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Forbidden triples and traceability: a characterization

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Abstract

Given a connected graph G, a family $\mathscr{F}$ of connected graphs is called a forbidden family if no induced subgraph of G is isomorphic to any graph in $\mathscr{F}$. If this is the case, G is said to be $\mathscr{F}$-free. In earlier papers the authors identified four distinct families of triples of subgraphs that imply traceability when they are forbidden in sufficiently large graphs. In this paper the authors introduce a fifth family and show these are all such families.

1. Background and notation

The graphs discussed here are simple graphs. For terms not defined here, see [3].

Let G be a connected graph and let $\mathscr{F}$ be a family of connected graphs. We say that $\mathscr{F}$ is a family of *forbidden subgraphs* (or a *forbidden family*) if no induced subgraph of G is isomorphic to any graph in $\mathscr{F}$. If this is the case, G is said to be $\mathscr{F}$*-free*. If $\mathscr{F}$ consists of a single graph, say H, we say that G is H-free. A graph is said to be *traceable* if it contains a path that spans the vertex set.

In two previous papers [4,5] four distinct families of triples of subgraphs were shown to imply traceability when forbidden in sufficiently large graphs. The families are as follows (refer to Fig. 1 for the graphs themselves):

1. $\{K_{1,m}, Y_l, Z_1\}$ $(m \geqslant 4, l \geqslant 4)$.
2. $\{K_{1,m}, P_4, V_r\}$ $(m \geqslant 4, r \geqslant 3)$.
3. $\{K_{1,3}, E_r, Z_2\}$ $(r \geqslant 4)$.
4. $\{K_{1,m}, P_l, Q_r\}$ $(m \geqslant 4, l \geqslant 5, r \geqslant 3)$.

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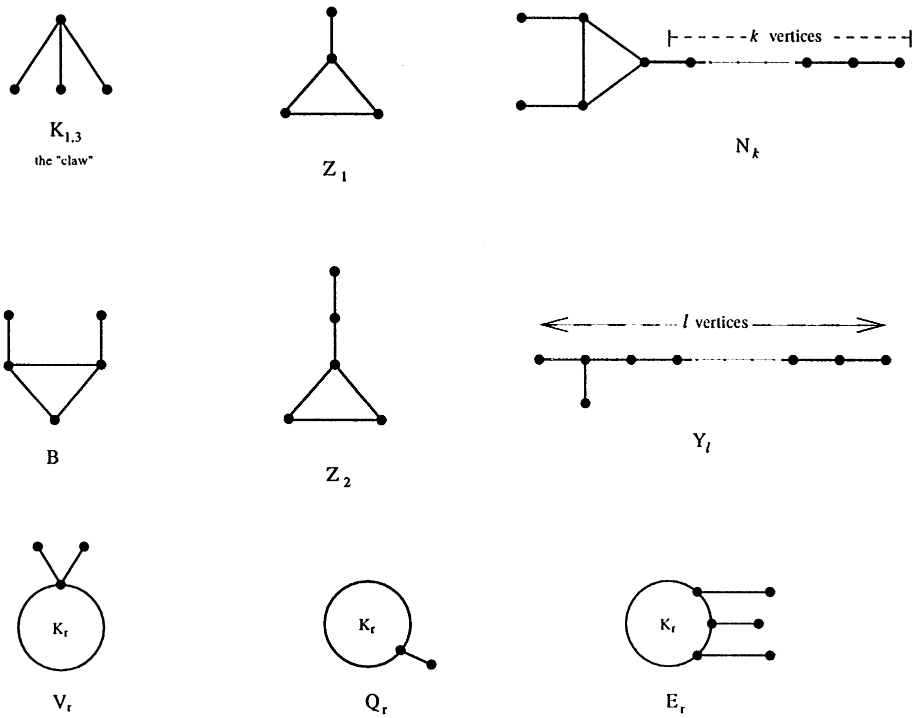


Fig. 1. Graphs involved in forbidden triples.

Characterizations have been discovered for all the single graphs and all the pairs of graphs that imply traceability when forbidden in connected graphs (see [2]). It should be noted that if any of these graphs (the single or the pairs) are contained in a triple $\mathscr{T} = \{A, B, C\}$, then certainly a connected graph that is $\mathscr{T}$-free will be traceable. The single and the pairs are described in Section 3 of this paper.

In Section 2 we identify an additional family, $\{K_{1,3}, Q_r, N_k\}$, that enjoys the property of implying traceability in sufficiently large graphs. In Section 3 we show that this family, along with the previous four, are the only nontrivial families of triples do this (that is, the only families not containing the single graph or one of the pairs mentioned above).

Regarding notation, given two vertices v and w of a graph G, we let $d_G(v,w)$ denote the distance (the length of a shortest path) in G from v to w. If A is a subset of the vertices of G, we let $\langle A \rangle$ denote the subgraph of G induced by A. Also, given a vertex v, we let $N_A(v)$ denote the set of vertices in A that are adjacent to v. Finally, in a graph G, suppose we have internally disjoint paths $P_1 : a_1, a_2, \ldots, a_i$ and $P_2 : b_1, b_2, \ldots, b_j$. If the edge $a_i b_1$ exists, then the path P in G described by $P : a_1, \ldots, a_i,\ b_1, \ldots, b_j$ will be denoted as $[a_1, a_i]_{P_1}, [b_1, b_j]_{P_2}$. In a similar fashion, if $a_i = b_1$, then the notation given by $[a_1, a_i]_{P_1}, (b_1, b_j]_{P_2}$ will represent the path S in G given by $S : a_1, \ldots, a_i,\ b_2, \ldots, b_j$.

2. The family: $\{K_{1,3}, Q_r, N_k\}(r \geqslant 4, k \geqslant 2)$

We begin this section by stating a result from Sumner (see [6, p. 142]) that we will use later. Note that $\kappa(G)$ represents the connectivity of G.

Theorem A (Sumner [6]). *If G is a claw-free graph of order n, and if $\kappa(G) \geqslant n/4$, then G is hamiltonian.*

Theorem 2.1. *Let $r \geqslant 4$ and $k \geqslant 2$ be fixed integers. Let G be a connected graph of order n that is $\{K_{1,3}, Q_r, N_k\}$-free. If n is sufficiently large, then G is traceable.*

Proof. Let T be a minimum cut set of G, let $v \in T$, and let $S = T \setminus \{v\}$. (It is possible that $S = \emptyset$.) We know that $\langle V(G) \setminus T \rangle$ is either disconnected or a single vertex. If $\langle V(G) \setminus T \rangle$ is a single vertex, then $|T| = |V(G)| - 1$, and hence G is a complete graph, and is certainly traceable. Therefore, assume that $\langle V(G) \setminus T \rangle$ is disconnected.

Since T is minimum, it must be that $\langle V(G) \setminus S \rangle$ is 1-connected and has v as a cut vertex. Now, if $\langle V(G) \setminus T \rangle$ has more than two components, then there exist vertices $a, b, c \in N(v)$ that are pairwise nonadjacent, and then we will have a claw: $\langle \{a, b, c, v\} \rangle$. Thus, $\langle V(G) - T \rangle$ must have exactly two components, say A and B.

We partition the vertices of A and B as follows. For $i = 1, 2, \ldots$, define $A_i = \{u \in V(A) : d(u, v) = i\}$ and $B_i = \{u \in V(B) : d(u, v) = i\}$. Further, define $A_0 = B_0 = \{v\}$. Note that since G is finite, there exists an integer $l \geqslant 1$ such that $A_l \neq \emptyset$ and $A_i = \emptyset$ for $i > l$. Also, there must exist an integer $m \geqslant 1$ such that $B_m \neq \emptyset$ and $B_i = \emptyset$ for $i > m$.

We now make several Notes, each of which is easily verified:

Note (a): Each vertex of S is adjacent to at least one vertex of A and to at least one vertex of B.

Note (b): No vertex of A is adjacent to any vertex of B.

Note (c): (i) $N(A_i) \cap A_j = \emptyset$ for each $i \in 1, \ldots, l$ and for each $j \neq i-1, i, i+1$; (ii) $N(B_i) \cap B_j = \emptyset$ for each $i \in 1, \ldots, m$ and for each $j \neq i-1, i, i+1$.

Note (d): For $i \geqslant 1$, if $x \in A_i$ (resp. B_i), then x is adjacent to some vertex of A_{i-1} (resp. B_{i-1}).

Note (e): If x and y are nonadjacent vertices of A_i (resp. B_j), then x and y have no common neighbors in A_{i-1} (resp. B_{j-1}).

Note (f): The subgraphs $\langle A_1 \rangle$ and $\langle B_1 \rangle$ are complete.

Note (g): If x is a vertex of A_i, then there exists an induced path P: $x, a_{i-1}, a_{i-2}, \ldots, a_1, v$ where x and v are the endpoints and $a_j \in A_j$ for $j = 1, 2, \ldots, i-1$.

We make a definition: given $i \in \{1, \ldots, l-1\}$, some vertices of A_i are adjacent to vertices of A_{i+1}, while some vertices may not be. That is, some vertices of A_i "continue on" to A_{i+1}, and some do not continue. We will call a vertex $x \in A_i$ a *continuer* if it is adjacent to some vertex of A_{i+1}. Otherwise, we call x a *noncontinuer*. The terms *continuer* and *noncontinuer* will have similar meanings in B.

Note (h): Each of $A_0, A_1, A_2, \ldots, A_{l-1}, B_1, B_2, \ldots, B_{m-1}$ contains at least one continuer.

Claim 2.1. *For each* $i\in\{1,2,\dots,l\}, |A_i|<(r-1)^i$, *and for each* $j\in\{1,2,\dots,m\}$, $|B_j|<(r-1)^j$.

Proof. We will prove the bound on $|A_i|$ by induction. The argument for $|B_j|$ is almost identical.

From Note (f) above we know that $\langle A_1\rangle$ is complete. If we suppose that $|A_1|\geqslant r-1$, and we let b_1 be a vertex of B_1, then we see that $\langle A_1\cup\{v\}\cup\{b_1\}\rangle$ contains an induced Q_r. Thus, $|A_1|<r-1$.

Now, suppose the claim is true for A_{i-1} where $i\geqslant 2$. Let a_{i-1} be a vertex of A_{i-1}, let $a_{i-2}\in A_{i-2}$ be a neighbor of a_{i-1}, and consider the vertices of $N_{A_i}(a_{i-1})$.

If vertices $a_i, a_i'\in N_{A_i}(a_{i-1})$ are nonadjacent, then $\langle\{a_i,a_i',a_{i-1},a_{i-2}\}\rangle$ is an induced $K_{1,3}$. Therefore, a_i and a_i' must be adjacent, and we can then conclude that $\langle N_{A_i}(a_{i-1})\rangle$ must be complete. Thus, if $|N_{A_i}(a_{i-1})|\geqslant r-1$, we again have a subgraph $(\langle N_{A_i}(a_{i-1})\cup\{a_{i-1}\}\cup\{a_{i-2}\}\rangle)$ which contains an induced Q_r. Hence $|N_{A_i}(a_{i-1})|<r-1$. Thus we have that

$$|A_i|\leqslant\left|\bigcup_{x\in A_{i-1}} N_{A_i}(x)\right|<(r-1)(r-1)^{i-1}=(r-1)^i,$$

and the claim is proved. □

Given the integers r and k, we let

$$\gamma=2\sum_{i=1}^{2k}(r-1)^i,$$

and we take $n\geqslant\frac{4}{3}\gamma$. If we suppose that $|V(A)|+|V(B)|\leqslant\gamma$, then we have that $|T|=|V(G)|-|V(A)|-|V(B)|\geqslant|V(G)|-\gamma=n-\gamma\geqslant n-\frac{3}{4}n=\frac{1}{4}n$. Therefore $\kappa(G)\geqslant n/4$, and by Theorem A, G is Hamiltonian, and thus clearly traceable.

Thus, we can assume that $|V(A)|+|V(B)|>\gamma$. By the definition of γ, this implies that one of l or m is at least $2k+1$. We suppose without loss of generality that $l\geqslant 2k+1$.

Claim 2.2. *Each of* $\langle A_1\rangle,\dots,\langle A_l\rangle$, $\langle B_1\rangle,\dots,\langle B_m\rangle$ *is complete.*

Proof. From Note (f) we know that $\langle A_1\rangle$ and $\langle B_1\rangle$ are both complete. Let i be the least integer such that $\langle A_i\rangle$ is not complete ($i\geqslant 2$), and suppose $a_i,a_i'\in A_i$ are nonadjacent. From Note (e) above, a_i and a_i' have distinct neighbors in A_{i-1}. Let them be a_{i-1} and a_{i-1}', respectively. Since $\langle A_{i-1}\rangle$ is complete, a_{i-1} and a_{i-1}' are adjacent. Furthermore, let $a_{i-2}\in A_{i-2}$ be a neighbor of a_{i-1}. If $a_{i-1}'a_{i-2}\notin E(G)$, then $\langle\{a_i,a_{i-1},a_{i-1}',a_{i-2}\}\rangle$ would be an induced claw, so $a_{i-1}'a_{i-2}\in E(G)$. Let $a_{i-3}\in A_{i-3}$ be a neighbor of a_{i-2} (if $i=2$, then let a_{i-3} be some vertex of B_1).

Suppose that $i>k$. We know there exists a path $a_{i-3},a_{i-4},\dots,\ a_2,a_1,v,b_1$ such that $a_j\in A_j$ for $j=1,2,\dots,i-3$ and $b_1\in B_1$. This, however, produces an induced N_k (see Fig. 2): $\langle\{a_i,a_i',a_{i-1},a_{i-1}',a_{i-2},a_{i-3},\dots,a_1,v,b_1\}\rangle$.

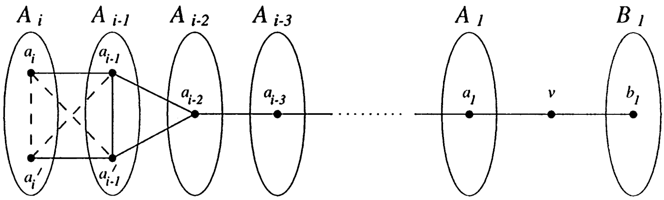


Fig. 2.

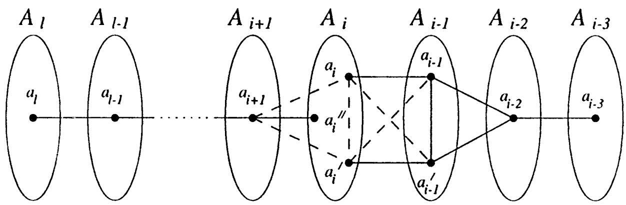


Fig. 3.

Therefore, let us assume that $i \leqslant k$. Since $A_l \neq \emptyset$, there must exist a path $a_l, a_{l-1}, \ldots, a_{i+1}$ where for each $j \in \{i+1, \ldots, l\}$, $a_j \in A_j$.

If both a_i and a_i' are adjacent to a_{i+1}, then $\langle \{a_{i+2}, a_{i+1}, a_i, a_i'\} \rangle$ forms an induced claw. Further, if exactly one of a_i and a_i' (say a_i) is adjacent to a_{i+1}, then since $i \leqslant k$ and $l \geqslant 2k+1$, the subgraph $\langle \{a_{i+1}, a_{i+2}, \ldots, a_l, a_i, a_i', a_{i-1}, a_{i-1}', a_{i-2}, a_{i-3}\} \rangle$ contains an induced N_k.

Therefore we assume that neither a_i nor a_i' is adjacent to a_{i+1}. If this is true, then there must exist some other vertex, say a_i'', in A_i that *is* adjacent to a_{i+1} (see Fig. 3). Now, if either of a_i or a_i' is nonadjacent to a_i'', then the argument in the preceding paragraph applies, and it produces a contradiction. Further, if both a_i and a_i' are adjacent to a_i'', then, again, we have an induced $K_{1,3}$: $\langle \{a_{i+1}, a_i'', a_i, a_i'\} \rangle$. Hence, no such integer i exists, and it must be that each of $\langle A_1 \rangle, \ldots, \langle A_l \rangle$ is complete.

Now, suppose that j is the least integer such that $\langle B_j \rangle$ is not complete ($j \geqslant 2$), and let $b_j, b_j' \in B_j$ be nonadjacent vertices. Again from Note (e) we see that b_j and b_j' must have distinct neighbors in B_{j-1}. Let these neighbors be b_{j-1} and b_{j-1}', respectively. Further, let $b_{j-2} \in B_{j-2}$ be a neighbor of b_{j-1}. Since G is claw-free, it must be that $b_{j-2} b_{j-1}' \in E(G)$. Moreover, due to the nature of the partitions of A and B, there must exist a path $b_{j-2}, b_{j-3}, \ldots, v, a_1, a_2, \ldots, a_l$ in G such that $b_t \in B_t$ and $a_t \in A_t$ for each t. This provides an induced subgraph that contains an induced N_k. Therefore, it must be that no such integer j exists. Thus, each of $\langle B_1 \rangle, \langle B_2 \rangle, \ldots, \langle B_m \rangle$ is complete, and so is the proof of the claim. $\square$

Given $i \in \{1, 2, \ldots, l-1\}$, suppose a_i is some continuer in A_i, and let P be a path that satisfies the following conditions:

(i) $V(P) \subseteq V(A)$;

(ii) $V(P) \cap A_i = \{a_i\}$;

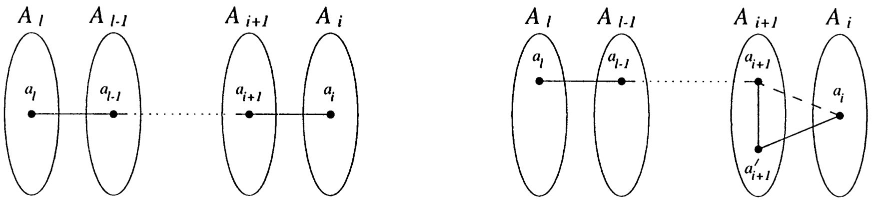


Fig. 4. The two kinds of continuing paths.

(iii) $1 \leqslant |V(P) \cap A_{i+1}| \leqslant 2$;
(iv) $|V(P) \cap A_j| = 1$ for $j = i+2, i+3, \ldots, l$;
(v) $|V(P) \cap A_j| = 0$ for $j < i$;
(vi) P is an induced path.
We will call such a path a *continuing path from* a_i. Fig. 4 shows examples of continuing paths.

Claim 2.3. *If* $a_i \in A_i$ *is a continuer, then there exists a continuing path from* a_i *in* G.

Proof. Given $a_l \in A_l$, let P': $a_l, a_{l-1}, \ldots, a_{i+1}$ be a path where $a_j \in A_j$ for each $j = i+1, \ldots, l$. If a_i is adjacent to a_{i+1}, then the path P given by $a_i, [a_{i+1}, a_l]_{P'}$ is the desired continuing path.

Suppose then that a_i is not adjacent to a_{i+1}. Then since a_i is a continuer, there exists some $a'_{i+1} \in A_{i+1}$ such that $a_i a'_{i+1} \in E(G)$. Further, $a_{i+1} a'_{i+1} \in E(G)$ since $\langle A_{i+1} \rangle$ is complete. If a'_{i+1} is adjacent to a_{i+2}, then the continuing path is given by $a_i, a'_{i+1}, [a_{i+2}, a_l]_{P'}$. If a'_{i+1} is not adjacent to a_{i+2}, then the desired continuing path is $a_i, a'_{i+1}, [a_{i+1}, a_l]_{P'}$. □

We now turn our attention to the vertices of $S = T \setminus \{v\}$. If $S = \emptyset$ then some of the claims that follow will be vacuous.

Let $s \in S$. From Note (a) we know that s is incident with at least one of $A_1, A_2, \ldots, A_l$, and at least one of $B_1, B_2, \ldots, B_m$. Suppose that s is adjacent to $a_i \in A_i$ and $a_j \in A_j$, and suppose that $|i - j| > 1$. If b is a vertex of B adjacent to s, then $\langle \{a_i, a_j, s, b\} \rangle$ is an induced $K_{1,3}$, a contradiction. Therefore, the following claim holds:

Claim 2.4. *If* $s \in S$ *is incident with two distinct sets* A_i *and* A_j, *then* $|i - j| = 1$. *Consequently,* s *is incident with at most two sets from* $A_1, A_2, \ldots, A_l$.

Claim 2.5. *If* p *is the greatest integer such that* $s \in S$ *is incident with* A_p, *then* $p \in \{1, 2, l\}$.

Proof. Suppose the claim is false and consider two cases.

Case 1: Suppose $3 \leqslant p \leqslant k+1$.

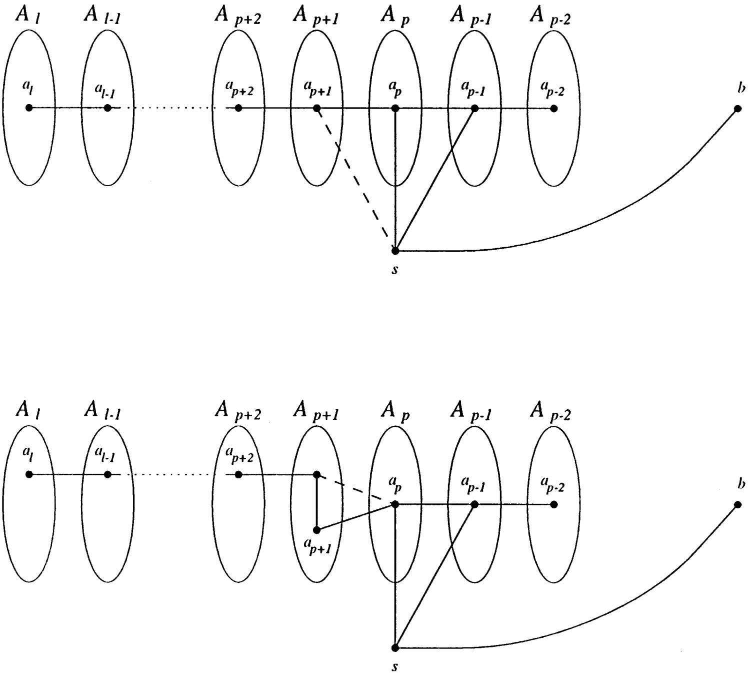


Fig. 5.

Subcase 1.1: Suppose s is adjacent to some continuer, say a_p, in A_p.

Let $a_{p-1} \in A_{p-1}$ be a neighbor of a_p, let $a_{p-2} \in A_{p-2}$ be a neighbor of a_{p-1}, and let $b \in V(B)$ be a neighbor of s. From Claim 2.3 we know there is a continuing path P from the continuer a_p. Let a_{p+1} be the vertex of A_{p+1} that is adjacent to a_p on P. Because of the maximality of p, s is not adjacent to a_{p+1}, and since G is claw-free, the edge sa_{p-1} must be present. Then from Claim 2.4 s is incident with A_p and A_{p-1}, and s is not incident with A_i for $i \neq p,\ p-1$.

On our continuing path P, let $\{a_j\} = V(P) \cap A_j$ for $j = p+2, p+3, \ldots, l$. Then, depending on the value of $|V(P) \cap A_{p+1}|$ (recall that it can be 1 or 2), we have one of the two situations depicted in Fig. 5.

Since $3 \leqslant p \leqslant k+1$ and since $l \geqslant 2k+1$, each of these possibilities contains an induced N_k, which provides a contradiction for this Subcase.

Subcase 1.2: Suppose s is not adjacent to any continuer of A_p.

Let $a_p \in A_p$ be a neighbor (necessarily a non-continuer) of s. The set A_p must contain a continuer, so let a'_p be a continuer in A_p. Further, let $a_{p-1} \in A_{p-1}$ be a neighbor of a'_p, and let $a_{p-2} \in A_{p-2}$ be a neighbor of a_{p-1}. From Claim 2.3, there exists a continuing path P from the continuer a'_p. Let a_{p+1} be the vertex of A_{p+1} that is adjacent to a'_p on P. Again, for $j \in \{p+2, p+3, \ldots, l\}$, let $\{a_j\} = V(P) \cap A_j$.

Now, since a_p is not a continuer, $a_p a_{p+1} \notin E(G)$. Thus, since G is claw-free, a_p must be adjacent to a_{p-1}. Furthermore, since $sa_{p-2}, sa'_p \notin E(G)$, s cannot be adjacent

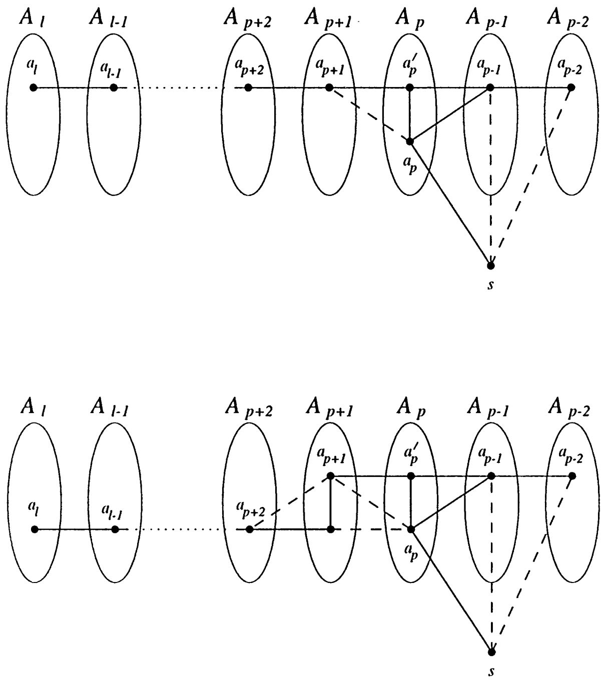


Fig. 6.

to a_{p-1}, or else $\langle\{a'_p, a_{p-1}, a_{p-2}, s\}\rangle$ would be an induced $K_{1,3}$. Therefore, again depending on the value of $|V(P) \cap A_{p+1}|$, we have one of the two situations shown in Fig. 6. Since $3 \leqslant p \leqslant k+1$ and since $l \geqslant 2k+1$, we see that each of these possibilities leads to an induced N_k, providing a contradiction in this Subcase. Thus p is not in the interval $3 \leqslant p \leqslant k+1$.

Case 2: Suppose $k+2 \leqslant p \leqslant l-1$.

Let a_p be a continuer in A_p, let $a_{p+1} \in A_{p+1}$ and $a_{p-1} \in A_{p-1}$ be neighbors of a_p, and let $b \in V(B)$ be some neighbor of s. Further, let $a_{p-2}, a_{p-3}, \ldots, a_1$ be vertices such that $a_i \in A_i$ for each i, and such that the subgraph induced by the vertices $a_{p-1}, a_{p-2}, \ldots, a_1$ is a path.

Suppose first that s is adjacent to a_p. If this is the case, then s must also be adjacent to a_{p-1}, since otherwise $\langle\{a_{p+1}, a_p, a_{p-1}, s\}\rangle$ would be a claw. But if s is adjacent to a_{p-1}, we get an induced N_k, which is a contradiction.

Therefore it cannot be that s is adjacent to a_p. By a similar argument, we can show that s is nonadjacent to all continuers in A_p. Let $a'_p \in A_p$ be a neighbor of s (a'_p is necessarily a noncontinuer). The vertex a'_p must be adjacent to a_{p-1}, since otherwise $\langle\{a_{p-1}, a_p, a_{p+1}, a'_p\}\rangle$ would be an induced $K_{1,3}$. Now, if $sa_{p-1} \in E(G)$, then $\langle\{a_{p-1}, a_p, a_{p-2}, s\}\rangle$ is an induced claw, which is a contradiction. Further, if $sa_{p-1} \notin E(G)$, then we obtain an induced N_k, another contradiction.

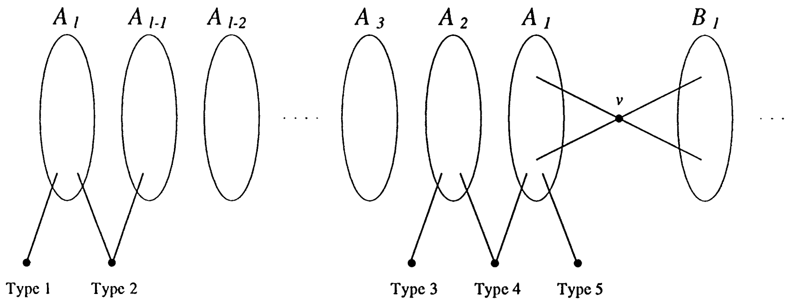


Fig. 7. Examples of each type.

We have contradicted the assumption that $k+2 \leqslant p \leqslant l-1$, and we conclude that $p \in \{1,2,l\}$. □

In the previous claim, s was an arbitrary element of S. It follows, then, that each vertex of S can be classified as one of five types, according to its adjacencies in A:

$$S_1 = \{v \in S\colon N_{A_l}(v) \neq \emptyset, \text{ and } N_{A_j}(v) = \emptyset \text{ for } j \neq l\},$$
$$S_2 = \{v \in S\colon N_{A_i}(v) \neq \emptyset \text{ for } i = l, l-1; N_{A_j}(v) = \emptyset \text{ for } j \neq l, l-1\},$$
$$S_3 = \{v \in S\colon N_{A_2}(v) \neq \emptyset, \text{ and } N_{A_j}(v) = \emptyset \text{ for } j \neq 2\},$$
$$S_4 = \{v \in S\colon N_{A_i}(v) \neq \emptyset \text{ for } i = 1,2, \text{ and } N_{A_j}(v) = \emptyset \text{ for } j \neq 1,2\},$$
$$S_5 = \{v \in S\colon N_{A_1}(v) \neq \emptyset, \text{ and } N_{A_j}(v) = \emptyset \text{ for } j \neq 1\}.$$

A typical vertex of each type is shown in Fig. 7.

The following claims (2.6–2.10) are now straightforward to prove.

Claim 2.6. *Each vertex of S_2 is adjacent to every vertex of A_l.*

Claim 2.7. *Each vertex of S_3 is adjacent to v and to every vertex of B_1.*

Claim 2.8. *Each vertex of S_5 that is not adjacent to v is adjacent to every vertex of B_1.*

Now, define the set

$$S_5^B = \{s \in S_5\colon sv \notin E(G)\}.$$

From Claim 2.8 we know that each vertex of S_5^B is adjacent to all of B_1.

Claim 2.9. *Each vertex of S_5^B is adjacent to all vertices of B_2.*

We now partition the set

$$S_5 \setminus S_5^B = \{s \in S_5\colon sv \in E(G)\}$$

into two sets S_{5c}, S_{5n} where

$$S_{5c} = \{w \in S_5 \setminus S_5^B\colon w \text{ is adjacent to some continuer in } A_1\},$$

$$S_{5n} = \{w \in S_5 \setminus S_5^B\colon w \text{ is nonadjacent to all continuers in } A_1\}.$$

Clearly then, S_5 is the disjoint union of sets S_5^B, S_{5c}, and S_{5n}.

Also, let us partition the vertices of S_4 into two sets S_{4c}, S_{4n} where

$$S_{4c} = \{w \in S_4\colon w \text{ is adjacent to some continuer in } A_2\},$$

$$S_{4n} = \{w \in S_4\colon w \text{ is nonadjacent to all continuers in } A_2\}.$$

Claim 2.10. (a) *Each vertex of S_{4c} is adjacent to all noncontinuers in A_2.*
(b) *Each vertex of S_{5c} is adjacent to all noncontinuers in A_1.*

Claim 2.11. *Each of the sets $S_1, S_2, S_3, S_{4c}, S_{4n}, S_{5c}, S_{5n}$, and S_5^B induces a complete subgraph of G.*

Proof. (I) *Consider S_1 and S_2*: In order to prove that $\langle S_1\rangle$ and $\langle S_2\rangle$ are complete, it will be helpful for us to generalize, since the proofs are very similar. Let the ordered pair (i,j) be one of the members of the set $\{(1,l),(2,l-1)\}$. We will prove that $\langle S_i\rangle$ is complete.

Let s_i and s_i' be nonadjacent vertices of S_i. If s_i and s_i' have a common neighbor in A_j, say a_j, then $\langle\{a_j, a_{j-1}, s_i, s_i'\}\rangle$ is an induced $K_{1,3}$, where $a_{j-1} \in A_{j-1}$ is a neighbor of a_j.

Thus, we suppose that s_i and s_i' have no common neighbors in A_j. Say that a_j and a_j' in A_j are neighbors of s_i and s_i', respectively. Then if we let $a_{j-1} \in A_{j-1}$ be a neighbor of a_j, we must have that a_j' is adjacent to a_{j-1}, or else $\langle\{a_j, a_j', s_i, a_{j-1}\}\rangle$ is an induced claw.

Let $a_{j-2} \in A_{j-2}$ be a neighbor of a_{j-1}. By Note (g), there is an induced path with k vertices in A beginning from a_{j-2}. Hence we have an induced N_k. Thus, we have a contradiction, and so s_i and s_i' are adjacent. We therefore can conclude that $\langle S_i\rangle$ is complete for both $i=1$ and $i=2$.

(II) *Consider S_3*: Let s_3, s_3' be nonadjacent vertices of S_3. Again, if they have a common neighbor in A_2, say a_2, we get an induced claw: $\langle s_3, s_3', a_2, a_1\rangle$, where $a_1 \in N_{A_1}(\{a_2\})$. Thus assume they have distinct neighbors in A_2, say a_2 and a_2', respectively. Note here that a_2 is not a continuer. If it were, then $\langle\{s_3, a_2, a_1, a_3\}\rangle$ would be an induced $K_{1,3}$ (where $a_1 \in N_{A_1}(\{a_2\})$ and $a_3 \in N_{A_3}(\{a_2\})$. Similarly, a_2' is not a continuer.

So, neither a_2 nor a_2' is a continuer. Let a_2'' be a continuer in A_2. From Claim 2.3, there is a continuing path P from the continuer a_2''. Let a_3 be the vertex of A_3 that is adjacent to a_2'' on P. Since s_3 and s_3' have no common neighbors in A_2, at most one of them is adjacent to a_2''. But if we suppose for the moment that s_3 is adjacent to a_2'', we see that $\langle\{a_3, a_2'', s_3, a_2'\}\rangle$ is an induced $K_{1,3}$. We reach a similar conclusion if s_3' is adjacent to a_2''. Thus, it must be that neither of s_3, s_3' is adjacent to a_2''. But then we have an induced N_k, which is a contradiction.

(III) *Consider* S_{4c} *and* S_{5c}: Once again, the proofs for these two sets are very similar, so we generalize: let the ordered pair (i,j) be one of the members of the set $\{(4,2),(5,1)\}$. We show that $\langle S_{ic}\rangle$ is complete.

Let $s_{ic}, s_{ic}' \in S_{ic}$ be nonadjacent. By definition, both of these vertices are adjacent to continuers in A_j. If there is a continuer in A_j, say a_j, that is adjacent to both s_{ic} and s_{ic}', then $\langle\{s_{ic}, s_{ic}', a_j, a_{j+1}\}\rangle$ is a claw (where $a_{j+1} \in A_{j+1}$ is a neighbor of a_j).

So, it must be that s_{ic} and s_{ic}' are adjacent to distinct continuers in A_j; call them a_j and a_j', respectively.

We now claim that a_j and a_j' must have identical adjacencies in A_{j+1}. If this were not true, then there would exist an $x \in A_{j+1}$ which was adjacent to one of a_j, a_j' (say a_j) and nonadjacent to the other. This, though, would imply the existence of an induced claw: $\langle\{x, a_j, a_j', s_{ic}\}\rangle$. Thus we can conclude that $N_{A_{j+1}}(a_j) = N_{A_{j+1}}(a_j')$.

Case 1: Suppose there exists a vertex $a_{j+1} \in A_{j+1}$ which is a continuer and which is adjacent to a_j (and a_j').

From Claim 2.3, there is a continuing path from a_{j+1}, and this yields an induced N_k (see Fig. 8(a)).

Case 2: Suppose that a_j and a_j' are only adjacent to noncontinuers in A_{j+1}.

Let $a_{j+1} \in A_{j+1}$ be a noncontinuer that is a neighbor of a_j and a_j', and let a_{j+1}' be a continuer in A_{j+1}. Once again, from Claim 2.3, there is a continuing path from a_{j+1}', and this also produces an induced N_k (Fig. 8(b)), again a contradiction.

(IV) *Consider* S_{4n} *and* S_{5n}: Again, we handle these cases simultaneously. Let the ordered pair (i,j) be one of the members of the set $\{(4,2),(5,1)\}$, and suppose that vertices $s_{in}, s_{in}' \in S_{in}$ are nonadjacent.

If s_{in}, s_{in}' have a common adjacency in A_j, say a_j, then

$$\langle\{a_j', a_j, s_{in}, s_{in}'\}\rangle$$

is an induced claw, where a_j' is any continuer in A_j.

So it must be that s_{in} and s_{in}' have distinct neighbors in A_j. Let two such neighbors be a_j and a_j', respectively (they are both necessarily noncontinuers, since $s_{in}, s_{in}' \in S_{in}$). If a_j'' is a continuer in A_j, then we know there is a continuing path from a_j'', and therefore we have an induced N_k.

Again, we have reached a contradiction, and so it must be that s_{in} is adjacent to s_{in}'. Therefore, $\langle S_{in}\rangle$ is complete for both $i=4$ and $i=5$.

(V) *Consider* S_5^B: Recall that S_5^B is the set of vertices of Type 5 that are not adjacent to v.

Suppose that $s_5, s_5' \in S_5^B$ are nonadjacent. If these two vertices have a common neighbor in A_1, say a_1, then $\langle\{s_5, s_5', a_1, v\}\rangle$ is an induced claw. Therefore we will assume that they have distinct adjacencies in A_1. Let two such neighbors be a_1 and a_1', respectively.

Suppose that neither a_1 nor a_1' is a continuer, and let a_1'' be a continuer in A_1. Let P be a continuing path from a_1'' and let a_2 be the vertex of A_2 that is

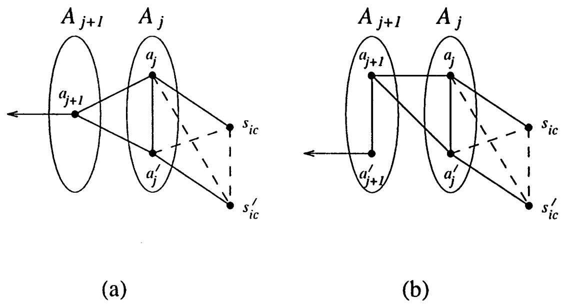


Fig. 8.

adjacent to a_1'' on P. Since s_5, s_5' do not share a neighbor in A_1, at most one of them is adjacent to a_1''. However, if s_5 is adjacent to a_1'', then $\langle\{a_2, a_1'', a_1', s_5\}\rangle$ is an induced claw. We reach a similar conclusion if s_5' is adjacent to a_1''. Thus neither s_5 nor s_5' is adjacent to a_1''. But this implies that we have an induced N_k, which is a contradiction.

Therefore we must assume that at least one of a_1 or a_1' is a continuer. We claim now that a_1 and a_1' have identical neighbors in A_2 (so, in fact, they are both continuers). If we suppose that this is not the case, and we let $x \in A_2$ be a neighbor of one of them, say a_1 and a non-neighbor of the other, a_1', then we will have an induced claw: $\langle\{x, a_1, a_1', s_5\}\rangle$. It must be, then that a_1, a_1' have identical neighbors in A_2. We now consider two cases.

Case 1: Suppose there is a continuer a_2 in A_2, which is adjacent to a_1 and a_1'.

From Claim 2.3, there is a continuing path from a_2, and so we have an induced N_k (Fig. 9(a)).

Case 2: Suppose that a_1, a_1' are only adjacent to noncontinuers in A_2.

Let $a_2 \in A_2$ be such a noncontinuer, and let a_2' be a continuer in A_2. Again, from Claim 2.3, there is a continuing path from a_2' and this produces an induced N_k (Fig. 9(b)).

Having reached a contradiction in each case, we can conclude that s_5 and s_5' must be adjacent. Thus $\langle S_5^B\rangle$ is also complete.

We have therefore completed the proof of Claim 2.11. □

We have now established enough structure to be able to show that G is in fact traceable. The vertices of G have been partitioned into several subsets:

$$A_1, A_2, \ldots, A_l, B_1, B_2, \ldots, B_m, \{v\}, S_1, S_2, S_3, S_{4c}, S_{4n}, S_{5c}, S_{5n}, S_5^B,$$

each of which induces a complete subgraph of G. Each of these complete subgraphs is clearly traceable, so all that remains is to demonstrate a way to "trace through" these complete subgraphs, forming a Hamiltonian path in G. To accomplish this we will establish a series of claims, each of which will provide a Hamiltonian path through a portion of G. Once these paths are established, we will then attach them end-to-end to

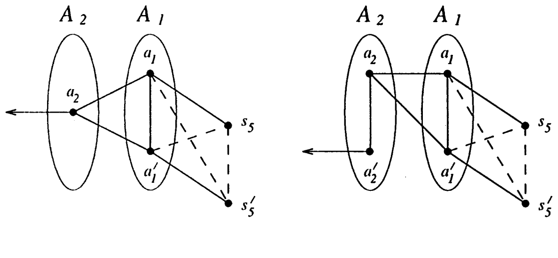


Fig. 9.

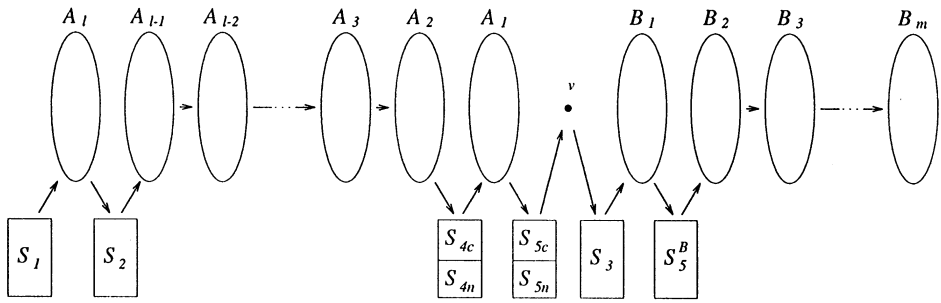


Fig. 10. General order of the tracing.

form the Hamiltonian path for G. Claims 2.12–2.14 can be easily verified, and so the proofs are omitted.

Fig. 10 gives a general idea of the order in which we will trace through the complete subgraphs. In the claims that follow, we consider the possibilities that one or more of the sets is empty.

Claim 2.12. *There exists a path W_1 in G such that $V(W_1) = A_l \cup S_1$ and such that at least one of the end vertices of W_1 is in A_l.*

Claim 2.13. *Let a_l be a vertex of A_l. There exists a path W_2 in G such that $V(W_2) = \{a_l\} \cup A_{l-1} \cup S_2$ and such that the end vertices of W_2 are a_l and some vertex of A_{l-1}.*

Claim 2.14. *Let a_{l-1} be a vertex of A_{l-1}. There exists a path W_3 in G such that*

$$V(P_3) = \{a_{l-1}\} \cup A_{l-2} \cup A_{l-3} \cup A_{l-4} \cup \cdots \cup A_3$$

and such that the end vertices of W_3 are a_{l-1} and some vertex of A_3.

The proofs of the next two claims are conceptually quite simple. However, due to the various possible sizes of the sets involved, there are a number of cases and subcases. For this reason, we include only the proof of Claim 2.16 for the case where S_{5c} and

S_{5n} are both nonempty. The proof of this case is typical of those of the other cases in these two claims.

For notational convenience let us partition the set A_1 into two disjoint sets $A_1^{\rm c}$ and $A_1^{\rm n}$, where $A_1^{\rm c}$ is the set of all continuers in A_1, and $A_1^{\rm n}$ is the set of all noncontinuers in A_1. Since $\langle A_1\rangle$ is complete, it is clear that both $A_1^{\rm c}$ and $A_1^{\rm n}$ induce complete subgraphs. Also, given a complete induced subgraph $\langle R\rangle$ of G, and given vertices $a,b\in R$, let $H_R[a,b]$ denote a Hamiltonian path for $\langle R\rangle$ which has end vertices a and b, and let $H_R[a,\bigstar]$ denote a Hamiltonian path for $\langle R\rangle$ that has a as one of its end vertices.

Claim 2.15. *Let a_3 be a vertex of A_3. There exists a path W_4 in G such that $V(W_4)=\{a_3\}\cup A_2\cup S_{4c}\cup S_{4n}$ and such that the end vertices of W_4 are a_3 and some vertex of $A_2\cup S_4$.*

Claim 2.16. *Let p be a vertex in $A_2\cup S_4$. There exists a path W_5 in G such that $V(W_5)=\{p\}\cup A_1\cup S_{5c}\cup S_{5n}\cup\{v\}$ and such that the end vertices of W_5 are p and v.*

Proof. Suppose $S_{5c}\neq\emptyset$ and $S_{5n}\neq\emptyset$.

Note that this case implies that $A_1^n\neq\emptyset$ (and we know already that $A_1^{\rm c}\neq\emptyset$).

Case 1: Suppose that p is adjacent to a vertex, say s_{5n}, of S_{5n}.

Let s_{5c} be a vertex of S_{5c} and let $a_1\in A_1^{\rm c}$ be a neighbor of s_{5c}. Further, let T be a Hamiltonian path for $\langle S_{5n}\rangle$ with end vertices s_{5n} and some x, and let $a_1'\in A_1^n$ be a neighbor of x.

Let the path W_5 (Fig. 11(a)) be described as follows:

$$p,[s_{5n},x]_T,H_{A_1}[a_1',a_1],H_{S_{5c}}[s_{5c},\bigstar],v.$$

Case 2: Suppose that p is adjacent to a vertex, say s_{5c}, of S_{5c}. (Note that Cases 1 and 2 may both be true. If this is the case, though, then either argument will suffice to give us the desired path.)

Let s_{5n} be a vertex of S_{5n}, and let $a_1\in A_1^n$ be a neighbor of s_{5n}. If $|S_{5c}|>1$, then let s_{5c}' be a vertex of S_{5c} that is different from s_{5c}. Otherwise, let $s_{5c}'=s_{5c}$. Further, let $a_1'\in A_1^{\rm c}$ be a neighbor of s_{5c}'.

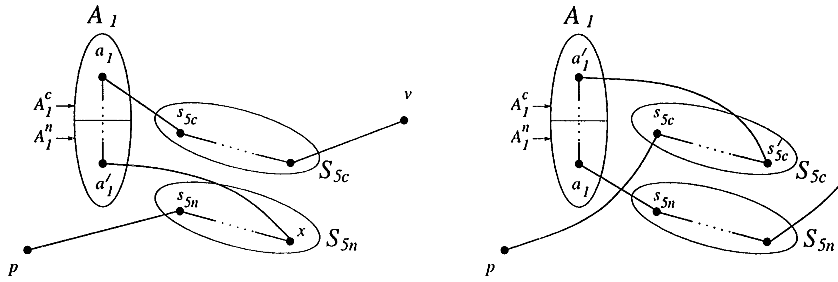
Let W_5 (Fig. 11(b)) be described by

$$p,H_{S_{5c}}[s_{5c},s_{5c}'],H_{A_1}[a_1',a_1],H_{S_{5n}}[s_{5n},\bigstar],v.$$

Case 3: Suppose p is nonadjacent to all of S_{5c} and S_{5n}.

Subcase 3.1. Suppose that p and a vertex of S_{5n}, say s_{5n}, share a neighbor in A_1, say a_1. Note that a_1 is necessarily in $A_1^{\rm n}$.

Let x be an arbitrary element of S_{5c}. We know from Claim 2.10 that x is adjacent to a_1. We also know that neither the edge px nor the edge ps_{5n} is present. Therefore, since G is claw-free, it must be that the edge xs_{5n} is present. Therefore, since $x\in S_{5c}$ was arbitrary, we can conclude that s_{5n} is adjacent to every member of S_{5c}.



(a) (b)

Fig. 11.

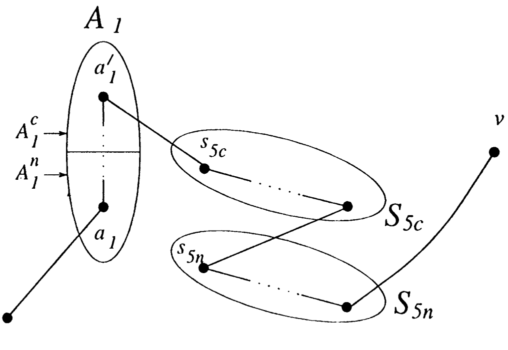


Fig. 12.

If we let s_{5c} be a vertex of S_{5c}, and we let $a_1' \in A_1^{\mathrm{c}}$ be one of its neighbors, then let the path W_5 (Fig. 12) be represented by

$$p, H_{A_1}[a_1, a_1'], H_{S_{5c}}[s_{5c}, \bigstar], H_{S_{5n}}[s_{5n}, \bigstar], v.$$

Subcase 3.2. Suppose that p does not share a neighbor in A_1 with any vertex of S_{5n}.

Let $a_1 \in A_1$ be a neighbor of p, and let $a_1' \in A_1^{\mathrm{n}}$ be a neighbor of $s_{5n} \in S_{5n}$ (thus, $a_1 \neq a_1'$).

Subcase 3.2.1. Suppose there exists a vertex $s_{5c} \in S_{5c}$ such that $|N_{A_1^{\mathrm{c}}}(s_{5c}) \cup \{a_1\}| > 1$. Let $a_1'' \in A_1^{\mathrm{c}}$ be a neighbor of s_{5c} that is different from a_1, and let W_5 be the following path: $p, H_{A_1 \setminus \{a_1'\}}[a_1, a_1''], H_{S_{5c}}[s_{5c}, \bigstar], a_1', H_{S_{5n}}[s_{5n}, \bigstar], v$.

Subcase 3.2.2. Suppose that $|N_{A_1^{\mathrm{c}}}(x) \cup \{a_1\}| = 1$ for all $x \in S_{5c}$. That is, a_1 is necessarily a continuer, and it is the only one adjacent to an element of S_{5c}.

Let $s_{5c} \in S_{5c}$ be a neighbor of a_1.

Subcase 3.2.2.1. Suppose that $|A_1^{\mathrm{c}}| = 1$ and that $|A_1^{\mathrm{n}}| = 1$. That is, $A_1^{\mathrm{c}} = \{a_1\}$ and $A_1^{\mathrm{n}} = \{a_1'\}$.

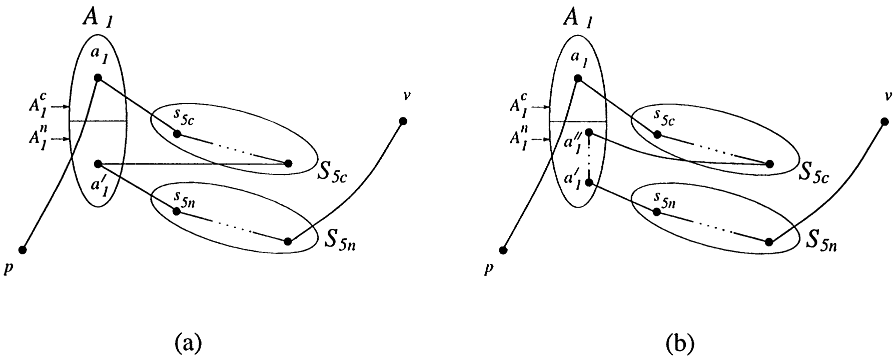


Fig. 13.

Let W_5 (Fig. 13(a)) be as follows:

$$p, a_1, H_{S_{5c}}[s_{5c}, \bigstar], a_1', H_{S_{5n}}[s_{5n}, \bigstar], v.$$

Subcase 3.2.2.2. Suppose that $|A_1^{\mathrm{c}}| = 1$ and that $|A_1^{\mathrm{n}}| > 1$.

Let $a_1'' \in A_1^{\mathrm{n}}$ be a vertex different from a_1', and let W_5 (Fig. 13(b)) be

$$p, a_1, H_{S_{5c}}[s_{5c}, \bigstar], H_{A_1 \setminus \{a_1\}}[a_1'', a_1'], H_{S_{5n}}[s_{5n}, \bigstar], v.$$

Subcase 3.2.2.3. Suppose that $|A_1^{\mathrm{c}}| > 1$.

Let $a_1''' \in A_1^{\mathrm{c}}$ be a vertex different from a_1. Since ps_{5c} and $a_1''' s_{5c}$ are not edges of G, it must be that pa_1''' is an edge of G, since otherwise $\langle \{a_1, p, s_{5c}, a_1'''\} \rangle$ would be an induced claw.

Let the path W_5 be as follows: $p, H_{A_1 \setminus \{a_1'\}}[a_1''', a_1], H_{S_{5c}}[s_{5c}, \bigstar], a_1', H_{S_{5n}}[s_{5n}, \bigstar], v$.

Thus the proof of this case is complete. The other cases are similar. □

Claim 2.17. *There exists a path W_6 in G such that*

$$V(W_6) = \{v\} \cup S_3 \cup B_1 \cup S_5^B \cup B_2 \cup B_3 \cup \cdots \cup B_m$$

and such that v is an end vertex of W_6.

Proof. Again we present a proof for a particular case, namely the case where both S_3 and S_5^B are nonempty. The other cases can be easily verified.

Let W_6^m be a Hamiltonian path for $\langle B_m \rangle$, and suppose its end vertices are b_m and b_m'. Let b_{m-1}' be a vertex of B_{m-1} which is adjacent to b_m, and let W_6^{m-1} be a Hamiltonian path for $\langle B_{m-1} \rangle$ that has b_{m-1}' as an end vertex. Let b_{m-1} be the other end vertex of W_6^{m-1}.

We continue in this fashion to obtain paths W_6^i and pairs of vertices b_i, b_i' where for each $i \in \{1, 2, \ldots, m\}$, the path W_6^i is a Hamiltonian path for $\langle B_i \rangle$ with end vertices b_i and b_i' and where for each $i \in \{2, 3, \ldots, m\}$, b_i is adjacent to b_{i-1}'.

Let W_6' be a Hamiltonian path for $\langle S_5^B \rangle$, say with endpoints s_5 and s_5' ($s_5 = s_5'$ if $|S_5^B| = 1$), and let W_6'' be a Hamiltonian path for $\langle S_3 \rangle$, with endpoints s_3 and s_3'.

From Claim 2.9, we know that $s_5' \in S_5^B$ is adjacent to $b_2 \in B_2$. Further, it follows from the definition of the set S_5^B and from Claim 2.8 that s_5 is adjacent to b_1'. Moreover, it follows from Claim 2.7 that b_1 is adjacent to s_3' and that s_3 is adjacent to v.

Let the path W_6 be described by

$$W_6\colon v, [s_3, s_3']_{W_6''}, [b_1, b_1']_{W_6^1}, [s_5, s_5']_{W_6'}, [b_2, b_2']_{W_6^2}, \ldots, [b_m, b_m']_{W_6^m}.$$

This path W_6 is the path we seek for this case. □

Having completed this series of claims, we can now proceed to "piece together" our Hamiltonian path for G.

Let W_1 be a path as described in Claim 2.12. Let a_l be an end vertex that is in A_l, and let x be the other end vertex. Given this $a_l \in A_l$, let W_2 be a path that satisfies the statement of Claim 2.13, and let $a_{l-1} \in A_{l-1}$ be the other end vertex of W_2. Given this $a_{l-1} \in A_{l-1}$, let W_3 be a path with the properties given in Claim 2.14, and let $a_3 \in A_3$ be the other end vertex of W_3. Next, given this $a_3 \in A_3$, let W_4 be a path that fits the description given in Claim 2.15, and let $p \in A_2 \cup S_4$ be the other end vertex of W_4. Applying Claim 2.16 to this vertex p, let W_5 be a path as described in the statement of the claim. Finally, let W_6 be a path with the properties given in Claim 2.17, with end vertices v and some y.

A Hamiltonian path for G is then given by

$$[x, a_l]_{W_1}, (a_l, a_{l-1}]_{W_2}, (a_{l-1}, a_3]_{W_3}, (a_3, p]_{W_4}, (p, v]_{W_5}, (v, y]_{W_6}.$$

G is therefore traceable, and the proof of the theorem is complete. □

Note here that

$$n \geqslant \frac{4}{3}\left(2\sum_{i=1}^{2k}(r-1)^i\right)$$

suffices in the theorem. Also note that if $r < 4$ and/or $k < 2$, the result follows from the pairs work in [2].

Corollary 2.2. *Let $r \geqslant 4$ and $k \geqslant 2$ be fixed integers. Let R, S and T be connected induced subgraphs of $K_{1,3}$, Q_r, and N_k, respectively. If G is a connected graph of order n that is $\{R, S, T\}$-free, and if n is sufficiently large, then G is traceable.*

3. The characterization

In this section we give a characterization of the triples of subgraphs that imply traceability when forbidden. Note here that since being P_3-free implies completeness (and thus traceability), any pair or triple that involves P_3 will of course also imply traceability. In [2] five other pairs are shown to imply traceability when forbidden. Each pair consists of the claw and one of the five following graphs: N_1, B, Z_1, K_3, P_4 (see Fig. 1). If a triple contains any of these pairs then that triple will also (trivially)

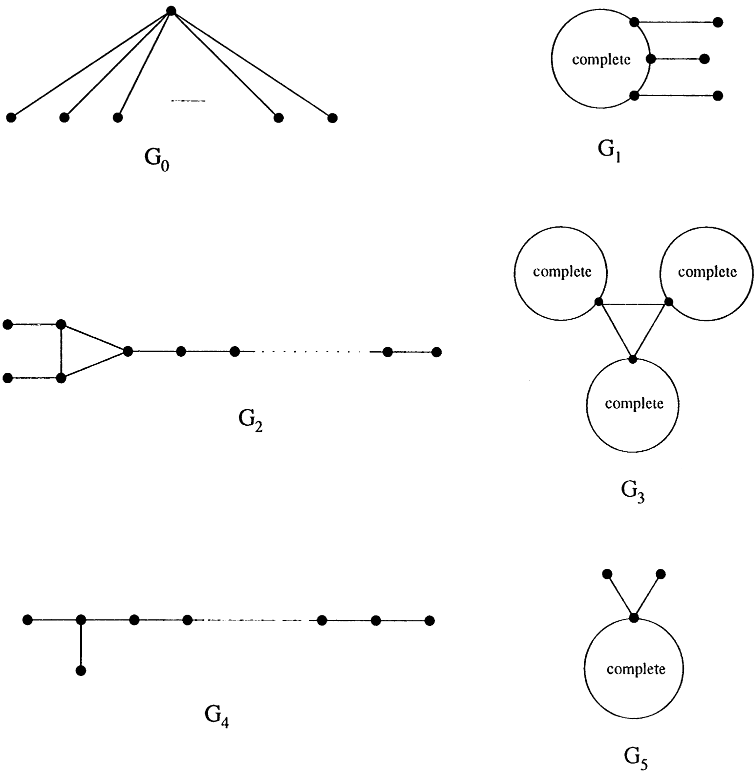


Fig. 14. Infinite families of nontraceable graphs.

imply the existence of a Hamiltonian path. In view of this, the theorem in this section gives a characterization of all "nontrivial" families of triples that enjoy this property.

In what follows, CI(G) denotes the set of connected induced subgraphs of a given graph G.

Theorem 3.1. *Let R, S, T ($\neq P_3$) be connected graphs such that no forbidden pair of them implies traceability. If being $\{R,S,T\}$-free implies traceability, then one of the following is true* (*up to the ordering of R, S, and T*):

1. $R = K_{1,m}$, $S = Y_l$, $T \in \mathrm{CI}(Z_1)$ *for some* $m \geqslant 4$, $l \geqslant 4$;
2. $R = K_{1,m}$, $S = P_4$, $T \in \mathrm{CI}(V_r)$ *for some* $m \geqslant 4$, $r \geqslant 3$;
3. $R = K_{1,m}$, $S = P_l$, $T \in \mathrm{CI}(Q_r)$ *for some* $m \geqslant 4$, $l \geqslant 5$, $r \geqslant 3$;
4. $R = K_{1,3}$, $S \in \mathrm{CI}(Q_r)$, $T \in \mathrm{CI}(N_k)$ *for some* $r \geqslant 4$, $k \geqslant 2$;
5. $R = K_{1,3}$, $S \in \mathrm{CI}(E_r)$, $T = Z_2$ *for some* $r \geqslant 4$.

Proof. Suppose that being $\{R,S,T\}$-free implies traceability, and consider the infinite families of nontraceable graphs in Fig. 14.

Case 1: Suppose that none of R, S, or T is isomorphic to $K_{1,3}$.

Consider the graph G_0. It is nontraceable, and so it must be that G_0 contains one of R, S, or T as an induced subgraph. Suppose, without loss of generality, that G_0 contains R.

Then $R=K_{1,r}$ for some $r\geqslant 4$ (if $r=2$ or $r=3$, then $R=P_3$ or $R=K_{1,3}$, respectively, and each of these is a contradiction to our assumptions).

We see that the graph G_5 is nontraceable and R-free, and so G_5 must contain either S or T as an induced subgraph. Assume without loss that G_5 contains T. Therefore $T\in \mathrm{CI}(V_m)$ for some $m\geqslant 3$ (again, if $m<3$ we get contradictions).

At this point we can see that the graph G_4 is nontraceable and $\{R,T\}$-free, and so it must be that G_4 contains S. Thus, $S\in \mathrm{CI}(G_4)$. That is, either $S=Y_l$ or $S=P_l$ for some $l\geqslant 4$ ($l<4$ again contradicts our assumptions).

Subcase 1.1: Suppose $S=Y_l$ for some $l\geqslant 4$.

The graph G_1 is also nontraceable and $\{R,S\}$-free, and so it must be that T is contained in G_1. This means that $T\in \mathrm{CI}(E_m)$ for some $m\geqslant 3$. Since it is also true that $T\in \mathrm{CI}(V_m)$ for some $m\geqslant 3$, we can conclude that $T\in \mathrm{CI}(Q_m)$ for some $m\geqslant 3$.

Since G_2 is also nontraceable and $\{R,S\}$-free, T must also be an induced subgraph of G_2. Therefore it must be that $T\in \mathrm{CI}(Z_1)$.

So, in this subcase we have

$$R=K_{1,r}, S=Y_l, T\in \mathrm{CI}(Z_1).$$

Subcase 1.2: Suppose $S=P_l$ for some $l\geqslant 4$.

Subcase 1.2.1: Suppose $S=P_4$.

In this case, we simply have

$$R=K_{1,r}, S=P_4, T\in \mathrm{CI}(V_m).$$

Subcase 1.2.2: Suppose that $S=P_l$ for some $l\geqslant 5$.

Consider the graph G_1. It is both nontraceable and $\{R,S\}$-free. Therefore T must be an induced subgraph of G_1. Since we also know that T is an induced subgraph of V_m for some $m\geqslant 3$, it must be that $T\in \mathrm{CI}(Q_m)$ for some $m\geqslant 3$.

Thus in this subcase we have

$$R=K_{1,r}, S=P_l, T\in \mathrm{CI}(Q_m).$$

Case 2: Suppose that one of R, S, or T is $K_{1,3}$.

Suppose without loss that $R=K_{1,3}$.

Consider the graph G_1. It is nontraceable and $K_{1,3}$-free, so it must be that G_1 contains one of S or T. Suppose, again without loss, that G_1 contains S. Then $S\in \mathrm{CI}(E_r)$ for some $r\geqslant 4$ (note that $r\neq 3$ since then $E_r=N$, and then R,S would be a forbidden pair that implied traceability). More specifically, we can say that $S\in \mathrm{CI}(E_r)\setminus \mathrm{CI}(N)$ for some $r\geqslant 4$.

Now consider the graph G_2. G_2 is nontraceable and $\{R,S\}$-free. Thus G_2 must contain T as an induced subgraph. Hence, $T\in \mathrm{CI}(N_k)\setminus \mathrm{CI}(N)$ for some $k\geqslant 2$ (note that $k=1$ would again yield an N).

Subcase 2.1: Suppose $T\neq Z_2$.

Consider the graph G_3. G_3 is nontraceable and $\{R,T\}$-free, so S must be an induced subgraph of G_3. But we know from before that $S \in \mathrm{CI}(E_r)\backslash\mathrm{CI}(N)$ for some $r \geqslant 4$. Thus, S is an induced subgraph of both G_1 and G_3. Hence we can conclude that $S \in \mathrm{CI}(Q_r)$ for some $r \geqslant 4$.

Therefore, in this subcase, we have

$$R = K_{1,3},\ S \in \mathrm{CI}(Q_r),\ T \in \mathrm{CI}(N_k).$$

Subcase 2.2: Suppose $T = Z_2$.

In this case, we simply have

$$R = K_{1,3},\ S \in \mathrm{CI}(E_r),\ T = Z_2.$$

The proof of the theorem is complete. □

4. For further reading

The following reference is also of interest to the reader: [1]

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