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# An integrable SIS model

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**Abstract**

We provide a demonstration of the integrability of a classical model of an infectious disease which neither kills nor induces autoimmunity by means of the Painlevé analysis and use the Lie theory of transformation groups to present an explicit solution.


*Keywords:* SIS mathematical model; Painlevé property; Lie symmetries

## 1. Introduction

A standard model for the evolution in time of a system results in a set of first-order nonlinear ordinary differential equations. The first of such models dates back to the eighteenth century's proposal by the Englishman, Thomas Malthus, that shortly we would all be standing upon each other's toes, the motivation for such a prediction possibly engendered by the dramatic increase in European populations in that century which had not generally experienced the serious encounters with the bubonic plague that had caused major reductions in population—decimation is not the adequate word—over vast swathes of the Europe of previous centuries. Some fifty years later (1837) the Hollander, P.F. Verhulst, suggested that a little natural restraint would be more appropriate and devised the logistic equation as a model of such restraint.

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An attraction of both first-order differential equations used in the models of Malthus and Verhulst was that they were singular and integrable in closed form by a variety of elementary procedures. Unfortunately the diversity of their applicability was somewhat limited. Some 70–80 years ago there was the beginning of serious studies of models of divers processes in biology, chemistry and, of course, epidemiology. Lotka [1] and Volterra [2] are commemorated in the names of the systems they proposed which added to the ideas of Malthus [3] and Verhulst [4] terms describing interaction between various 'objects,' whether they be chemicals in a reactor, beasts on the savannah or the victims of diseases in a population. Lotka–Volterra systems have been supplemented by quadratic [5] and higher-order systems, not to mention more irregularly nonlinear systems. In epidemiology a class of Lotka–Volterra equations was proposed by Kermack and McKendrick [6] to provide a mathematical description of such gross events as the Great Plague of London in 1665–1666. When it comes to such events, the statisticians' law of large numbers permits the application of deterministic differential equations to describe situations which are, in their fine detail, of a more statistical nature. It really is the same thing as fluid models of the Universe.

One of the sad features of these more accurate mathematical models of natural phenomena is that the sets of first-order ordinary differential equations arising in the modelling process have a tendency to be quite nonintegrable even in the almost simplest of models. This is in curious contrast situation in physics wherein the paradigms—say the Kepler problem and the simple harmonic oscillator—are so integrable that there seems to be no end to the interpretations of their mathematical properties. This characteristic was recognised, almost before the field was initiated, by Henri Poincaré who inaugurated the discipline of dynamical systems to be able to say something useful about these apparently otherwise intractable systems.

Curiously in the same epoch, in which Poincaré was establishing dynamical systems as a useful tool for the analysis of these difficult systems, two different approaches to the establishment of criteria for integrability were developed. In approximately the last quarter of the nineteenth century the Norwegian, Sophus Lie, devised his theory of continuous groups, the full practical effects of which have only begun to be appreciated almost a century later following the pioneering works of Laurentiev Ovsiannikov in serious application to the solution of differential equations arising in mathematical physics. About the same time Paul Painlevé and his School elucidated the singularity properties of ordinary differential equations and their relationship with integrability which had been used so effectively by Sophie Kowalevski to find the third integrable case of the Euler equations describing the motion of a top.

For some strange reason the symmetry methods of Lie and the singularity analysis of Painlevé have not found the same degree of application to the systems of ordinary differential equations arising in the mathematical modelling of biological, ecological and chemical systems, let alone the precise area of epidemiology of interest in this note, as the theory of dynamical systems of Poincaré and his successors. This is unfortunate as the resolution of equations arising in the modelling process is the critical part of the analysis, not the adjective used to describe the particular brand of analysis being used.

In this note we examine a simple epidemiological model from the viewpoint of both Lie and Painlevé. The model is a particular case of the classical SIS model introduced by

Kermack and McKendrick [6]. We assume that recovery from the nonfatal infective disease does not confer immunity. The two first-order ordinary differential equations are [7]

$$\begin{aligned}\dot{S} &= -\beta SI - \mu S + \gamma I + \mu K,\\ \dot{I} &= \beta SI - (\mu+\gamma)I,\end{aligned} \tag{1}$$

in which the overdot denotes differentiation with respect to time, $S(t)$ is the susceptible component of the population, $I(t)$ is the infected component of the population, $\mu K$ represents a constant birth rate, $\mu$ is the proportionate death rate, $\beta$ is the infectivity coefficient of the typical Lotka–Volterra interaction term and $\gamma$ the recovery coefficient. We emphasise that the disease is assumed to be nonfatal so that the standard term removing deceased infectives ($-\alpha I$ in [7]) is omitted.

In Section 2 we subject the system (1) to the Painlevé analysis in its present state as a raw dynamical system, as it were, and also as a single second-order ordinary differential equation which has been 'sanitised' by the removal of mathematically distracting parameters. In Section 3 we examine the derived second-order ordinary differential equation for Lie symmetries and find them somewhat obviously lacking given the integrability of the system already established by the Painlevé analysis. However, this apparent lack of symmetry is shown to be a problem of representation dependence and system is actually trivially integrable. We conclude with some observations of the delicate dependence of the integrability of a nonlinear system on its precise form, its balance as it were, in the presence of nontrivial parameters.

## 2. Painlevé analysis

The application of the Painlevé test is fairly routine—although see Géronimi et al. [8] for an examination of concepts underlying the Painlevé test—and the reader unfamiliar with the details of the method is referred to a standard exposition such as that of Tabor [9] or Ramani et al. [10]. We seek the parameters in the leading-order behaviour of $S(t)$ and $I(t)$ in (1) by writing

$$S = a\tau^{p}, \qquad I = b\tau^{q}, \tag{2}$$

where $a$, $b$, $p$ and $q$ are constants to be determined and $\tau = t - t_0$ with $t_0$ being the location of the presumed movable pole, and substituting this into (1) to obtain

$$\begin{aligned}ap\tau^{p-1} &= -\beta ab\tau^{p+q} - \mu a\tau^{p} + \gamma b\tau^{q} - \mu K,\\ bq\tau^{q-1} &= \beta ab\tau^{p+q} - (\mu+\gamma)\tau^{q},\end{aligned} \tag{3}$$

from which it is evident that the dominant terms are those on the left and the first on the right. We see that $p = q = -1$ and $-a = b = 1/\beta$ so that the leading-order behaviour is

$$S = -\frac{1}{\beta}\tau^{-1}, \qquad I = \frac{1}{\beta}\tau^{-1}. \tag{4}$$

To determine the resonance at which the required second arbitrary constant occurs we set

$$S = -\frac{1}{\beta}\tau^{-1} + m\tau^{r-1}, \qquad I = \frac{1}{\beta}\tau^{-1} + n\tau^{r-1} \tag{5}$$

in the dominant terms already determined by the analysis of leading-order behaviour. Collecting the terms linear in $m$ and $n$ we find that

$$\begin{pmatrix} r & 1 \\ 1 & r \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = 0 \tag{6}$$

for which a nontrivial solution exists if $r = \pm 1$. Thus the second arbitrary constant enters at the second term in the Laurent expansion for the solution of (1). To check for consistency with the nondominant terms all we need do is to substitute (5) into (1). At $\tau^{-1}$ we need

$$\begin{aligned} 0 &= -\beta(an + bm) - \mu a + \gamma b, \\ 0 &= \beta(an + bm) - (\mu + \gamma)b. \end{aligned} \tag{7}$$

Given the values of $a$ and $b$ the two equations in (7) are identical and so we have consistency. We can take $m$ as the arbitrary constant and set

$$n = m - \frac{\mu + \gamma}{\beta}. \tag{8}$$

The system (1) passes the Painlevé test and is integrable in the sense of Poincaré, i.e., in terms of functions analytic away from the movable singularity the location of which is fixed by the initial conditions. The formal Laurent series may be obtained by substituting

$$S = \sum_{i=0}^{\infty} a_i \tau^{i-1}, \qquad I = \sum_{i=0}^{\infty} b_i \tau^{i-1} \tag{9}$$

into (1).

## 3. Lie analysis

In January 2001 the first Whiteman prize for notable exposition on the history of mathematics was awarded to Thomas Hawkins by the American Mathematical Society. In the citation, published in the Notices of Amer. Math. Soc. 48 (2001) 416–417, one reads that Thomas Hawkins "... has written extensively on the history of Lie groups. In particular he has traced their origins to Lie's work in the 1870s on differential equations... the idée fixe guiding Lie's work was the development of a Galois theory of differential equations... Hawkins' book [11] highlights the fascinating interaction of geometry, analysis, mathematical physics, algebra and topology...."

In the Introduction of his book [12] Olver wrote that "it is impossible to overestimate the importance of Lie's contribution to modern science and mathematics. Nevertheless anyone who is already familiar with (it)... is perhaps surprised to know that its original inspirational source was the field of differential equations."

Lie's monumental work on transformation groups [13–15] and in particular contact transformations [16], led him to achieve his goal [17].

Many books have been dedicated to this subject and its generalisations [12,18–26].

Lie group analysis is indeed the most powerful tool to find the general solution of ordinary differential equations. Any known integration technique can be shown to be a particular case of a general integration method based on the derivation of the continuous group of symmetries admitted by the differential equation, i.e., the Lie symmetry algebra.

In particular Bianchi's theorem [12,27] states that, if an admitted $n$-dimensional solvable Lie symmetry algebra is found, then the general solution of the corresponding $n$th-order system of ordinary differential equations can be obtained by quadratures. The admitted Lie symmetry algebra can be easily derived by a straightforward although lengthy procedure. As computer algebra software becomes widely used, the integration of systems of ordinary differential equations by means of Lie group analysis is becoming easier to perform.

A major drawback of Lie's method is that systems of first-order ordinary differential equations do not lend themselves kindly to analysis for the possession of symmetry since there exists an infinite number of Lie symmetries. For a practical resolution of the problem there are two approaches possible. In one of them an Ansatz is made of the structure of the coefficient functions. Although this approach is open to the fundamental objection that the Ansatz is more likely to be based upon the imagination of the person making the analysis than of the inherent features of the system under consideration, there have been occasions when this is a very fruitful approach [28]. In the second approach the method of reduction of order [29] effectively replaces a first-order system by one containing at least one second-order equation which reduces the number of symmetries from infinity to a finite number, which one hopes is not zero. This idea has been successfully applied in several instances [29–35]. Consequently we replace the system (1) by a single second-order ordinary differential equation.

From (1b),

$$S = \frac{\dot{I}}{\beta I} + \frac{\mu + \gamma}{\beta} \tag{10}$$

and (1a) is then

$$I\ddot{I} - \dot{I}^2 + \beta I^2 \dot{I} + \mu I \dot{I} + \beta\mu I^3 + \mu(\mu + \gamma - \beta K)I^2 = 0, \tag{11}$$

an equation of which it could fairly be remarked that therein is a plethora of constants. Since (11) does not possess a rescaling symmetry, we may achieve a cosmetic simplification by means of the rescalings

$$I = \frac{\mu}{\beta} y, \qquad t = \frac{x}{\mu} \tag{12}$$

and the relabelling

$$1 + \frac{\gamma - \beta K}{\mu} = k, \tag{13}$$

videlicet the somewhat grotesque albeit autonomous nonlinear second-order ordinary differential equation

$$yy'' - y'^2 + y^2 y' + yy' + y^3 + ky^2 = 0, \tag{14}$$

an equation quite unknown to Kamke [36].

The integrability of (14) is a consequence of the integrability of the system (1) as revealed by the Painlevé analysis. A minor disadvantage of the Painlevé analysis is that a successful application leads to a Laurent series and series representations of solutions are not always immediately recognisable in terms of known functions, if indeed they are series representations of known functions. The attraction of the Lie approach is that the possession of a suitable number of symmetries of the right type leads naturally to reduction to quadratures and even solution in closed form.

Equation (14) is obviously autonomous. The zeroth-order and first-order differential invariants are given by the two independent solutions of the associated Lagrange's system of the symmetry $\partial_x$, videlicet

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy'}{0}, \tag{15}$$

i.e., $u = y$ and $v = y'$. The reduced form of (14) is

$$uv\frac{dv}{du} - v^2 + (u^2 + u)v + u^3 + ku^2 = 0 \tag{16}$$

which is an Abel's of the second kind and hence not to be expected to give the joy of a solution in closed form.

The symmetry $G_1 = \partial_x$ is the only Lie point symmetry of (14) for general values of the parameter $k$. We note that an analysis of (11) for Lie point symmetries via the well-known interactive code developed by Nucci [37,38] reveals a certain branching structure, i.e., either there is a constraint on a parameter or parameters in the equation. Thus one has the choice $\beta = 0$ or $\beta \neq 0$. The former cannot realistically be accepted since it removes an essential term from the model system (1). A second choice proffered is $\gamma = \beta K$ or $\gamma \neq \beta K$. This choice has no effect upon the formal integrability of (11)/(14), but the former does introduce a second Lie point symmetry, videlicet

$$G_2 = e^{\mu t}(\partial_t - \mu I \partial_I) \sim e^x(\partial_x - y\partial_y), \tag{17}$$

in which we give the representations for both (11) and (14). (The structure of the second symmetry (17) and the way it arises when there is a constraint on the parameters was observed in another context by Torrisi et al. [30].) We note that $[G_1, G_2] = (\mu)G_2$ so that reduction of (14) should be by $G_2$ rather than the more usual $G_1$ and the equation is a member of Lie's Type III second-order ordinary differential equations with two point symmetries. The standard form of the equation [17] is not one which is linearisable by means of a point transformation except for some very particular forms [39].

The associated Lagrange's system for the zeroth and first-order invariants of $G_2$ in the (14) representation is

$$\frac{dx}{1} = \frac{dy}{-y} = \frac{dy'}{-2y' - y} \tag{18}$$

so that

$$u = ye^x, \qquad v = \frac{y'}{y^2} + \frac{1}{y}. \tag{19}$$

To render the reduced first-order ordinary differential equation more amenable to direct quadrature we consider the form of the second symmetry $\partial_x$ in these variables. This is $u\partial_u$ and so a good set of new variables is

$$X = \log y + x, \qquad Y = \frac{y'}{y^2} + \frac{1}{y} \tag{20}$$

since then the first-order differential equation is autonomous. We find that

$$\frac{dY}{dX} + Y + 1 = 0 \tag{21}$$

which is readily integrated to

$$(Y+1)e^X = B \quad \Leftrightarrow \quad \frac{y'}{y} + 1 + y = Be^{-x} \tag{22}$$

which is a standard Riccati equation. We integrate (22) to obtain

$$y = \frac{BCe^{-x}}{A\exp^{Be^{-x}} + C} \tag{23}$$

which is just the solution to be obtained for general $k$, (35), with $k = 1$.

Although, as a scalar second-order ordinary differential equation (14) possesses a Lagrangian, its determination is not obvious and the possibility that $\partial_x$ would also be a Noetherian symmetry leading to a first integral and so formal integrability, if not explicit integration, is just that. Essentially one is no further advanced than the level of knowledge provided by the Painlevé analysis. One could contemplate a more general search for Lie symmetries by not imposing any condition on the functional dependence of the coefficient functions in

$$G = \xi\partial_x + \eta\partial_y. \tag{24}$$

Fortunately one can, in analogy to Boyer's theorem regarding unconstrained Noether's symmetries [40], set one or other of $\xi$ and $\eta$ at zero to reduce the complexity of calculation. The latter choice is an attractive one for (14) since, as the equation is autonomous, the equation for $\xi$ is necessarily linear of the first order in $\xi'$ and consequently amenable to formal solution. (The former choice $\xi = 0$ presents a linear second-order ordinary differential equation for $\eta$ which is impenetrable.)

With $\eta = 0$ the action of the second extension of (24), videlicet $G^{[2]} = \xi\partial_x - y'\xi'\partial_{y'} - (2y''\xi' + y'\xi'')\partial_{y''}$, on (14) gives

$$y^2\xi'' + \xi'\left[2yy'' - 2y'^2 + y'(y^2 + y)\right] = 0 \tag{25}$$

which is formally integrated to give

$$\xi = A + B\int \exp\left[-y - \frac{2y'}{y}\right]\frac{dx}{y}. \tag{26}$$

The coefficient of the constant $A$ gives the obvious symmetry $\partial_x$ that of $B$ a symmetry which is powerfully nonlocal.

The Lie point symmetry structure of (14) is at odds with its established integrability in terms of analytic functions. This incompatibility has at times [41–43] been demonstrated to be due to an 'inappropriate' choice of variables. A suitable choice of variables requires a knowledge of the symmetries of the system and, if these be not point (contact for systems of higher order), their determination in general is not easy and even under restricting Ansätze limited in the extent of the applicability of the results [44]. One technique to obviate this problem is to change the order of the equation. The reduction given above does not present attractive potential. Consequently we resort to an increase in order. This procedure is really the very opposite to reduction of order. For the latter one uses a symmetry to generate a transformation based on the invariants of the symmetry to reduce the order of the equation. In the case of the former one adopts a transformation which increases the order of the equation and the very choice of transformation implies that the higher-order equation has the requisite symmetry for its generation. The choice of a suitable symmetry is indeed quite daunting. However, there does exist one, the homogeneity symmetry, which makes no assumptions about the solution of the higher-order equation, but simply imposes homogeneity. The actual increase in order is not generally going to aid the process of solution since both order and number of symmetries have increased by one. However, in the general procedure of reduction of order symmetries without the requisite Lie bracket relations with the reducing symmetry become exponential nonlocal symmetries [12]. (Interestingly our search for nonlocal symmetries of (14) did not reveal any of these.) One can hope that the higher-order equation has one, dare one hope several, point hidden symmetries of Type I [45], i.e., point symmetries which appear from nowhere, as it seems, on increase of the order of a differential equation.

It is in this spirit that we increase the order of (14) by means of the Riccati transformation

$$y = \rho \frac{w'}{w}, \tag{27}$$

where $\rho$ is to be just a constant since the second-order ordinary differential equation is already autonomous instead of $\rho(x)$ as one would normally use in the generalised Riccati transformation. When the transformation (27) is applied to (24), we obtain after some rearrangement of terms

$$\rho^2\left(\frac{w'w''' - w''^2}{w^2}\right) + (\rho^3 - \rho^2)\frac{w'^2 w''}{w^3} + (\rho^2 - \rho^3)\frac{w'^4}{w^4} + \rho^2\frac{w'w''}{w^2} + (\rho^2 - \rho^3)\frac{w'^3}{w^3} + k\rho^2\frac{w'^2}{w^2} = 0 \tag{28}$$

from which it is quite obvious that the choice $\rho = 1$ provides a considerable simplification to

$$w'w''' - w''^2 + w'w'' + kw'^2 = 0. \tag{29}$$

Equation (29) has the obvious point symmetries

$$G_1 = \partial_x, \qquad G_2 = \partial_w, \qquad G_3 = w\partial_w, \tag{30}$$

the first being inherited from (14) and the third being due to the transformation (27). The middle symmetry, which has no point counterpart in (14), is consequent upon the choice

$\rho = 1$. It is this symmetry which we choose for the reduction of (29) to a second-order ordinary differential equation. The obvious variables for the reduction using $G_2$ are

$$X = x, \qquad W = \log w' \tag{31}$$

so that we obtain the linear second-order ordinary differential equation

$$W'' + W' + k = 0 \tag{32}$$

which is essentially a first-order ordinary differential equation in $W'$. The solution for $y$ follows from that of (32). Thus we have

$$W(X) = -kX + \log A + Be^{-X}, \tag{33}$$

$$w(x) = A \int \exp[-kx + Be^{-x}]\,dx + C, \tag{34}$$

$$y(x) = \frac{A \exp[-kx + Be^{-x}]}{A \int \exp[-kx + Be^{-x}]\,dx + C}, \tag{35}$$

where $A$, $B$ and $C$ are constants of integration, whence the solution of the original system (1) is

$$I(t) = \frac{1}{\beta} \frac{A \exp[-(\mu + \gamma - \beta K)t + Be^{-\mu t}]}{A\mu \int \exp[-(\mu + \gamma - \beta K)t + Be^{-\mu t}]\,dt + C}, \tag{36}$$

$$\begin{aligned} S(t) = {} & \frac{\mu + \gamma}{\beta} + \frac{1}{\beta} \frac{A \exp[-(\mu + \gamma - \beta K)t + Be^{-\mu t}]}{A\mu \int \exp[-(\mu + \gamma - \beta K)t + Be^{-\mu t}]\,dt + C} \\ & \times \left\{ \mu + \gamma - \beta K - \mu B e^{-\mu t} \right. \\ & \left. + \frac{1}{\beta} \frac{A \exp[-(\mu + \gamma - \beta K)t + Be^{-\mu t}]}{A\mu \int \exp[-(\mu + \gamma - \beta K)t + Be^{-\mu t}]\,dt + C} \right\} \end{aligned} \tag{37}$$

which does seem to be inordinately complicated for what has the appearance of a somewhat simple system.

Since (32) is a linear second-order ordinary differential equation, it has eight Lie point symmetries [17] with the algebra sl(3, $R$). These symmetries are

$$\begin{aligned} &\Gamma_1 = \partial_X, && \Gamma_5 = -e^{-X}\big[\partial_X + (W - 3kX)\partial_W\big], \\ &\Gamma_2 = e^X[\partial_X - k\partial_W], && \Gamma_6 = (W + kX)\partial_W, \\ &\Gamma_3 = \partial_W, && \Gamma_7 = (W + kX)\partial_X + [4k^2X - 2kXW - W^2]\partial_W, \\ &\Gamma_4 = e^{-X}\partial_W, && \Gamma_8 = e^X(W + kX)[\partial_X - k\partial_W]. \end{aligned} \tag{38}$$

Naturally these symmetries have expressions in terms of the original coordinates and so give symmetries for (14) and hence (11) and the system (1). The transformation connecting (14) and (32) is

$$X = x, \qquad W = \log y + \int y\,dx. \tag{39}$$

Table 1
The destinations of the point symmetries of the linear second-order equation (32) in the third-order equation (29) and the original second-order equation (14)

| | | |
|---|---|---|
| $\Gamma_1$ | $\partial_x$ | $\partial_x$ |
| $\Gamma_2$ | $e^x\partial_x+[(k-1)\int w'e^x\,dx]\partial_w$ | $e^x\partial_x+\{(k-1)[e^{-\int y\,dx}\int ye^{(x+\int y\,dx)}\,dx]'-ye^x\}\partial_y$ |
| $\Gamma_3$ | $w\partial_w$ | lost in the reduction |
| $-$ | $\partial_w$ | $-ye^{-\int y\,dx}\partial_y$ |
| $\Gamma_4$ | $-\int w'e^{-x}\,dx\partial_w$ | $-[e^{-\int y\,dx}\int y'e^{(-x+\int y\,dx)}\,dx]'\partial_y$ |
| $\Gamma_5$ | $e^{-x}\partial_x+[\int w'e^{-x}(\log w'-3kx-1)\,dx]\partial_w$ | $e^{-x}\partial_x+\{ye^{-x}+[e^{-\int y\,dx}\int ye^{(-x+\int y\,dx)}$ $\times(\log ye^{\int y\,dy}-3kx-1)\,dx]'\}\partial_y$ |
| $\Gamma_6$ | $[\int w'(\log w'+kx)\,dx]\partial_w$ | $[e^{-\int y\,dx}\int ye^{-\int y\,dx}(\log y+\int y\,dx+kx)\,dx]'\partial_y$ |
| $\Gamma_7$ | $(\log w'+kx)\partial_x+[w'+kw+\int(4k^2xw'$ $-2kxw'\log w'-w'\log^2 w')\,dx]\partial_w$ | $(kx+\log y+\int y\,dx)\partial_x+\{-(k+y)y+[e^{-\int y\,dx}$ $\times\int(4k^2xye^{\int y\,dx}(\log y+\int y\,dx)-ye^{-\int y\,dx}$ $\times(\log y+\int y\,dx)^2)\,dx]'\}\partial_y$ |
| $\Gamma_8$ | $(\log w'+kx)e^x\partial_x+\{(w'+kw)e^x$ $+\int[-(w'+kw)e^x-w'(\log w'$ $+kx)e^x(k-1)]\,dx\}\partial_w$ | $(\log y+\int y\,dx+kx)e^x\partial_x-\{[e^{-\int y\,dx}\int[(y+k)$ $\times e^{(\int y\,dx+x)}$ $+ye^{(x+\int y\,dx)}(\log y+\int y\,dx+kx)(k-1)]\,dx]'$ $+y(\log y+\int y\,dx+kx)e^x-(y+k)(1-y)e^x\}\partial_y$ |

To express the symmetries in (38) we relate the coefficient functions through the intermediate equation in $w$ and $x$. The symmetries descend from $(x,w)$-space to $(X,W)$-space according to

$$\xi\partial_x+\eta\partial_w+\zeta_{w'}\quad\rightarrow\quad\xi\partial_X+\frac{\zeta}{w'}\partial_W \tag{40}$$

and from $(x,w)$-space to $(x,y)$-space according to

$$\xi\partial_x+\eta\partial_w+\zeta_{w'}\quad\rightarrow\quad\xi\partial_x+\left(\frac{\zeta}{w}-\frac{\eta w'}{w^2}\right)\partial_y. \tag{41}$$

We list the various forms of the symmetries in Table 1.

The symmetries $\Gamma_1$ and $\Gamma_3$ become the point symmetries $\partial_x$ and $w\partial_w$ of (29). In the reduction from (29) to (32) $\partial_w$ is lost. Likewise $w\partial_w$ is lost in the reduction from (29) to (14), since it is the generator of the Riccati transformation (27), but $\partial_x$ persists. Of the other symmetries we note that $\Gamma_2$ is quite nonlocal for general values of $k$. However, for the special value $k=1$, corresponding to $\gamma=\beta K$, the second point symmetry of (14) noted above is recovered. This is the only symmetry for which setting $k$ to one removes the nonlocality.

## 4. Conclusion

We have shown that a simple SIS model for a nonfatal infectious disease is integrable firstly from the viewpoint of singularity analysis and, encouraged by this indication, secondly by integration of the second-order ordinary differential equation (14), derived from the system (1). The route for integration, the raising and lowering of order by means of nonpoint transformations of the dependent variable is based on Lie symmetries.

In the case of a fatal disease one must add a term, $-\alpha I$, to (1b). Performance of the Painlevé analysis immediately demands that $\alpha$ be zero for the system to be integrable in terms of analytic functions. The second-order ordinary differential equation of the system (1) equivalent to (14) is

$$yy'' - y'^2 + y^2y' + ay^3 + yy' + ky^2 = 0, \tag{42}$$

where now

$$a = 1 + \frac{\alpha}{\mu}, \qquad k = 1 + \frac{\alpha + \gamma - \beta K}{\mu}. \tag{43}$$

Not only does (42) fail the Painlevé test but the nice route for reduction to a linear second-order differential equation is, not surprisingly, lost. It would seem that a fatal disease which this models is also not good for mathematics!

In the system, when $\alpha \neq 0$, there are two essential constants. Our analysis shows that the value of one, $\alpha$, is critical for successful analysis of the system by both the Painlevé and Lie approaches. The value of the other, the collection labelled $k$, has no effect upon the integrability of the system, but, when it takes the value one corresponding to the constraint $\gamma = \beta K$, the process of explicit integration is somewhat less circuitous than for general values of $k$.

The main feature of the work discussed here is that the Lie and Painlevé analyses throw up critical values of parameters, and yield to solution in closed form. This type of result has already been observed in the Lie analysis of a mathematical model which describes HIV transmission in male homosexual/bisexual communities [30], a core group model for sexually transmitted diseases [34], and a SIRI disease transmission model [35].

Here we have sought to promote the use of Lie and Painlevé analyses for mathematical models in epidemiology and more generally in the biosciences as a standard routine. These analyses complement the results obtained through the methods of dynamical systems and consequently offer the prospect of providing greater information about the evolution in time of the system under consideration.

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