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Journal of Approximation Theory 162 (2010) 1945–1963

Journal of Approximation Theory

www.elsevier.com/locate/jat

Asymptotics for Jacobi–Sobolev orthogonal polynomials associated with non-coherent pairs of measures

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Received 20 September 2009; received in revised form 13 February 2010; accepted 12 May 2010
Available online 21 May 2010

Communicated by Andrei Martinez-Finkelshtein

Abstract

We consider the Sobolev inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}\psi^{(\alpha,\beta)}(x) + \int f'(x)g'(x)\mathrm{d}\psi(x),$$

where $\mathrm{d}\psi^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x$ with $\alpha, \beta > -1$, and ψ is a measure involving a rational modification of a Jacobi weight and with a mass point outside the interval $(-1, 1)$. We study the asymptotic behaviour of the polynomials which are orthogonal with respect to this inner product on different regions of the complex plane. In fact, we obtain the outer and inner strong asymptotics for these polynomials as well as the Mehler–Heine asymptotics which allow us to obtain the asymptotics of the largest zeros of these polynomials. We also show that in a certain sense the above inner product is also equilibrated.

Keywords: Orthogonal polynomials; Sobolev orthogonal polynomials; Asymptotics

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2010.05.003

1. Introduction

The theory of Sobolev orthogonal polynomials has been developed in many aspects during the last two decades. Different topics for these nonstandard orthogonal polynomials such as algebraic relations, zeros, differential equations or asymptotics have been treated in many papers. The surveys [9,11] can give an overview about the results obtained in some topics of this theory, especially those ones related to asymptotic properties of the Sobolev orthogonal polynomials.

The *continuous* Sobolev polynomials are orthogonal with respect to an inner product of the form

$$\langle f, g\rangle_S = \sum_{i=0}^{p} \int_{I_i} f^{(i)}(x) g^{(i)}(x) \mathrm{d}\psi_i(x), \tag{1}$$

where $\psi_i(x)$ are measures supported on the interval $I_i \subseteq \mathbb{R}$ with absolutely continuous parts different from zero.

When $p \geq 1$ the orthogonal polynomials with respect to (1) are nonstandard, that is, they do not satisfy Favard's theorem. Therefore, there is no three-term recurrence relation and the nice and well-known properties of the standard polynomials do not hold anymore. The existence of the derivatives in the inner product (1) makes a general approach to study this inner product not be easy. For example, up to we know there is not a Riemann–Hilbert approach to study these families of orthogonal polynomials. The use of this powerful technique for Sobolev orthogonal polynomials is a very interesting open question.

The most studied case in the literature is when $p = 1$ in (1). In this paper, we deal with a particular case related to a Jacobi weight. In fact, we consider the inner product

$$\langle f, g\rangle_S = \int_{-1}^{1} f(x) g(x) (1-x)^{\alpha} (1+x)^{\beta} \mathrm{d}x + \int_{I_1} f'(x) g'(x) \mathrm{d}\psi_1(x), \quad \alpha, \beta > -1, \tag{2}$$

where ψ_1 is a measure related to a rational modification of a Jacobi weight with a mass point outside $(-1, 1)$. Another paper in this direction is [6] where a Gegenbauer weight is considered. That case is not a symmetrization of the one considered in this paper.

Our objective is to study asymptotic properties of the polynomials orthogonal with respect to the inner product given in (2). However, it is very important to emphasize two points:

- we choose the measure ψ_1 in such a way that the pair (ψ_0, ψ_1) is not a coherent pair according to Meijer's classification (see [13]) because this case has been widely studied in the literature. Furthermore, ψ_1 has a mass point outside $(-1, 1)$. Outer strong asymptotics for general Sobolev orthogonal polynomials have been obtained by Martínez–Finkelshtein in [10] when the measures are absolutely continuous and supported on a smooth Jordan closed curve;
- from the point of view of asymptotics properties the measure in (2) playing the most important role is ψ_1 because we have a quadratic factor n^2 out of the derivatives when we work with monic polynomials (see the surveys [9,11]). Here, this situation is also true but we will show that the inner product (2) is equilibrated in some sense. Notice that the technique used here is different from the one used in [1,2] where varying Sobolev orthogonal polynomials are considered.

More precisely, we consider

$$\langle f, g\rangle_S = \int_{-1}^{1} f(x) g(x) (1-x)^{\alpha} (1+x)^{\beta} \mathrm{d}x + \int_{-1}^{1} f'(x) g'(x) \frac{\kappa(\kappa_1+\kappa_2) - \kappa_1 x}{\kappa - x} \times (1-x)^{\alpha+1} (1+x)^{\beta+1} \mathrm{d}x + \kappa_2 \kappa_3 f'(\kappa) g'(\kappa),$$

that can be rewritten as

$$\langle f, g\rangle_S = \langle f, g\rangle_{\psi^{(\alpha,\beta)}} + \kappa_1\langle f', g'\rangle_{\psi^{(\alpha+1,\beta+1)}} + \kappa_2\langle f', g'\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}}, \tag{3}$$

where

$$\langle f, g\rangle_{\psi^{(\alpha,\beta)}} := \int_{-1}^{1} f(x)g(x)\mathrm{d}\psi^{(\alpha,\beta)}(x),$$

with $\mathrm{d}\psi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta\mathrm{d}x, \alpha, \beta > -1$, and

$$\langle f, g\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}} = \int_{-1}^{1} f(x)g(x)\mathrm{d}\psi^{(\alpha,\beta,\kappa)}(x) + \kappa_3\left[f(\kappa)g(\kappa)\right], \tag{4}$$

with $\mathrm{d}\psi^{(\alpha,\beta,\kappa)}(x) = \frac{\kappa}{\kappa-x}\mathrm{d}\psi^{(\alpha+1,\beta+1)}(x)$.

Under the restrictions $|\kappa| \geq 1, \kappa_2 \geq 0, \kappa_3 \geq 0$ and $\kappa_1 \geq -\frac{|\kappa|}{1+|\kappa|}\kappa_2$, the Eq. (3) is an inner product. The orthogonal polynomials with respect to (3) were introduced in [5]. Very recently, in [4] the authors have studied the behaviour of the zeros of these orthogonal polynomials for $\kappa_1 \neq 0$.

Thus, we give a new look at (3) and we can observe two points according to the above comments about the election of the measure ψ_1:

- if $\kappa_1 \neq 0$, the pair of measures $\{\mathrm{d}\psi^{(\alpha,\beta)}, \kappa_1\mathrm{d}\psi^{(\alpha+1,\beta+1)} + \kappa_2\mathrm{d}\psi^{(\alpha,\beta,\kappa,\kappa_3)}\}$ does not form a coherent pair according to Meijer's classification given in [13];
- the classical Jacobi orthogonal polynomials associated with the measure $\psi^{(\alpha,\beta)}$ are also orthogonal with respect to

$$\langle f, g\rangle_{\psi^{(\alpha,\beta)}} + \kappa_1\langle f', g'\rangle_{\psi^{(\alpha+1,\beta+1)}}. \tag{5}$$

  Then, when we consider monic polynomials a quadratic factor n^2 appears in (5) and now the two parts,

$$\langle f, g\rangle_{\psi^{(\alpha,\beta)}} + \kappa_1\langle f', g'\rangle_{\psi^{(\alpha+1,\beta+1)}} \quad \text{and} \quad \kappa_2\langle f', g'\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}},$$

  in the inner product (3) play a similar role in the asymptotic behaviour of the Sobolev polynomials. In Section 3 we will show a result (Theorem 3.3) which is similar to Theorem 1 in [1] where the authors balance the inner product (see Remark 3.3).

As we have commented before our objective in this paper is to obtain asymptotic results for the Sobolev polynomials orthogonal with respect to (3) which allow us to describe these polynomials in all the complex plane. Notice that the case $\kappa_2 = 0$ is trivial because the Sobolev polynomials are the Jacobi polynomials. Thus, along this paper we assume

$$|\kappa| \geq 1, \qquad \kappa_1 \geq 0, \qquad \kappa_2 > 0, \quad \text{and} \quad \kappa_3 \geq 0. \tag{6}$$

We denote by $\left\{\mathcal{S}_n = \mathcal{S}_n^{(\alpha,\beta,\kappa,\kappa_1,\kappa_2,\kappa_3)}\right\}_{n=0}^{\infty}$ the sequence of orthogonal polynomials with respect to (3) that we call Jacobi–Sobolev orthogonal polynomials.

The structure of the paper is the following: in Section 2 we give a background with properties of classical Jacobi polynomials and of other polynomials related to them. In Section 3 some properties of the polynomials $\mathcal{S}_n$ are obtained as well as their outer strong asymptotics. In Section 4 we introduce our results about Mehler–Heine type formulas and their consequences in the asymptotic behaviour of the zeros of $\mathcal{S}_n$. Finally, in Section 5 we also give the inner strong asymptotics for $\mathcal{S}_n$, that is, the asymptotics on $(-1, 1)$ where we use some techniques from Section 4.

2. Brief background about Jacobi polynomials and related polynomials

We denote the sequence of monic classical Jacobi polynomials by $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$. They are orthogonal with respect to the inner product

$$\langle f, g\rangle_{\psi^{(\alpha,\beta)}} := \int_{-1}^{1} f(x)g(x)\mathrm{d}\psi^{(\alpha,\beta)}(x) = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x, \quad \alpha, \beta > -1,$$

and they can also be defined by Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x) = (-1)^n \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}(1-x)^{-\alpha}(1+x)^{-\beta} \times \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \tag{7}$$

The monic Jacobi polynomials satisfy the following three-term recurrence relation

$$P_{n+1}^{(\alpha,\beta)}(x) = (x - \beta_{n+1}^{(\alpha,\beta)})P_n^{(\alpha,\beta)}(x) - \alpha_{n+1}^{(\alpha,\beta)} P_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 1,$$

where $\beta_{n+1}^{(\alpha,\beta)}$, $n \geq 0$, are real numbers and

$$\alpha_{n+1}^{(\alpha,\beta)} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)} > 0, \quad n \geq 1. \tag{8}$$

As a consequence, the n zeros of $P_n^{(\alpha,\beta)}$ are simple and lie inside $(-1, 1)$. For $x = 1$, we have

$$P_n^{(\alpha,\beta)}(1) = \frac{2^n\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(2n+\alpha+\beta+1)}. \tag{9}$$

Another algebraic relation that we use along this paper is

$$P_n^{(\alpha,\beta+1)}(x) = P_n^{(\alpha+1,\beta+1)}(x) - \frac{2n(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n \geq 1. \tag{10}$$

Many other properties of Jacobi polynomials can be found, for example, in [7,8,15].

We also need asymptotics properties of the Jacobi polynomials. They can be found in [8,14,15]. If we denote by $\rho_n^{(\alpha,\beta)} = \langle P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)}\rangle_{\psi^{(\alpha,\beta)}}$, then

$$\lim_{n\to\infty} \frac{\rho_n^{(\alpha,\beta)}}{\rho_{n+1}^{(\alpha,\beta)}} = \lim_{n\to\infty} \frac{1}{\alpha_{n+2}^{(\alpha,\beta)}} = 4 \quad \text{and} \quad \lim_{n\to\infty} \frac{\rho_{n+1}^{(\alpha,\beta)}}{\rho_n^{(\alpha+1,\beta+1)}} = 1. \tag{11}$$

We consider the complex function

$$\varphi(z) = z + \sqrt{z^2-1}, \quad \text{for } z \in \mathbb{C} \setminus [-1, 1],$$

where $\sqrt{z^2-1}>0$ when $z>1$, and $\varphi(+\infty)=+\infty$. Then, we have the ratio asymptotic

$$\lim_{n\to\infty}\frac{P_{n+1}^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(x)}=\frac{\varphi(x)}{2}, \tag{12}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1,1]$.

To obtain our results we will need some properties of the sequence of polynomials $\{P_n^{(\alpha,\beta,\kappa,\kappa_3)}\}_{n=0}^{\infty}$ orthogonal with respect to the inner product (4). Expanding $P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)$ in terms of Jacobi polynomials, we have

$$P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)=P_n^{(\alpha+1,\beta+1)}(x)+d_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa)P_{n-1}^{(\alpha+1,\beta+1)}(x),\quad n\geq 1, \tag{13}$$

where $d_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa)=-\frac{\rho_n^{(\alpha,\beta,\kappa,\kappa_3)}}{\kappa\rho_{n-1}^{(\alpha+1,\beta+1)}}$, with $\rho_n^{(\alpha,\beta,\kappa,\kappa_3)}=\langle P_n^{(\alpha,\beta,\kappa,\kappa_3)},P_n^{(\alpha,\beta,\kappa,\kappa_3)}\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}}$.

We can find in [14] (see also [12]) that

$$\lim_{n\to\infty}\frac{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)}{P_n^{(\alpha,\beta,\kappa,0)}(x)}=\frac{1}{2}\,\frac{(\varphi(x)-\varphi(\kappa))^2}{(x-\kappa)\varphi(x)},\quad \text{if } \kappa_3>0,$$

and

$$\lim_{n\to\infty}\frac{P_n^{(\alpha+1,\beta+1)}(x)}{P_n^{(\alpha,\beta,\kappa,0)}(x)}=\frac{1}{2}\frac{\varphi(x)-\varphi(\kappa)}{x-\kappa},$$

both uniformly on compact subsets of $\mathbb{C}\setminus([-1,1]\cup\{\kappa\})$.

Taking into account these results, in [12] the authors established the following result.

Lemma 2.1. *The following limit holds*

$$\lim_{n\to\infty}\frac{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)}{P_n^{(\alpha,\beta)}(x)}=\begin{cases}\dfrac{\varphi'(x)}{2}\left(1-\dfrac{\varphi(\kappa)}{\varphi(x)}\right), & \textit{if } \kappa_3>0,\\[2ex] \dfrac{\varphi'(x)}{2}\left(1-\dfrac{1}{\varphi(\kappa)\varphi(x)}\right), & \textit{if } \kappa_3=0,\end{cases} \tag{14}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1,1]$.

Now, using the previous results we are ready to give the asymptotic behaviour of the coefficients $d_n^{(\alpha,\beta,\kappa_3)}(\kappa)$ in (13) in a straightforward way.

Lemma 2.2. *For the coefficients* $d_n^{(\alpha,\beta,\kappa_3)}(\kappa)$ *in* (13) *it holds*

$$\lim_{n\to\infty}d_n^{(\alpha,\beta,\kappa_3)}(\kappa)=\begin{cases}-\dfrac{\varphi(\kappa)}{2}, & \textit{if } \kappa_3>0,\\[2ex] -\dfrac{1}{2\,\varphi(\kappa)}, & \textit{if } \kappa_3=0.\end{cases}$$

From (13) and Lemma 2.2 we obtain

Corollary 2.1.

$$\lim_{n\to\infty} \frac{\rho_n^{(\alpha,\beta,\kappa,\kappa_3)}}{\rho_{n-1}^{(\alpha+1,\beta+1)}} = \begin{cases} \dfrac{\kappa\varphi(\kappa)}{2}, & \textit{if } \kappa_3 > 0, \\ \dfrac{\kappa}{2\varphi(\kappa)}, & \textit{if } \kappa_3 = 0. \end{cases} \tag{15}$$

3. Outer asymptotics for Jacobi–Sobolev orthogonal polynomials

We recall that Jacobi–Sobolev polynomials are orthogonal with respect to the nonstandard inner product (3), i.e.,

$$\langle f, g\rangle_S = \langle f, g\rangle_{\psi^{(\alpha,\beta)}} + \kappa_1\langle f', g'\rangle_{\psi^{(\alpha+1,\beta+1)}} + \kappa_2\langle f', g'\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}}.$$

In [5, Corollary 4.1.2] the authors established the following recurrence relation for the polynomials $\mathcal{S}_n$

$$\mathcal{S}_{n+1}(x) + a_n(\kappa,\kappa_1,\kappa_2,\kappa_3)\mathcal{S}_n(x) = P_{n+1}^{(\alpha,\beta)}(x) + b_n^{(\alpha,\beta,\kappa_3)}(\kappa)P_n^{(\alpha,\beta)}(x), \quad n \geq 1, \tag{16}$$

with $\mathcal{S}_0(x) = P_0^{(\alpha,\beta)}(x) = 1$, $\mathcal{S}_1(x) = P_1^{(\alpha,\beta)}(x)$, $b_n(\kappa) := b_n^{(\alpha,\beta,\kappa_3)}(\kappa) = \frac{n+1}{n} d_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa)$, and

$$a_n := a_n(\kappa,\kappa_1,\kappa_2,\kappa_3) = \frac{\rho_n^{(\alpha,\beta)} + \kappa_1 n^2 \rho_{n-1}^{(\alpha+1,\beta+1)}}{\rho_n^S} b_n(\kappa), \quad n \geq 1, \tag{17}$$

where $\rho_n^S := \langle \mathcal{S}_n, \mathcal{S}_n\rangle_S$. The coefficients a_n, $n \geq 2$, can be generated recursively by

$$a_n = \frac{\nu_n(\kappa_1)\alpha_n{}^{(\alpha+1,\beta+1)}}{\nu_n(\kappa_1)\alpha_n{}^{(\alpha+1,\beta+1)} + b_{n-1}(\kappa)\left\{-n(n-1)\kappa_2\kappa + \nu_{n-1}(\kappa_1)[b_{n-1}(\kappa) - a_{n-1}]\right\}} b_n(\kappa), \tag{18}$$

with initial condition $a_1 = \frac{\nu_1(\kappa_1)}{\nu_1(\kappa_1)+\kappa_2\rho_0^{(\alpha,\beta,\kappa,\kappa_3)}/\rho_0^{(\alpha+1,\beta+1)}} b_1(\kappa)$, and $\nu_n(\kappa_1) = n^2\kappa_1 + \frac{n}{n+\alpha+\beta+1}$.

Clearly, from the definition of the sequence $b_n(\kappa)$, we get

$$\lim_{n\to\infty} d_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa) = \lim_{n\to\infty} b_n(\kappa) = b(\kappa) = \begin{cases} -\dfrac{\varphi(\kappa)}{2}, & \text{if } \kappa_3 > 0, \\ -\dfrac{1}{2\varphi(\kappa)}, & \text{if } \kappa_3 = 0. \end{cases} \tag{19}$$

On the other hand, we can observe that $\operatorname{sgn}(d_n^{(\alpha,\beta,\kappa_3)}(\kappa)) = \operatorname{sgn}(b_n(\kappa)) = \operatorname{sgn}(a_n) = -\operatorname{sgn}(\kappa)$.

The following theorem gives us lower and upper bounds for the square of norms $\|\mathcal{S}_n\|_S = [\rho_n^S]^{1/2}$.

Theorem 3.1. *We have*

$$\gamma_n^{(\alpha,\beta)}(\kappa_1) + n^2\kappa_2\rho_{n-1}^{(\alpha,\beta,\kappa,\kappa_3)} \leq \rho_n^S \leq \gamma_n^{(\alpha,\beta)}(\kappa_1) + n^2\kappa_2\rho_{n-1}^{(\alpha,\beta,\kappa,\kappa_3)} + b_{n-1}^2(\kappa)\gamma_{n-1}^{(\alpha,\beta)}(\kappa_1),$$

where $\gamma_n^{(\alpha,\beta)}(\kappa_1) = \rho_n^{(\alpha,\beta)} + \kappa_1 n^2\rho_{n-1}^{(\alpha+1,\beta+1)}$, $n \geq 1$.

Proof. Using the extremal property of the norms we get

$$\begin{aligned}\rho_n^S &= \langle \mathcal{S}_n, \mathcal{S}_n \rangle_S = \langle \mathcal{S}_n, \mathcal{S}_n \rangle_{\psi^{(\alpha,\beta)}} + \kappa_1 \langle \mathcal{S}_n', \mathcal{S}_n' \rangle_{\psi^{(\alpha+1,\beta+1)}} + \kappa_2 \langle \mathcal{S}_n', \mathcal{S}_n' \rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}} \\ &\geq \rho_n^{(\alpha,\beta)} + n^2 \kappa_1 \rho_{n-1}^{(\alpha+1,\beta+1)} + n^2 \kappa_2 \rho_{n-1}^{(\alpha,\beta,\kappa,\kappa_3)}.\end{aligned}$$

To prove the right-hand side inequality, it is enough to use

$$\rho_n^S = \langle \mathcal{S}_n, \mathcal{S}_n \rangle_S \leq \langle P_n^{(\alpha,\beta)} + b_{n-1}(\kappa) P_{n-1}^{(\alpha,\beta)}, P_n^{(\alpha,\beta)} + b_{n-1}(\kappa) P_{n-1}^{(\alpha,\beta)} \rangle_S. \quad \square$$

Now, we are going to establish the asymptotic behaviour of the sequence a_n. First, we define

$$\tilde{\kappa} := \begin{cases} \dfrac{\kappa(\kappa_1+\kappa_2)}{\kappa_1}, & \text{if } \kappa_1 > 0, \\ +\infty, & \text{if } \kappa_1 = 0, \kappa \geq 1, \\ -\infty, & \text{if } \kappa_1 = 0, \kappa \leq -1. \end{cases} \tag{20}$$

Observe that the previous definition has sense because

$$\lim_{\kappa_1 \to 0^+} \tilde{\kappa} = \lim_{\kappa_1 \to 0^+} \frac{\kappa(\kappa_1+\kappa_2)}{\kappa_1} = \begin{cases} +\infty, & \text{if } \kappa \geq 1, \\ -\infty, & \text{if } \kappa \leq -1. \end{cases}$$

Thus, we have $\varphi(\tilde{\kappa}) = \varphi(\pm\infty) = \pm\infty$ when $\kappa_1 = 0$.

Theorem 3.2. *The coefficients $a_n = a_n(\kappa, \kappa_1, \kappa_2, \kappa_3)$ in (16) satisfy*

$$a(\tilde{\kappa}) := \lim_{n\to\infty} a_n = -\frac{1}{2\varphi(\tilde{\kappa})}, \tag{21}$$

where $\tilde{\kappa}$ is given in (20).

Proof. Notice that when $\kappa_1 = 0$, i.e., the coherence case, we have to prove that $\lim_n a_n = 0$ according to the previous comments, but this was proved in [12]. Then, we are going to prove the result for $\kappa_1 > 0$. From previous theorem and (17) we get

$$b_n(\kappa)[b_n(\kappa) - a_n] \geq 0 \quad \text{and} \quad |a_n| \leq \frac{\gamma_n^{(\alpha,\beta)}(\kappa_1)}{\gamma_n^{(\alpha,\beta)}(\kappa_1) + n^2 \kappa_2 \rho_{n-1}^{(\alpha,\beta,\kappa,\kappa_3)}} |b_n(\kappa)|.$$

If $a(\tilde{\kappa}) = \lim_{n\to\infty} a_n$ exists, then from the inequality in Theorem 3.1, Eqs. (11) and (15), and Lemma 2.2,

$$0 \leq |a(\tilde{\kappa})| \leq \frac{\kappa_1}{\kappa_1 - 4\kappa_2 \kappa b(\kappa)} |b(\kappa)|. \tag{22}$$

From (18), we obtain

$$a_n = \frac{\eta_n \alpha_n^{(\alpha+1,\beta+1)}}{\eta_n \alpha_n^{(\alpha+1,\beta+1)} + b_{n-1}(\kappa) \frac{n-1}{n} \left[-\kappa_2 \kappa + \eta_{n-1} \frac{n-1}{n} (b_{n-1}(\kappa) - a_{n-1}) \right]} b_n(\kappa), \tag{23}$$

where $\eta_n = \nu_n(\kappa_1)/n^2$. Observe that, when $n \to \infty$, $\eta_n \to \kappa_1$ and, from (8), $\alpha_n^{(\alpha+1,\beta+1)} \to 1/4$.

Thus, if $a(\tilde{\kappa}) = \lim_{n\to\infty} a_n$ exists, then from (23),

$$a^2(\tilde{\kappa}) - \left[\frac{1}{4b(\kappa)} + b(\kappa) - \frac{\kappa_2 \kappa}{\kappa_1} \right] a(\tilde{\kappa}) + \frac{1}{4} = 0. \tag{24}$$

The two possible values for $b(\kappa)$ given in Lemma 2.2 lead to

$$a^2(\tilde{\kappa}) + \kappa\left(1 + \frac{\kappa_2}{\kappa_1}\right) a(\tilde{\kappa}) + \frac{1}{4} = 0.$$

Hence, choosing the solution that satisfies the restriction given in (22), we obtain

$$a(\tilde{\kappa}) = -\frac{1}{2}\left[\varphi\left(\frac{\kappa(\kappa_1+\kappa_2)}{\kappa_1}\right)\right]^{-1}.$$

We now confirm that $\lim_{n\to\infty} a_n = a(\tilde{\kappa})$ as given before. From (23), we obtain

$$\begin{aligned}
|a_n - a(\tilde{\kappa})| &\le \left|\frac{\eta_n\alpha_n^{(\alpha+1,\beta+1)}(b_n(\kappa)-a(\tilde{\kappa})) + \frac{n-1}{n}\vartheta_n b_{n-1}(\kappa)a(\tilde{\kappa})}{\eta_n\alpha_n^{(\alpha+1,\beta+1)} + b_{n-1}(\kappa)\frac{n-1}{n}\left[-\kappa_2\kappa + \eta_{n-1}\frac{n-1}{n}(b_{n-1}(\kappa)-a_{n-1})\right]}\right| \\
&\quad + \left|\frac{\eta_{n-1}b_{n-1}(\kappa)a(\tilde{\kappa})(a_{n-1}-a(\tilde{\kappa}))}{\eta_n\,\alpha_n^{(\alpha+1,\beta+1)} + b_{n-1}(\kappa)\frac{n-1}{n}\left[-\kappa_2\kappa + \eta_{n-1}\frac{n-1}{n}(b_{n-1}(\kappa)-a_{n-1})\right]}\right| \\
&\le \left|\frac{\eta_n\,\alpha_n^{(\alpha+1,\beta+1)}(b_n(\kappa)-a(\tilde{\kappa})) + \frac{n-1}{n}\,\vartheta_n b_{n-1}(\kappa)a(\tilde{\kappa})}{\eta_n\alpha_n^{(\alpha+1,\beta+1)} - \frac{n-1}{n}\,\kappa_2\kappa b_{n-1}(\kappa)}\right| \\
&\quad + \left|\frac{\eta_{n-1}b_{n-1}(\kappa)a(\tilde{\kappa})}{\eta_n\alpha_n^{(\alpha+1,\beta+1)} - \frac{n-1}{n}\kappa_2\kappa b_{n-1}(\kappa)}\right| |a_{n-1} - a(\tilde{\kappa})|,
\end{aligned}$$

where $\vartheta_n = \kappa_2\kappa - \frac{n-1}{n}\eta_{n-1}(b_{n-1}(\kappa) - a(\tilde{\kappa}))$.

We have used $b_n(\kappa)[b_n(\kappa) - a_n] \ge 0$ to prove the last inequality. From (24),

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\eta_n\alpha_n^{(\alpha+1,\beta+1)}(b_n(\kappa)-a(\tilde{\kappa})) + \frac{n-1}{n}\,\vartheta_n b_{n-1}(\kappa)a(\tilde{\kappa})\right\} \\
&\quad = \kappa_1 b(\kappa)\left[a^2(\tilde{\kappa}) - \left(\frac{1}{4b(\kappa)} - \frac{\kappa_2\kappa}{\kappa_1} + b(\kappa)\right)a(\tilde{\kappa}) + \frac{1}{4}\right] = 0.
\end{aligned}$$

Therefore,

$$\limsup |a_n - a(\tilde{\kappa})| \le \left|\frac{\kappa_1}{\kappa_1 - 4\kappa_2\kappa b(\kappa)}\right| |4b(\kappa)a(\tilde{\kappa})| \limsup |a_{n-1} - a(\tilde{\kappa})|.$$

Thus, the convergence of a_n is established if we prove that $\left|\frac{\kappa_1}{\kappa_1-4\kappa_2\kappa b(\kappa)}\right| |4b(\kappa)a(\tilde{\kappa})| < 1$. Since $\mathrm{sgn}(b(\kappa)) = -\mathrm{sgn}(\kappa)$, $\left|\frac{\kappa_1}{\kappa_1-4\kappa_2\kappa b(\kappa)}\right| < 1$. Now,

$$|4b(\kappa)a(\tilde{\kappa})| = \begin{cases} \left|\dfrac{\varphi(\kappa)}{\varphi(\tilde{\kappa})}\right|, & \text{if } \kappa_3 > 0, \\[2ex] \left|\dfrac{1}{\varphi(\kappa)\varphi(\tilde{\kappa})}\right|, & \text{if } \kappa_3 = 0. \end{cases}$$

Since $|\tilde{\kappa}| > |\kappa| \ge 1$, then $|\varphi(\tilde{\kappa})| > |\varphi(\kappa)| \ge 1$, and thus $|4b(\kappa)a(\tilde{\kappa})| < 1$. Hence, the theorem is proved. $\square$

Using the expression (17) together with the results in (11), Lemma 2.2 and Theorem 3.2, since $b_n(\kappa) = -[(n+1)\rho_n^{(\alpha,\beta,\kappa,\kappa_3)}]/[n\kappa\rho_{n-1}^{(\alpha+1,\beta+1)}]$, we obtain

$$\lim_{n\to\infty} \frac{n^2 \rho_n^{(\alpha,\beta,\kappa,\kappa_3)}}{\rho_n^S} = \frac{\kappa}{2\kappa_1 \varphi(\tilde{\kappa})}.$$

Theorem 3.3. *For $\kappa_1 \geq 0$, we get the following relative asymptotics between the Sobolev polynomials $\mathcal{S}_n$ and Jacobi polynomials*

$$\lim_{n\to\infty} \frac{\mathcal{S}_n(x)}{P_n^{(\alpha,\beta)}(x)} = \begin{cases} \dfrac{\varphi(x)-\varphi(\kappa)}{\varphi(x)-1/\varphi(\tilde{\kappa})}, & \text{if } \kappa_3 > 0, \\ \dfrac{\varphi(x)-1/\varphi(\kappa)}{\varphi(x)-1/\varphi(\tilde{\kappa})}, & \text{if } \kappa_3 = 0, \end{cases} \tag{25}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$.

Proof. From the recurrence relation (16), we can write

$$f_{n+1}(x) = 1 + g_n(x) + h_n(x) f_n(x), \tag{26}$$

where

$$f_n(x) = \frac{\mathcal{S}_n(x)}{P_n^{(\alpha,\beta)}(x)}, \qquad g_n(x) = b_n(\kappa)\frac{P_n^{(\alpha,\beta)}(x)}{P_{n+1}^{(\alpha,\beta)}(x)} \quad \text{and} \quad h_n(x) = -a_n \frac{P_n^{(\alpha,\beta)}(x)}{P_{n+1}^{(\alpha,\beta)}(x)}$$

are analytic functions on $\mathbb{C} \setminus [-1, 1]$. Then, using Theorem 3.2, and Eqs. (12) and (19), we get

$$\lim_{n\to\infty} g_n(x) = g(x) = \frac{2b(\kappa)}{\varphi(x)} \quad \text{and} \quad \lim_{n\to\infty} h_n(x) = h(x) = -\frac{2a(\tilde{\kappa})}{\varphi(x)} = \frac{1}{\varphi(x)\varphi(\tilde{\kappa})}, \tag{27}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$.

Let K be an arbitrary compact subset of $\mathbb{C} \setminus [-1, 1]$. Notice that $|h(x)| < 1$ and that $|g(x)|$ is also bounded for all $x \in \mathbb{C} \setminus [-1, 1]$. Hence there exist positive constants $M_0 < 1$ and M_1, and a positive integer N such that for all $n \geq N$,

$$|h_n(x)| \leq M_0 < 1 \quad \text{and} \quad |g_n(x)| \leq M_1 \quad \text{when} \quad x \in K.$$

Notice that N depends on K but M_0 and M_1 do not. Hence from (26),

$$|f_{N+1}(x)| \leq 1 + M_1 + M_0 |f_N(x)|,$$
$$\vdots$$
$$|f_{N+i}(x)| \leq \frac{(1+M_1)(1-M_0^i)}{1-M_0} + M_0^i |f_N(x)| < \frac{1+M_1}{1-M_0} + |f_N(x)|.$$

Therefore, f_n is uniformly bounded on K, and since K is an arbitrary compact subset of $\mathbb{C} \setminus [-1, 1]$ we get that the sequence f_n is uniformly bounded on compact subsets of $\mathbb{C} \setminus [-1, 1]$.

Now, we show that the sequence $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$. First, if the limit $f(x)$ exists, then from (26) it satisfies

$$f(x) = 1 + g(x) + h(x) f(x). \tag{28}$$

Thus, from (27),

$$f(x) = \frac{\varphi(x) + 2b(\kappa)}{\varphi(x) + 2a(\tilde{\kappa})}. \tag{29}$$

Using (26) and (28),

$$|f_{n+1}(x) - f(x)| \leq |g_n(x) - g(x)| + |h_n(x) - h(x)|\,|f_n(x)| + |h(x)|\,|f_n(x) - f(x)|,$$

and since f_n is uniformly bounded on compact subsets of $\mathbb{C} \setminus [-1, 1]$, we get

$$\limsup_{n\to\infty} |f_{n+1}(x) - f(x)| \leq M_0 \limsup_{n\to\infty} |f_n(x) - f(x)|.$$

Since $0 < M_0 < 1$, the convergence of $f_n(x)$ to $f(x)$ follows. Finally, it only remains to use Lemma 2.2, Theorem 3.2, and (29) to prove the theorem. □

Since we can write

$$\frac{\mathcal{S}_n(x)}{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)} = \frac{\mathcal{S}_n(x)}{P_n^{(\alpha,\beta)}(x)} \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)},$$

using the above theorem and (14) we obtain:

Corollary 3.1. *For $\kappa_1 \geq 0$ and $\kappa_3 \geq 0$, it holds*

$$\lim_{n\to\infty} \frac{\mathcal{S}_n(x)}{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)} = \frac{2}{\varphi'(x)} \frac{1}{1 - 1/(\varphi(x)\varphi(\tilde{\kappa}))},$$

uniformly on compact subsets of $\mathbb{C} \setminus \operatorname{supp}(\psi^{(\alpha,\beta,\kappa,\kappa_3)})$.

Remark 3.1. Observe that, from this theorem, we can deduce the outer strong asymptotics for the Jacobi–Sobolev orthogonal polynomials taking into account the corresponding one for the classical Jacobi polynomials (see, for example, [15, Th. 8.21.7]).

Remark 3.2. We can recover some results appearing in [12] for coherent pairs of Jacobi type. For example, taking $\kappa_1 = 0$ in Corollary 3.1 we get

$$\lim_{n\to\infty} \frac{\mathcal{S}_n(x)}{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)} = \frac{2}{\varphi'(x)},$$

uniformly on compact subsets of $\mathbb{C} \setminus \operatorname{supp}(\psi^{(\alpha,\beta,\kappa,\kappa_3)})$.

Remark 3.3. The most interesting case is when $\kappa_1 > 0$ because we are not in the framework of coherence. According to the comments in Section 1, when $\kappa_1 > 0$ the two parts, $\langle f, g\rangle_{\psi^{(\alpha,\beta)}} + \kappa_1\langle f', g'\rangle_{\psi^{(\alpha+1,\beta+1)}}$ and $\kappa_2\langle f', g'\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}}$, in the inner product (3) have relevance in the asymptotic behaviour of the Sobolev polynomials. In this sense, we can say that (3) is an equilibrated inner product. We are going to explain this affirmation. Of course, the ideas exposed here are different from the method developed in [1] and later in [2], but we can establish a relation between Theorem 3.3 and Theorem 1 in [1].

In [1] the authors consider the varying Sobolev inner product

$$\langle p, q\rangle_S = \int pq\,\mathrm{d}\mu_0 + \lambda_n \int p'q'\,\mathrm{d}\mu_1,$$

where (μ_0, μ_1) is a coherent pair and λ_n is a decreasing sequence of real positive numbers satisfying some properties, specially $\lim_{n\to\infty} n^2\lambda_n = L \in [0, +\infty]$. To equilibrate the inner product we choose $L \in (0, +\infty)$. They denote by P_n and T_n the orthogonal polynomials with respect to the measure μ_0 and μ_1, respectively, by Q_{n,λ_n} the Sobolev orthogonal polynomials, and

$$\pi_n = \int P_n^2 \mathrm{d}\mu_0, \qquad k_n(\lambda_n) = \langle Q_{n,\lambda_n}, Q_{n,\lambda_n}\rangle_S.$$

Then, they prove in Theorem 1 that

$$\lim_{n\to\infty} \frac{Q_{n,\lambda_n}(x)}{P_n(x)} = \frac{\Psi(x)}{k(L)\,\Psi(x) + (1-k(L))\varphi'(x)/2},$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$, where

$$k(L) := \lim_{n\to\infty} \frac{\pi_n}{k_n(\lambda_n)} \in [0, 1], \qquad \Psi(x) = \lim_{n\to\infty} \frac{T_n(x)}{P_n(x)} = \frac{\varphi'(x)}{2}\left(1 - \frac{2\sigma}{\varphi(x)}\right),$$

with σ being a constant. Thus, the above limit can be rewritten after straightforward computations as

$$\lim_{n\to\infty} \frac{Q_{n,\lambda_n}(x)}{P_n(x)} = \frac{\varphi(x) - 2\sigma}{\varphi(x) - 2\sigma k(L)}, \tag{30}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1, 1]$. Observe that this asymptotic behaviour is analogous to the one obtained in Theorem 3.3. Furthermore, it is possible to establish a relation between the constants in (25) and (30). We can observe that the asymptotic behaviour of the norms

$$k(L) = \lim_{n\to\infty} \frac{\pi_n}{k_n(\lambda_n)} \tag{31}$$

considered in [1, Theorem 1] can be expressed in our framework as

$$\lim_{n\to\infty} \frac{\kappa_1 n^2 \rho_n^{\alpha,\beta}}{\rho_n^S} = \begin{cases} \dfrac{1}{\varphi(\kappa)\varphi(\tilde{\kappa})}, & \text{if } \kappa_3 > 0,\\[2mm] \dfrac{\varphi(\kappa)}{\varphi(\tilde{\kappa})}, & \text{if } \kappa_3 = 0.\end{cases}$$

The factor n^2 now appearing in the above limit is natural since in (31) it is implicitly considered because the simplest choice of λ_n to balance the inner product is $\lambda_n = 1/n^2$.

The key to this situation is the term $\kappa_1\langle f', g'\rangle_{\psi^{(\alpha+1,\beta+1)}}$ which in this case plays a similar role to the one played by the varying sequence λ_n in [1]. However, we point out that the techniques developed to balance Sobolev inner products in [1,2] are more general.

4. Mehler–Heine type asymptotics for Jacobi–Sobolev polynomials

In [4] the authors have shown that under the conditions

$$\kappa_2 \geq 2\kappa_1 \geq 0, \qquad \kappa_3 \geq 0, \qquad \alpha + \beta > 2 \quad \text{and} \quad \begin{cases} \alpha \leq \beta, & \text{if} \kappa \leq -1,\\ \alpha \geq \beta, & \text{if } \kappa \geq 1,\end{cases} \tag{32}$$

the polynomial $\mathcal{S}_n$ has n different real zeros and at least $n-1$ of them lie inside $(-1, 1)$, although numerical experiments lead the authors to conjecture that this happens for all valid values of the parameters. If we denote the zeros of $\mathcal{S}_n$ by $s_{n,i}$, $i = 1, 2, \ldots, n$, in a decreasing order, i.e., $s_{n,n} < s_{n,n-1} < \cdots < s_{n,1}$, then we can have all zeros inside $(-1, 1)$ or $s_{n,1} > 1$ or $s_{n,n} < -1$.

Choosing $x = \kappa$ with $|\kappa| \geq 1$ in (25), since $P_n^{(\alpha,\beta)}(\kappa) \neq 0$, we can conclude that, for $\kappa_3 > 0$,

$$\lim_{n\to\infty} s_{n,1} = \kappa, \quad \text{if } \kappa \geq 1,$$
$$\lim_{n\to\infty} s_{n,n} = \kappa, \quad \text{if } \kappa \leq -1.$$

Mehler–Heine type formulas are a good technique to analyze the asymptotic behaviour of the zeros of families of orthogonal polynomials for which these formulas are known. Mehler–Heine formula for monic Jacobi orthogonal polynomials is (see [15, Th. 8.1.1])

$$\lim_{n\to\infty} \frac{2^n P_n^{(\alpha,\beta)}(\cos(x/(n+j)))}{n^{\alpha+\frac{1}{2}}} = \frac{\sqrt{\pi}}{2^\beta} x^{-\alpha} J_\alpha(x) \tag{33}$$

uniformly on compact subsets of the complex plane $\mathbb{C}$, where $j \in \mathbb{Z}$ is fixed and J_α is the Bessel function of the first kind.

We define the polynomials

$$R_n^{(\alpha,\beta,\kappa,\kappa_3)}(x) := P_n^{(\alpha,\beta)}(x) + b_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa) P_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 1, \tag{34}$$

and $R_0^{(\alpha,\beta,\kappa,\kappa_3)}(x) = 1$. These polynomials appear on the right-hand side of relation (16). From (33) we can obtain a Mehler–Heine type formula for $R_n^{(\alpha,\beta,\kappa,\kappa_3)}$.

Lemma 4.1. *We have for a fixed $j \in \mathbb{Z}$ and $\alpha, \beta > -1$,*

$$\lim_{n\to\infty} \frac{2^n R_n^{(\alpha,\beta,\kappa,\kappa_3)}(\cos(x/(n+j)))}{n^{\alpha+\frac{1}{2}}} = (1 + 2b(\kappa)) \frac{\sqrt{\pi}}{2^\beta} x^{-\alpha} J_\alpha(x),$$

uniformly on compact subsets of $\mathbb{C}$.

Proof. From (34) we get

$$\frac{2^n R_n^{(\alpha,\beta,\kappa,\kappa_3)}(\cos(x/(n+j)))}{n^{\alpha+\frac{1}{2}}} = \frac{2^n P_n^{(\alpha,\beta)}(\cos(x/(n+j)))}{n^{\alpha+\frac{1}{2}}} + 2b_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa) \times \frac{(n-1)^{\alpha+\frac{1}{2}}}{n^{\alpha+\frac{1}{2}}} \frac{2^{n-1} P_{n-1}^{(\alpha,\beta)}(\cos(x/(n+j)))}{(n-1)^{\alpha+\frac{1}{2}}}.$$

The result follows from (19) and (33). □

Later we will need the following technical result that has been proved in [3].

Lemma 4.2. *Let $\{c_n\}_{n=0}^\infty$ be a sequence of real numbers such that $\lim_{n\to\infty} c_n = c$ and $|c| < 1$. For $n \geq 0$, and $i = 1, 2, \ldots, n$, let $t_i^{(n)} = \prod_{j=1}^i c_{n-j}$ and $t_0^{(n)} = 1$. Then, there exist constants P and r, with $P > 1$ and $0 < r < 1$, such that $|t_i^{(n)}| < P r^i$ for all $n \geq 0$ and $0 \leq i \leq n$.*

Now we can obtain a Mehler–Heine type formula for Jacobi–Sobolev orthogonal polynomials, $\mathcal{S}_n$.

Theorem 4.1. *We have for* $\alpha, \beta > -1$ *and* $\kappa_1 \geq 0$,

$$\lim_{n\to\infty} \frac{2^n \mathcal{S}_n(\cos(x/n))}{n^{\alpha+\frac{1}{2}}} = \frac{1+2b(\kappa)}{1+2a(\tilde{\kappa})} \frac{\sqrt{\pi}}{2^\beta} x^{-\alpha} J_\alpha(x), \tag{35}$$

uniformly on compact subsets of the complex plane, where $a(\tilde{\kappa})$ *and* $b(\kappa)$ *are given in* (19) *and* (21), *respectively.*

Proof. From (34) we can write $\mathcal{S}_n(x) = R_n^{(\alpha,\beta,\kappa,\kappa_3)}(x) - a_{n-1}\mathcal{S}_{n-1}(x), n \geq 1$. Then, applying this relation recursively we get

$$\mathcal{S}_n(x) = \sum_{i=0}^{n} (-1)^i u_i^{(n)} R_{n-i}^{(\alpha,\beta,\kappa,\kappa_3)}(x), \quad n \geq 1,$$

where $u_i^{(n)} = \prod_{j=1}^{i} a_{n-j}$ for $i = 1, 2, \ldots, n$, and $u_0^{(n)} = 1$.

Thus, we can write

$$\frac{2^n \mathcal{S}_n(\cos(x/n))}{n^{\alpha+\frac{1}{2}}} = \sum_{i=0}^{n} r_{n,i}(\cos(x/n)),$$

where

$$r_{n,i}(\cos(x/n)) := (-1)^i 2^i u_i^{(n)} \frac{(n-i)^{\alpha+\frac{1}{2}}}{n^{\alpha+\frac{1}{2}}} \frac{2^{n-i} R_{n-i}^{(\alpha,\beta,\kappa,\kappa_3)}(\cos(x/n))}{(n-i)^{\alpha+\frac{1}{2}}}.$$

We use Theorem 3.2 and Lemma 4.2 with $t_i^{(n)} = 2^i u_i^{(n)}$ and we get

$$|2^i u_i^{(n)}| = \left| 2^i \prod_{j=1}^{i} a_{n-j} \right| = |2a_{n-1} 2a_{n-2} \cdots 2a_{n-i}| < Pr^i, \tag{36}$$

where $0 < r < 1$ and $P > 1$. Then, given a compact subset $K \subset \mathbb{C}$, from (36) and Lemma 4.1, there exists a constant $\tilde{P}$, depending only on K, such that

$$|r_{n,i}(\cos(x/n))| < \tilde{P} r^i, \quad i = 0, 1, \ldots, n \text{ and } x \in K.$$

This result allows us to use the Lebesgue dominated convergence theorem and we obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{2^n \mathcal{S}_n(\cos(x/n))}{n^{\alpha+1/2}} &= \sum_{i=0}^{\infty} \lim_{n\to\infty} r_{n,i}(\cos(x/n)) \\ &= (1+2b(\kappa)) \frac{\sqrt{\pi}}{2^\beta} x^{-\alpha} J_\alpha(x) \sum_{i=0}^{\infty} (-2a(\tilde{\kappa}))^i, \end{aligned}$$

from where the result follows. □

Remark 4.1. According to (19) when $\kappa = 1$ we obtain $\lim_{n\to\infty} d_{n-1}^{(\alpha,\beta,\kappa_3)}(1) = \lim_{n\to\infty} b_n^{(\alpha,\beta,\kappa_3)}(1) = b(1) = -1/2$. In this case the above theorem does not provide asymptotic information because the value of the limits in the previous theorem is equal to zero. On the other hand, when $\kappa \neq 1$ Theorem 4.1 is useful to obtain the asymptotic behaviour of the largest zeros of these polynomials. Applying Hurwitz's Theorem in (35) we get the following result.

Corollary 4.1. *Under the assumptions (32) with $\kappa \notin (-1, 1]$, let $s_{n,m} < s_{n,m-1} < \cdots < s_{n,2} < s_{n,1}$ be the m zeros of $\mathcal{S}_n$ inside $(-1, 1)$. Then,*

$$\lim_{n\to\infty} n \arccos(s_{n,i}) = j_i^{(\alpha)},$$

where $0 < j_1^{(\alpha)} < j_2^{(\alpha)} < \cdots < j_m^{(\alpha)}$ denote the first m positive zeros of Bessel function of the first kind J_α.

4.1. Mehler–Heine type asymptotics for Jacobi–Sobolev orthogonal polynomials when $\kappa = 1$

As we have just observed, we need to find another Mehler–Heine type formula for $\mathcal{S}_n$ when $\kappa = 1$. First of all we need to find a Mehler–Heine type formula for the polynomial $P_n^{(\alpha,\beta,1,\kappa_3)}$ orthogonal with respect to the inner product

$$\langle f, g\rangle_{\psi^{(\alpha,\beta,1,\kappa_3)}} = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta+1}\mathrm{d}x + \kappa_3 f(1)g(1),$$

with $\kappa_3 \geq 0$ and $\alpha, \beta > -1$.

Using (10) and (13) we can write

$$P_n^{(\alpha,\beta,1,\kappa_3)}(x) = P_n^{(\alpha,\beta+1)}(x) + A_n^{(\alpha,\beta,\kappa_3)}(1)P_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n \geq 1, \tag{37}$$

where

$$A_n^{(\alpha,\beta,\kappa_3)}(1) = \frac{2n(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} + d_{n-1}^{(\alpha,\beta,\kappa_3)}(1) := \frac{B_n^{(\alpha,\beta,\kappa_3)}(1)}{C_n^{(\alpha,\beta,\kappa_3)}(1)}. \tag{38}$$

But now, for any value of κ, using (13), we can rewrite $d_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa)$ as

$$d_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa) = -\frac{I_n^{(\alpha,\beta)}(\kappa) + \kappa_3 P_n^{(\alpha+1,\beta+1)}(\kappa)}{I_{n-1}^{(\alpha,\beta)}(\kappa) + \kappa_3 P_{n-1}^{(\alpha+1,\beta+1)}(\kappa)}, \quad n \geq 1, \tag{39}$$

with

$$I_n^{(\alpha,\beta)}(\kappa) = \int_{-1}^{1} P_n^{(\alpha+1,\beta+1)}(x)\frac{\kappa}{\kappa - x}(1-x)^{\alpha+1}(1+x)^{\beta+1}\mathrm{d}x. \tag{40}$$

Then, taking into account Eqs. (10), (38) and (40) we get

$$\begin{aligned} B_n^{(\alpha,\beta,\kappa_3)}(1) &= \frac{2n(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} I_{n-1}^{(\alpha,\beta)}(1) - I_n^{(\alpha,\beta)}(1) \\ &\quad + \kappa_3 \left\{ \frac{2n(n+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} - P_n^{(\alpha+1,\beta+1)}(1) \right\} \\ &= -\kappa_3 P_n^{(\alpha,\beta+1)}(1), \end{aligned} \tag{41}$$

$$C_n^{(\alpha,\beta,\kappa_3)}(1) = I_{n-1}^{(\alpha,\beta)}(1) + \kappa_3 P_{n-1}^{(\alpha+1,\beta+1)}(1). \tag{42}$$

Now, substituting $P_n^{(\alpha+1,\beta+1)}(x)$ by its Rodrigues' formula (7) in the integral in (40) and integrating by parts n times, we obtain for $\kappa = 1$

$$\begin{aligned} I_n^{(\alpha,\beta)}(1) &= n!2^{n+\alpha+\beta+2}\frac{\Gamma(n+\alpha+\beta+3)}{\Gamma(2n+\alpha+\beta+3)}\int_0^1 (1-t)^{\alpha}t^{n+\beta+1}\mathrm{d}t \\ &= n!\,2^{n+\alpha+\beta+2}\frac{\Gamma(n+\alpha+\beta+3)}{\Gamma(2n+\alpha+\beta+3)}\mathcal{B}(n+\beta+2,\alpha+1), \end{aligned}$$

where $\mathcal{B}(x, y)$ is the well-known beta function. Using that $\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ (see [15]), we can write

$$I_n^{(\alpha,\beta)}(1) = n!\,2^{n+\alpha+\beta+2}\frac{\Gamma(\alpha+1)\Gamma(n+\beta+2)}{\Gamma(2n+\alpha+\beta+3)}.$$

From (9) and previous equation we obtain

$$\lim_{n\to\infty}\frac{I_{n-1}^{(\alpha,\beta)}(1)}{P_{n-1}^{(\alpha+1,\beta+1)}(1)} = 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{nP_n^{(\alpha,\beta+1)}(1)}{P_{n-1}^{(\alpha+1,\beta+1)}(1)} = \alpha+1. \tag{43}$$

Now, we are ready to prove the following Mehler–Heine type formula for the polynomial $P_n^{(\alpha,\beta,1,\kappa_3)}$.

Theorem 4.2. *We have for $\alpha > -1$ and $\beta > -1$,*

$$\lim_{n\to\infty}\frac{2^nP_n^{(\alpha,\beta,1,\kappa_3)}(\cos(x/n))}{n^{\alpha+\frac{1}{2}}} = \begin{cases} -\dfrac{\sqrt{\pi}}{2^{\beta+1}}x^{-\alpha}J_{\alpha+2}(x), & \text{if } \kappa_3 > 0,\\ \dfrac{\sqrt{\pi}}{2^{\beta+1}}x^{-\alpha}J_{\alpha}(x), & \text{if } \kappa_3 = 0, \end{cases}$$

uniformly on compact subsets of the complex plane.

Proof. If $\kappa_3 = 0$ we have $P_n^{(\alpha,\beta,1,0)}(x) = P_n^{(\alpha,\beta+1)}(x)$, then $A_n^{(\alpha,\beta,0)}(1) = 0$, and the result follows from (33).

Consider now $\kappa_3 > 0$. From (41) and (42) we can write $A_n^{(\alpha,\beta,\kappa_3)}(1)$ as

$$A_n^{(\alpha,\beta,\kappa_3)}(1) = -\frac{\kappa_3P_n^{(\alpha,\beta+1)}(1)}{I_{n-1}^{(\alpha,\beta)}(1)+\kappa_3P_{n-1}^{(\alpha+1,\beta+1)}(1)} = -\kappa_3\frac{P_n^{(\alpha,\beta+1)}(1)/P_{n-1}^{(\alpha+1,\beta+1)}(1)}{I_{n-1}^{(\alpha,\beta)}(1)/P_{n-1}^{(\alpha+1,\beta+1)}(1)+\kappa_3}.$$

From (43) we obtain

$$\lim_{n\to\infty} nA_n^{(\alpha,\beta,\kappa_3)}(1) = -(\alpha+1). \tag{44}$$

Using (37) we can write

$$\begin{aligned} \frac{2^nP_n^{(\alpha,\beta,1,\kappa_3)}(\cos(x/n))}{n^{\alpha+1/2}} &= \frac{2^nP_n^{(\alpha,\beta+1)}(\cos(x/n))}{n^{\alpha+1/2}} \\ &\quad + 2nA_n^{(\alpha,\beta,\kappa_3)}(1)\left(\frac{n-1}{n}\right)^{\alpha+3/2}\frac{2^{n-1}P_{n-1}^{(\alpha+1,\beta+1)}(\cos(x/n))}{(n-1)^{\alpha+3/2}}. \end{aligned}$$

Applying (33) and (44) we get

$$\lim_{n\to\infty}\frac{2^nP_n^{(\alpha,\beta,1,\kappa_3)}(\cos(x/n))}{n^{\alpha+1/2}} = \frac{\sqrt{\pi}}{2^{\beta+1}}\,x^{-\alpha}\left(J_\alpha(x) - 2(\alpha+1)x^{-1}J_{\alpha+1}(x)\right).$$

The result of the theorem follows using the property of Bessel function $J_{\alpha+2}(x) + J_\alpha(x) = 2(\alpha+1)x^{-1}J_{\alpha+1}(x)$ (see [15, p. 15]). □

Considering (13) and (34) we can rewrite $R_n^{(\alpha,\beta,1,\kappa_3)}$ in terms of the polynomials $P_n^{(\alpha-1,\beta-1,1,\kappa_3)}$ and $P_{n-1}^{(\alpha,\beta)}$ as follows

$$R_n^{(\alpha,\beta,1,\kappa_3)}(x) = P_n^{(\alpha-1,\beta-1,1,\kappa_3)}(x) + \left[b_{n-1}^{(\alpha,\beta,\kappa_3)}(1) - d_{n-1}^{(\alpha-1,\beta-1,\kappa_3)}(1)\right] P_{n-1}^{(\alpha,\beta)}(x),$$

with $\alpha > 0$ and $\beta > -1$ since $\langle f, g\rangle_{\psi^{(\alpha,\beta,1,\kappa_3)}} = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha-1}(1+x)^\beta \mathrm{d}x + \kappa_3 f(1)g(1)$. Therefore, we have

$$\begin{aligned}\frac{2^n R_n^{(\alpha,\beta,1,\kappa_3)}(\cos(x/n))}{n^{\alpha-1/2}} &= \frac{2^n P_n^{(\alpha-1,\beta-1,1,\kappa_3)}(\cos(x/n))}{n^{\alpha-1/2}} \\ &\quad + 2nD_{n-1}^{(\alpha,\beta,\kappa_3)}(1)\left(\frac{n-1}{n}\right)^{\alpha+1/2} \frac{2^{n-1}P_{n-1}^{(\alpha,\beta)}(\cos(x/n))}{(n-1)^{\alpha+1/2}},\end{aligned}$$

with $D_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa) = b_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa) - d_{n-1}^{(\alpha-1,\beta-1,\kappa_3)}(\kappa)$. To obtain a Mehler–Heine type formula for the polynomials $R_n^{(\alpha,\beta,1,\kappa_3)}$ we need to know the behaviour of $nD_n^{(\alpha,\beta,\kappa_3)}(1)$ when n tends to infinity. For this purpose let us obtain $\lim_{n\to\infty} n\left[1 + 2d_{n-1}^{(\alpha,\beta,\kappa_3)}(1)\right]$.

Using (9), (39) and (40), we get

$$\begin{aligned}1 + 2d_{n-1}^{(\alpha,\beta,\kappa_3)}(1) &= \frac{I_{n-1}^{(\alpha,\beta)}(1) - 2I_n^{(\alpha,\beta)}(1) + \kappa_3\left[P_{n-1}^{(\alpha+1,\beta+1)}(1) - 2P_n^{(\alpha+1,\beta+1)}(1)\right]}{I_{n-1}^{(\alpha,\beta)}(1) + \kappa_3 P_{n-1}^{(\alpha+1,\beta+1)}(1)} \\ &= \frac{\tilde{A}_n^{(\alpha,\beta)} I_{n-1}^{(\alpha,\beta)}(1)/P_{n-1}^{(\alpha+1,\beta+1)}(1) + \kappa_3 \tilde{B}_n^{(\alpha,\beta)}}{I_{n-1}^{(\alpha,\beta)}(1)/P_{n-1}^{(\alpha+1,\beta+1)}(1) + \kappa_3}, \quad n \geq 1,\end{aligned}$$

with

$$\begin{aligned}\tilde{A}_n^{(\alpha,\beta)} &= 1 - \frac{4n(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \\ \tilde{B}_n^{(\alpha,\beta)} &= 1 - \frac{4(n+\alpha+1)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}.\end{aligned}$$

It is clear that

$$\lim_{n\to\infty} n\tilde{A}_n^{(\alpha,\beta)} = \alpha + 1/2 \quad \text{and} \quad \lim_{n\to\infty} n\tilde{B}_n^{(\alpha,\beta)} = -(\alpha+3/2).$$

From this and (43) it immediately follows that

$$\lim_{n\to\infty} n\left(1 + 2d_{n-1}^{(\alpha,\beta,\kappa_3)}(1)\right) = \begin{cases} -(\alpha+3/2), & \text{if } \kappa_3 > 0, \\ \alpha + 1/2, & \text{if } \kappa_3 = 0. \end{cases}$$

In the same way, we get

$$\lim_{n\to\infty} n\left(1 + 2b_n^{(\alpha,\beta,\kappa_3)}(1)\right) = \begin{cases} -(\alpha+5/2), & \text{if } \kappa_3 > 0, \\ \alpha - 1/2, & \text{if } \kappa_3 = 0, \end{cases}$$

and, finally, we obtain

$$\lim_{n\to\infty} n D_{n-1}^{(\alpha,\beta,\kappa_3)}(1) = \lim_{n\to\infty} n\left(b_{n-1}^{(\alpha,\beta,\kappa_3)}(1) - d_{n-1}^{(\alpha-1,\beta-1,\kappa_3)}(1)\right) = \begin{cases} -1, & \text{if } \kappa_3 > 0, \\ 0, & \text{if } \kappa_3 = 0. \end{cases}$$

Thus, we can obtain a Mehler–Heine type formula for the polynomials $R_n^{(\alpha,\beta,1,\kappa_3)}$.

Theorem 4.3. *For $\alpha > 0$, $\beta > -1$, it holds*

$$\lim_{n\to\infty} \frac{2^n R_n^{(\alpha,\beta,1,\kappa_3)}(\cos(x/n))}{n^{\alpha-\frac{1}{2}}} = \begin{cases} -\dfrac{\sqrt{\pi}}{2^\beta}\, x^{-\alpha}\left[x J_{\alpha+1}(x) + 2J_\alpha(x)\right], & \text{if } \kappa_3 > 0, \\ \dfrac{\sqrt{\pi}}{2^\beta} x^{-(\alpha-1)} J_{\alpha-1}(x), & \text{if } \kappa_3 = 0, \end{cases}$$

uniformly on compact subsets of the complex plane.

Now, using the above theorem and the same arguments as in the proof of Theorem 4.1, we can obtain a Mehler–Heine type formula for the Jacobi–Sobolev orthogonal polynomials, $\mathcal{S}_n$, when $\kappa = 1$.

Theorem 4.4. *For $\alpha > 0$, $\beta > -1$ and $\kappa_1 \geq 0$, uniformly on compact subsets of the complex plane it holds*

- *for $\kappa_3 > 0$,*

$$\lim_{n\to\infty} \frac{2^n \mathcal{S}_n(\cos(x/n))}{n^{\alpha-\frac{1}{2}}} = -\frac{1}{1+2a(\tilde{\kappa})} \frac{\sqrt{\pi}}{2^\beta} x^{-\alpha}\left[x J_{\alpha+1}(x) + 2J_\alpha(x)\right],$$

- *for $\kappa_3 = 0$,*

$$\lim_{n\to\infty} \frac{2^n \mathcal{S}_n(\cos(x/n))}{n^{\alpha-\frac{1}{2}}} = \frac{1}{1+2a(\tilde{\kappa})} \frac{\sqrt{\pi}}{2^\beta}\, x^{-(\alpha-1)} J_{\alpha-1}(x),$$

where $a(\tilde{\kappa})$ is given in (21).

When $\kappa = 1$ the above theorem is useful to obtain the asymptotic behaviour of the largest zeros of the polynomials $\mathcal{S}_n$. Applying Hurwitz's Theorem we get the following result.

Corollary 4.2. *Under the assumptions (32) and $\kappa = 1$, let $s_{n,m} < s_{n,m-1} < \cdots < s_{n,2} < s_{n,1}$ be the m zeros of $\mathcal{S}_n$ inside $(-1, 1)$. Then,*

- *if $\kappa_3 > 0$ then*

$$\lim_{n\to\infty} n \arccos(s_{n,i}) = h_i^{(\alpha)},$$

where $0 < h_1^{(\alpha)} < h_2^{(\alpha)} < \cdots < h_m^{(\alpha)}$ denote the first m positive zeros of the function $H_\alpha(x) = x J_{\alpha+1}(x) + 2J_\alpha(x)$,

- *if $\kappa_3 = 0$ then*

$$\lim_{n\to\infty} n \arccos(s_{n,i}) = j_i^{(\alpha-1)},$$

where $0 < j_1^{(\alpha-1)} < j_2^{(\alpha-1)} < \cdots < j_m^{(\alpha-1)}$ denote the first m positive zeros of Bessel function $J_{\alpha-1}(x)$.

5. Inner strong asymptotics for Jacobi–Sobolev orthogonal polynomials

In this section we establish the asymptotics for the Jacobi–Sobolev polynomials $\mathcal{S}_n$ inside $(-1,1)$. For this purpose we need some properties of monic Jacobi polynomials that we can find, for example, in [8] or [15].

Proposition 5.1. *Let α, β be arbitrary real numbers, $\theta \in (0,\pi)$ and $N = n + (\alpha+\beta+1)/2$. Then, the Jacobi orthogonal polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy*

- $$2^n P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\sqrt{\pi}}{2^{\alpha+\beta}} \sigma_n^{(\alpha,\beta)} k(\theta)\cos(N\theta+\gamma) + O(n^{-1}), \tag{45}$$

  where

  $$\sigma_n^{(\alpha,\beta)} = \frac{2^{2n+\alpha+\beta}\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}{\sqrt{n\pi}\,\Gamma(2n+\alpha+\beta+1)},$$

  $$k(\theta) = \pi^{-\frac{1}{2}} \left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}} \quad \text{and} \quad \gamma = -\frac{\left(\alpha+\frac{1}{2}\right)\pi}{2}.$$

- *If $\alpha > -1$,*

  $$\frac{2^n P_n^{(\alpha,\beta)}(\cos\theta)}{\sqrt{n}} = \frac{\sqrt{\pi}}{2^{\alpha+\beta}}\, \delta_n^{(\alpha,\beta)} k_1(\theta) J_\alpha(N\theta) + O(n^{-\frac{3}{2}}), \tag{46}$$

  where

  $$\delta_n^{(\alpha,\beta)} = \frac{2^{2n+\alpha+\beta}\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)N^{-\alpha}}{\sqrt{n\pi}\,\Gamma(2n+\alpha+\beta+1)},$$

  and

  $$k_1(\theta) = \left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}} \left(\sin\frac{\theta}{2}\right)^{-\alpha} \left(\cos\frac{\theta}{2}\right)^{-\beta}.$$

Both formulas hold uniformly in $[\varepsilon, \pi-\varepsilon]$ with $\varepsilon > 0$.

Remark 5.1. It is important to observe that

$$\lim_{n\to\infty} \sigma_n^{(\alpha,\beta)} = \lim_{n\to\infty} \delta_n^{(\alpha,\beta)} = 1.$$

From (34), we have for $\theta \in (0,\pi)$

$$2^n R_n^{(\alpha,\beta,\kappa,\kappa_3)}(\cos\theta) = 2^n P_n^{(\alpha,\beta)}(\cos\theta) + 2b_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa) 2^{n-1} P_{n-1}^{(\alpha,\beta)}(\cos\theta), \quad n \ge 1,$$

and using (19) and (45) we get $2^n R_n^{(\alpha,\beta,\kappa,\kappa_3)}(\cos\theta)$ is uniformly bounded in $(0,\pi)$. Thus, taking into account (19), (45) and (46) we obtain

$$\begin{aligned}\frac{2^n R_n^{(\alpha,\beta,\kappa,\kappa_3)}(\cos\theta)}{\sqrt{n}} &= \frac{2^n P_n^{(\alpha,\beta)}(\cos\theta)}{\sqrt{n}} + \frac{2b_{n-1}^{(\alpha,\beta,\kappa_3)}(\kappa)}{\sqrt{n}} 2^{n-1} P_{n-1}^{(\alpha,\beta)}(\cos\theta) \\ &= \frac{\sqrt{\pi}}{2^{\alpha+\beta}} \delta_n^{(\alpha,\beta)} k_1(\theta) J_\alpha(N\theta) + O(n^{-\frac{1}{2}}),\end{aligned} \tag{47}$$

uniformly in $[\varepsilon, \pi-\varepsilon]$ with $\varepsilon > 0$. Thus, we establish

Theorem 5.1. *Under the assumptions (6) it holds*

$$\frac{2^n \mathcal{S}_n(\cos\theta)}{\sqrt{n}} = \frac{\sqrt{\pi}}{2^{\alpha+\beta}}\, \delta_n^{(\alpha,\beta)} k_1(\theta) J_\alpha(N\theta) + O(n^{-\frac{1}{2}}),$$

uniformly in $[\varepsilon, \pi-\varepsilon]$ *with* $\varepsilon > 0$*, where* $\delta_n^{(\alpha,\beta)}$, $k_1(\theta)$*, and* N *are given in Proposition 5.1 with* $\lim_{n\to\infty} \delta_n^{(\alpha,\beta)} = 1$.

Proof. We proceed as in Theorem 4.1 proving that $2^n \mathcal{S}_n(\cos\theta)$ is uniformly bounded in $(0,\pi)$ and taking into account the inner strong asymptotics for the polynomials $R_n^{(\alpha,\beta,\kappa,\kappa_3)}$ given in (47). □

Acknowledgments

We thank the anonymous referees for the revision of the manuscript. Their constructive comments and suggestions helped in improving this article. The first and second authors' work was partially supported by grants from CAPES, CNPq and FAPESP of Brazil. The third and fourth authors' work was partially supported by MICINN of Spain under Grant MTM2008-06689-C02-01 and Junta de Andalucía (FQM229 and P06-FQM-1735).

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