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A new class of bivariate distributions and its mixture

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Abstract

A new class of bivariate distributions is presented in this paper. The procedure used in this paper is based on a latent random variable with exponential distribution. The model introduced here is of Marshall–Olkin type. A mixture of the proposed bivariate distributions is also discussed. The results obtained here generalize those of the bivariate exponential distribution present in the literature.

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1. Introduction

The generalized exponential distribution was introduced recently by Gupta and Kundu [1]. They observed that it can be used quite effectively in analyzing many lifetime data, especially in place of gamma and Weibull distributions. The primary reason for this is that the family of generalized exponential distributions does include models with increasing and decreasing failure rates. Gupta and Kundu [2] studied the maximum likelihood estimation of the parameters of generalized exponential distribution. These maximum likelihood estimates have been compared with other estimators by Gupta and Kundu [3]. Raqab and Ahsanullah [10] used order statistics

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to estimate the location and scale parameters of generalized exponential distribution. Recently, the ratio of the maximized likelihoods was used by Gupta et al. [5] to discriminate between two overlapping families of distributions, viz. gamma versus generalized exponential or Weibull versus generalized exponential; see also Gupta and Kundu [4].

In many practical problems, multivariate lifetime data arise frequently, and in these situations it is important to consider different multivariate models that could be used to model such multivariate lifetime data. For an encyclopedia treatment on various multivariate models and their properties and applications, one may refer to the book by Kotz et al. [7]. In this paper, we propose a class of bivariate distributions and also discuss their mixtures.

The construction of the new bivariate distribution is given in Section 2. The derivation of the probability density function of this distribution is also given in this section. The marginal and conditional probability density functions are obtained in Section 3. We also present in this section the expectations of the marginal distributions, the conditional expectations and the joint moment generating function. We show that the marginal and conditional expectations and the joint moment generating function for the case of the bivariate exponential distribution can be derived as special cases of the results presented in this section. Finally, the mixture of the new bivariate distributions is discussed in Section 4.

2. The new bivariate distribution

In this section, we define a new version of bivariate distributions, shortly denoted by NBD. We start with the joint survival function of the distribution and then derive the corresponding joint probability density function.

2.1. The joint survival function

In what follows, we present the model that produces the NBD. Let U_1, U_2 and U_3 be mutually independent random variables with the following distributions:

$$U_i \sim GED(1, \theta_i), \quad i = 1, 2 \quad \text{and} \quad U_3 \sim E(\theta_0),$$

where $GED(1, \theta_i)$ denotes the generalized exponential distribution with parameters $(1, \theta_i)$, while $E(\theta_0)$ denotes the exponential distribution with parameter θ_0. That is, the random variable U_i $(i = 1, 2)$ has a generalized exponential distribution with distribution function

$$G_i(t) = \left(1 - e^{-t}\right)^{\theta_i}, \quad t \geqslant 0, \quad \theta_i > 0 \ (i = 1, 2)$$

and the variable U_3 has an exponential distribution with a constant failure rate $\theta_0 > 0$ with distribution function

$$G_3(t) = 1 - e^{-\theta_0 t}, \quad t \geqslant 0, \quad \theta_0 > 0.$$

The survival functions of U_i $(i = 1, 2, 3)$ are

$$\bar{G}_i(t) = 1 - \left(1 - e^{-t}\right)^{\theta_i}, \quad t \geqslant 0, \quad \theta_i > 0 \ (i = 1, 2), \tag{2.1}$$

$$\bar{G}_3(t) = e^{-\theta_0 t}, \quad t \geqslant 0, \quad \theta_0 > 0. \tag{2.2}$$

Define the random variables X_1 and X_2 as

$$X_i = \min(U_i, U_3), \quad i = 1, 2. \tag{2.3}$$

It is evident that the random variables X_1 and X_2 in (2.3) are dependent because of the common random (latent) variable U_3.

We now study the joint distribution of the random variables X_1 and X_2. The following lemma gives the joint survival function of X_1 and X_2, which is the survival function of the NBD.

Lemma 2.1. *The joint survival function of X_1 and X_2 is*

$$\bar{F}_{X_1,X_2}(x_1,x_2) = e^{-\theta_0 z}\left\{1-\left(1-e^{-x_1}\right)^{\theta_1}\right\}\left\{1-\left(1-e^{-x_2}\right)^{\theta_2}\right\}, \tag{2.4}$$

where $z = \max(x_1, x_2)$.

Proof. Since

$$\bar{F}_{X_1,X_2}(x_1,x_2) = P\Big(X_1 > x_1, X_2 > x_2\Big),$$

we have

$$\begin{aligned}\bar{F}_{X_1,X_2}(x_1,x_2) &= P\Big(\min(U_1,U_3) > x_1, \min(U_2,U_3) > x_2\Big)\\ &= P\Big(U_1 > x_1, U_3 > x_1, U_2 > x_2, U_3 > x_2\Big)\\ &= P\Big(U_1 > x_1, U_2 > x_2, U_3 > \max(x_1,x_2)\Big).\end{aligned}$$

As U_i $(i = 1, 2, 3)$ are mutually independent, we readily obtain

$$\begin{aligned}\bar{F}_{X_1,X_2}(x_1,x_2) &= P(U_1 > x_1)P(U_2 > x_2)P(U_3 > \max(x_1,x_2))\\ &= \bar{G}_1(x_1)\bar{G}_2(x_2)\bar{G}_3(z).\end{aligned}$$

Substituting from (2.2) and (2.1) into the above equation, we obtain (2.4), which completes the proof of the lemma. □

2.2. The joint probability density function

The following theorem gives the joint probability density function of the NBD.

Theorem 2.1. *If the joint survival function of* (X_1, X_2) *is as in* (2.4), *the joint probability density function of* (X_1, X_2) *is given by*

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & \text{if } x_1 > x_2 > 0,\\ f_2(x_1,x_2) & \text{if } x_2 > x_1 > 0,\\ f_0(x_1,x_1) & \text{if } x_1 = x_2 > 0,\end{cases} \tag{2.5}$$

where

$$\begin{aligned}f_1(x_1,x_2) = {}&\theta_2 e^{-(\theta_0 x_1 + x_2)}\left(1-e^{-x_2}\right)^{\theta_2-1}\\ &\times\left\{\theta_0 - \theta_0\left(1-e^{-x_1}\right)^{\theta_1} + \theta_1 e^{-x_1}\left(1-e^{-x_1}\right)^{\theta_1-1}\right\},\end{aligned}$$

$$f_2(x_1, x_2) = \theta_1 e^{-(\theta_0 x_2 + x_1)} \left(1 - e^{-x_1}\right)^{\theta_1 - 1} \times \left\{\theta_0 - \theta_0 \left(1 - e^{-x_2}\right)^{\theta_2} + \theta_2 e^{-x_2} \left(1 - e^{-x_2}\right)^{\theta_2 - 1}\right\}$$

and

$$f_0(x_1, x_1) = \theta_0 e^{-\theta_0 x_1} \prod_{i=1}^{2} \left[1 - \left(1 - e^{-x_1}\right)^{\theta_i}\right].$$

Proof. Let us first assume that $x_1 > x_2$. In this case, $\bar{F}_{X_1,X_2}(x_1, x_2)$ in (2.4) becomes

$$\bar{F}_{X_1,X_2}(x_1, x_2) = e^{-\theta_0 x_1} \left\{1 - \left(1 - e^{-x_1}\right)^{\theta_1}\right\} \left\{1 - \left(1 - e^{-x_2}\right)^{\theta_2}\right\}.$$

Then, upon differentiation, we obtain the expression of $f_{X_1,X_2}(x_1, x_2) = \frac{\partial^2 \bar{F}_{X_1,X_2}(x_1,x_2)}{\partial x_1 \partial x_2}$ to be $f_1(x_1, x_2)$ given above. Similarly, we find the expression of $f_{X_1,X_2}(x_1, x_2)$ to be $f_2(x_1, x_2)$ when $x_1 < x_2$. But, $f_0(x_1, x_1)$ cannot be derived in a similar way. For this reason, we use the identity

$$\int_0^\infty f_0(x_1, x_1)\, dx_1 + \int_0^\infty \int_0^{x_1} f_1(x_1, x_2)\, dx_2\, dx_1 + \int_0^\infty \int_0^{x_2} f_2(x_1, x_2)\, dx_1\, dx_2 = 1. \tag{2.6}$$

One can verify that

$$\begin{aligned} I_1 &= \int_0^\infty \int_0^{x_1} f_1(x_1, x_2)\, dx_2\, dx_1 \\ &= \int_0^\infty \left\{\theta_0 \left(1 - e^{-x_1}\right)^{\theta_2} e^{-\theta_0 x_1} - \theta_0 \left(1 - e^{-x_1}\right)^{\theta_1 + \theta_2} e^{-\theta_0 x_1} \right. \\ &\quad \left. + \theta_1 \left(1 - e^{-x_1}\right)^{\theta_1 + \theta_2 - 1} e^{-(\theta_0 + 1) x_1}\right\} dx_1 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} I_2 &= \int_0^\infty \int_0^{x_2} f_2(x_1, x_2)\, dx_1\, dx_2 \\ &= \int_0^\infty \left\{\theta_0 \left(1 - e^{-x_2}\right)^{\theta_2} e^{-\theta_0 x_2} - \theta_0 \left(1 - e^{-x_2}\right)^{\theta_1 + \theta_2} e^{-\theta_0 x_2} \right. \\ &\quad \left. + \theta_2 \left(1 - e^{-x_2}\right)^{\theta_1 + \theta_2 - 1} e^{-(\theta_0 + 1) x_2}\right\} dx_2. \end{aligned} \tag{2.8}$$

Using the transformation $u = e^{-x_i}$ $(i = 1, 2)$ in (2.7) and (2.8), respectively, we can see that

$$I_1 + I_2 = -\theta_0 \left\{B(\theta_0, \theta_1 + \theta_2 + 1) - B(\theta_0, \theta_1 + 1) - B(\theta_0, \theta_2 + 1)\right\}, \tag{2.9}$$

where $B(m, n)$ denotes the complete beta function defined by

$$B(m, n) = \int_0^1 u^{m-1}(1 - u)^{n-1}\, du, \quad m, n > 0. \tag{2.10}$$

From (2.6) and (2.9), we then get

$$\int_0^\infty f_0(x_1, x_1)\, dx_1 = 1 + \theta_0 \{B(\theta_0, \theta_1 + \theta_2 + 1) - B(\theta_0, \theta_1 + 1) - B(\theta_0, \theta_2 + 1)\}. \tag{2.11}$$

Since (using the transformation $u = e^{-s}$) we can write (2.10) as

$$B(m, n) = \int_0^\infty e^{-ms} \left(1 - e^{-s}\right)^{n-1} ds, \quad m, n > 0, \tag{2.12}$$

we obtain from (2.11) and (2.12) that

$$\int_0^\infty f_0(x_1, x_1)\, dx_1 = \theta_0 \int_0^\infty e^{-\theta_0 x_1} \left\{1 - \left(1 - e^{-x_1}\right)^{\theta_1} - \left(1 - e^{-x_1}\right)^{\theta_2} + \left(1 - e^{-x_1}\right)^{\theta_1+\theta_2}\right\} dx_1 \tag{2.13}$$

which readily yields

$$f_0(x_1, x_1) = \theta_0 e^{-\theta_0 x_1} \left\{1 - \left(1 - e^{-x_1}\right)^{\theta_1}\right\} \left\{1 - \left(1 - e^{-x_1}\right)^{\theta_2}\right\}.$$

This completes the proof of the theorem. □

Lemma 2.2. *The joint probability density function of the bivariate exponential distribution is*

$$f(x_1, x_2) = \begin{cases} (1 + \theta_0)e^{-(\theta_0 x_1 + x_1 + x_2)} & \text{if } x_1 > x_2 > 0, \\ (1 + \theta_0)e^{-(\theta_0 x_2 + x_1 + x_2)} & \text{if } x_2 > x_1 > 0, \\ \theta_0 e^{-(\theta_0 + 2)x_1} & \text{if } x_1 = x_2 > 0. \end{cases} \tag{2.14}$$

Proof. The result is obtained immediately from Theorem 2.1 upon setting $\theta_1 = \theta_2 = 1$. □

3. Marginal and conditional probability density functions

In this section, we derive the marginal density functions of X_i and the conditional density functions of $X_i|X_j$, $i \neq j = 1, 2$. We then present the marginal expectations of X_i and the conditional expectations of $X_i|X_j$, $i \neq j = 1, 2$. We also present the joint moment generating function of X_1 and X_2.

3.1. Marginal probability density functions

Theorem 3.1. *The marginal pdf of* X_i $(i = 1, 2)$ *is given by*

$$f_{X_i}(x_i) = e^{-\theta_0 x_i} \left\{\theta_0 - \theta_0 \left(1 - e^{-x_i}\right)^{\theta_i} + \theta_i e^{-x_i} \left(1 - e^{-x_i}\right)^{\theta_i - 1}\right\}, \quad x_i > 0, \quad i = 1, 2. \tag{3.1}$$

Proof. First, we shall derive $f_{X_1}(x_1)$. From the fact that

$$f_{X_1}(x_1) = \int_0^\infty f_{X_1,X_2}(x_1, x_2)\, dx_2,$$

we can express

$$f_{X_1}(x_1) = \phi(x_1) + \psi(x_1) + f_0(x_1, x_1), \tag{3.2}$$

where

$$\phi(x_1) = \int_0^{x_1} f_1(x_1, x_2)\, dx_2 \quad \text{and} \quad \psi(x_1) = \int_{x_1}^\infty f_2(x_1, x_2)\, dx_2.$$

Using the expressions of $f_1(x_1, x_2)$ and $f_1(x_1, x_2)$ given in Theorem 2.1, we can show that

$$\phi(x_1) = e^{-\theta_0 x_1}\left(1 - e^{-x_1}\right)^{\theta_2}\left\{\theta_0 - \theta_0\left(1 - e^{-x_1}\right)^{\theta_1} + \theta_1 e^{-x_1}\left(1 - e^{-x_1}\right)^{\theta_1 - 1}\right\} \tag{3.3}$$

and

$$\psi(x_1) = \theta_1 e^{-x_1}\left(1 - e^{-x_1}\right)^{\theta_1 - 1} \times \left\{e^{-\theta_0 x_1} - \theta_0 B_{e^{-x_1}}(\theta_0, \theta_2 + 1) + \theta_2 B_{e^{-x_1}}(\theta_0 + 1, \theta_2)\right\}, \tag{3.4}$$

where $B_x(p, q)$ is the incomplete beta function defined by

$$B_x(p, q) = \int_0^x t^{p-1}(1 - t)^{q-1}\, dt, \quad (0 \leqslant x \leqslant 1).$$

Upon considering $B_{e^{-x_1}}(\theta_0 + 1, \theta_2)$ and integrating by parts, we have

$$B_{e^{-x_1}}(\theta_0 + 1, \theta_2) = -\frac{1}{\theta_2} e^{-\theta_0 x_1}\left(1 - e^{-x_1}\right)^{\theta_2} + \frac{\theta_0}{\theta_2} B_{e^{-x_1}}(\theta_0, \theta_2 + 1). \tag{3.5}$$

Now using the expression in (3.5) into (3.4) and simplifying, we get

$$\psi(x_1) = \theta_1 e^{-x_1}\left(1 - e^{-x_1}\right)^{\theta_1 - 1}\left\{1 - \left(1 - e^{-x_1}\right)^{\theta_2}\right\}. \tag{3.6}$$

Substituting for (3.3) and (3.6) into (3.2) and using the form of $f_0(x_1, x_1)$, we obtain $f_{X_1}(x_1)$ given in (3.1). Proceeding similarly, we can derive $f_{X_2}(x_2)$ as given in (3.1), which completes the proof of the theorem. □

Note that the marginal pdf of X_i can be derived in another way. For this, we first derive the marginal survival function of X_i, say $\bar{F}_{X_i}(x_i)$, as follows:

$$\bar{F}_{X_i}(x_i) = P\Big(X_i > x_i\Big) = P\Big(\min(U_i, U_3) > x_i\Big) = P\Big(U_i > x_i, U_3 > x_i\Big)$$

and since U_i is independent of U_3, we simply have

$$\bar{F}_{X_i}(x_i) = e^{-\theta_0 x_i}\left\{1 - \left(1 - e^{-x_i}\right)^{\theta_i}\right\}$$

from which we readily derive the pdf of X_i, $f_{X_i}(x_i) = -\frac{\partial}{\partial x_i}\bar{F}_{X_i}(x_i)$, as in (3.1).

Based on the above theorem, we can prove the following lemma.

Lemma 3.1. *The marginal pdf of* X_i *in the case of the bivariate exponential distribution, with joint pdf as in* (2.14), *is*

$$f_{X_i}(x_i) = (\theta_0 + 1)e^{-(\theta_0+1)x_i}, \quad x_i > 0. \tag{3.7}$$

3.2. Conditional probability density functions

Having obtained the marginal probability density functions of X_1 and X_2, we can now derive the conditional probability density functions as presented in the following theorem.

Theorem 3.2. *The conditional pdf of* X_i, *given* $X_j = x_j$, *denoted by* $f_{i|j}(x_i|x_j)$ $(i \neq j = 1, 2)$, *is given by*

$$f_{i|j}(x_i|x_j) = \begin{cases} f^{(1)}_{i|j}(x_i|x_j) & \text{if } x_i > x_j, \\ f^{(2)}_{i|j}(x_i|x_j) & \text{if } x_i < x_j, \\ f^{(0)}_{i|j}(x_i|x_j) & \text{if } x_i = x_j, \end{cases} \tag{3.8}$$

where

$$f^{(1)}_{i|j}(x_i|x_j) = \frac{\theta_j e^{-(\theta_0 x_i + x_j)}\left(1-e^{-x_j}\right)^{\theta_j-1}\left\{\theta_0-\theta_0\left(1-e^{-x_i}\right)^{\theta_i}+\theta_i e^{-x_i}\left(1-e^{-x_i}\right)^{\theta_i-1}\right\}}{e^{-\theta_0 x_j}\left\{\theta_0-\theta_0\left(1-e^{-x_j}\right)^{\theta_j}+\theta_j e^{-x_j}\left(1-e^{-x_j}\right)^{\theta_j-1}\right\}},$$

$$f^{(2)}_{i|j}(x_i|x_j) = \theta_i e^{-x_i}\left(1-e^{-x_i}\right)^{\theta_i-1},$$

and

$$f^{(0)}_{i|j}(x_i|x_j) = \frac{\theta_0\left\{1-\left(1-e^{-x_i}\right)^{\theta_i}\right\}\left\{1-\left(1-e^{-x_i}\right)^{\theta_j}\right\}}{\left\{\theta_0-\theta_0\left(1-e^{-x_i}\right)^{\theta_j}+\theta_j e^{-x_i}\left(1-e^{-x_i}\right)^{\theta_j-1}\right\}}.$$

Proof. The theorem follows readily upon substituting for the joint pdf of (X_1, X_2) in (2.5) and the marginal pdf of X_i $(i = 1, 2)$ in (3.1), in the relation

$$f_{i|j}(x_i|x_j) = \frac{f_{X_i,X_j}(x_i, x_j)}{f_{X_j}(x_j)}. \qquad \square$$

Lemma 3.2. *For the case of the bivariate exponential distribution, we obtain upon setting* $\theta_1 = \theta_2 = 1$ *in* (3.8)

$$f_{i|j}(x_i|x_j) = \begin{cases} e^{-(\theta_0+1)x_i+\theta_0 x_j} & \text{if } x_i > x_j, \\ e^{-x_i} & \text{if } x_i < x_j, \\ \dfrac{\theta_0}{\theta_0+1}e^{-x_i} & \text{if } x_i = x_j. \end{cases} \tag{3.9}$$

3.3. Mathematical expectations

Based on the results presented in the last two subsections, we can derive the mathematical expectations of X_i, the second moments of X_i, and the conditional expectations of $X_1|X_2$ and of $X_2|X_1$.

Theorem 3.3. *The expectation of* X_i $(i = 1, 2)$ *is given by*

$$E[X_i] = \frac{1}{\theta_0} + \theta_0\kappa(\theta_0, \theta_i) - (\theta_0 + \theta_i)\kappa(\theta_0 + 1, \theta_i), \tag{3.10}$$

where

$$\kappa(\alpha, \beta) = B(\alpha, \beta)\left\{\psi(\alpha) - \psi(\alpha + \beta)\right\},$$

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is the digamma function, and

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du, \quad x > 0$$

is the complete gamma function.

Proof. Starting with

$$E[X_i] = \int_0^\infty x_i f_{X_i}(x_i)\,dx_i$$

and substituting for $f_{X_i}(x_i)$ from (3.1), we get

$$E[X_i] = \frac{1}{\theta_0} - \theta_0\int_0^\infty xe^{-\theta_0 x}\left(1 - e^{-x}\right)^{\theta_i}\,dx + \theta_i\int_0^\infty xe^{-(1+\theta_0)x}\left(1 - e^{-x}\right)^{\theta_i-1}\,dx. \tag{3.11}$$

Since

$$\int_0^\infty xe^{-\theta_0 x}\left(1 - e^{-x}\right)^{\theta_i}\,dx = \int_0^\infty xe^{-\theta_0 x}\left(1 - e^{-x}\right)^{\theta_i-1}\,dx - \int_0^\infty xe^{-(\theta_0+1)x}\left(1 - e^{-x}\right)^{\theta_i-1}\,dx,$$

we have

$$E[X_i] = \frac{1}{\theta_0} - \theta_0\int_0^\infty xe^{-\theta_0 x}\left(1 - e^{-x}\right)^{\theta_i-1}\,dx + (\theta_0 + \theta_i)\int_0^\infty xe^{-(1+\theta_0)x}\left(1 - e^{-x}\right)^{\theta_i-1}\,dx.$$

Setting $u = e^{-x}$ in the above integrals, we get

$$E[X_i] = \frac{1}{\theta_0} + \theta_0 \int_0^1 u^{\theta_0 - 1}(1-u)^{\theta_i - 1} \ln u \, du$$
$$-(\theta_0 + \theta_i) \int_0^1 u^{\theta_0}(1-u)^{\theta_i - 1} \ln u \, du. \tag{3.12}$$

Let

$$\kappa(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta_1 - 1} \ln u \, du.$$

Using the Euler's psi function (see Gradshteyn and Ryzhik [6, p. 538; 4.253.1]), we have

$$\kappa(\alpha, \beta) = B(\alpha, \beta)\big[\psi(\alpha) - \psi(\alpha+\beta)\big], \quad \alpha, \beta > 0.$$

Upon using this expression for the integrals in (3.12), we derive the expression in (3.10), which completes the proof of this theorem. □

Theorem 3.4. *The second moment of* X_i $(i = 1, 2)$ *is given by*

$$E\left[X_i^2\right] = \frac{2}{\theta_0^2} - \theta_0 \mu(\theta_0, \theta_i) + (\theta_0 + \theta_i)\mu(\theta_0 + 1, \theta_i), \tag{3.13}$$

where

$$\mu(\alpha, \beta) = B(\alpha, \beta)\left\{\psi'(\alpha) - \psi'(\alpha+\beta) + \big[\psi(\alpha) - \psi(\alpha+\beta)\big]^2\right\}. \tag{3.14}$$

Proof. Starting with

$$E\left[X_i^2\right] = \int_0^\infty x_i^2 f_{X_i}(x_i)\, dx_i,$$

substituting for $f_{X_i}(x_i)$ from (3.1) and setting $u = e^{-x_i}$, we get

$$E\left[X_i^2\right] = \frac{2}{\theta_0^2} - \theta_0 \int_0^1 u^{\theta_0 - 1}(1-u)^{\theta_i - 1}(\ln u)^2 \, du$$
$$+(\theta_0 + \theta_i) \int_0^1 u^{\theta_0}(1-u)^{\theta_i - 1}(\ln u)^2 \, du. \tag{3.15}$$

Denoting

$$\mu(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta_1 - 1}(\ln u)^2 \, du$$

and using the trigamma function (see Gradshteyn and Ryzhik [6, p. 541; 4.261.17]), we have

$$\mu(\alpha, \beta) = B(\alpha, \beta)\left\{\big[\psi(\alpha) - \psi(\alpha+\beta)\big]^2 + \psi'(\alpha) - \psi'(\alpha+\beta)\right\}.$$

Upon using this expression for the integrals in (3.15), we derive the expression in (3.13), which completes the proof of this theorem. □

Lemma 3.3. *For the bivariate exponential distribution*, *we have*

$$E[X_i] = \frac{1}{\theta_0 + 1}. \tag{3.16}$$

Proof. By setting $\theta_1 = \theta_2 = 1$ in (3.10), $E[X_i]$ becomes

$$E[X_i] = \frac{1}{\theta_0} + \theta_0 \kappa(\theta_0, 1) - (\theta_0 + 1)\kappa(\theta_0 + 1, 1). \tag{3.17}$$

Now, using the recurrence relation $\psi(z) = \psi(z-1) + \frac{1}{z-1}$, we get

$$\kappa(\theta_0, 1) = -\frac{1}{\theta_0^2}, \quad \kappa(\theta_0 + 1, 1) = -\frac{1}{(\theta_0 + 1)^2}. \tag{3.18}$$

When the expressions in (3.18) are substituted into (3.17) and simplified, we obtain the expression in (3.16). □

The conditional expectation of X_i, given $X_j = x_j$ $(i \neq j = 1, 2)$, are presented in the following theorem.

Theorem 3.5. *The conditional expectation of* X_i, *given* $X_j = x_j$ $(i \neq j = 1, 2)$, *is given by*

$$E[X_i|X_j = x_j] = \frac{x_j L_j(x_j)}{\ell_j(x_j)} - I_j(x_j) + k_j(x_j)\left\{\frac{e^{-\theta_0 x_j}}{\theta_0} - \theta_0 \beta_{x_j}(\theta_0, \theta_j) + (\theta_0 + \theta_j)\beta_{x_j}(\theta_0 + 1, \theta_j)\right\}, \tag{3.19}$$

where

$$\beta_x(m, n) = \int_x^\infty u e^{-mu}\left(1 - e^{-u}\right)^{n-1} du,$$

$$I_j(x_j) = \int_0^{x_j} \left(1 - e^{-x}\right)^{\theta_j} dx,$$

$$\ell_j(x_j) = \theta_0 - \theta_0\left(1 - e^{-x_j}\right)^{\theta_j} + \theta_j e^{-x_j}\left(1 - e^{-x_j}\right)^{\theta_j - 1},$$

$$L_j(x_j) = \ell_j(x_j) + \theta_j e^{-x_j}\left(1 - e^{-x_j}\right)^{\theta_1 + \theta_2 - 1},$$

$$k_j(x_j) = \frac{\theta_j e^{-(1-\theta_0)x_j}\left(1 - e^{-x_j}\right)^{\theta_j - 1}}{\ell_j(x_j)}.$$

Proof. Starting with

$$E[X_i|X_j = x_j] = \int_0^\infty x_i f_{i|j}(x_i|x_j)\, dx_i, \tag{3.20}$$

substituting for $f_{i|j}(x_i|x_j)$ from (3.8) into (3.20) and simplifying the resulting expression, we obtain (3.19). $\square$

If we assume that θ_1 and θ_2 are positive integers, then using integration by parts and binomial expansion, we can derive the expression

$$\beta_x(\tau, \theta_j) = \sum_{i=0}^{\theta_j - 1} \binom{\theta_j - 1}{i} \frac{(-1)^i}{\tau + i} \left[x + \frac{1}{\tau + i} \right] e^{-(\tau+i)x}, \tag{3.21}$$

where $\tau = \theta_0$ and $\theta_0 + 1$ for $\beta_x(\theta_0, \theta_j)$ and $\beta_x(\theta_0 + 1, \theta_j)$, respectively, and $I_j(x_j)$ in this case becomes

$$I_j(x_j) = x_j + \sum_{l=1}^{\theta_j} \binom{\theta_j}{l} \frac{(-1)^l}{l} \left[1 - e^{-l x_j} \right]. \tag{3.22}$$

Lemma 3.4. *For the bivariate exponential distribution, we have*

$$E[X_i | X_j = x_j] = 1 - \frac{\theta_0(\theta_0 + 2)}{(\theta_0 + 1)^2} e^{-x_j}. \tag{3.23}$$

Proof. Setting $\theta_1 = \theta_2 = 1$, we readily have

$$\ell_j(x_j) = (\theta_0 + 1) e^{-x_j}, \quad k_j(x_j) = \frac{1}{(\theta_0 + 1) e^{-\theta_0 x_j}},$$

$$L_j(x_j) = e^{-x_j} \left\{ \theta_0 + 2 - e^{-x_j} \right\}.$$

From (3.21), we have

$$\beta_{x_j}(\theta_0, \theta_j) = \left\{ \frac{x_j}{\theta_0} + \frac{1}{\theta_0^2} \right\} e^{-\theta_0 x_j},$$

$$\beta_{x_j}(\theta_0 + 1, \theta_j) = \left\{ \frac{x_j}{\theta_0 + 1} + \frac{1}{(\theta_0 + 1)^2} \right\} e^{-(\theta_0 + 1) x_j}.$$

Also, from (3.22), we have $I_j(x_j) = x_j + e^{-x_j} - 1$. Substituting for all these in (3.19), we get $E[X_i | X_j = x_j]$ as given in (3.23). $\square$

3.4. Moment generating functions

In this subsection, we present the joint moment generating function of (X_1, X_2) and the marginal moment generating function of X_i $(i = 1, 2)$.

Lemma 3.5. *The moment generating function of X_i $(i = 1, 2)$ is given by*

$$M_{X_i}(t_i) = \frac{\theta_0}{\theta_0 + t_i} - \theta_0 B(\theta_0 + t_i, \theta_i) + (\theta_0 + \theta_i) B(\theta_0 + t_i + 1, \theta_i). \tag{3.24}$$

Proof. Starting with

$$M_{X_i}(t_i) = E\left[e^{-t_i X_i}\right] = \int_0^\infty e^{-t_i x_i} f_{X_i}(x_i)\,dx_i$$

and substituting for $f_{X_i}(x_i)$ from (3.1), we get

$$M_{X_i}(t_i) = \int_0^\infty e^{-(\theta_0+t_i)x}\left\{\theta_0 - \theta_0\left(1-e^{-x}\right)^{\theta_i} + \theta_i e^{-x}\left(1-e^{-x}\right)^{\theta_i-1}\right\}dx$$

from which we readily derive the expression of $M_{X_i}(t_i)$ given in (3.24). □

Note that the moment generating function $M_{X_i}(t_i)$ can be used, instead of the marginal pdf $f_{X_i}(x_i)$, to derive the marginal expectation of X_i as

$$E[X_i] = -\frac{d}{dt_i} M_{X_i}(t_i)|_{t_i=0}.$$

From (3.24), we obtain

$$-\frac{d}{dt_i} M_{X_i}(t_i) = \frac{\theta_0}{(\theta_0+t_i)^2} + \theta_0 \kappa(\theta_0+t_i,\theta_i) - (\theta_0+\theta_i)\kappa(\theta_0+t_i+1,\theta_i) \tag{3.25}$$

in which if we set $t_i = 0$, we obtain $E[X_i]$ as given in (3.10).

Similarly, the second moment of X_i, given in (3.13), can be derived from $M_{X_i}(t_i)$ as its second derivative at $t_i = 0$.

For the special case when θ_1 and θ_2 are positive integers, the following lemma gives the marginal moment generating functions.

Lemma 3.6. *When θ_1 and θ_2 are positive integers, then for $i = 1, 2$*

$$M_{X_i}(t_i) = \frac{\theta_0}{\theta_0+t_i} + (-1)^{\theta_i-1}\sum_{k=0}^{\theta_i-1}\binom{\theta_i-1}{k} \times \left\{\frac{(-1)^k(\theta_0+\theta_i)}{\theta_i+\theta_0+t_i-k} - \frac{(-1)^k\theta_0}{\theta_i+\theta_0+t_i-1-k}\right\}. \tag{3.26}$$

Proof. The proof of this lemma follows from (3.24) with the use of the following relation (see Gradshteyn and Ryzhik [6, p. 333; 3.432])

$$B(m,n) = (-1)^{n-1}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-1)^k}{n+m-1-k}, \quad n = 1, 2, \ldots \quad \square \tag{3.27}$$

The expression for the function $M_{X_i}(t_i)$ in (3.26) can be used to derive the rth moment of X_i as given below.

Lemma 3.7. *If* θ_1 *and* θ_2 *are positive integers, then for* $r = 1, 2, \ldots$

$$E\left[X_i^r\right] = \frac{\theta_0 r!}{\theta_0^r} + (-1)^{\theta_i - 1} r! \sum_{k=0}^{\theta_i - 1} \binom{\theta_i - 1}{k} (-1)^k \times \left\{ \frac{(\theta_0 + \theta_i)}{(\theta_i + \theta_0 - k)^{r+1}} - \frac{\theta_0}{(\theta_i + \theta_0 - 1 - k)^{r+1}} \right\}. \tag{3.28}$$

The following theorem gives the joint moment generating function of (X_1, X_2).

Theorem 3.6. *The joint moment generating function of* (X_1, X_2) *is given by*

$$\begin{aligned} M(t_1, t_2) = {} & \theta_0 \left\{ \frac{1}{a} - B(a, \theta_1 + 1) - B(a, \theta_2 + 1) + B(a, \theta_1 + \theta_2 + 1) \right\} \\ & + \theta_1 B(t_1 + 1, \theta_1) \left\{ \frac{\theta_0}{\theta_0 + t_2} - \theta_0 B(\theta_0 + t_2, \theta_2) \right. \\ & \left. + (\theta_0 + \theta_2) B(\theta_0 + t_2 + 1, \theta_2) \right\} \\ & + \theta_2 B(t_2 + 1, \theta_2) \left\{ \frac{\theta_0}{\theta_0 + t_1} - \theta_0 B(\theta_0 + t_1, \theta_1) \right. \\ & \left. + (\theta_0 + \theta_1) B(\theta_0 + t_1 + 1, \theta_1) \right\} \\ & - \frac{\theta_1}{t_1 + 1} \left\{ \frac{\theta_0}{a + 1} \, {}_3F_2(a + 1, t_1 + 1, 1 - \theta_1; t_1 + 2, a + 2; 1) \right. \\ & - \theta_0 B(a + 1, \theta_2) {}_3F_2(a + 1, t_1 + 1, 1 - \theta_1; t_1 + 2, a + \theta_2 + 1; 1) \\ & + (\theta_0 + \theta_2) B(a + 2, \theta_2) {}_3F_2(a + 2, t_1 + 1, 1 - \theta_1; \\ & \left. t_1 + 2, a + \theta_2 + 2; 1) \right\} \\ & - \frac{\theta_2}{t_2 + 1} \left\{ \frac{\theta_0}{a + 1} \, {}_3F_2(a + 1, t_2 + 1, 1 - \theta_2; t_2 + 2, a + 2; 1) \right. \\ & - \theta_0 B(a + 1, \theta_1) {}_3F_2(a + 1, t_2 + 1, 1 - \theta_2; t_2 + 2, a + \theta_1 + 1; 1) \\ & + (\theta_0 + \theta_1) B(a + 2, \theta_1) {}_3F_2(a + 2, t_2 + 1, 1 - \theta_2; t_2 \\ & \left. + 2, a + \theta_1 + 2; 1) \right\}, \end{aligned} \tag{3.29}$$

where

$$a = \theta_0 + t_1 + t_2, \quad {}_pF_q(b_1, \ldots, b_p; c_1, \ldots, c_q; u) = \sum_{i=0}^{\infty} \frac{(b_1)_i \ldots (b_p)_i}{(c_1)_i \ldots (c_q)_i} \frac{u^i}{i!}$$

and $(b)_i = b(b+1)\ldots(b+i-1) = \frac{\Gamma(b+i)}{\Gamma(b)}$ $(b \neq 0,\ i = 1, 2, \ldots)$, *and p and q are nonnegative integers*.

Proof. The joint moment generating function of (X_1, X_2) is given by

$$\begin{aligned} M(t_1, t_2) = E\left[e^{-t_1x_1-t_2x_2}\right] &= \int_0^\infty \int_0^\infty e^{-t_1x_1-t_2x_2} f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 \\ &= \int_0^\infty \int_0^{x_1} e^{-t_1x_1-t_2x_2} f_1(x_1, x_2)\, dx_2\, dx_1 \\ &\quad + \int_0^\infty \int_0^{x_2} e^{-t_1x_1-t_2x_2} f_2(x_1, x_2)\, dx_1\, dx_2 \\ &\quad + \int_0^\infty e^{-(t_1+t_2)x_1} f_0(x_1, x_1)\, dx_1. \end{aligned} \tag{3.30}$$

Upon substituting from (2.5), using the fact that

$$B_x(\alpha, \beta) = \frac{x^\alpha}{\alpha}\, {}_2F_1(\alpha, 1-\beta; \alpha+1; x),$$

and the identity (see Mathai [8, p. 119])

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\, {}_2F_1(c, d; \rho; u)\, du = B(\alpha, \beta)\, {}_3F_2(\alpha, c,\ d; \rho, \alpha+\beta; 1),$$
$$\text{for } \alpha, \beta > 0 \quad \text{and} \quad d+\beta-\alpha-c > 0,$$

we can derive the expression for $M(t_1, t_2)$ given in (3.29). □

The following lemma gives the joint moment generating function for the case when θ_1 and θ_2 are positive integers.

Lemma 3.8. *If* θ_1 *and* θ_2 *are positive integers*, *then*

$$\begin{aligned} M(t_1, t_2) = \theta_0 \Bigg\{ &\frac{1}{(\theta_0+t_1+t_2)} - \sum_{\ell \in \{\theta_1, \theta_2\}} \sum_{i=0}^{l} \frac{(-1)^i \binom{\ell}{i}}{(\theta_0+i+t_1+t_2)} \\ &+ \sum_{i=0}^{\theta_1+\theta_2} \frac{(-1)^i \binom{\theta_1+\theta_2}{i}}{(\theta_0+i+t_1+t_2)} + \sum_{\ell=1}^{2} \frac{\theta_\ell}{\theta_0+t_\ell} \sum_{j=0}^{\theta_\ell-1} \frac{(-1)^j \binom{\theta_\ell-1}{j}}{(\theta_0+t_1+t_2+j+1)} \Bigg\} \\ &+ \sum_{j=0}^{\theta_2-1} \sum_{i=0}^{\theta_1-1} (-1)^{i+j} \binom{\theta_1-1}{i} \binom{\theta_2-1}{j} \end{aligned}$$

$$\times\sum_{\ell=0}^{1}\left\{\frac{(-1)^{\ell}}{(\theta_0+i+j+2+t_1+t_2-\ell)}\right.$$
$$\left.\times\left[\frac{\theta_2(\theta_0+(1-\ell)\theta_1)}{(\theta_0+t_1+j+1-\ell)}+\frac{\theta_1(\theta_0+(1-\ell)\theta_2)}{(\theta_0+t_2+i+1-\ell)}\right]\right\}. \tag{3.31}$$

Proof. The proof follows from (3.29) upon using the relation in (3.27) and the known properties of hypergeometric functions. □

Lemma 3.9. *For the bivariate exponential distribution, we have*

$$M(t_1,t_2)=\frac{1}{\theta_0+t_1+t_2+2}\left\{\theta_0+\frac{\theta_0+1}{(\theta_0+1+t_1)}+\frac{\theta_0+1}{(\theta_0+1+t_2)}\right\}. \tag{3.32}$$

Proof. Setting $\theta_1=\theta_2=1$ in (3.29), we get

$$M(t_1,t_2)=\frac{\theta_0}{a+2}+\frac{\theta_0+1}{(t_1+1)(\theta_0+t_2+1)}+\frac{\theta_0+1}{(t_1+1)(\theta_0+t_2+1)}$$
$$-\frac{\theta_0+1}{a+2}\left\{\frac{1}{t_1+1}\,{}_3F_2(a+2,t_1+1,0;t_1+1,a+3;1)\right.$$
$$\left.+\frac{1}{t_2+1}\,{}_3F_2(a+2,t_2+1,0;t_2+1,a+3;1)\right\}.$$

Since

$${}_3F_2(a+2,t+1,0;t+1,a+3;1)=1,$$

the above expression for $M(t_1,t_2)$ reduces to the form in (3.32). □

Note that Lemma 3.9 can also be proved by setting $\theta_1=\theta_2=1$ in (3.31).

It needs to be mentioned here that the joint moment generating function of the bivariate exponential distribution in formula (2.8) of Patra and Dey [9] seems to be in error.

Lemma 3.10. *If θ_1 and θ_2 are positive integers, then*

$$E[X_1X_2]=\frac{2}{\theta_0^2}-2\theta_0\sum_{\ell\in\{\theta_1,\theta_2\}}\sum_{i=0}^{l}\frac{(-1)^i\binom{\ell}{i}}{(\theta_0+i)^3}+2\theta_0\sum_{i=0}^{\theta_1+\theta_2}\frac{(-1)^i\binom{\theta_1+\theta_2}{i}}{(\theta_0+i)^3}$$
$$+\sum_{\ell=1}^{2}\sum_{j=0}^{\theta_\ell-1}\frac{(-1)^j\binom{\theta_\ell-1}{j}\theta_\ell(3\theta_0+j+1)}{(\theta_0+j+1)^3}$$
$$+\sum_{j=0}^{\theta_2-1}\sum_{i=0}^{\theta_1-1}\sum_{\ell=0}^{1}\sum_{k=0}^{1}\left\{\frac{(-1)^{i+j+\ell}\binom{\theta_1-1}{i}\binom{\theta_2-1}{j}(3-k)}{(\theta_0+i+j+2-\ell)^{3-k}}\right.$$

$$\left.\times\left[\frac{\theta_2(\theta_0+(1-\ell)\theta_1)}{(\theta_0+j+1-\ell)^{k+1}}+\frac{\theta_1(\theta_0+(1-\ell)\theta_2)}{(\theta_0+i+1-\ell)^{k+1}}\right]\right\}. \tag{3.33}$$

Proof. Calculating the second partial derivative of $M(t_1,t_2)$ from (3.31), with respect to t_1 and t_2 and then setting $t_1=t_2=0$, we readily obtain (3.33). □

4. Mixture of the new bivariate distributions

Let us now assume that X_{ij} ($i=1,2$, and $j=1,2$) are independent random variables having generalized exponential distributions with shape parameter θ_{ij}, viz. $GED(1,\theta_{ij})$. That is, the probability density function of X_{ij} is

$$f_{X_{ij}}(x)=\theta_{ij}e^{-x}\left(1-e^{-x}\right)^{\theta_{ij}-1},\quad \theta_{ij}>0,\ \ x>0.$$

Consider Y_i as a mixture of two generalized exponential random variables X_{i1} and X_{i2} ($i=1,2$), viz.

$$Y_i\sim a_i\,GED(1,\theta_{i1})+(1-a_i)\,GED(1,\theta_{i2}),\quad 0\leqslant a_i\leqslant 1.$$

Let W be a random variable independent of X_{ij} for all i and j. Then, Y_i ($i=1,2$) are independent of W. Let us also assume that the random variable W has an exponential distribution with pdf

$$f_Z(z)=\theta_0 e^{-\theta_0 z},\quad \theta_0>0,\ \ z\geqslant 0.$$

Let us now define new random variables W_i as

$$W_i=\min(Y_i,Z),\quad i=1,2.$$

Then, in the random vector $W=(W_1,W_2)$, W_1 and W_2 are dependent because of the common latent variable Z.

The following theorem gives the joint survival function of W_1 and W_2.

Theorem 4.1. *The joint survival function of* W_1 *and* W_2 *is*

$$\begin{aligned}\bar{F}_{W_1,W_2}(w_1,w_2)=\;&p_{00}\left\{1-\left(1-e^{-w_1}\right)^{\theta_{11}}\right\}\left\{1-\left(1-e^{-w_2}\right)^{\theta_{21}}\right\}e^{-\theta_0 w_0}\\&+p_{01}\left\{1-\left(1-e^{-w_1}\right)^{\theta_{11}}\right\}\left\{1-\left(1-e^{-w_2}\right)^{\theta_{22}}\right\}e^{-\theta_0 w_0}\\&+p_{10}\left\{1-\left(1-e^{-w_1}\right)^{\theta_{12}}\right\}\left\{1-\left(1-e^{-w_2}\right)^{\theta_{21}}\right\}e^{-\theta_0 w_0}\\&+p_{11}\left\{1-\left(1-e^{-w_1}\right)^{\theta_{12}}\right\}\left\{1-\left(1-e^{-w_2}\right)^{\theta_{22}}\right\}e^{-\theta_0 w_0},\end{aligned}\tag{4.1}$$

where $w_0=\max(w_1,w_2)$ *and* $p_{ij}=a_i^{1-i}a_j^{1-j}(1-a_i)^i(1-a_j)^j$, $i,j\in\{0,1\}$.

Proof. Since

$$\bar{F}_{W_1,W_2}(w_1,w_2)=P(W_1>w_1,W_2>w_2),$$

we have

$$\bar{F}_{W_1,W_2}(w_1, w_2) = P(\min(Y_1, Z) > w_1, \min(Y_2, Z) > w_2)$$
$$= P(Y_1 > w_1, Y_2 > w_2, Z > \max(w_1, w_2)).$$

Since Y_1, Y_2 and Z are mutually independent, we readily have

$$\bar{F}_{W_1,W_2}(w_1, w_2) = P(Y_1 > w_1)P(Y_2 > w_2)P(Z > \max(w_1, w_2))$$
$$= e^{-\theta_0 w_0} \prod_{i=1}^{2} \Big[a_{i1} \left\{ 1 - \left(1 - e^{-w_i}\right)^{\theta_{i1}} \right\}$$
$$+ (1 - a_{i1}) \left\{ 1 - \left(1 - e^{-w_i}\right)^{\theta_{i2}} \right\} \Big]$$

which can be expressed as in (4.1). □

Note that, since $p_{00}, p_{01}, p_{10}, p_{11} \geqslant 0$, $p_{00} + p_{01} + p_{10} + p_{11} = 1$ and each function in (4.1) is a survival function of the new bivariate distribution, the function $\bar{F}_{W_1,W_2}(w_1, w_2)$ is a survival function of a mixture of the new bivariate distributions. Consequently, it can be rewritten as

$$\bar{F}_{W_1,W_2}(w_1, w_2) = \sum_{i=1}^{4} b_i \bar{S}_i(\lambda_i, \varepsilon_i, \theta_0), \tag{4.2}$$

where $\bar{S}_i$ is the survival function of a NBD$(\lambda_i, \varepsilon_i, \theta_0)$, and $b_1 = p_{00}, b_2 = p_{01}, b_3 = p_{10}, b_4 = p_{11}$, $\lambda_1 = \lambda_2 = \theta_{11}$, $\lambda_3 = \lambda_4 = \theta_{12}$, $\varepsilon_1 = \varepsilon_3 = \theta_{21}$, and $\varepsilon_2 = \varepsilon_4 = \theta_{22}$.

The following theorem presents the bivariate probability density function $f_{W_1,W_2}(w_1, w_2)$ of (W_1, W_2).

Theorem 4.2. *The joint density function of* (W_1, W_2) *is*

$$f_{W_1,W_2}(w_1, w_2) = \begin{cases} f_1(w_1, w_2) & \text{if } w_1 > w_2, \\ f_2(w_1, w_2) & \text{if } w_1 < w_2, \\ f_0(w_1, w_1) & \text{if } w_1 = w_2, \end{cases} \tag{4.3}$$

where

$$f_1(w_1, w_2) = \sum_{i=1}^{4} b_i \varepsilon_i e^{-(\theta_0 w_1 + w_2)} \left(1 - e^{-w_2}\right)^{\varepsilon_1} \Big\{ \theta_0 - \theta_0 \left(1 - e^{-w_1}\right)^{\lambda_i}$$
$$+ \lambda_i e^{-w_1} \left(1 - e^{-w_1}\right)^{\lambda_i - 1} \Big\},$$

$$f_2(w_1, w_2) = \sum_{i=1}^{4} b_i \lambda_i e^{-(\theta_0 w_1 + w_2)} \left(1 - e^{-w_2}\right)^{\lambda_i} \Big\{ \theta_0 - \theta_0 \left(1 - e^{-w_1}\right)^{\varepsilon_i}$$
$$+ \varepsilon_i e^{-w_1} \left(1 - e^{-w_1}\right)^{\varepsilon_i - 1} \Big\},$$

and

$$f_0(w_1, w_1) = \sum_{i=1}^{4} b_i \theta_0 e^{-\theta_0 w_1} \left\{1 - \left(1 - e^{-w_1}\right)^{\lambda_i}\right\} \left\{1 - \left(1 - e^{-w_1}\right)^{\varepsilon_i}\right\}.$$

Proof. The proof follows along the same lines as of Theorem 2.1. □

The marginal probability density functions of W_1 and W_2 can be derived from $f_{W_1,W_2}(w_1, w_2)$ in (4.3) as follows.

Lemma 4.1. *The marginal density functions of* W_1 *and* W_2 *are, respectively,*

$$\begin{aligned} f_{W_1}(w_1) = &\, a_1 e^{-\theta_0 w_1} \left\{\theta_0 - \theta_0 \left(1 - e^{-w_1}\right)^{\theta_{11}} + \theta_{11} e^{-w_1} \left(1 - e^{-w_1}\right)^{\theta_{11}-1}\right\} \\ &+ (1 - a_1) e^{-\theta_0 w_1} \left\{\theta_0 - \theta_0 \left(1 - e^{-w_1}\right)^{\theta_{12}} + \theta_{12} e^{-w_1} \left(1 - e^{-w_1}\right)^{\theta_{12}-1}\right\}, \\ & w_1 > 0, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} f_{W_2}(w_2) = &\, a_2 e^{-\theta_0 w_2} \left\{\theta_0 - \theta_0 \left(1 - e^{-w_2}\right)^{\theta_{21}} + \theta_{21} e^{-w_2} \left(1 - e^{-w_2}\right)^{\theta_{21}-1}\right\} \\ &+ (1 - a_2) e^{-\theta_0 w_2} \left\{\theta_0 - \theta_0 \left(1 - e^{-w_2}\right)^{\theta_{22}} + \theta_{22} e^{-w_2} \left(1 - e^{-w_2}\right)^{\theta_{22}-1}\right\}, \\ & w_2 > 0. \end{aligned} \tag{4.5}$$

From the marginal densities, we can derive the marginal moment generating functions of W_i as follows.

Lemma 4.2. *The moment generating functions of* W_1 *and* W_2 *are, respectively,*

$$\begin{aligned} M_{W_1}(t_1) = 1 + a_1 &\Big\{(\theta_0 + \theta_{11}) B(\theta_0 + t_1 + 1, \theta_{11}) - \theta_0 B(\theta_0 + t_1, \theta_{11})\Big\} \\ &+ (1 - a_1)\Big\{(\theta_0 + \theta_{12}) B(\theta_0 + t_1 + 1, \theta_{12}) - \theta_0 B(\theta_0 + t_1, \theta_{12})\Big\} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} M_{W_2}(t_2) = 1 + a_2 &\Big\{(\theta_0 + \theta_{21}) B(\theta_0 + t_2 + 1, \theta_{21}) - \theta_0 B(\theta_0 + t_2, \theta_{21})\Big\} \\ &+ (1 - a_2)\Big\{(\theta_0 + \theta_{22}) B(\theta_0 + t_2 + 1, \theta_{22}) - \theta_0 B(\theta_0 + t_2, \theta_{22})\Big\}. \end{aligned} \tag{4.7}$$

Lemma 4.3. *From* (4.6) *and* (4.7), *we readily have*

$$\begin{aligned} E[W_1] = 1 + a_1 &\Big\{\theta_0 \kappa(\theta_0, \theta_{11}) - (\theta_0 + \theta_{11}) \kappa(\theta_0 + 1, \theta_{11})\Big\} \\ &+ (1 - a_1)\Big\{\theta_0 \kappa(\theta_0, \theta_{12}) - (\theta_0 + \theta_{12}) \kappa(\theta_0 + 1, \theta_{12})\Big\} \end{aligned} \tag{4.8}$$

and

$$E[W_2] = 1 + a_2\Big\{\theta_0\kappa(\theta_0, \theta_{21}) - (\theta_0 + \theta_{21})\kappa(\theta_0 + 1, \theta_{21})\Big\}$$
$$+(1 - a_2)\Big\{\theta_0\kappa(\theta_0, \theta_{22}) - (\theta_0 + \theta_{22})\kappa(\theta_0 + 1, \theta_{22})\Big\}. \tag{4.9}$$

Lemma 4.4. *From* (4.6) *and* (4.7), *we also have*

$$E\left[W_1^2\right] = 1 + a_1\Big\{(\theta_0 + \theta_{11})\mu(\theta_0 + t_1 + 1, \theta_{11}) - \theta_0\mu(\theta_0 + t_1, \theta_{11})\Big\}$$
$$+(1 - a_1)\Big\{(\theta_0 + \theta_{12})\mu(\theta_0 + t_1 + 1, \theta_{12}) - \theta_0\mu(\theta_0 + t_1, \theta_{12})\Big\} \tag{4.10}$$

and

$$E\left[W_2^2\right] = 1 + a_2\Big\{(\theta_0 + \theta_{21})\mu(\theta_0 + t_2 + 1, \theta_{21}) - \theta_0\mu(\theta_0 + t_2, \theta_{21})\Big\}$$
$$+(1 - a_2)\Big\{(\theta_0 + \theta_{22})\mu(\theta_0 + t_2 + 1, \theta_{22}) - \theta_0\mu(\theta_0 + t_2, \theta_{22})\Big\}. \tag{4.11}$$

The following lemma presents the joint moment generating function of W_1 and W_2, denoted by $M_{W_1,W_2}(t_1, t_2)$.

Lemma 4.5. *The joint moment generating function of* (W_1, W_2) *is given by*

$$M_{W_1,W_2}(t_1, t_2) = \sum_{i=1}^{4} b_i M_i(t_1, t_2), \tag{4.12}$$

where $M_i(t_1, t_2)$ *can be obtained from* (3.29) *by replacing* θ_1, θ_2 *by* λ_i, ε_i, *respectively.*

Proof. One can establish this lemma from (4.2) and (3.29). □

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