MATHEMATICS

A CLASS OF INFINITE MATRICES

BY

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1. We introduce below the class of Σ-matrices, which can be used to represent polynomial forms and infinite series. As an application, a theorem is given, in § 3, on the "product" of two infinite series.

We define Σ-matrices as infinite lower semi-matrices of the type

\[
A = \begin{bmatrix}
    a_0, & 0, & 0, & 0, & \ldots \\
    a_1, & a_0, & 0, & 0, & \ldots \\
    a_2, & a_1, & a_0, & 0, & \ldots \\
    a_3, & a_2, & a_1, & a_0, & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

(1.1)

where the elements \(a_i\) are complex numbers.

Theorem 1, I. Σ-matrices form an integral domain.

If \(A, B\) are Σ-matrices, the first column of \(AB\) is

\[
\{a_0 b_0, a_1 b_0 + a_0 b_1, \ldots, a_n b_0 + a_{n-1} b_1 + \ldots + a_0 b_n, \ldots\}
\]

(1.2)

It is easily seen that \(AB\) is a Σ-matrix, and in fact that Σ-matrices form a commutative ring. It remains to prove that the "cancellation law" holds, i.e., \(AB=AC\) implies \(B=C\), if \(A \neq 0\). If \(AB=AC\), we have

\[
a_0 b_0 = a_0 c_0, a_1 b_0 + a_0 b_1 = a_1 c_0 + a_0 c_1, a_2 b_0 + a_1 b_1 + a_0 b_2 = a_2 c_0 + a_1 c_1 + a_0 c_2, \ldots,
\]

whence if some \(a_i \neq 0\), it follows that \(b_0 = c_0\) and thence that \(b_i = c_i\) \((i=1, 2, \ldots)\). Thus the result is established.

If \(a_0 \neq 0\), the Σ-matrix (1.1) has a unique two-sided Σ-reciprocal \([1, 22]\), which we shall call its inverse, and thus the sub-class of Σ-matrices for which \(a_0 \neq 0\) forms a field.

Theorem 1, II. There is an isomorphism between the class of column-finite Σ-matrices and the class of polynomial forms in an indeterminate \(x\), over the field of complex numbers.

Let \(B_{(m)}\) be the Σ-matrix whose leading column is

\[
\{b_0, b_1, b_2, \ldots, b_n, 0, 0, \ldots\}
\]

let \(C_{(m)}\) be similarly defined, and let \(b(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n, c(x) = c_0 + c_1 x + \ldots + c_m x^m\). We also define \(b_j = 0\) \((j > n)\), \(c_k = 0\) \((k > m)\). Then if \(B_{(m)} \leftrightarrow b(x), C_{(m)} \leftrightarrow c(x)\), we have at once \(B_{(m)} + C_{(m)} \leftrightarrow b(x) + c(x)\). Also the \((r+1)\)-th element in the leading column of \(B_{(m)} C_{(m)}\) is \(b_r c_0 + b_{r-1} c_1 + + \ldots + b_0 c_r\), i.e., the coefficient of \(x^r\) in \(b(x), c(x)\). Finally, \(k B_{(m)} \leftrightarrow k b(x)\), where \(k\) is any scalar, and the isomorphism is thus established.
2. If \( a_0 \neq 0 \), the \( \Sigma \)-matrix \( A \), defined by (1.1), has a \( \Sigma \)-inverse \( A^{-1} \) whose leading column is \( \{x_0, x_1, x_2, \ldots\} \), where

\[
a_0 x_0 = 1, \quad a_1 x_0 + a_0 x_1 = a_2 x_0 + a_1 x_1 + a_0 x_2 = \ldots = 0.
\]

Using (2.1), we have \((a_0 + a_1 x + a_2 x^2 + \ldots \cdot (x_0 + x_1 x + x_2 x^2 + \ldots) = 1\), if both series converge. Conversely, the inverse of \( A \) could be deduced from consideration of the two infinite series.

Example 1. Let \( A \) be the \( \Sigma \)-matrix defined by the leading column \( \{1, -2, 1, 0, 0, \ldots\} \). Comparing with \((1-x)\pm_2\), the leading column of \( A^{-1} \) is \( \{1, 2, 3, 4, \ldots\} \).

Example 2. From the expansions for \( e^x \), \( e^{-x} \), if

\[
B = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & 1 & 1 & \ldots \\
\frac{1}{2!} & 1 & 1 & 1 & 1 & \ldots \\
\frac{1}{3!} & \frac{1}{2!} & 1 & 1 & 1 & \ldots \\
\frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{bmatrix}
\]

then \( B^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \ldots \\
-1 & 1 & 0 & 0 & 0 & \ldots \\
\frac{1}{2!} & -1 & 1 & 0 & 0 & \ldots \\
\frac{1}{3!} & \frac{1}{2!} & -1 & 1 & 0 & \ldots \\
\frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & -1 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{bmatrix} \)

Example 3. To find \( p_i \) such that \((1+x+x^2)^{-1}=p_0+p_1 x+p_2 x^2+\ldots \), when the right-hand side converges.

Let \( A \) be the \( \Sigma \)-matrix whose leading column is \( \{1, 1, 0, 1, 0, 0, \ldots\} \); the leading column of \( A^{-1} \) is computed term by term as

\[
\{1, -1, 1, -2, 3, -4, 6, -9, 13, \ldots\}
\]

Thus \( p_0 = 1, p_1 = -1, p_2 = 1, p_3 = -2, \ldots \), and in fact \( p_n = -(p_{n-1} + p_{n-3}) \), \( n \geq 3 \).

3. The following two theorems on the "product" of two convergent series are well-known (see, for example, [2], 31).

Theorem 3, I. (Cauchy-Mertens). If \( \Sigma u_n \) and \( \Sigma v_n \) converge to the values \( u, v \) respectively, and at least one of the series is absolutely convergent, then the series \( u_0 v_0 + (u_0 v_1 + u_1 v_0) + (u_0 v_2 + u_1 v_1 + u_2 v_0) + \ldots \) converges to the value \( uv \).

Theorem 3, II. (Abel). If \( \Sigma u_n \) converges to \( u \) and \( \Sigma v_n \) converges to \( v \), then if the series \( \Sigma (u_0 v_0 + u_1 v_1 + \ldots + u_n v_0) \) converges, its sum is \( uv \).

We shall prove below

Theorem 3, III. If \( \Sigma v_n \) is any conditionally convergent series, there exists a conditionally convergent series \( \Sigma u_n \) such that

\[
\Sigma (u_0 v_0 + u_1 v_1 + \ldots + u_n v_0)
\]

diverges.
We first need a lemma. \( B = (b_{i,j}) \) \((i, j = 1, 2, \ldots)\) is a \( \delta \)-matrix if the convergence of \( \Sigma u_k \) implies that of \( \Sigma_{n} \Sigma_{i} b_{n,k} u_k \). Necessary and sufficient conditions are (i) \( \Sigma_{n} b_{n,k} \) is convergent for all \( k \), (ii) 
\[
\sum_{k=1}^{\infty} \left| \sum_{i=1}^{n} (b_{i,k} - b_{i,k+1}) \right| \leq M
\]
for all \( n \), [3]. Applied to \( \Sigma \)-matrices, these conditions give the following

Lemma. Let \( B \) be the \( \Sigma \)-matrix \((b_{n,k})\) whose leading column is \( \{b_0, b_1, b_2, \ldots\} \), so that \( b_{n,k} = b_{n-k} \) \((n > k)\), \( b_{n,k} = 0 \) \((n < k)\). Then a necessary and sufficient condition for \( B \) to be a \( \delta \)-matrix is the absolute convergence of \( \Sigma b_{n,k} \).

For
\[
\sum_{k=1}^{\infty} \left| \sum_{i=1}^{n} (b_{i,k} - b_{i,k+1}) \right| = \sum_{k=1}^{\infty} \left| (b_{1,k} - b_{1,k+1}) + (b_{2,k} - b_{2,k+1}) + \ldots + (b_{n,k} - b_{n,k+1}) \right|
\]
\[
= \sum_{k=1}^{\infty} |b_{n,k}|, \text{ since } b_{n,k} = b_{n+1,k+1} \text{ and } b_{1,k+1} = 0,
\]
\[
= |b_n| + |b_{n-1}| + \ldots + |b_0|.
\]
Thus the second condition for \( \delta \)-matrices is satisfied for all \( n \) if and only if \( \Sigma b_{n,k} \) is absolutely convergent, and condition (i) then also holds.

We can now prove Theorem 3, III. Let \( A \) be the \( \Sigma \)-matrix defined by \( a_{n,k} = v_{n-k} \) \((n > k)\), \( a_{n,k} = 0 \) \((n < k)\), where \( \Sigma v_i \) is conditionally convergent, and let \( \Sigma u_i \) be a convergent series. Then
\[
q_n = \Sigma_k a_{n,k} u_k = u_0 v_n + u_1 v_{n-1} + \ldots + u_n v_0.
\]
Now if \( \Sigma q_n \) were to converge for all convergent series \( \Sigma u_k \), \( A \) would be a \( \delta \)-matrix, which is impossible, by the lemma, since by hypothesis \( \Sigma v_i \) is conditionally convergent. It follows that there is at least one convergent series \( \Sigma u_k \) such that \( \Sigma(u_0 v_n + \ldots + u_n v_0) \) diverges. Such a series \( \Sigma u_k \) cannot be absolutely convergent, by Theorem 3, I., and thus the theorem is proved.

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Postscript. Another proof of Theorem 3, III. is given by J. Schur, (Crelle, 151 (1921), 100, 101).

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REFERENCES