and assume that* B-mod *admits two distinct maximal sincere modules with the same exceptional vertex. Then B is a tilted algebra of type Δ, with $\Delta = \tilde{\mathbb{E}}_6$ or $\Delta = \tilde{\mathbb{E}}_7$ or $\Delta = \tilde{\mathbb{E}}_8$.*

Another consequence of the above lemma, which also follows immediately from [P], is that M is maximal sincere, then neither τM nor $\tau^- M$ is maximal sincere.

Summarizing our considerations, we will assume in the following, without stating it explicitly, that B is tilted from $A = k\Delta^*$, where Δ is of the type T_{pqr} and Δ has factorspace orientation, that $M(a)$ with $M(a)^\tau = b$ is the last maximal sincere B-module, that $T(a)$ with $(\underline{\dim}\, T(a))_b = 2$ and $(\underline{\dim}\, T(a))_x = 1$ for $x \neq b$ is an indecomposable direct summand of T, that if $\tau_B^r M(a)$ is maximal sincere, then $\tau_A^r T(a)$ is a direct summand of T, and that alll preinjective direct summands of T are predecessors of $\tau_A^r I_A(b)$.

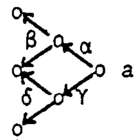
2. Upper Bounds for the Number of Maximal Sincere B-Modules if B Is Tilted from a Representation-Finite Algebra

2.1. Let Δ be of type $\mathbb{E}_6$. Then $\Gamma(A)$ is of the form

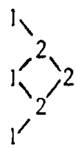
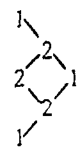


Observe that we only draw edges instead of arrows between the vertices of $\Gamma(A)$. As usual, the arrows go from the left to the right. The square in $\Gamma(A)$ corresponds to $T(a)$.

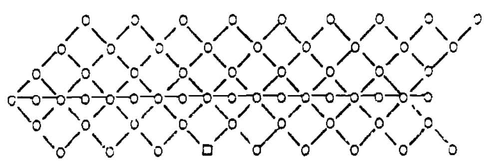
The only module in the τ-orbit of $T(a)$ which is the predecessor of $T(a)$ and has no extensions with $T(a)$ is the projective module. Hence there are at most two maximal sincere B-modules. On the other hand, if Λ is the quiver



and $\mathscr{I}$ is the ideal generated by $\alpha\beta - \gamma\delta$, then $B = k\Lambda/\mathscr{I}$ is an $\mathbb{E}_6$-tilted algebra admitting two maximal sincere modules $M(a)$ and $M(c)$ with

$\underline{\dim}$ M(a) =  and $\underline{\dim}$ M(c) = 

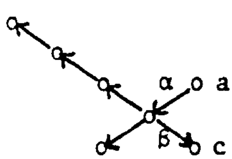
2.2. Let Δ be of type $\mathbb{E}_7$. Then $\Gamma(A)$ is of the form



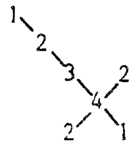
Again, the square corresponds to $T(a)$.

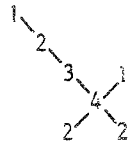
There are two predecesors of $T(a)$ in $\mathscr{O}(T(a))$, the τ-orbit of $T(a)$, which do not extend with $T(a)$, namely $\tau_A^2 T(a)$ and $\tau_A^3 T(a)$. Since $\mathrm{Ext}_A^1(\tau_A^2 T(a), \tau_A^3 T(a)) \neq 0$, only one of them can be a direct summand of T, implying that there are at most two maximal sincere B-modules.

On the other hand, if Λ is the quiver

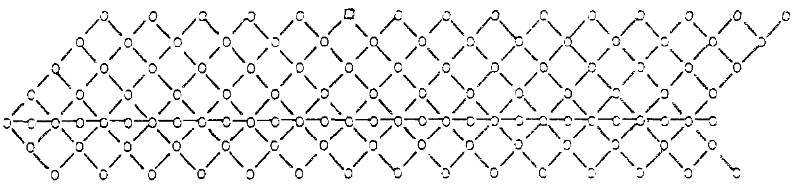


and $\mathscr{I} = \langle \alpha\beta \rangle$, then $B = k\Lambda/\mathscr{I}$ is an $\mathbb{E}_7$-tilted algebra with two maximal sincere B-modules $M(a)$ and $M(c)$, where

$\underline{\dim}$ M(a) = 

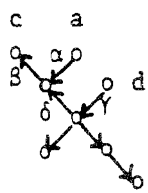
and $\underline{\dim}$ M(c)= 

2.3. Let Λ be of type $\mathbb{E}_8$, and consider $\Gamma(A)$:

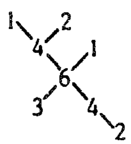


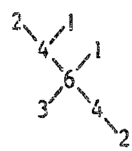
Again, the square corresponds to $T(a)$.

The modules $\tau_A^s T(a)$ with $2 \leqslant s \leqslant 5$ are predecessors of $T(a)$ in $\mathscr{C}(T(a))$ and do not extend with $T(a)$. But at most two of them do not extend with each other. Hence there are at most three maximal sincere B-modules. If Λ is the quiver

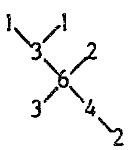


and $\mathscr{I} = \langle \alpha\beta, \gamma\delta \rangle$, then $B = k\Lambda/\mathscr{I}$ is an $\mathbb{E}_8$-tilted algebra admitting three maximal sincere modules $M(a)$, $M(c)$, and $M(d)$, with

$\underline{\dim}$ M(a) = 

, $\underline{\dim}$ M(c) = 

and

$\underline{\dim}$ M(d) = 

2.4. Recall that a vertex v of Δ is called a tip, if v has exactly one neighbour in Δ. If x and y are vertices of Δ, then the distance $d(x, y)$ between x and y is the number of edges between x and y. We say that the vertex x belongs to a branch of Δ, if $\Delta\setminus\{x\}$ is either connected or has a component of type $\mathbb{A}_n$. The length of a branch Δ' of Δ is the distance $d(b, v)$ between the branching point b and the tip v in Δ'.

Let $\mathbf{\Delta}$ be of type $\mathbb{D}_n$, and let v be the tip of the longest branch of Δ. Let v' be the neighbour of v. Then $T(a)=\tau_A^- P_A(v')$, hence there is no predecessor of $T(a)$ in $\mathscr{O}(T(a))$ not extending with $T(a)$. This implies that there is exactly one maximal sincere B-module.

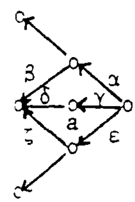
3. Upper Bounds for the Number of Maximal Sincere B-Modules if B Is Tilted from a Tame Algebra

In the remaining sections we will use the following notation. If Δ is of the form T_{pqr}, then the tip of the shortest branch will be denoted by z, the tip of the second longest branch by u, and the tip of the longest branch by v.

3.1. Let $\mathbf{\Delta}$ be of type $\tilde{\mathbb{E}}_6$. Then $T(a)$ is simple regular fo period two [DR], implying that there is exactly one indecomposable summand of T in $\mathscr{O}(T(a))$. This implies that all maximal sincere B-modules have the same exceptional vertex, and that only modules of the form $\tau_B^{2m}M(a)$ for $m\geq 0$ can be maximal.

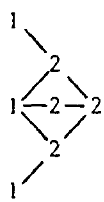
The replication number (for the definition see [BB]) for B is six [BB], and it is achieved, if the maximal B-projective module P_B in the order given by $M\leq N$ if there is a path from M to N in $\Gamma(B)$, is a tip of $\mathscr{O}(\Gamma(B))$.

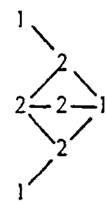
This implies that there are at most four modules M in $\Gamma(B)$ with $M^\tau=b$ satisfying that $\mathscr{S}(M\rightarrow)$ and $\mathscr{S}(\rightarrow M)$, the slice having M as the only sink, are complete slices. But this has to be satisfied by a sincere module. Hence, there are at most two maximal sincere B-modules, and this number is achieved for the $\tilde{\mathbb{E}}_6$-tilted algebra

$B = k\vec{\Lambda}/I$ with $\vec{\Lambda}$: 

and $I = \langle\alpha\beta-\gamma\delta,\ \varepsilon\zeta-\gamma\delta\rangle$

The maximal sincere modules are $M(a)$ and $M'(a)$ with

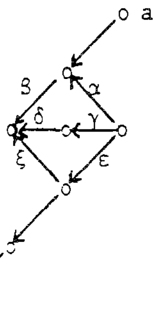
$\underline{\dim}\, M(a)$ = 

and $\underline{\dim}\, M(a)'$ = 

3.2. Let Δ be of type $\tilde{\mathbb{E}}_7$. Then $T(a)$ is simple regular of period three [DR], again implying that there is only one indecomposable summand of T in $\mathscr{C}(T(a))$, and if $M(a)$ is maximal sincere, then only the modules $\tau_B^{3m}M(a)$ for some $m \geqslant 0$ can be maximal as well. The replication number for B is 12 [BB] and direct computation shows that the maximal number of modules M with $M^\tau = b$ and $\mathscr{S}(\rightarrow M)$ and $\mathscr{S}(M\rightarrow)$ complete slices is achieved if the maximal B-projective module P_B is the vertex v in $\mathscr{C}(\Gamma(B))$. Then there are nine modules M with the above properties, implying that there are at most three maximal sincere B-modules.

On the other hand, if $B = k\Lambda/\mathscr{I}$, where

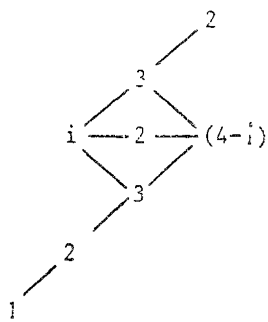
$\vec{\Lambda}$:



and $I = \langle \alpha\beta - \gamma\delta, \gamma\delta - \varepsilon\xi \rangle$

then B is an $\tilde{\mathbb{E}}_7$-tilted algebra with three maximal sincere modules $M(a)^i$, with $1 \leqslant i \leqslant 3$, given by the dimension vectors

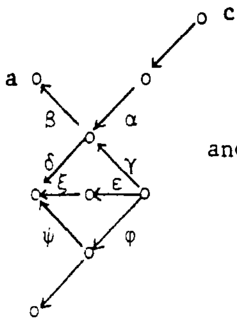
$\underline{\dim}\, M(a)^i =$



3.3. Let Δ be of type $\tilde{\mathbb{E}}_8$. Then $T(a)$ is simple regular of period five [DR], implying that there are at most two indecomposable summands of T in $\mathscr{C}(T(a))$, namely either $T(a)$ and $\tau_A^2 T(a)$ or $T(a)$ and $\tau_A^3 T(a)$. This yields that if $M(a)$ is maximal sincere, only the modules $\tau_B^{5m}M(a)$ and $\tau_B^{5m+2}M(a)$ or $\tau_B^{5m}M(a)$ and $\tau_B^{5m+3}M(a)$ for $m \geqslant 0$ can be maximal sincere.

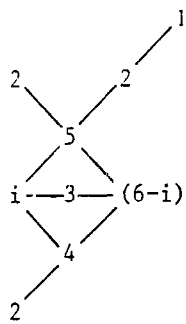
The replication number for B is at most 29, and again direct computation shows that the maximal number of modules M with $M^\tau = b$ and

$\mathscr{S}(M\to)$ and $\mathscr{S}(\to M)$ complete slices is 25, and it is achieved if the maximal projective satisfies $P_B^{\tau} = v$. This implies that there are at most 10 maximal sincere B-modules, and they are achieved for $B = k\Lambda/\mathscr{I}$ with

$\vec{\Lambda}$: 

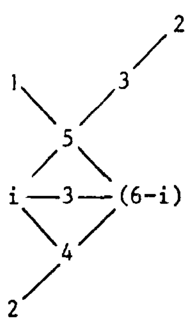
and $I = \langle \alpha\beta,\ \gamma\delta - \varepsilon\xi,\ \varepsilon\xi - \varphi\psi \rangle$

The maximal sincere B-modules have either the exceptional vertex a, and then they are given by the dimension vectors

$\underline{\dim}\, M(a)^i =$ 

for $1 \le i \le 5$

or they have the exceptional vertex c, and then they are given by

$\underline{\dim}\, M(c)^i =$ 

for $1 \le i \le 5$

4. Upper Bounds for the Number of Maximal Sincere B-Modules if B Is Tilted from a Wild Algebra

4.1. If B is a representation-finite tilted algebra, and A is representation-infinite, then T has A-preprojective and -preinjective summands.

The following two lemmas show that there are tips x and y of Δ such that there is a direct summand of T in $\mathcal{O}(I_A(x))$ and a direct summand of T in $\mathcal{O}(P_A(y))$.

Lemma 1. *Let d be a vertex in a branch of a quiver $\mathbf{Q}$ and assume this branch also contains the tip x. Let $A = k\mathbf{Q}^*$.*

Let $T(d) = \tau_A^s I_A(d)$ be a direct summand of an A-tilting module T. If there is a sectional path w from $T(d)$ to a module in $\mathcal{O}(I_A(x))$, then T has an indecomposable direct summand in $\mathcal{O}(I_A(x))$ which is either a sectional predecessor or a sectional successor of $T(d)$ or which is incomparable with $T(d)$.

Proof. Consider the sectional path w: $T(d) \to X(1) \to \cdots \to X(n)$, where $X(n) \in \mathcal{O}(I_A(x))$. By assumption, no $X(i)$ lies in the τ-orbit of $I_A(b)$, where b denotes a branching point of $\mathbf{Q}$. We will prove the lemma by induction on the number of arrows in w.

If $T(d) = X(n)$, there is nothing to show. Hence, assume that $T(d) \neq X(n)$.

A nonzero map f: $T(d) \to X(n)$ is an epimorphism, and the kernel of f is $\tau X(1)$.

Consider the exact sequence η: $0 \to \tau_A X(1) \to T(d) \to X(n) \to 0$. Applying the functor $\mathrm{Hom}_A(T, -)$ to it, we get that $\mathrm{Ext}_A^1(T, X(n)) = 0$. Hence, if $X(n)$ is not a direct summand of T, there is an indecomposable direct summand $T(c)$ of T with $\mathrm{Ext}_A^1(X(n), T(c)) \neq 0$, that is, $\mathrm{Hom}_A(T(c), \tau_A X(n)) \neq 0$. Applying $\mathrm{Hom}_A(-, T(c))$ to η yields that then also $\mathrm{Hom}_A(\tau_A X(1), T(c)) \neq 0$. But then $T(c)$ is a module in the sectional path from $\tau_A X(1)$ to $\tau_A X(n)$ and this path has fewer arrows than w, and the assertion follows by induction hypothesis.

Lemma 2. *Let d be a vertex in a branch of a quiver $\mathbf{Q}$, and assume that this branch also contains the tip x. Let $A = k\mathbf{Q}^*$, and let T be an A-tilting module, having $T(d) = \tau_A^{-s} P_A(d)$ as a direct summand.*

If T has no direct summand in $\mathcal{O}(P_A(x))$ which is either a sectional predecessor or a sectional successor of $T(d)$ or which is incomparable with $T(d)$, then $\mathrm{Hom}_A(T, I_A(b))$ *is for all branching points b of $\mathbf{Q}$ a nonsincere* End_A *T-module.*

Proof. Let $\mathscr{R}$ be the set of indecomposable preprojective direct summands of T satisfying the following condition:

If $T(d) \in \mathscr{R}$ and $w: T(d) \to X(1) \to \cdots \to X(n)$ with $X(n) \in \mathcal{O}(P_A(x))$ and x a tip of Q is a sectional path and no $X(i)$ is in the τ-orbit of $P_A(b)$, then T does not contain an indecomposable direct summand in $\mathcal{O}(P_A(x))$ which is the sectional predecessor, sectional successor, or incomparable with $T(d)$.

Let $T(d)$ in $\mathscr{R}$ be chosen in such a way that the sectional path w is of minimal length. We claim that $X(1)$ is projective, and then obviously also $T(d)$ is projective.

Assume $X(1)$ is not projective. Then the irreducible map $f: \tau_A X(1) \to T(d)$ is a monomorphism, and $X(n)$ is isomorphic to the cokernel of f. Since $X(n)$ is not a summand of T, similar arguments as above force the existence of an indecomposable direct summand $T(c)$ of T which is a successor of $\tau_A X(i)$ and a predecessor of $\tau_A X(n)$. But then $T(c) \in \mathscr{R}$ and the corresponding sectional path is of smaller length, a contradiction. By induction we get that all $X(i)$ for $1 \leqslant i \leqslant n$ are projective. Hence $T(d) = P_A(d)$ and $X(1) = P_A(c)$ are projective, and the irreducible map $g: P_A(d) \to P_A(c)$ is a monomorphism, whose cokernel is isomorphic to $\tau_A^{n-1} I_A(x)$ and there is no path from $\tau_A^{n-1} I_A(x)$ to an injective A-module $I_A(b)$, where b is a branching point of $\mathbf{Q}$.

Since by assumption $P_A(c)$ is not a direct summand of T, there is an indecomposable direct summand $T(e)$ of T with $\operatorname{Ext}_A^1(T(e), P_A(c)) \neq 0$. Applying $\operatorname{Hom}_A(T(e), -)$ to the sequence $0 \to P_A(d) \to P_A(c) \to \tau_A^{n-1} I_A(x) \to 0$ we get that $\operatorname{Ext}_A^1(T(e), \tau_A^{n-1} I_A(x)) \neq 0 \neq \operatorname{Hom}_A(\tau_A^{n-1} I_A(x), \tau T(e))$, hence $T(e)$ is a successor of $\tau_A^{n-2} I_A(x)$, implying that for all branching points b of $\mathbf{Q}$ we have $\operatorname{Hom}_A(T(e), I_A(b)) = 0$, the desired result.

Applying the above lemmas to the situation we are considering, we obtain:

COROLLARY. *If B-mod admits a maximal sincere module M with $M^\tau = b$, then there exist tips x and y of Δ such that there is a direct summand in the τ-orbit of $I_A(x)$ and a direct summand in the τ-orbit of $P_A(y)$.*

4.2. The following result has been proven in [HU].

PROPOSITION. *Let $\mathbf{Q}$ be a wild quiver with more than two vertices, and let a be a vertex in $\mathbf{Q}$ which has exactly one neighbour. Assume $\mathbf{Q}' = \mathbf{Q} \setminus \{a\}$ is representation-finite.*

Let $\mathscr{C}$ and $\mathscr{C}'$ denote the preinjective components of $k\mathbf{Q}$ and $k\mathbf{Q}'$, respectively, and let $k\mathscr{C}$ and $k\mathscr{C}'$ be the corresponding mesh categories.

Let $\mathscr{I}(a)$ be the ideal generated by the residue classes of paths in $K\mathscr{C}$ factoring over a module in $\mathcal{O}(I_{k\mathbf{Q}}(a))$. Then $k\mathscr{C}'$ is isomorphic to $k\mathscr{C}/\mathscr{I}(a)$.

As a consequence of the above proposition we obtain results concerning homomorphisms between indecomposable preinjective modules over one-point extensions (for the definition and notations we refer to [R]) $C = C'[P_{C'}(a')]$ of a representation-infinite algebra $C' = k\mathbf{Q}'$ by an indecomposable C'-projective module $P_{C'}(a')$. Then $C = k\mathbf{Q}$ is hereditary and $\mathbf{Q}'$ is a subquiver of $\mathbf{Q}$. We enumerate the vertices of $\mathbf{Q}'$ according to the vertices of $\mathbf{Q}$, and the extension vertex will be denoted by a.

An easy consequence of the above propoposition is

COROLLARY. *Let z and z' be vertices of $\mathbf{Q}'$.*

(i) *If* $\dim_k \operatorname{Hom}_{C'}(\tau^r_{C'} I_{C'}(z), \tau^s_{C'} I_{C'}(z')) = n$, *then*

$$\dim_k \operatorname{Hom}_C(\tau^r_C I_C(z), \tau^s_C I_C(z')) \geqslant n.$$

(ii) *If* $\operatorname{Hom}_{C'}(\tau^r_{C'} I_{C'}(z), \tau^s_{C'} I_{C'}(a')) \neq 0$, *then*

$$\operatorname{Hom}_C(\tau^r_C I_C(z), \tau^s_C I_C(a)) \neq 0.$$

(iii) *If* $\operatorname{Hom}_{C'}(\tau^r_{C'} I_{C'}(a'), \tau^s_{C'} I(z)) \neq 0$, *then*

$$\operatorname{Hom}_C(\tau^{r+1}_C I_C(a), \tau^s_C I_C(z)) \neq 0.$$

(iv) *If* $\operatorname{Hom}_{C'}(\tau^r_{C'} I_{C'}(a'), \tau^s_{C'} I_{C'}(a')) \neq 0$ *for* $r > s$, *then*

$$\operatorname{Hom}_C(\tau^{r+1}_C I_C(a), \tau^s_C I_C(a)) \neq 0.$$

Remarks. (a) It follows from the description of the Auslander–Reiten sequences in C-mod which are lifted from C'-mod [R] that if M is an indecomposable C'-module with $\dim_k \operatorname{Hom}_{C'}(\tau_{C'} M, I_{C'}(a')) = n$, then the ath entry of $\underline{\dim}\, \tau_C M$ is n, and all other entries coincide with those of $\underline{\dim}\, \tau_{C'} M$. In particular, if $\dim_k \operatorname{Hom}_{C'}(\tau_{C'} M, I_{C'}(a')) = 0$, then $\tau_{C'} M = \tau_C M$.

(b) Let Δ' be a representation-infinite star, say $\Delta' = T_{pqr}$, and a' be the tip of Δ' in the branch of length r, and assume that Δ' has factorspace orientation. Let $A' = k\mathbf{\Delta}'^*$. Then $A'[P_{A'}(a')] = A = k\mathbf{\Delta}^*$, where $\Delta = T_{pqr+1}$ and $\mathbf{\Delta}$ has factorspace orientation as well. A and A' obviously satisfy the above proposition and its corollary.

4.3. The following results, which have been proven in [U], will be needed in the sequel.

LEMMA 1. *Let Δ be a wild star containing $\tilde{\mathbb{E}}_6$, and let $C = k\mathbf{\Delta}^*$ for some orientation $\mathbf{\Delta}$ of Δ. Let x and y be different tips of Δ. If X is a module in $\mathscr{C}(I_C(x))$ and Y is a sectional successor of X in $\mathscr{C}(I_C(y))$, then* $\operatorname{Hom}_C(\tau^{n+2}_C X, Y) \neq 0$ *for all* $n \geqslant 0$.

Recall that an epimorphism $f\colon X\to Y$ is called τ^+-stable if $\tau^n f\colon \tau^n X\to\tau^n Y$ is an epimorphism for all $n\geqslant 0$.

LEMMA 2. *Let Δ be a wild star containing $\tilde{\mathbb{E}}_6$, and let $C=k\mathbf{\Delta}^*$ for some orientation $\mathbf{\Delta}$ of Δ. Let x be a tip of Δ. Assume that X is a module in $\mathcal{O}(I_C(x))$, and that there is a sectional path $w\colon X\rightsquigarrow V\rightsquigarrow Y$, where V is a module in $\mathcal{O}(I_C(b))$ and there are at least two arrows between V and Y. Then a nonzero morphism from X to Y is a τ_C^+-stable epimorphism.*

PROPOSITION. *Let B be a sincere directed algebra which is tilted from $A=k\mathbf{\Delta}^*$, and assume that Δ contains $\tilde{\mathbb{E}}_6$ as a proper subgraph. Then B-*mod *admits exactly one maximal sincere B-module.*

Proof. As a sincere directed algebra, B possesses at least one maximal sincere module. Assume B-mod admits more than one maximal sincere module.

According to 4.1 there are tips x and y of Δ such that $T(x)=\tau_A^s I_A(x)$ and $T'(y)=\tau_A^{s'}P_A(y)$ are indecomposable direct summands of T. By 1.2, $T(x)$ is a predecessor of $\tau_A^2 I_A(b)$, hence $s\geqslant r+2$, where r denotes the distance $d(b,x)$ between b and x. We have that $\mathrm{Ext}_A^1(\tau_A^s I_A(x),\tau_A^{-s'}P_A(y))=0$, and this can be rephrased into the condition

$$\mathrm{Hom}_A(\tau_A^{s+s'+1}I_A(x), I_A(y))=0. \qquad (*)$$

Lemma 1 states that $(*)$ cannot be satisfied if x and y are different tips of Δ, hence $y=x$.

Assume that there is a tip z different from x with $d(b,z)>2$. Then it follows from Lemma 2 that there is a τ_A^+-stable epimorphism $f\colon\tau_A^{r+2}I_A(x)\to X$, where X is the sectional successor of $\tau^{r+2}I(x)$ in the τ_A-orbit of $I_A(a)$, where a lies in the branch containing z, and $d(b,a)=2$. Since a is not a tip of Δ, it can be shown easily that $\mathrm{Hom}_A(\tau_A^n X, I(x))\neq 0$ for all $n\geqslant 0$, implying that $\mathrm{Hom}_A(\tau_A^{r+2+n}I(x), I(x))\neq 0$ for all $n\geqslant 0$. Hence, condition $(*)$ cannot be satisfied.

The only case which remains to be considered is $\Delta=T_{22r}$, where $r>2$, and x with $T(x)\in\mathcal{O}(I_A(x))$ and $T'(x)\in\mathcal{O}(P_A(x))$ is the tip of the branch of length r.

According to [U], $\tau_A^{r+4}I_A(x)$ is τ_A^+-stable faithful, hence condition $(*)$ can only be satisfied if $T(x)=\tau_A^{r+2}I_A(x)$ and $T'(x)=P_A(x)$. If $\Delta=T_{223}$, direct computation shows that $\mathrm{Hom}_A(\tau_A^{r+3}I_A(x), I(i))\neq 0$ for all $i\neq x$, and if we apply Corollary 4.2 inductively, we obtain the same result for $\Delta=T_{22r}$ and $r\geqslant 3$. This implies that $\tau_A T$ is an $A'=k\mathbf{\Delta}'^*$ tilting module, where $\mathbf{\Delta}'=\mathbf{\Delta}\setminus\{x\}$, and since A' is representation-infinite, $\tau_A T$ must have an A'-preprojective summand.

$\tau_A T(x)$ is a direct summand of T, and we claim that $\tau_A T(x)\simeq T_{A'}I_{A'}(b)$.

Proof of the Claim. The Cartan matrix of A is

$$\begin{bmatrix}
1 & 00 & 00 & 00 & \cdots & 0 \\
1 & 10 & 00 & 00 & \cdots & 0 \\
1 & 11 & 00 & 00 & \cdots & 0 \\
\hline
1 & 00 & 10 & 00 & \cdots & 0 \\
1 & 00 & 11 & 00 & \cdots & 0 \\
\hline
1 & 00 & 00 & 1 & & \\
1 & 00 & 00 & 11 & & \bigcirc \\
\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot & \\
\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \\
\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \\
1 & 00 & 00 & 11 & \cdots & 1
\end{bmatrix}$$

The Coxeter matrix $\Phi_A = -C_A^{-\prime} C_A$ of A is

$$\begin{bmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & \cdots & -1 \\
\hline
1 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\hline
1 & 1 & 1 & 0 & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & & \bigcirc \\
\cdot & \cdot & \cdot & \cdot & \cdot & & 1 & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \bigcirc & & \ddots & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & 1 & \\
0 & 0 & 0 & 0 & 0 & & & & 0
\end{bmatrix}$$

$$\underline{\dim}\, \tau_{A'} I_{A'}(b) = (1 \mid 1,\ 1 \mid 1,\ 1 \mid 1, \ldots, 1)\Phi_{A'} = (2 \mid 2,\ 1 \mid 2,\ 1 \mid 2, \ldots, 2, 1)$$

$$\underline{\dim}\, \tau_A^{r+3} I_A(x) = (0 \mid 0,\ 0 \mid 0,\ 0 \mid 0, \ldots, 0, 1)\Phi_A^{r+3} = (2 \mid 2,\ 1 \mid,\ 1 \mid,\ 2, \ldots, 2, 1, 0).$$

Since $\tau_A^{r+3} I_A(x)$ is preinjective, then $\tau_A^{r+3} I_A(x)$ and $\tau_{A'} I_{A'}(b)$ are isomorphic. But there is no A'-preprojective module X satisfying that $\operatorname{Ext}^1_{A'}(\tau_{A'} I_{A'}(b), X) = 0$, hence $\operatorname{End}_A \tau T$ and therefore $\operatorname{End}_A T = B$ are representation-infinite, a contradiction. This finishes the proof of the proposition.

4.4. In this section we want to determine upper bounds for the number of maximal sincere B-modules if B is a tilted algebra of $A = k\mathbf{\Delta}^*$, and $\Delta = T_{1pq}$ containing $\tilde{\mathbb{E}}_7$ properly. Again we need some preliminary results.

LEMMA 1. *Let $\Delta = T_{1pq}$ and assume that $\tilde{\mathbb{E}}_7$ is a proper subgraph of Δ. Let z be the tip of the branch of length 1. Then $\tau_A^3 I_A(z)$ is a τ_A^+-stable faithful module.*

Proof. If $\Delta = T_{34}$ the result can be proven directly. The general asumption follows by applying Corollary 4.2 inductively.

Similarly we prove:

LEMMA 2. *Let $\Delta = T_{1pq}$ containing $\tilde{\mathbb{E}}_7$ properly. Assume that $p > 3$, and let x be the tip of the branch of length q. Then $\tau_A^{q+3} I_A(x)$ is a τ_A^+-stable faithful module.*

Note that in the previous lemma we did not assume that $p \leqslant q$.

PROPOSITION. *Let B be sincere directed of type $\mathbf{\Delta}$, where Δ contains $\tilde{\mathbb{E}}_7$ properly. Then* B-mod *admits exactly one maximal sincere B-module.*

The proof follows from Lemma 1 and Lemma 2 exactly as in 4.3.

4.5. The remaining part of the article is devoted to the case where B is a sincere directed algebra of type $\mathbf{\Delta}$, where $\Delta = T_{12r}$ and $r \geqslant 6$. Again we denote the tip of the branch of length 1 by z, the tip of the branch of length 2 by u, and the tip of the branch of length r by v.

The objective of this section is to prove that if $T(x) = \tau_A^s I_A(x)$ and $T(y) = \tau_A^{-s} P_A(y)$ are direct summands of T, and B has more than one maximal sincere module, then x and y are vertices in the branch containing v.

Let us assume first that $\Delta = T_{126}$. Then $\operatorname{Hom}_A(\tau_A^{2+n} I_A(z), I_A(z)) \neq 0$, $\operatorname{Hom}_A(\tau_A^{3+n} I_A(z), I_A(u)) \neq 0$, $\operatorname{Hom}_A(\tau_A^{4+n} I(u), I(z)) \neq 0$, and $\operatorname{Hom}_A(\tau_A^{5+n} I_A(u), I_A(u)) \neq 0$ for all $n \geqslant 0$. Then obviously Corollary 4.2 gives the same result for $A = k\mathbf{\Delta}$ and $\Delta = T_{12r}$ with $r \geqslant 6$.

Now assume that B-mod admits more than one maximal sincere module. Then the above considerations prove that it is not possible to have $T(x) \in \mathcal{C}(I_A(z))$ or $T(x) \in \mathcal{C}(I_A(u))$ and at the same time $T(y) \in \mathcal{C}(P_A(z))$ or $T(y) \in \mathcal{C}(P_A(u))$. Hence the following cases remain to be considered:

(a) $T(x) \in \mathcal{C}(I_A(z))$ and $T(y) \in \mathcal{C}(P_A(v))$ and

(b) $T(x) \in \mathcal{C}(I_A(u))$ and $T(y) \in \mathcal{C}(P_A(v))$.

The cases where $T(x) \in \mathcal{C}(I_A(v))$ and $T(y) \in \mathcal{C}(P_A(z))$ and $T(x) \in \mathcal{C}(I_A(v))$ and $T(y) \in \mathcal{C}(P_A(u))$ are dual to (a) and (b).

(a) Assume that $T(x) \in \mathscr{C}(I_A(z))$ and $T(y) \in \mathscr{C}(P_A(v))$. Since $\tau_A^5 I_A(z)$ is τ_A^+-stable faithful and since the immediate predecessor of $\tau_A^2 I_A(b)$ in $\mathscr{C}(I_A(z))$ is $\tau_A^3 I_A(z)$, the only possibility for $T(x)$ is $\tau_A^3 I_A(z)$, and then $T(y) = P_A(v)$ is the only projective summand of T. Then $\tau_A T$ is a complete $A' = k\mathbf{\Delta}' = k(\mathbf{\Delta} \setminus \{v\})$ tilting module, and $\tau_A^4 I_A(z)$ is a summand of $\tau_A T$.

It can be proven by the same methods as in Proposition 4.3 that $\tau_A^4 I_A(z) \simeq \tau_{A'}^2 I_{A'}(j)$, where j is the neighbour of b in the branch containing v and this is a $\tau_{A'}^+$-stable faithful module. Then there is no A'-projective summand of $\tau_A T$, a contradiction.

(b) Assume that $T(x) \in \mathscr{C}(I_A(u))$ and $T(y) \in \mathscr{C}(P_A(v))$. Since $\tau_A^7 I_A(u)$ is τ_A^+-stable faithful, and since $\tau_A^4 I_A(u)$ is the sectional predecessor of $\tau_A^2 I_A(b)$, the only possibilities for $T(x)$ are $T(x) = \tau_A^4 I_A(u)$ or $T(x) = \tau_A^5 I_A(u)$.

But since $\tau_A^5 I_A(u)$ is faithful, we may again assume without loss of generality that $T(x) = \tau_A^5 I_A(u)$, and then $T(y) = P_A(v)$ is the only projective summand of T. But then $\tau_A^6 I_A(u) \simeq \tau_{A'}^4 I(i)$ is a direct summand of the $A' = k(\mathbf{\Delta} \setminus \{v\})$ tilting module $\tau_A T$, and i is the vertex in the branch of Δ containing v, and $d(b, i) = 2$. But $\tau_{A'}^4 I_{A'}(i)$ is $\tau_{A'}^+$-stable faithful, a contradiction.

4.6. PROPOSITION. *If $\Delta = T_{12r}$ with $r > 6$, then B-*mod *admits exactly one maximal sincere module.*

Proof. Assume that there is more than one maximal sincere B-module. By [R], $r \leqslant 7$, therefore $\Delta = T_{127}$.

There is a direct summand $T(x)$ of T with $T(x) \in \mathscr{C}(I_A(v))$ and a direct summand $T(y) \in \mathscr{C}(P_A(v))$. Moreover $T(x) = \tau_A^s I_A(v)$ is a predecessor of $\tau_A^2 I_A(b)$ and therefore $s \geqslant r + 2 = 9$. Since $\tau_A^{r+7} I_A(v)$ is τ_A^+-stable faithful [U], then $9 \leqslant s \leqslant r + 5 = 12$. A direct calculation gives a contradiction.

4.7. PROPOSITION. *Let $\Delta = T_{126}$ and assume B-*mod *admits more than one maximal sincere module. Then $\Gamma(B)$ does not contain a module M with $M^\tau = b$ and $\mathscr{S}(M \rightarrow)$ and $\mathscr{S}(\rightarrow \tau^l M)$ complete slices for $0 \leqslant l \leqslant 3$.*

Proof. Since B-mod admits more than one maximal sincere module, we know by 4.5 that T does not have a direct summand $\tau^s I_A(a)$ or $\tau^{-s'} P_A(a)$, where a is a vertex in a branch containing z or u, and T has a summand $T(x) \in \mathscr{C}(I_A(v))$ and $T(y) \in \mathscr{C}(P_A(v))$.

If we assume that $\Gamma(B)$ has a module M with $M^\tau = b$ and $\mathscr{S}(M \rightarrow)$ and $\mathscr{S}(\rightarrow \tau^l M)$ complete slices for $0 \leqslant l \leqslant 3$, we get furthermore that all preinjective summands of T are prececessors of $\tau^3 I_A(b)$.

Analogously to the arguments in 4.6, we obtain that the only cases to consider are:

(a) $T(x) = \tau_A^{14} I_A(v)$ and $T(y) = P_A(v)$ and

(b) $T(x)=\tau_A^{11}I_A(v)$ and $T(y)=P_A(v)$ and

(c) $T(x)=\tau_A^{9}I_A(v)$ and $T(y)=P_A(v)$.

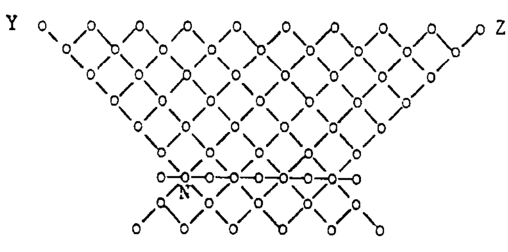
(a) If $T(x)=\tau_A^{14}I_A(v)$, then $P_A(v)$ is the only projective summand of T, and $\tau_A T$ is an $A'=k\Delta'^*=k(\Delta^*\setminus\{v\})$ tilting module. Direct computation shows that $\tau_A^{15}I_A(v)\simeq\tau_{A'}^3 I_{A'}(a)$, where a is the neighbour of b in Δ in the branch containing v. But $\tau_{A'}^3 I_{A'}(a)$ is $\tau_{A'}^+$-stable faithful, a contradiction.

(b) If $T(x)=\tau_A^{11}I_A(v)$, then again $\tau_A T$ is an $A'=k\Delta'^*=k(\Delta^*\setminus\{v\})$ tilting module, and $\tau_A^{12}I_A(v)\simeq\tau_{A'}^4 I_{A'}(c)$ with c the vertex in the branch containing v and $d(b,c)=3$ is a direct summand of τT. The only A'-preprojective module X with $\mathrm{Ext}^1_{A'}(\tau_{A'}^4 I_{A'}(c),X)=0$ is $X=\tau_{A'}^{-2}P_{A'}(v')$, where again v' denotes the neighbour of v. But $\tau_{A'}^{-2}P_{A'}(v')=\tau_A^{-2}P_A(v)$, hence $\tau_A^{-3}P(v)$ is a summand of T, and this implies that there are no summands $\tau^s I_A(i)$ of T with i a vertex in the branch containing v which are successors of $\tau_A^{11}I_A(v)$ and predecessors of $\tau_A^4 I_A(b)$. Hence $\tau_A^{12}I_A T$ is an A'-tilting module, and $\tau_A^{-15}P_A(v)$ is a summand of $\tau_A^{-12}T$. But for similar reasons, $\tau_A^{-15}P_A(v)=\tau_{A'}^{-7}P_{A'}(a)$, with a the neighbour of b in the branch containing v, cannot be extended to a tilting module with a representation-finite endomorphism-ring.

(c) Assume that $\tau_A^9 I_A(v)=T(x)$ and $P_A(v)=T(y)$ are summands of T. The only predecessors of $\tau_A^3 I_A(b)$ which are successors of $\tau_A^9 I_A(v)$ not of the form $\tau_A^s I_A(i)$, i a vertex in a branch containing z or u, satisfying $\mathrm{Ext}^1_A(\tau_A^s I_A(i), P_A(v))=0$ are the immediate successor $\tau_A^8 I_A(v')$ of $\tau_A^9 I_A(v)$ and $\tau_A^7 I_A(v'')$, where v'' is the vertex with $d(v,v'')=2$.

If $\tau_A^7 I_A(v'')$ is a direct summand of T, then τT is a $k\Delta^*\setminus\{v\}=A'$ tilting module, and $\tau_A^8 I_A(v'')=\tau_{A'}^2 I_{A'}(b)$ is a summand of τT, which cannot be extended to a tilting module with representation-finite endomorphism-ring, a contradiction.

If $\tau_A^8 I_A(v')$ is a summand of T, then $\Gamma(B)$ contains the subtranslation quiver



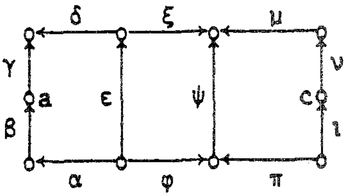
and the module Y corresponds to a projective vertex in $\Gamma(B)$. Calculating the indicator set for Y (for the definition see [BB]), we get that $\mathscr{S}(Z\to)$ is a complete slice in $\Gamma(B)$ [BB] and $\tau_B^{-10}N$ is a module in $\mathscr{S}(Z\to)$. But then $\dim_k \operatorname{Hom}_B(Y, \tau^{-10}N) = 7$, implying that B is not representation-finite.

Hence $\tau_A^{-10}T$ is a complete A'-tilting module and $\tau_A^{-10}P_A(v) = \tau_{A'}^{-8}P_{A'}(a)$ is a direct summand of $\tau_A^{-10}T$, and the only A'-preinjective modules X with $\operatorname{Ext}^1_{A'}(X, \tau_{A'}^{-8}P_{A'}(u)) = 0$ are

$$\text{(i)}\quad X = I_{A'}(v') \qquad \text{or} \qquad \text{(ii)}\quad X = \tau^3_{A'}I_{A'}(v').$$

If $\tau^3_{A'}I_{A'}(v) = \tau^4_A I_A(v)$ is a direct summand of $\tau_A^{-10}T$, then $\tau_A^{14}I_A(v)$ is a direct summand of T, and this was disproved in (a). Hence, $I_{A'}(v') = \tau_A I_A(v)$ is a direct summand of $\tau^{-10}T$, hence $\tau_A^{11}I_A(v)$ and $\tau_A^9 I_A(v)$ are direct summands of T. But this contradicts (b).

4.8. THEOREM. *Let B be a sincere directed algebra which is tilted from $A = k\Delta$, and A is of wild representation type. B-*mod *admits more than one maximal sincere module, if and only if $B \simeq k\Lambda/\mathscr{I}$, where Λ is the quiver*



and $\mathscr{I} = \langle \alpha\beta\gamma - \varepsilon\delta,\ \varepsilon\xi - \varphi\psi,\ \pi\psi - \iota\nu\mu \rangle$.

There are exactly two maximal sincere B-modules $M(a)$ and *$M(c)$ given by the dimensionvectors*

$$\underline{\dim}\, M(a) = \begin{array}{ccccccc} 1 & - & 2 & - & 1 & - & 1 \\ | & & | & & | & & | \\ 2 & & | & & | & & 1 \\ | & & | & & | & & | \\ 2 & - & 2 & - & 2 & - & 1 \end{array} \qquad \text{and} \qquad \underline{\dim}\, M(c) = \begin{array}{ccccccc} 1 & - & 2 & - & 2 & - & 2 \\ | & & | & & | & & | \\ 1 & & | & & | & & 2 \\ | & & | & & | & & | \\ 1 & - & 1 & - & 2 & - & 1 \end{array}$$

and they are dominated by $P_B(a)$ and $P_B(c)$, respectively.

Proof. Due to our previous considerations we know that if B has more than one maximal sincere module, then B is tilted from $A = k\Delta^*$, where $\Delta = T_{126}$. We already know four indecomposable summands of T, namely $\tau_A^8 I_A(v)$, $P_A(v)$, $T(a)$, and $\tau_A^2 T(a)$. Recall that $T(a)$ was given the dimensionvector $(\underline{\dim}\, T(a))_b = 2$ and $(\underline{\dim}\, T(a))_x = 1$ for all $x \neq b$. We have also proven that T has no direct summand $T(i)$ with $T(i) \in \mathcal{O}(I_A(i))$ or $T(i) \in \mathcal{O}(P_A(i))$, where i is a vertex in a branch containing z or u.

Direct computation shows that the only possibility for preinjective module X which is predecessor of $\tau_A^2 I_A(b)$ and successor of $\tau_A^8 I_A(v)$ with

$\mathrm{Ext}^1_A(X, P_A(v)) = 0$ is $X = \tau^7_A I_A(v')$, where v' is the neighbour of v. But $\mathrm{Ext}^1_A(\tau^7_A I_A(v'), \tau^2_A T(a)) \neq 0$, hence $\tau^7_A I_A(v')$ is not a direct summand of T.

Also $\mathrm{Ext}^1_A(\tau^7_A I_A(v''), \tau^2_A T(a)) \neq 0$, where $v'' \neq v$ is the neighbour of v', implying that $\tau^7_A I_A(v'')$ is also not a diremct summand of T. The assumption that $\tau^8_A I_A(v')$ is a direct summand of T was contradicted in 4.7(c). Hence, $\tau_A^{-9} T$ is an $A' = k(\Delta^* \setminus \{v\})$ tilting module, and $\tau_A^{-9} P_A(v) \simeq \tau_{A'}^{-7} P_{A'}(u)$ is a summand of $\tau_A^{-9} T$. The only A'-preinjective modules X satisfying $\mathrm{Ext}^1_{A'}(X, \tau_{A'}^{-7} P_{A'}(u)) = 0$ are $\tau_{A'} I_{A'}(v') = \tau^2_A I_A(v)$ and $\tau^4_{A'} I_{A'}(v') = \tau^5_A I_A(v)$.

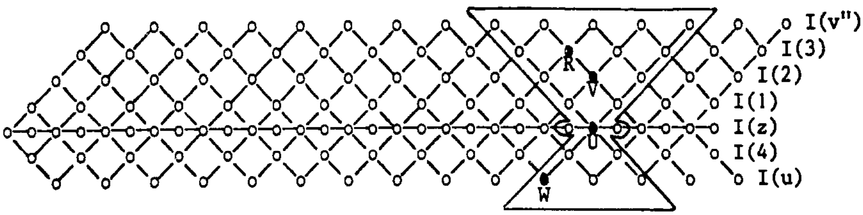
If $\tau^5_A I_A(v)$ is a direct summand of $\tau_A^{-9} T$, then $\tau^{14}_A I_A(v)$ is a direct summand of T, and this was contradicted in 4.7(a). Hence $\tau^{11}_A I_A(v)$ is a direct summand of T.

It has been shown in 4.7(b) that the only possibility to extend $\tau^{11}_A I_A(v)$ to a tilting module whose endomorphism-ring is representation-finite is if $\tau_A^{-3} P_A(v)$ is a direct summand of T.

Direct computation shows that $\tau_A^{-2} P_A(v) = \tau_{A'}^{-2} P_{A'}(v')$, and furthermore that there is no A'-preinjective successor of $\tau^{11}_A I_A(v)$ which is different from $\tau^8_A I(v)$ and a summand of T. Summarizing, we must have the following direct summand of T: $\tau^{11}_A I_A(v) \oplus \tau^8_A I_A(v) \oplus T(a) \oplus \tau^2_A T(a) \oplus P_A(v) \oplus \tau_A^{-3} P_A(v)$. Since $\mathrm{Hom}_A(\tau^{12}_A I(v), I(i)) \neq 0$ for all $i \neq v$, we get that $\tau_A T$ is a complete A'-tilting module, and $\tau^{12}_A I_A(v) \oplus \tau^9_A I_A(v) \oplus \tau_A T(a) \oplus \tau^3_A T(a) \oplus \tau_A^{-2} P_A(v)$ is a direct summand of $\tau_A T$.

Since $\tau_{A'}(\tau^{12}_A I_A(v))$, $\tau^2_{A'}(\tau^{12}_A I_A(v))$, and $\tau^3_A(\tau^{12}_A I_A(v))$ are A'-faithful modules, we obtain that $\tau^3_{A'}(\tau_A T) = T'$ is an $A'' = k(\Delta \setminus \{v, v'\})$ tilting module.

A'' is of type $\mathbb{E}_8$, and the direct sum of $\tau^3_{A'}(\tau^{12}_A I_A(v)) = U$, $\tau^3_{A'}(\tau^9_A I_A(v)) = V$, $\tau^3_{A'}(\tau_A T(a)) = W$, and $\tau^3_{A'}(\tau^3_A T(a)) = R$ is a direct summand of T'. The position of these modules in $\Gamma(A'')$ is



Since $\mathrm{Hom}_{A''}(M, \tau^2_{A''} I_{A''}(b)) \neq 0 \neq \mathrm{Hom}_{A''}(\tau^2_{A''} I_{A''}(b), N)$ for all predecessors M of $\tau^2_{A''} I_{A''}(b)$ and all successors N of $\tau^2_{A''} I_{A''}(b)$, we get that all direct summands of T' lie in the encircled part of $\Gamma(A'')$. All successors of U in $\Gamma(A'')$ are not direct summands of T', since otherwise there would be a successor of $\tau^{11}_A I_A(v)$ or $\tau^8_A I_A(v)$ which would be a summand of T, and this was excluded before. This implies that $\tau^3_{A''} I_{A''}(z)$, $\tau^3_{A''} I_{A''}(1)$, and $\tau^3_{A''} I_{A''}(4)$ are direct summands of T'.

Finally, if $\tau^4_{A''} I_{A''}(v'') = \tau^5_{A'} I_{A'}(v')$, then $\tau^2_A I_A(v)$ would be a summand of T, a contradiction. Hence $\tau^5_{A''} I_{A''}(v'')$ is a summand of T', and all indecomposable summands of T, and therefore B, are uniquely determined.

Calculating $\mathrm{End}_A T$ we get the asserted quiver with relations, and calculating $\Gamma(B)$ gives the converse implication and the additional assertions.

Acknowledgment

I thank the referee for pointing out an argument that shortens the proof of Proposition 4.3.

References

[BB] R. Bautista and S. Brenner, Replication numbers for non-Dynkin sectional subgraphs in finite Auslander–Reiten quivers and some properties of Weyl roots, *Proc. London Math. Soc. (3)* **43** (1983), 429–462.

[B1] K. Bongartz, Treue einfach zusammenhängende Algebren, I., *Comment. Math. Helv.* **57** (1982), 282–330.

[B2] K. Bongartz, Algebras and quadratic forms, *J. London Math. Soc.* **28** (1983), 461–469.

[DR] V. Dlab and C. M. Ringel, Indecomposable representations of graphs and algebras, *Mem. Amer. Math. Soc.* **173** (1976).

[G] P. Gabriel, Auslander–Reien sequences and representation-finite algebras, *in* "Representation Theory I," pp. 1–71, Lecture Notes in Mathematics, Vol. 831, Springer-Verlag, New York/Berlin, 1980.

[HR] D. Happel and C. M. Ringel, Tilted algebras, *Trans. Amer. Math. Soc.* **274** (1982), 399–443.

[HU] D. Happel and L. Unger, Faxtors of conceales algebras, *Math. Z.* **201** (1989), 477–483.

[P] J. A. de la Peña, On omnipresent modules in simply connected algebras, *J. London Math. Soc. (2)* **36** (1987), 385–392.

[R] C. M. Ringel, Tame algebras and integral quadratic forms, *in* "Lecture Notes in Mathematics, Vol. 1099," Springer-Verlag, New York/Berlin, 1984.

[U] L. Unger, Lower bounds for faithful, preinjective modules, *Manuscripta Math.* **57** (1986), 1–31.