A diffusion defined on a fractal state space

William B. Krebs

Department of Statistics, Florida State University, Tallahassee, FL 32306-3033, USA

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In the plane, we define a fractal known as the Vicsek snowflake in terms of a family of affine contractions in $\mathbb{R}^2$. We show that the Vicsek snowflake is a nested fractal in the sense of Lindström (1990). We define random walks on the Vicsek snowflake and explicitly find an invariant probability for random walk. From this invariant probability, we construct a Brownian motion on the Vicsek snowflake. We show that this Brownian motion is the unique diffusion limit under weak convergence of rescaled random walks with any probability parameter. We show that Brownian motion on the Vicsek snowflake has a scaling property reminiscent of Brownian motion in $\mathbb{R}^1$. Using a coupling argument, we show that our Brownian motion has transition densities with respect to Hausdorff measure on the snowflake.

diffusions * fractals

1. Introduction

Construct a subset of the unit square by the following recursive procedure: Let $\mathcal{G}_0$ denote the unit square. Construct $\mathcal{G}_1$ by deleting from $\mathcal{G}_0$ four squares, each with edge length $\frac{1}{3}$, centered along the four edges of $\mathcal{G}_0$. $\mathcal{G}_1$ will consist of five squares with edge $\frac{1}{3}$ whose corners overlap. For $n = 2, 3, \ldots$, construct $\mathcal{G}_n$ from $\mathcal{G}_{n-1}$, by taking each square $F$ in $\mathcal{G}_{n-1}$, and deleting the four squares centered along the edges of $F$ with edges of length $3^{-n}$. $\mathcal{G}_N$ then consists of $5^n$ squares with edges of length $3^{-n}$.

Take $\Gamma = \bigcap_{n=1}^{\infty} \mathcal{G}_n$. Some easy topology shows that $\Gamma$ is a closed connected set with Lebesgue measure 0. In fact, it is not hard to show that $\Gamma$ has finite Hausdorff log,5-dimensional measure. In the terminology of Mandelbrot, $\Gamma$ is a fractal with starter polygon $\mathcal{G}_1$. Extensive treatments of such fractal sets have been given by various authors. (See, for example, Hutchinson [7] or Barnsley and Demko [4]).

A number of authors have treated the problem of constructing Brownian motion on nested fractals. Particular attention has been paid to Brownian motion on the Sierpinski gasket, a fractal constructed from a unit equilateral triangle by successively deleting 'middle' triangles. Goldstein [6] and Kusuoka [8] constructed a Brownian motion on the Sierpinski gasket, using a decimation-invariance property. Barlow and Perkins [3] have studied this Brownian motion comprehensively. Brownian motion on the Sierpinski gasket is broadly similar to Brownian motion on the Vicsek snowflake, and the results of these authors are generally similar to those in the present work. I fully acknowledge the priority of their results. More recently,
Lindstrøm [9] has constructed a Brownian motion on any fractal set satisfying a general set of nesting axioms from a sequence of random walks, provided that the distribution of the random walk satisfies a non-degeneracy condition.

The first objective of this paper is to construct Brownian motion on the Vicsek snowflake, starting from a non-degenerate random walk model. In an important respect, the problem of defining Brownian motion on the snowflake is more complicated than defining diffusions on the Sierpinski gasket. On the snowflake, one can define a variety of random walk models that are symmetric under the natural symmetries of the square. A natural question is whether one can construct Brownian motion for any such model. Another is whether such a Brownian motion on the fractal is unique. The snowflake seems to be the simplest nested fractal where such questions arise. For the snowflake, the answer is that if the random walk is not degenerate then a unique diffusion limit exists whatever the underlying random walk model. The corresponding problem for general nested fractals remains unsolved at the time of this writing.

A further objective of this paper is to demonstrate the relation between the weak convergence and scaling results and the properties of the diffusion. The author feels that such relationships may be useful in solving more complicated diffusion problems on fractals.

In the first section of this paper, we construct the diffusion on \( I \), following Lindstrøm [9]. We first show that \( I \) satisfies Lindstrøm's nesting axioms. If we treat the polygons \( \mathcal{G}_n \) as graphs, we can then find a random walk \( \{X_t\} \) that is invariant under changes in the scale of our graph. This gives us a Brownian motion on \( I \). A more detailed study of the relationship between random walks on \( \mathcal{G}_n \) for different values of \( n \) shows that this Brownian motion is, in fact, unique.

In the second section of this paper, the embedded random walks are applied to study the properties of the sample paths of our diffusion. One useful consequence of the embedding is that the law of the diffusion possesses a scaling property similar to that of Brownian motion. Another is that the law of the diffusion at time \( t \) converges in total variation to its stationary distribution. Using these properties together, we show that the invariant measure of the process is Hausdorff \( \log_{153} \)-dimensional measure, restricted to \( I \) and that Brownian motion on \( I \) has transition densities with respect to this Hausdorff measure.

2. Constructing the diffusion

Consider the following system of transformations:

\[
M_1: x \mapsto \frac{1}{3}x, \quad M_2: x \mapsto \frac{1}{3}x + (\frac{1}{3}, \frac{2}{3}), \quad M_3: x \mapsto \frac{1}{3}x + (\frac{2}{3}, 0), \\
M_4: x \mapsto \frac{1}{3}x + (0, \frac{2}{3}), \quad M_5: x \mapsto \frac{1}{3}x + (\frac{1}{3}, \frac{1}{3})..
\]

(2.1)

By inspection, \( M_1, \ldots, M_5 \) are strict contractions, with fixed points

\[
x_1 = (0, 0), \quad x_2 = (1, 1), \quad x_3 = (1, 0), \quad x_4 = (0, 1), \quad x_5 = (\frac{1}{3}, \frac{1}{3}).
\]

(2.2)
respectively. For \( i, j \neq 5 \), it is easy to see that \( M_X = M_X \). We say that \( x_1, x_2, x_3 \), and \( x_4 \) are essential fixed points for this system, while \( x_5 \) is an inessential fixed point.

Let \( F \) denote the essential fixed points of the transformations \( \{ M_1, \ldots, M_5 \} \).

For bounded subsets \( A \) of \( \mathbb{R}^2 \), define \( M(A) = \bigcup_{i=1}^5 M_i(A) \). In Hutchinson [7, Theorem 3.13] it was proved that there is a unique compact set in \( \mathbb{R}^2 \) invariant under \( M \). We will call this set the Vicsek snowflake, and denote it by \( I \).

In accordance with Lindstrøm [9, Chapter IV] we establish the following terminology. For any \( A \subset \mathbb{R}^2 \), let \( A^{(n)} = A \) and let \( A^{(n)} = M(A^{(n-1)}) \), \( n = 1, 2, \ldots \). Say that \( F^{(n)} \) is the set of \( n \)-points of \( I \). If \( M_1, \ldots, M_5 \) is any sequence of transformations in \( \{ M_1, \ldots, M_5 \} \), call \( F_{i_1 \ldots i_n} = M_{i_1} \circ \cdots \circ M_{i_n}(F) \) an \( n \)-cell. Say that \( \Gamma_{i_1 \ldots i_n} = M_{i_1} \circ \cdots \circ M_{i_n}(I) \) is the associated \( n \)-complex.

After Lindstrøm [9], we say that \( I \) is a nested fractal if it satisfies the following four conditions:

(i) Any two \( 1 \)-cells \( C \) and \( C' \) are connected by a sequence of \( 1 \)-cells.

(ii) For \( x, y \in F \), let \( l_{xy} \) be the line midway between \( x \) and \( y \), and let \( R_{xy} \) be reflection about \( l_{xy} \). Then \( R_{xy} \) maps \( n \)-cells into \( n \)-cells, and any \( n \)-cell containing points on both sides of \( l_{xy} \) is mapped into itself.

(iii) If \( i_1, \ldots, i_n \) and \( j_1, \ldots, j_n \) are distinct sequences, then \( F_{i_1 \ldots i_n} \neq F_{j_1 \ldots j_n} \) and \( \Gamma_{i_1 \ldots i_n} \cap \Gamma_{j_1 \ldots j_n} = F_{i_1 \ldots i_n} \cap F_{j_1 \ldots j_n} \).

(iv) There exists a bounded open set \( U \) such that \( M_1(U), \ldots, M_5(U) \) are disjoint and \( \bigcup_{i=1}^5 M_i(U) \subset U \).

**Proposition 2.1.** \( I \) is a nested fractal.

**Proof.** Let \( S = [0, 1]^2 \). Then, property (i) is obvious from inspection of Figure 1 (see Section 4).

We will prove (ii) by induction. For \( n = 1 \), property (ii) is obvious from inspection of Figure 1. So, suppose property (ii) holds for \( n = k \), let \( F_{i_1 \cdots i_k} \) be a \( k + 1 \) cell, and let \( x, y \in F \). If \( l_{xy} \) intersects \( M_i(S) \), then it is obvious from Figure 1 that \( l_{xy} = l_{uv} \) with \( u, v \in F_i \). Then, (ii) follows from the inductive hypothesis. Alternatively, suppose that \( l_{xy} \) does not intersect \( M_i(S) \). Further inspection of Figure 1 shows that \( R_{xy} \circ M_i = M_x \), for some \( M_i \in \{ M_1, \ldots, M_5 \} \). Then \( R_{xy}(F_{i_1 \cdots i_k}) = F_{i_1 \cdots i_k} \), and, again (ii) follows.

To prove property (iii), first note that the essential fixed points \( F \) are the corners of \( S \). If \( n = 1 \), property (iii) is also obvious from Figure 1. So, suppose \( n > 1 \), and let \( M_{i_1}, \ldots, M_{i_n} \) and \( M_{j_1}, \ldots, M_{j_n} \) be sequences of the transformations \( M_1, \ldots, M_5 \). Without loss of generality, suppose \( i_1 \neq j_1 \). Then

\[
M_{i_1} \circ \cdots \circ M_{i_n}(S) \cap M_{j_1} \circ \cdots \circ M_{j_n}(S) \subset M_{i_1}(S) \cap M_{j_1}(S) = M_{i_1}(F) \cap M_{j_1}(F).
\]

(2.3)

Unless \( M_{i_2} \circ \cdots \circ M_{i_n}(S) \cap F \neq \emptyset \) and \( M_{j_2} \circ \cdots \circ M_{j_n}(S) \cap F \neq \emptyset \), then

\[
M_{i_1} \circ \cdots \circ M_{i_n}(S) \cap M_{j_1} \circ \cdots \circ M_{j_n}(S) = \emptyset.
\]
But
\[ M_i \circ \cdots \circ M_n(S) \cap F = M_i \circ \cdots \circ M_n(F) \cap F \]
and
\[ M_j \circ \cdots \circ M_n(S) \cap F = M_j \circ \cdots \circ M_n(F) \cap F, \]
so it follows that
\[ M_i \circ \cdots \circ M_n(S) \cap M_j \circ \cdots \circ M_n(S) \]
\[ = M_i \circ \cdots \circ M_n(F) \cap M_j \circ \cdots \circ M_n(F), \]
which is property (iii).

Finally, to show that property (iv) is satisfied, it suffices to observe that property (iii) shows that \( M_i(S^0), \ldots, M_j(S^0) \) are disjoint, while \( \cup_{i=1}^{5} M_i(S^0) \subset S^0 \), where \( S^0 \) denotes the interior of \( S \).

From Hutchinson [7, Theorem 5.3.1] the Hausdorff dimension of \( \Gamma \) may now be computed as \( \log_5 5 \). For the moment, regard \( M(S) \) as a graph, which we will denote by \( \mathcal{U} \). The vertices of \( \mathcal{U} \) will be \( F^{(1)} \). Any two points lying in a common 1-complex \( C \) will be joined by an edge.

As a first step towards defining a diffusion on \( \Gamma \), we will define a random walk \( X \) on \( \mathcal{U} \). For \( n = 0, 1, \ldots \), let \( W_n \) be the \( n \)th vertex visited by the walk. Let \( x \) and \( y \) be vertices of \( \mathcal{U} \), let \( N_x \) be the number of vertices adjacent to \( x \) in \( \mathcal{U} \), and let \( p \) be an arbitrary number in \( [0, 1) \).

Suppose \( W_n = x \). If \( N_x = 3 \), let
\[ P[W_{n+1} = y] = \begin{cases} p & \text{if } x \text{ and } y \text{ are diagonally adjacent,} \\ \frac{1}{3}(1 - p) & \text{if } x \text{ and } y \text{ are vertically or horizontally adjacent;} \end{cases} \]
if \( N_x = 6 \), let
\[ P[W_{n+1} = y] = \begin{cases} \frac{1}{3}p & \text{if } x \text{ and } y \text{ are diagonally adjacent,} \\ \frac{2}{3}(1 - p) & \text{if } x \text{ and } y \text{ are vertically or horizontally adjacent.} \end{cases} \]
To complete the definition of \( X \), we will introduce distributions for the time required to cross edges of \( \mathcal{U} \). Let \( \tau_{vh} \) and \( \tau_d \) be arbitrary distributions on \((0, \infty)\). Set \( T_0 = 0 \) and suppose \( T_1 - T_0, T_2 - T_1, \ldots \) are independent random variables such that
\[ T_n - T_{n-1} \sim \tau_{vh} \text{ if } W_{n-1} \text{ and } W_n \text{ are joined by a vertical or horizontal edge,} \]
\[ T_n - T_{n-1} \sim \tau_d \text{ if } W_{n-1} \text{ and } W_n \text{ are joined by a diagonal edge.} \]
Define the process \( \{X_t\} \) by
\[ X_0 = x, \]
\[ X_t = W_n, \quad T_{n-1} < t \leq T_n, \quad n = 1, 2, \ldots \]
We will call \( \{X_t\} \) a random walk on \( \mathcal{U} \) starting from \( x \), with parameters \((p, \tau_{vh}, \tau_d)\).

In the same fashion, we can define a graph and a random walk on the \( n \)-points of \( \Gamma \) for any \( n \). Let \( X_t^{(n)} \) denote a random walk on \( F^{(n)} \) with arbitrary parameters \((p, \tau_{vh}, \tau_d)\). From the strong Markov property of \( W_j \), it is clear that if \( k \leq n \) and we
observe \( X^{(n)} \) at the times when it visits vertices in \( F^{(k)} \), then we have another random walk on \( \mathcal{U}_k \), with some parameters \((p', \tau_{vh}, \tau_d)\). Let \((p^{(n)}, \tau_{vh}^{(n)}, \tau_d^{(n)})\) be the parameters of the random walk \( X^{(n)} \) induces on \( \mathcal{U}_1 \).

In general, \( p \neq p^{(n)} \), \( \tau_{vh} \neq \tau_{vh}^{(n)} \), and \( \tau_d \neq \tau_d^{(n)} \). To construct a Brownian motion on \( \Gamma \) we must first find a specification \((p, \tau_{vh}, \tau_d)\) which is unaffected by such changes in scale. Say that the specification \((p, \tau_{vh}, \tau_d)\) is invariant if \( p = p^{(n)} \) and there exists \( \lambda > 0 \) such that \( \tau_{vh}^{(n)} = \lambda^{-n} \tau_{vh} \) and \( \tau_d^{(n)} = \lambda^{-n} \tau_d \) for all \( n \geq 0 \).

**Proposition 2.2.** There exists a distribution \( \phi \) on \((0, \infty)\) such that \((\frac{1}{2}, \phi, \phi)\) is an invariant specification for a random walk on \( \Gamma \).

**Proof.** Suppose that \( p = \frac{1}{2}(1 - p) = \frac{1}{3} \); then the probability of any path between a pair of vertices in \( F \) will depend only on the number of edges in it. Observe that the graph \( \mathcal{U} \) is invariant under any mapping that exchanges a pair of the vertices in \( F \), but leaves the other two fixed. Thus, this symmetry argument shows that \( \tau_{vh} = \tau_d = \tau \).

Now, let \( N \) denote the number of steps required by \( \{W_n\} \) to pass between vertices in \( F \), the outer corners of \( \mathcal{U} \), when \( p = \frac{1}{3} \). Let \( f(t) = E_t^N \). We can calculate that

\[
f(t) = \frac{t^3}{(3-2t)(12-12t+t^2)}, \quad EN = 15. \tag{2.6}
\]

(See Section 4 for details.) If \( Z_n \) is a branching process with \( f \) as the generating function of its offspring distribution, then well-known results show that \( 15^{-n} \). \( Z_n \) converges almost surely to some random variable \( T \) and the Laplace transform \( L(u) = E e^{-uT} \) satisfies the relation

\[
L(u) = f(L(\frac{1}{3}u)). \tag{2.7}
\]

Let \( \phi \) denote the law of \( T \). By inspection, \( f(L(u)) \) is the Laplace transform of \( \phi^1 \), so eq. (2.7) shows that \( \phi \) is an invariant measure with \( \lambda = 15 \). □

**Theorem 2.3.** There exists a Brownian motion \( Y \), with state space \( \Gamma \) which is a rescaled limit of random walks on \( \mathcal{U}_n \) defined by the parameters \((\frac{1}{3}, \phi, \phi)\).

**Proof.** Since \((\frac{1}{3}, \phi, \phi)\) is an invariant probability specification, the theorem follows from Theorem VII.8 in Lindström [9]. □

For application in the next proposition, we cite the following lemma about multitype branching processes.

**Lemma 2.4.** Let \( \{Z_n\} \) be a supercritical branching process with \( k \) types of particles and mean matrix \( M \). Let \( \rho > 1 \) be the largest eigenvalue of \( M \), with corresponding left eigenvector \( u \). Then \( V_n = (u \cdot M)^n \rho^{-n} \) is a non-negative martingale, so that \( V = \lim_{n \to \infty} V_n \) exists a.s.
Proof. This is Theorem VI.6.4 in Athreya and Ney [2]. □

**Proposition 2.5.** \( \left( \frac{1}{3}, \phi, \phi \right) \) is the unique invariant probability specification for random walk and \( \{ Y_t \} \) is the unique Brownian motion on \( F \). If \( (p^n, \tau^n_{v_h}, \tau^n_d) \) is any other probability specification for a random walk on \( \mathcal{U} \), then \( p^n \to \frac{1}{3} \) and \( (\tau^n_{v_h}, \tau^n_d) \to (\phi, \phi) \).

Proof. First, we show that \( p = \frac{1}{3} \) is the unique invariant probability for the random walk \( \{ W_n \} \) without transit times. To do this, consider a random walk \( \{ W_n \} \) with parameter \( p \) on the basic lattice \( \mathcal{U} \). Recall that \( p \) is the probability of crossing a diagonal edge. For an arbitrary vertex \( v \) in \( F \), let \( A \) denote the vertex in \( F \) diagonal to \( v \) and let \( B \) denote the two vertices in \( F \) vertically and horizontally adjacent to \( v \). (See Figure 1 for an example.) If \( W_0 = v \), let \( T_A \) be the first time \( \{ W_n \} \) reaches \( A \) and let \( T_B \) be the first time \( \{ W_n \} \) visits a vertex \( B \). Then, \( p \to P[T_A < T_B] = r(p) \).

By straightforward calculations, we can compute \( r(p) \) explicitly as

\[
r(p) = \frac{1}{4 - 3p}. \tag{2.8}
\]

\( p = \frac{1}{3} \) is a fixed point for (2.8). Since \( r(p) \) is convex on \( [0, 1] \), \( \frac{1}{3} \) is the unique stable fixed point for \( r \), and if \( 0 < p < 1 \), \( p^n \to \frac{1}{3} \) as \( n \to \infty \). Next, we assume that \( p = \frac{1}{3} \); we will find the unique invariant transition time distribution. Suppose that \( (\tau_{v_h}, \tau_d) \) is an arbitrary set of transition time distributions. Let \( U_1, U_2, \ldots, U_j \) and \( V_1, V_2, \ldots, V_j \) be independent sequences of independent random variables. Then for each \( n \), \( \tau_{v_h}^{(n)} \) and \( \tau_d^{(n)} \) satisfy

\[
\sum_{i=1}^{j_{v_h}^{(n)}} U_i + \sum_{i=1}^{j_{v_h}^{(n)}} V_i \sim \tau_{v_h}^{(n)}, \quad \sum_{i=1}^{j_d^{(n)}} U_i + \sum_{i=1}^{j_d^{(n)}} V_i \sim \tau_d^{(n)}. \tag{2.9}
\]

\( J_{v_h}^{(n)} \) is the number of vertical or horizontal steps a random walk on \( \mathcal{U} \) makes while crossing between vertically or horizontally adjacent vertices in \( F^{(0)} \). Similarly, \( J_d^{(n)} \) is the number of vertical or horizontal steps made while crossing between diagonally adjacent vertices in \( F^{(0)} \). \( K_{v_h}^{(n)} \) and \( K_d^{(n)} \) are the corresponding numbers of diagonal steps. We omit the superscripts when \( n = 1 \).

It is not hard to see that \( \{ (J_{v_h}^{(n)}, K_{v_h}^{(n)}) \} \) and \( \{ (J_d^{(n)}, K_d^{(n)}) \} \) are two-type branching processes with initial values \( (1, 0) \) and \( (0, 1) \), respectively. The number of ‘offspring’ of a vertical or horizontal step is equal in distribution to \( (J_{v_h}, K_{v_h}) \); the offspring distribution of diagonal steps equals the distribution of \( (J_d, K_d) \).

Let \( G_{v_h}(u, v) = Eu^{J_{v_h}^{(n)}} e^{K_{v_h}^{(n)}} \) and \( G_d(u, v) = Eu^{J_d^{(n)}} e^{K_d^{(n)}} \) be the respective generating functions of the offspring distributions. By solving an appropriate system of linear equations we can compute \( G_{v_h}(u, v) \) and \( G_d(u, v) \). (See Section 4 for details.) The mean matrix for the offspring distributions of our branching process will be the matrix of partial derivatives of \( G_{v_h} \) and \( G_d \),

\[
M = \begin{bmatrix}
\frac{\partial G_{v_h}}{\partial u} & \frac{\partial G_{v_h}}{\partial v} \\
\frac{\partial G_d}{\partial u} & \frac{\partial G_d}{\partial v}
\end{bmatrix} = \begin{bmatrix}
\frac{93}{9} & \frac{88}{9} \\
\frac{44}{9} & \frac{47}{9}
\end{bmatrix}. \tag{2.10}
\]
The eigenvalues of $M$ are 15 and $\frac{1}{3}$. The left eigenvector corresponding to 15 is $u = [1, 1]$.

From Lemma 2.4, it follows that $15^{-n}(J^{(n)}_{vh} + K^{(n)}_{vh}) \to W$ and $15^{-n}(J^{(n)}_{ad} + K^{(n)}_{ad}) \to W$. Direct calculation (see Section 4) shows that $G_n(u, u) = G_d(u, u) = f(u)$. An easy inductive argument extends this to show $G_n(u, u) = (f_n(u), f_n(u))$. We can then conclude that
\[ E \exp[-\mu \cdot 15^{-n} \cdot (J^{(n)} + K^{(n)})] = f_n(e^{-15^{-n} \mu}). \quad (2.11) \]

Let $\psi(u) = E e^{-uW}$. As for a one type Galton-Watson process, it follows that $\psi(\mu) = f(\psi(\frac{1}{3} \mu))$. We have already mentioned that $L(u)$ is a solution to this equation. In Seneta [10] it is shown that the solution to this functional equation is unique, within a choice of scale of $u$.

Again, let $U_1, U_2, \ldots \sim \tau_{vh}$ and $V_1, V_2, \ldots \sim \tau_d$ be independent sequences of independent random variables. Suppose $m_{vh}$ and $m_d$ are the means of $\tau_{vh}$ and $\tau_d$, respectively. Then
\[
\frac{1}{15^n} \left( \sum_{i=1}^{N_{vh}} U_i + \sum_{i=1}^{N_d} V_i \right) \to W \cdot (m_{vh} + m_d),
\]
\[
\frac{1}{15^n} \left( \sum_{i=1}^{M_{vh}} U_i + \sum_{i=1}^{M_d} V_i \right) \to W \cdot (m_{vh} + m_d).
\]

So, $(\tau_{vh}(15^{-n} \times \phi), \tau_d(15^{-n} \times \phi)) \to (\phi, \phi)$, within a choice of scale, making $(\phi, \phi)$ is the unique invariant transition time distribution for $p = \frac{1}{3}$.

Finally, let $(p, \tau_{vh}, \tau_d)$ be any specification for a random walk on $\mathcal{F}$. Then, $p^{(n)} \to \frac{1}{3}$, and $(\tau_{vh}^{(n)}, \tau_d^{(n)}) \to (\phi, \phi)$ by an argument similar to that given in the preceding paragraph. Proposition 2.4 follows. □

Theorem 2.6. $(Y_t)$ is the unique diffusion limit of random walks on $\Gamma$.

Proof. This follows from Proposition 2.4 and Theorem VIII.1 in Lindström [9]. □

3. Scaling properties

In this chapter, we will study some of the detailed sample path properties of Brownian motion on the snowflake.

Let $T_y = \inf \{ t : Y_t = y \}$ be the first hitting time of $Y_t$ at $y$; set $T_y = \infty$ if $\{ t : Y_t = y \}$ is empty. We begin by proving several technical lemmas.

Lemma 3.1. Suppose $x, y \in F^{(\infty)}$ lie in an $n$-complex $\Gamma_{i_1, \ldots, i_n}$. Then $E^{x}T_y < 369 \cdot 3^{-n}$.

Proof. For fixed $i_1, \ldots, i_n$, let $\bar{\Gamma} = \Gamma_{i_1, \ldots, i_n}$ and $\bar{F} = F_{i_1, \ldots, i_n}$. Suppose $x, y \in \bar{F} \cap F^{(\infty)}$. There exists $m \geq n$ such that $x, y \in F^{(m)}$. We prove the estimate by induction on $m$. 

Suppose that \( m = n \). Neccessarily, \( x, y \in \mathcal{F} \). Let \( T_1, T_2, \ldots \) be the times between successive visits to \( F^{(n)} \). Then, \( T_y = \sum_{j=1}^{N_i} T_j \) where

\[
N_y = \sum_{i=1}^{M} \left( \sum_{j=1}^{N_i} R_{i,j} + 1 \right). \tag{3.1}
\]

Here, \( M \) is the number of vertices in \( \mathcal{F} \) that \( Y_i \) visits before \( T_y \), \( N_i \) is the number of excursions to \( F^{(n)} \setminus \mathcal{F} \) between the time \( Y_i \) hits the \((i-1)st\) and \( ith\) distinct vertices in \( \mathcal{F} \), and \( R_{i,j} \) is the length of the \( jth\) such excursion.

The strong Markov property shows that \( M \) has a geometric distribution with parameter \( \frac{1}{2} \). For \( i = 1, 2, \ldots, N_i \) either is identically 0 or else has a geometric distribution with parameter \( \frac{1}{2} \). To estimate \( E R_{i,j} \), note that each excursion outside of \( \mathcal{F} \) is a random walk on a finite graph, and \( R_{i,j} \) is the number of steps the walk takes to return to its starting point. It is well-known that this expected time is equal to twice the total number of edges in the graph divided by the degree of the starting vertex. (See Gobel and Jagers [5, Theorem 1.11]) Thus, \( E R_{i,j} \) is proportional to the number of the edges in the graph cut out of \( \mathcal{F}_n \) by \( \mathcal{F} \). This, in turn, is less than the number of edges in \( F^{(n)} \). It is not hard to compute that \( F^{(n)} \) has \( 6 \cdot 5^{n+1} \) edges, so \( E R_{i,j} \approx 20 \cdot 5^n \).

Recall that \( E T_1 = 15^{-n} \), and apply Wald’s identity,

\[
E^x T_y = EM \cdot (EN(i) \cdot ER_{i,j} + 1) \cdot ET \leq 3 \cdot (2 \cdot 20 \cdot 5^n + 1) \cdot 15^{-n} \leq 41 \cdot 3^{-n+1}. \tag{3.2}
\]

We now proceed by induction. Suppose that if \( x, y \in F^{(m)} \cap \mathcal{F} \) and \( x, y \in F^{(m)} \) with \( |x - y| \leq 3^{-n} \), then \( E^x T_y \leq 82 \cdot \sum_{i=0}^{m-1} 3^{-i+1} \). Let \( x, y \in F^{(m+1)} \cap \mathcal{F} \). There exist points \( v \) and \( w \) in \( \mathcal{F} \cap F^{(m)} \) such that either \( x = v \) or \( x \) is adjacent to \( v \) in \( \mathcal{F} \) and either \( y = w \) or \( y \) is adjacent to \( w \) in \( \mathcal{F} \). The strong Markov property, the inductive hypothesis, and (3.2) give

\[
E^x T_y \leq E^x T_v + E^v T_w + E^w T_y \leq 41 \cdot 3^{-m} + 82 \cdot \sum_{i=0}^{m-1} 3^{-i+1} + 41 \cdot 3^{-m} = 82 \cdot \sum_{i=0}^{m+1} 3^{-i+1}, \tag{3.3}
\]

completing the inductive step.

Finally, if \( x, y \in F^{(\infty)} \cap \mathcal{F} \), then

\[
E^x T_y \leq 82 \cdot \sum_{i=0}^{\infty} 3^{-i+1} = 369 \cdot 3^{-n}. \tag{3.4}
\]

**Corollary 3.2.** If \( x, y \in F^{(\infty)} \) and \( |x - y| < 3^{-n} \), then \( F^x T_y \leq 738 \cdot 3^{-n} \).

**Proof.** If \( |x - y| < 3^{-n} \), then \( x \) and \( y \) are either in a common \( n \)-complex \( \mathcal{F} \) or in two adjoining \( n \)-complexes \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \). In either case, the bound follows from Lemma 3.1. \( \square \)
Recall the contractions \( \{M_1, \ldots, M_5\} \) used to define \( \Gamma \). Define the continuous mapping \( N: \Gamma \to \Gamma \) by
\[
N(x) = \begin{cases} 
M_1^{-1}(x), & x \in [0, \frac{1}{3}]^2, \\
M_2^{-1}(x), & x \in \left[\frac{1}{3}, \frac{2}{3}\right]^2, \\
M_3^{-1}(x), & x \in \left[\frac{1}{3}, \frac{2}{3}\right], \\
M_4^{-1}(x), & x \in [0, \frac{1}{3}] \times \left[\frac{1}{3}, \frac{2}{3}\right], \\
M_5^{-1}(x), & x \in \left[\frac{2}{3}, 1\right] \times [0, \frac{1}{3}].
\end{cases}
\] (3.5)

**Lemma 3.3.** Let \( x \in \Gamma \). Then, \( N(Y_t^x) \overset{D}{=} Y_{15t}^{N(x)} \).

**Proof.** Inspection will verify that if \( X_t^k \) is a random walk on \( \mathcal{U}_k \) starting from \( x \), then \( N(X_t^k) \) is a random walk on \( \mathcal{U}_{k-1} \) starting from \( N(x) \) for any \( k > 1 \). We have shown in Section 2 that \( X_{15t}^k \to Y_t \) weakly. Then
\[
N(X_{15t}^k) = N(X_{15t}^{k-1}(15t)) \to Y_{15t}
\] (3.6)
and since \( N \) is continuous, \( N(X_{15t}^k) \to N(Y_t) \). \( \square \)

We shall call this scaling property the fractal scaling law.

**Corollary 3.4.** \( P^{N(x)}[Y_{15t}^x \in A] = P^x[Y_t \in N^{-1}(A)] \) for \( x \in \Gamma \) and any measurable \( A \). \( \square \)

We next show that two independent copies \( Y \) meet in finite time with probability 1. For \( x, y \in \Gamma \), let \( d(x, y) \) denote the Euclidean distance between \( x \) and \( y \). Let \( D[0, \infty] \) denote the space of functions \( \omega: \mathbb{R}^+ \to \Gamma \) which are right continuous and have left limits for all \( t > 0 \).

**Lemma 3.5.** For any \( t, \varepsilon > 0 \) and any compact set \( K \), let
\[
H(t, \varepsilon) = \{\omega: \inf_d d(\omega(u), K) \geq \varepsilon, 0 < u < t + \varepsilon\}.
\] (3.7)
Then \( H(t, \varepsilon) \) is an open subset in the topology \( \mathcal{T} \) on \( D[0, \infty] \) defined by convergence in the Skorokhod metric on compact intervals \( [0, p] \).

**Proof.** Let \( \omega, \psi \in D[0, \infty] \). For any \( p > 0 \), let \( \rho_p(\mu, \nu) \) equal the infimum of those \( \varepsilon > 0 \) for which there exists a continuous, increasing function \( \lambda : [0, p] \to [0, p] \) such that
\[
\text{(i)} \quad \sup\{0 < t < p: |\lambda(t) - t| \leq \varepsilon, \\
\text{(ii)} \quad \sup\{0 < t < p: d(\lambda(t)), \nu(t)) \leq \varepsilon.
\]
It is easy to see that $p_p$ is a pseudo-metric and that the topology $\mathcal{T}$ is induced by the family of pseudo-metrics $\{p_p\}_{p>0}$.

Suppose $\omega \in H(t, \epsilon)$. Let $a = \inf\{d(\omega(u), K), 0 < u < t + \epsilon\}$. By hypothesis, $a - \epsilon = \delta > 0$. Choose $\nu \in D[0, \infty]$ with $p_\nu(\omega, \nu) < \frac{1}{2} \delta$. There exists a strictly increasing continuous function $\lambda : [0, t] \to [0, t]$ satisfying (i) and (ii). By the triangle inequality, for any $u \in [0, t]$,

$$d(\nu(u), K) \geq d(\omega(\lambda(u)), K) - d(\omega(\lambda(u)), \nu(u)) > \epsilon + \delta - \frac{1}{2} \delta > \epsilon; \quad (3.8)$$

since $\lambda : [0, t] \to [0, t]$, $d(\omega(\lambda(u)), K) > \epsilon + \delta$ for $0 < u < t$. Thus, $\nu \in H(t, \epsilon)$, so $H_\epsilon$ is open.

Let $T = \inf\{t: \omega(t) \in K\}$. For every $\epsilon > 0$, $H(t, \epsilon) \subseteq \{\omega: T(\omega) > t\}$. Let $C[0, \infty]$ denote the continuous functions from $\mathbb{R}^+$ to $\Gamma$.

**Lemma 3.6.** $T : C[0, \infty) \to [0, \infty)$ is a lower semicontinuous function.

**Proof.** It suffices to show $\{\omega \in C[0, \infty]: T(\omega) > t\} = \bigcup_{k=1}^{\infty} H(t, k^{-1})$. Suppose $\omega$ is continuous, and that for every $k > 0$, there exists $s_k \in [0, t]$ such that $d(\omega(s_k), K) \leq k^{-1}$. As $[0, t]$ is compact, there exists a convergent subsequence $s_{k_n}$ with $s_{k_n} \to s \in [0, t]$. As $\omega$ is continuous, $\omega(x_{k_n}) \to \omega(x)$. However, $d(\omega(x_{k_n}), K) \to 0$, so $\omega(s) \in K$, and $T(\omega) \leq s < t$. Thus, if $T(\omega) > t$ then $\inf\{d(\omega(s), K), 0 \leq s \leq t\} > k^{-1}$ for some $k$.

Using the continuity of $T$ again, there must be some $m$ such that $d(\omega(s), K) > m^{-1}, t \leq s < t + m^{-1}$. If not, there would exist a sequence $x_n \to t$ such that $d(\omega(x_n), K) \leq m^{-1}$, forcing $d(\omega(t), K) \leq m^{-1}$. Choose $N = \min\{k, m\}$ to give $\omega \in H_N^{-1}$. \(\square\)

**Theorem 3.7.** Let $Y_i$ and $Y'_i$ be two independent copies of Brownian motion on $\Gamma$. Let $T_M = \inf\{u: Y_u = Y'_u\}$. Then $T_M < \infty$ a.s.

**Proof.** In Aldous [1] it is shown that if $X_i$ and $X'_i$ are two independent copies of a random walk in continuous time on a finite graph $H$, then there exists some constant $D$ such that $E T_M \leq D \max_{i,j} E' T_j$, where the maximum is taken over all pairs of states $i, j$. By Markov's inequality, it follows that $P[T_M > t] \leq t^{-1} \cdot D \max_{i,j} E' T_j$.

Let $X^n_t$ be a random walk on $\mathcal{U}_n$ and let $T^{(n)}_M$ be the time when two independent copies of $X^n_t$ meet. By Lemma 3.1, if $x, y$ in $E^{(n)}$ are vertices such that $\|x - y\| < 3^{-i}$, then $E' T_j \leq 738 \cdot 3^{-i}$. So, $P[T^{(n)}_M > t] \leq t^{-1} \cdot 738 D$ for all $n$.

Let $\Delta = \{(x, x): x \in \Gamma\}$. Clearly, $T_M$ is the first hitting time on $\Delta$ for the process $(Y_i, Y'_i)$. As before, let $H_\epsilon = \{\omega: \inf d(\omega(u), \Delta) > \epsilon, 0 < u < t + \epsilon\}$. Since $H_\epsilon \subset \{T_M > t\}$, it follows that

$$P[Y_n(t) \in H_\epsilon] \leq P[T^{(n)}_M > t] \leq t^{-1} \cdot 738 D \quad (3.9)$$

for all $n$. As $H_\epsilon$ is open, Prokhorov's theorem and Lemma 3.1 show that $P[Y_i \in H_\epsilon] \approx \lim_{n \to \infty} P[Y_n(t) \in H_\epsilon] \leq t^{-1} \cdot 738 D$. 
As \( Y_t \) has a.s. continuous sample paths, the preceding equation and Lemma 3.6 show that \( P[T_M > t] = \lim \inf_{n \to \infty} P[Y_t \in H_n^{-1}] \leq t^{-1} \cdot 738D \). Letting \( t \to \infty \) gives \( P[T_M = \infty] = 0 \), which is what we proposed to prove. \( \square \)

We next consider invariant measures for \( Y_t \). Again, consider the random walks \( X_{\infty}^{(n)} \), with the associated discrete random walks \( W_{\infty}^{(n)} \). As each \( W_{\infty}^{(n)} \) is a random walk on a finite graph, it has a unique reversible stationary measure, which we shall call \( \mu_n \). Since \( \Gamma \) is compact, \( \{ \mu_n \} \) has a subsequence \( \{ \mu_{n'} \} \) converging weakly to some \( \mu \).

**Proposition 3.8.** \( Y_t \) has stationary distribution \( \mu \).

**Proof.** To show the proposition, we apply weak convergence. We have previously shown that \( X_{\infty}^{(n)}(15^{-n} \cdot t) \to Y_t \). Let \( X_{\infty}^{(n)} \) be the stationary version of \( X_{\infty}^{(n)} \). Standard results on weak convergence show that \( X_{\infty}^{(n)} \) converges weakly to a process \( Y_{\infty}^{(n)} \), where \( Y_{\infty}^{(n)} \) is a version of \( Y \), with stationary distribution \( \mu \). \( \square \)

**Theorem 3.9.** The distribution of \( Y_t \) converges \( \mu \) in total variation norm. \( \mu \) is normalized Hausdorff \( \log_3 \)-dimensional measure, restricted to \( \Gamma \), and is the unique stationary distribution for \( Y_t \).

**Proof.** Let \( \mu_t(Y, A) = P[Y_t \in A] \). Let \( Y'_t \) be an independent stationary diffusion on \( \Gamma \). Couple \( Y_t \) to \( Y'_t \) by letting \( Y \) and \( Y' \) move independently prior to \( T_M \) but specifying that they move identically afterwards. Then for any measurable \( A \),

\[
|\mu_t(A) - \mu(A)| \leq P[Y_t \neq Y'_t] \leq t^{-1} \cdot 738D
\]

and the inequality is uniform over all measurable \( A \). This gives convergence in total variation norm.

Let \( 0 \) denote \((0,0)\). To show that \( \mu \) is normalized Hausdorff measure, we apply the fractal scaling law to get \( P[\mu_t(Y_t \in N^{-1}(A)) = \mu(N^{-1}(A)) \) and \( P[\mu_t(Y_t \in N^{-1}(A)) = P_0[Y_t \in N^{-1}(A)] = \mu(N^{-1}(A)) \) as \( t \to \infty \). So, \( \mu \) satisfies the equation \( \mu(N^{-1}(A)) = \mu(N^{-1}(A)) \). Theorem 4.4.1 in Hutchinson [7] shows that there is a unique measure on \( \Gamma \) satisfying this equation. Since \( \rho \)-dimensional Hausdorff measure restricted to \( \Gamma \) does satisfy this equation, it follows that \( \mu \) is \( \rho \)-dimensional Hausdorff measure on \( \Gamma \). This argument also shows that \( \mu \) is unique. \( \square \)

Theorem 3.9, together with the Lemma 3.3, implies that \( Y_t \) has a transition density with respect to Hausdorff measure.

Recall the transformations \( M_1, \ldots, M_5 \) and \( N \) defined in (2.1) and (3.5).

**Theorem 3.10.** \( \mu_t \preceq \mu \) for all \( t > 0 \).

**Proof.** Let \( B \subset \Gamma \) be a set with \( \mu(B) = 0 \), and suppose that \( \mu_t(B) = q > 0 \). We observe in passing that \( \mu(B) = 0 \) iff \( \mu(N(B)) = 0 \), and that \( N^{-1}(N(B)) \supset B \). Let
$B_{\infty} = \bigcup_{k=1}^{\infty} N^k(B)$. Since $\mu(B) = 0$, $\mu(N^k(B)) = 0$ for all $k$, so $\mu(B_{\infty}) = 0$. On the other hand, $B \subset N^{-k}(B_{\infty})$ for all $k$. By the bounded fractal scaling law,

$$\mu_{15}(B_{\infty}) = \mu_* (N^{-k}(B_{\infty})) \geq \mu_*(B) = q.$$  \hspace{1cm} (3.11)

Thus, $\liminf_{k \to \infty} \mu_{15}(B_{\infty}) \geq q$. But this contradicts the fact that $\mu_* \to \mu$ in total variation norm as $t \to \infty$. By contradiction, $\mu_*(B) = 0$. Thus, $\mu_* \ll \mu$. \hspace{1cm} \Box

4. Computing generating functions

In this section, we wish to calculate the recurrence function $r(p)$ and the generating functions $f(u)$, $G_{vb}(u, v)$, and $G_{ab}(u, v)$ discussed in Section 2.

We first consider $r(p)$. Let $\{W_n\}$ be a random walk on $\mathcal{U}$, with $p$ as its probability of crossing diagonal edges. We wish to compute $P_p[T_A < T_B]$ as a function of $p$. Label the vertices of $\mathcal{U}$ as in Figure 1. This gives the following sets of points:

$$a = \{(0, 0)\}, \quad b = \{(0, 1), (\frac{1}{2}, 0)\}, \quad c = \{(\frac{1}{2}, \frac{1}{2})\}, \quad d = \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0), (0, 1)\}, \quad e = \{(\frac{1}{2}, 1), (1, 0)\}, \quad f = \{(\frac{1}{2}, 1), (1, \frac{1}{2})\}, \quad g = \{(\frac{1}{2}, 1), (1, \frac{1}{2})\}. \hspace{1cm} (4.1)$$

For each $i \in \{a, \ldots, g\}$, let $q_i$ be the probability that $W_n$ starts from vertex $i$ and reaches $(1, 1)$ before either $(0, 1)$ or $(1, 0)$. From the Markov property of $W_n$, we get the following system of equations for $q_i$,

$$q_a = (1 - p)q_b + pq_c, \quad q_b = \frac{1}{2}(1 - p)q_a + pq_b + \frac{1}{2}(1 - p)q_c,$$

$$q_c = \frac{1}{2}pq_a + \frac{1}{2}(1 - p)q_b + \frac{1}{2}(1 - p)q_d + pq_f,$$

$$q_d = \frac{1}{2}(1 - p)q_c + \frac{1}{2}pq_a + \frac{1}{2}(1 - p)q_e + \frac{1}{2}(1 - p)q_l,$$

$$q_e = \frac{1}{2}(1 - p)q_d + \frac{1}{2}pq_c + \frac{1}{2}(1 - p)q_a + \frac{1}{2}(1 - p)q_k,$$

$$q_f = \frac{1}{2}(1 - p)q_e + \frac{1}{2}pq_d + \frac{1}{2}(1 - p)q_b + \frac{1}{2}pq_l,$$

$$q_l = \frac{1}{2}(1 - p)q_f + pq_g + \frac{1}{2}(1 - p). \hspace{1cm} (4.2)$$

Fig. 1. The graph $\mathcal{U}$. 

\hspace{1cm}
By applying Cramer’s rule, we get the solution
\[ q_a = r(p) = \frac{(\frac{1}{2})^5 \cdot (1 - p)^3 \cdot (1 + p)^3}{(\frac{1}{2})^5 \cdot (1 - p)^3 \cdot (1 + p)^3 \cdot (4 - 3p)} = \frac{1}{4 - 3p}. \] (4.3)

By a similar computation, we can compute generating functions for the number of steps \( \{ W_n \} \) makes while crossing between vertices in \( F \). Recall \( J_{vh}, J_d, K_{vh}, \) and \( K_d \) as defined in Section 2. \( G_{vh}(u, v) = Eu^J_{vh}v^K_{vh} \) and \( G_d(u, v) = Eu^J_dv^K_d \). Set \( J^A = J_1[\tau_A < \tau_B], J^B = J_1[\tau_A < \tau_B], K^A = K_1[\tau_A < \tau_B] \) and \( K^B = K_1[\tau_A < \tau_B] \). For \( i \) in \( \{ a, \ldots, g \} \), define \( h_i(u, v) = Eu^J_i v^K_i \) and \( k_i(u, v) = Eu^K_i v^K_i \). To compute \( G_{vh} \) and \( G_d \), we find the functions \( h_i \) and \( k_i \) and condition on \( [\tau_A < \tau_B] \). Since \( P[\tau_A < \tau_B] = \frac{3}{5} \), we get \( G_{vh}(u, v) = \frac{3}{5} h_a(u, v) \) and \( G_d(u, v) = 3k_g(u, v) \).

Applying the Markov property of \( W_n \) again, we see that \( h \) and \( k \) satisfy the equations:

\[
\begin{align*}
  h_a &= \frac{3}{2} u_{h_a} + \frac{1}{2} v_{h_c}, & k_a &= \frac{3}{2} u_{k_b} + \frac{1}{2} v_{k_c}, \\
  h_b &= \frac{3}{2} u_{h_b} + \frac{1}{2} v_{h_b} + \frac{1}{2} u_{h_v}, & k_b &= \frac{3}{2} u_{k_a} + \frac{1}{2} v_{k_b} + \frac{3}{2} u_{k_c}, \\
  h_c &= \frac{3}{2} u_{h_c} + \frac{1}{2} u_{h_a} + \frac{1}{2} u_{h_b} + \frac{1}{2} u_{h_f}, & k_c &= \frac{1}{2} u_{k_a} + \frac{3}{2} u_{k_b} + \frac{1}{2} u_{k_c} + \frac{1}{2} v_{k_f}, \\
  h_d &= \frac{3}{2} u_{h_d} + \frac{1}{2} v_{h_d} + \frac{1}{2} u_{h_e} + \frac{1}{2} v_{h_e}, & k_d &= \frac{1}{2} u_{k_c} + \frac{3}{2} u_{k_d} + \frac{1}{2} u_{k_e} + \frac{1}{2} v_{k_f}, \\
  h_e &= \frac{3}{2} u_{h_e} + \frac{1}{2} v_{h_e} + \frac{1}{2} u_{h_f} + \frac{1}{2} v_{h_f}, & k_e &= \frac{1}{2} u_{k_d} + \frac{3}{2} u_{k_e} + \frac{1}{2} v_{k_e}, \\
  h_f &= \frac{3}{2} u_{h_f} + \frac{1}{2} v_{h_f} + \frac{1}{2} u_{h_g} + \frac{1}{2} v_{h_g}, & k_f &= \frac{1}{2} u_{k_c} + \frac{3}{2} u_{k_d} + \frac{3}{2} u_{k_e} + \frac{1}{2} v_{k_f}, \\
  h_g &= \frac{3}{2} u_{h_g} + \frac{1}{2} v_{h_g}, & k_g &= \frac{1}{2} u_{k_c} + \frac{1}{2} v_{k_e} + \frac{1}{2} u.
\end{align*}
\] (4.4)

We can also solve these equations by using Cramer’s rule.

\[
G_{vh}(u, v) = \frac{3}{5} h_a(u, v) = \frac{3u(v^6 - 3v^5 - 2v^4u^2 - 27v^4 - 12v^3u^2 + 135v^3 + 4v^2u^4 + 126v^2u^2)}{-162v^2 + 36u^4 - 216u^2 + 8u^6 - 2u^2}.
\] (4.5)

\[
G_d(u, v) = 3k_g(u, v) = \frac{3(-v^7 + 15v^6 + 8v^5u^2 - 81v^5 - 78v^4u^2 + 189v^4 - 20v^3u^4 + 252v^3u^2 + 8u^6v + 621u^4v - 135v^3 + 6v^2u^2 + 9v^2 + 96u^2)}{-162v^2 + 108v^4u^2 - 270v^2u^2 + 16u^4v - 144u^4 - 24u^2}.
\] (4.6)

Finally, recall that \( f(u) = Eu^N \), where \( N \) is the total number of steps taken by \( \{ W_n \} \) between successive visits to \( F \). Then \( N = J + K \), so \( Eu^N = Eu^{J+K} = G_{vh}(u, u) = G_d(u, u) \). Thus,

\[
f(t) = G_{vh}(u, u) = \frac{u^3}{36 - 60u + 27u^2 - 2u^3}.
\] (4.7)
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