Invariant functionals and the uniqueness of invariant norms

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Dedicated with love to my wife M. Ángeles

Abstract

Let $\tau$ be a representation of a compact group $G$ on a Banach space $(X, \| \cdot \|)$. The question we address is whether $X$ carries a unique invariant norm in the sense that $\| \cdot \|$ is the unique norm on $X$ for which $\tau$ is a representation. We characterize the uniqueness of norm in terms of the automatic continuity of the invariant functionals in the case when $X$ is a dual Banach space and $\tau$ is a $\sigma(X, X_*)$-continuous representation of $G$ on $X$ such that $\tau(G)$ consists of $\sigma(X, X_*)$-continuous operators. We illustrate the usefulness of this characterization by studying the uniqueness of the norm on the spaces $L^p(\Omega)$, where $\Omega$ is a locally compact Hausdorff space equipped with a positive Radon measure and $G$ acts on $\Omega$ as a group of continuous invertible measure-preserving transformations.

1. Introduction

Over the past years tremendous attention has been paid to the uniqueness-of-norm problem in the context of Banach algebras. The most important result in this area is the famous theorem by Johnson [11] that every semisimple Banach algebra carries a...
unique Banach algebra norm. This was a historically important question raised by C.E. Rickart in 1950. The state-of-the-art is well documented in the comprehensive account of Dales [3]. According to Johnson’s theorem, \( L^1(G) \) carries a unique Banach algebra norm for any locally compact group \( G \). Recently quite a lot of progress has been made in the understanding of the intimate connection between the translations and the norm of the classical Banach spaces related to a locally compact group \( G \), such as \( L^p(G) \) with \( 1 \leq p \leq +\infty \). Investigation about this subject started with the seminal paper by Jarosz [10] about the uniqueness of translation invariant norms on \( L^1(\mathbb{R}) \) and rotation invariant norms on \( L^p(\mathbb{T}) \) \( (1 \leq p \leq \infty) \) and it has been successfully carried out for arbitrary locally compact abelian groups [5,26] and for compact (non-abelian) groups [6].

Though the question of the uniqueness of translation invariant norms seems to be very recent, it turned out in [5,6] that it is closely related to the classical problem of determining whether or not there is a discontinuous translation invariant functional on \( L^p(G) \). The question which spaces \( X \) have discontinuous translation invariant functionals has been much studied; see [14,16] for surveys. As a matter of fact, motivated by the automatic continuity of the translation invariant functionals on \( L^p(G) \), Willis showed in [27, Theorem 3.2] that if \( G \) contains the free group \( \mathbb{F}_2 \) as a closed subgroup, \( X \) is a Banach space on which \( G \) acts, and \( S : X \rightarrow L^1(G) \) is a linear operator commuting with translations, then \( S \) is continuous. This entails that every complete norm \( \| \cdot \| \) on \( L^1(G) \) making all the left translation operators from \( (L^1(G), \| \cdot \|) \) into itself continuous is equivalent to the norm \( \| \cdot \|_1 \). We say that \( L^1(G) \) carries a unique topologically invariant norm. Likewise, it can be deduced from [12, Corollary 2] that if \( G \) is a locally compact group containing \( \mathbb{Z} \) as a closed subgroup and \( X \) is a Banach space on which \( G \) acts boundedly, then every linear operator \( S : X \rightarrow L^1(G) \) commuting with translations is continuous. This clearly forces that every complete norm \( \| \cdot \| \) on \( L^1(G) \) with the property that translation operators from \( (L^1(G), \| \cdot \|) \) onto itself are isometries is equivalent to the norm \( \| \cdot \|_1 \). In this case we say that \( L^1(G) \) carries a unique invariant norm.

In this paper we study the uniqueness-of-invariant-norm problem for an arbitrary Banach space \( (X, \| \cdot \|) \) on which a compact group \( G \) acts. The problem is to decide whether each complete norm \( \| \cdot \| \) on \( X \) making translations continuous is equivalent to the norm \( \| \cdot \| \). Our approach relies on the invariant functionals on \( X \). The paper is organized as follows. In Section 2 we introduce the terminology and some technical devices that we shall use throughout the paper. In Section 3 it is pointed out that the class of invariant functionals on \( X \) is too small to characterize uniqueness of invariant norm for an arbitrary Banach space \( X \) on which \( G \) acts. For this characterization we introduce the auxiliary spaces \( L(H_\pi) \otimes X \), where \( \pi \) ranges through the irreducible unitary representations of \( G \) and \( H_\pi \) is the representation space of \( \pi \). We define the notions of \( \pi \)-invariant element in \( L(H_\pi) \otimes X \) and of \( \pi \)-invariant functional on \( L(H_\pi) \otimes X \) and show that \( X \) does not carry a unique invariant norm when there exist, for some \( \pi \), discontinuous \( \pi \)-invariant functionals on \( L(H_\pi) \otimes X \) and a non-zero \( \pi \)-invariant element in \( L(H_\pi) \otimes X \). Section 4 contains the main theorem: if \( \tau \) is a \( \sigma(X, X_\tau) \)-continuous representation of \( G \) on a dual
Banach space $X$ such that $\tau(G)$ consists of $\sigma(X, X_*)$-continuous operators, then $X$ carries a unique invariant norm if and only if every $\pi$-invariant functional is continuous whenever $\pi$ is such that there exists a non-zero $\pi$-invariant element. Finally, Section 5 is devoted to illustrating the usefulness of our characterization when considering the Banach spaces $L^p(\Omega)$, where $\Omega$ is a locally compact Hausdorff space equipped with a positive Radon measure on which $G$ acts as a group of continuous invertible measure-preserving transformations. It turns out that the property that the classical norms $\| \cdot \|_p$ are the unique norms on $L^p(\Omega)$ which are well-behaved with respect to translations is equivalent to the transitivity of the action and the property that every translation invariant functional on $L^p(G)$ is a multiple of the Haar integral. We thus extend the results of [6] and we find the complete analogue of the result for $\mathbb{T}$ in [10] for all the $N$-dimensional Euclidean spheres $S^N$; for $N \geq 2$ every complete norm on either $L^p(S^N)$ with $1 < p$ or $C(S^N)$ making rotations continuous is equivalent to the classical norm on the space.

2. Preliminaries

In this section we fix the notation and recall some basic definitions.

From now on $G$ denotes a compact group with left Haar measure $\lambda_G$ normalized so that $\lambda_G(G) = 1$.

2.1. Representations on Banach spaces

Let $L(X)$ denote the Banach algebra of all continuous linear operators on a given nonzero complex Banach space $X$, and let $X^*$ be the topological dual space of $X$. Let $X_*$ be any linear subspace of $X^*$. As usual, $\sigma(X, X_*)$ stands for the coarsest topology on $X$ for which each of the functionals of $X_*$ is continuous. For every $T \in L(X)$, $T^* \in L(X^*)$ stands for the adjoint operator of $T$. We shall use the same symbol to denote the norm on both spaces $X$ and $L(X)$ and we shall say ‘norm’ for ‘complete norm’.

A representation of $G$ on a Banach space $(X, \| \cdot \|)$ is a group homomorphism $\tau: G \to L(X)$ from $G$ into the group of all invertible elements of $L(X)$. For all $t \in G$ and $x \in X$ we call $\tau(t)x$ the translate of $x$ by $t$. Note that we do not require any continuity of the homomorphism $\tau$. In [27] is said that $X$ is a Banach space on which $G$ acts. The representation is said to be bounded if there exists a constant $C$ such that $\|\tau(t)\| \leq C$ for each $t \in G$ and, in this situation, $X$ becomes a Banach $G$-module in the sense of [12]. It is worth pointing out that by defining $|x| = \sup_{t \in G} |\tau(t)x|$ for each $x \in X$ we obtain a norm on $X$ which is equivalent to $\| \cdot \|$ and, with respect to this new norm, $\tau(G)$ consists of isometries. Thus when considering a bounded representation $\tau$ of $G$ we shall always assume that $\tau(G)$ consists of isometries. The representation is said to be strongly continuous if the map $t \mapsto \tau(t)x$ is continuous for each $x \in X$. Let $X_*$ be a linear subspace of $X^*$. We call the representation $\sigma(X, X_*)$-continuous if the function $t \mapsto \xi(\tau(t)x)$ is continuous on $G$ for all $x \in X$ and $\xi \in X_*$. Of course, every
strongly continuous representation is \( \sigma(X, X') \)-continuous. It is worth pointing out that every \( \sigma(X, X') \)-continuous representation of \( G \) is automatically strongly continuous (see [22, Lemma 1.4.2]). A key fact we shall use throughout the paper is that, under a certain additional assumption on the space \( X' \), every \( \sigma(X, X') \)-continuous representation of \( G \) on \( X \) becomes integrable. This will follow from the next result.

**Lemma 2.1.** Let \( X \) be a Banach space, let \( X' \) be a linear subspace of \( X' \), and let \( f : G \rightarrow X \) be a \( \sigma(X, X') \)-continuous map. Suppose that one of the following conditions holds:

(i) \( X' = X' \);

(ii) \( X \) is a dual Banach space and \( X' \) is a predual of \( X \).

Then, for every \( \mu \in M(G) \), \( f \) is \( \mu \)-integrable in the sense that there exists exactly one element \( \int_G f(t) \, d\mu(t) \in X \) with the property that

\[
\xi \left( \int_G f(t) \, d\mu(t) \right) = \int_G \xi(f(t)) \, d\mu(t)
\]

for each \( \xi \in X' \).

**Proof.** On account of [21, Theorem 3.27 and its Remark], \( f \) is \( \mu \)-integrable for each \( \mu \in M(G) \) whenever the \( \sigma(X, X') \)-closed convex hull of every \( \sigma(X, X') \)-compact subset of \( X \) is \( \sigma(X, X') \)-compact. In the case (i), this assumption holds by the Krein–Šmulian theorem on weak compactness while in the case (ii), this assumption follows from the Banach–Alaoglu theorem.

In the representation theory of locally compact groups, the unitary representations play the predominant role. As usual, a **unitary representation** of \( G \) is a strongly continuous representation \( \pi \) of \( G \) on some Hilbert space \( H_\pi \) such that \( \pi(G) \) consists of unitary operators. \( \pi \) is said to be **irreducible** if \( \{0\} \) and \( H_\pi \) are the only closed subspaces of \( H_\pi \) that are invariant under \( \pi(G) \). Since \( G \) is compact, it is well-known that every irreducible unitary representation of \( G \) is finite-dimensional (see [8, Theorem 22.13] for example). In this situation, when an orthonormal basis \( \{e_i\}_{i=1}^n \) of \( H_\pi \) is fixed, \( \pi(t) \) is represented by a unitary matrix \( \{\pi_{ij}(t)\} \), where \( \pi_{ij}(t) = \langle \pi(t)(e_j), e_i \rangle \) for all \( i, j \in \{1, \ldots, n\} \) and \( t \in G \). Note that \( \pi_{ij}(t) \) is a continuous complex-valued function on \( G \). Two unitary representations \( \pi \) and \( \pi' \) of \( G \) are said to be **equivalent** if there is a unitary operator \( U : H_\pi \rightarrow H_\pi' \) such that \( U \pi(t) = \pi'(t) U \) for each \( t \in G \). We shall denote by \( \hat{G} \) the set of equivalence classes of irreducible unitary representations of \( G \) and we shall denote by \( [\pi] \) the class of an irreducible unitary representation \( \pi \) of \( G \).

Let \( M(G) \) denote the Banach space of all bounded complex-valued regular Borel measures on \( G \). Recall that \( M(G) \) is a Banach \( \ast \)-algebra with the product given by convolution \( \star \) and involution given by \( \mu^\ast(E) = \mu(E^{-1}) \) for all \( \mu \in M(G) \) and \( E \in G \).
measurable. Every unitary representation $\pi$ of $G$ yields a norm decreasing unital algebra $*$-homomorphism $\pi^{M(G)}$ from $M(G)$ into the $C^*$-algebra $L(H_{\pi})$ which is defined by
\[ \pi^{M(G)}(\mu) = \int_G \pi(t) d\mu(t) \]
for each $\mu \in M(G)$. To shorten notation, we continue to write $\pi$ for $\pi^{M(G)}$ when no confusion can arise.

2.2. Adjoint representations

Let $\tau$ be a representation of $G$ on a Banach space $X$. Then it is easily seen that the map $\tau^* : G \rightarrow L(X^*)$ defined by $\tau^*(t) = \tau(t^{-1})^*$ for each $t \in G$ is a representation of $G$ on $X^*$, which is $\sigma(X^*, X)$-continuous in the case where $\tau$ is $\sigma(X, X^*)$-continuous. We call $\tau^*$ the adjoint representation of $\tau$.

In order to establish the most important results of this paper we are required to restrict our attention to certain adjoint representations of $G$ on dual Banach spaces. In the next result we give a characterization of these representations.

**Lemma 2.2.** Let $\tau$ be a representation of $G$ on a dual Banach space $X$ and let $X_s$ be a predual of $X$. Then the following assertions are equivalent:

(i) There exists a strongly continuous representation $\tau_s$ of $G$ on $X_s$ such that $\tau = (\tau_s)^*$.

(ii) $\tau$ is $\sigma(X, X_s)$-continuous and $\tau(G)$ consists of $\sigma(X, X_s)$-continuous operators.

**Proof.** Of course we only need to show that (ii) implies (i). Let $t \in G$. Then $\tau(t^{-1})$ is a $\sigma(X, X_s)$-continuous operator on $X$ and therefore there exists exactly one operator $\tau_s(t) \in L(X_s)$ such that $\tau_s(t)^* = \tau(t^{-1})$. It is clear that $\tau_s$ is a representation of $G$ on $X_s$. In fact, from the $\sigma(X, X_s)$-continuity of $\tau$ it follows that $\tau_s$ is $\sigma(X_s, X)$-continuous and therefore it is strongly continuous. \qed

2.3. Example

Let $\Omega$ be a locally compact Hausdorff space on which $G$ acts. This means that there exists a continuous map $(t, \omega) \mapsto t\omega$ from $G \times \Omega$ into $\Omega$ such that $e\omega = \omega$ and $s(t\omega) = (st)\omega$ for all $s, t \in G$ and $\omega \in \Omega$. We define the translates of every function $f : \Omega \rightarrow \mathbb{C}$ by $(\tau(t)f)(\omega) = f(t^{-1}\omega)$ for all $\omega \in \Omega$ and $t \in G$.

Let $C_0(\Omega)$ be the Banach space of all complex-valued continuous functions on $\Omega$ vanishing at infinity. It is shown in [1, Proposition VIII, Section 2.5] that $\tau$ gives a strongly continuous representation of $G$ on $C_0(\Omega)$. The dual of $C_0(\Omega)$ can be identified with the Banach space $M(\Omega)$ of (bounded) complex-valued regular Borel measures on $\Omega$. The left translates of a measure $\mu \in M(\Omega)$ are defined by $\tau(t)\mu(E) = \mu(\{t^{-1}x : x \in E\})$ for each $t \in G$ and $E \subseteq \Omega$. For each $\mu \in M(\Omega)$ we define the strongly continuous representation $\pi^{M(\Omega)}(\mu)$ of $\Omega$ on $C_0(\Omega)$ by $\pi^{M(\Omega)}(\mu)(f) = \int\phi d\mu = \int \phi f d\mu$ for each $f \in C_0(\Omega)$ and $\phi \in C_0(\Omega)$.
\(\mu(t^{-1}E)\) so that we obtain a representation of \(G\) on \(M(\Omega)\). This is the adjoint of the representation of \(G\) on \(C_0(\Omega)\).

We now assume \(\Omega\) to carry a positive invariant Radon measure \(\lambda_\Omega\). We denote by \(L^p(\Omega)\) \((1 \leq p)\) the Banach space of all measurable complex-valued functions \(f\) (or rather, equivalence classes thereof) on \(\Omega\) with

\[
\|f\|_p = \left( \int_\Omega |f(\omega)|^p \, d\lambda_\Omega(\omega) \right)^{1/p} < \infty \quad (p < \infty)
\]

\[
\|f\|_\infty = \inf \left\{ \sup_{\omega \in \Omega, Z} |f(\omega)| : Z \text{ is locally null} \right\} < \infty.
\]

It is shown in [1, Proposition VIII, Section 2.9] that \(\tau\) gives a strongly continuous representation of \(G\) on \(L^p(\Omega)\) for each \(1 \leq p < \infty\). Of course the representation of \(G\) on \(L^1(\Omega)\) is nothing but the adjoint of the representation of \(G\) on \(L^0(\Omega)\). Furthermore, this identification commutes with translations so that we can think of \(L^1(\Omega)\) as being an invariant subspace of \(M(\Omega)\).

The action of \(G\) on itself and the action of the isometries on a Riemannian manifold are of special interest.

It is obvious that the translation operators are isometries from the Banach spaces \((C_0(\Omega), \|\cdot\|_\infty)\) and \((L^\infty(\Omega), \|\cdot\|_\infty)\) onto themselves. The invariance of the measure \(\lambda_\Omega\) entails that the translation operators are also isometries from \((L^p(\Omega), \|\cdot\|_p)\) onto itself for each \(1 \leq p\). The question we wish to address is whether the classical norms \(\|\cdot\|_p\) are the unique norms which are so well-behaved with respect to translations.

Until further notice \((X, \|\cdot\|)\) stands for a Banach space which is equipped with a \(\sigma(X, X^*)\)-continuous representation \(\tau\) of \(G\) on \((X, \|\cdot\|)\), where either \(X^* = X^*\) or \(X\) is a dual Banach space and \(X^*\) is a predual of \(X\). It should be noted that \(\tau\) is bounded and thus we can certainly assume that \(\tau(G)\) consists of isometries. Indeed, the continuity of the map \(t \mapsto \tilde{\xi}(t)\) from \(G\) into \(X\) together with the compactness of \(G\) imply that the map is bounded for all \(x \in X\) and \(\xi \in X^*\). The uniform boundedness theorem now shows that the subset \(\{\tau(t) : t \in G\}\) of \(L(X)\) is bounded. Thus \(X\) is renormalized in the following so that \(\tau\) is a representation by isometries.

The Banach space \(X\) is said to carry a unique invariant/topologically invariant norm if every norm \(\|\cdot\|\) on \(X\) such that \(\tau(G)\) consists of isometries/homeomorphisms of the Banach space \((X, \|\cdot\|)\) is necessarily equivalent to \(\|\cdot\|\).

### 2.4. Convolutions and useful operators

According to Lemma 2.1, for arbitrary \(\mu \in M(G)\) and \(x \in X\), we can define \(\mu \star x \in X\) by

\[
\mu \star x = \int_G \tau(t)x \, d\mu(t).
\]
It is then straightforward to check that $X$ turns into a Banach left $M(G)$-module with the operation given by $\star$ and that the map $x \mapsto \lambda_g \star x$ is a projection onto the invariant elements of $X$. As usual by an invariant element of $X$ we mean an $x \in X$ with the property that

$$\tau(t)x = x$$

for each $t \in G$.

Remark 2.1. It is a simple matter to see that in the case when $X = M(G)$ the above defined operation is nothing but the classical convolution. This is the reason why we have chosen the notation $\star$.

We now consider $[\pi] \in \hat{G}$. Let $L(H_\pi) \hat{\otimes} X$ denote the projective tensor product of $L(H_\pi)$ and $X$. It should be noted that the underlying space of $L(H_\pi) \hat{\otimes} X$ is nothing but the algebraic tensor product $L(H_\pi) \otimes X$, which can be identified with the space $M_n(X)$ of all $n \times n$ matrices with entries from $X$, where $n = \dim H_\pi$ and we shall change freely between speaking of tensor products and of the corresponding matrix spaces. The Banach space $L(H_\pi) \hat{\otimes} X$ turns into a Banach $L(H_\pi)$-bimodule with the operations given by

$$S \cdot (T \otimes x) = (ST) \otimes x = (S \otimes x) \cdot T$$

for all $S, T \in L(H_\pi)$ and $x \in X$. Given another finite-dimensional Hilbert space $H$ and linear operators $U : H_\pi \to H$ and $V : H \to H_\pi$ we define $U \cdot \zeta \in L(H, H_\pi) \otimes X$ and $\zeta \cdot V \in L(H_\pi, H) \otimes X$ for all $\zeta \in L(H_\pi) \otimes X$ and $x \in X$ through $U \cdot (S \otimes x) = (US) \otimes x$ and $(T \otimes x) \cdot V = (TV) \otimes x$ for all $S \in L(H_\pi), T \in L(H)$, and $x \in X$ (where the meaning of both $L(H, H_\pi)$ and $L(H_\pi, H)$ is clear enough). On the other hand, we define a representation of $G$ on $L(H_\pi) \hat{\otimes} X$ through

$$\tau(t)(T \otimes x) = T \otimes \tau(t)x$$

for all $T \in L(H_\pi), x \in X$, and $t \in G$. Moreover we have $\tau(t)(S \cdot \zeta \cdot T) = S \cdot \tau(t)\zeta \cdot T$ for all $S, T \in L(H_\pi)$ and $\zeta \in L(H_\pi) \otimes X$, and this property is used frequently in what follows. It should be pointed out that the map $x \mapsto I \otimes x$ is an isometry from $X$ into $L(H_\pi) \hat{\otimes} X$ which commutes with translations so that $X$ is thought of in the following as being an invariant closed subspace of $L(H_\pi) \hat{\otimes} X$. By abuse of notation, we write $x$ instead of $I \otimes x$ for $x \in X$. Taking into account the usual identification of $(L(H_\pi) \hat{\otimes} X)_\pi$ with the injective tensor product $L(H_\pi)^* \hat{\otimes} X_\pi$ we can see that the representation of $G$ on $L(H_\pi) \hat{\otimes} X$ is $\sigma(L(H_\pi) \hat{\otimes} X, (L(H_\pi) \hat{\otimes} X)_\pi)$-continuous. Furthermore, we define a new representation $\tau^\pi$ of $G$ on $L(H_\pi) \hat{\otimes} X$ by

$$\tau^\pi(t)\zeta = \pi(t) \cdot \tau(t)\zeta.$$
for each \( \zeta \in L(H_\pi) \otimes X \). The representation \( \tau^\pi \) is \( \sigma(L(H_\pi) \otimes X, (L(H_\pi) \otimes X)_o) \)-continuous. We define a linear operator \( \pi^\beta : L(H_\pi) \otimes X \to L(H_\pi) \otimes X \) by

\[
\pi^\beta(\zeta) = \lambda_G \star \pi^\zeta = \int_G \pi(t) \cdot \tau(t) \zeta \ d\lambda_G(t)
\]

for each \( \zeta \in L(H_\pi) \otimes X \).

**Remark 2.2.** When an orthonormal basis of \( H_\pi \) is fixed we can think of \( \pi^\beta((x_{ij})) \) as being the matrix of \( M_n(X) \) given by

\[
\pi^\beta((x_{ij})) = \left( \sum_{k=1}^n \pi_{ik} \star x_{kj} \right)
\]

for each \( (x_{ij}) \in M_n(X) \).

We now gather together a few properties of \( \pi^\beta \) that we shall use in the sequel.

**Lemma 2.3.** Let \( \pi \) be a unitary representation of \( G \). Then the following assertions hold:

(i) \( \pi^\beta \) is continuous with \( ||\pi^\beta|| \leq 1 \);
(ii) \( (\pi^\beta)^2 = \pi^\beta \);
(iii) \( \pi^\beta(\mu \star x) = \pi^\beta(x) \cdot \pi(\bar{\mu}) \) and \( \mu \star \pi^\beta(\zeta) = \pi(\bar{\mu}) \cdot \pi^\beta(\zeta) \) for all \( \mu \in M(G) \), \( x \in X \), and \( \zeta \in L(H_\pi) \otimes X \).

**Proof.** Assertions (i) and (ii) are immediate from the definition.

What is left is to prove (iii). For all \( \mu \in M(G) \), \( x \in X \), and \( \zeta \in L(H_\pi) \otimes X \), we have

\[
\pi^\beta(\mu \star x) = \pi^\beta \left( \int_G \tau(t) x \ d\mu(t) \right) = \int_G \pi^\beta(\tau(t) x) \ d\mu(t) \\
= \int_G \left( \int_G \pi(s) \otimes \tau(s)(\tau(t) x) \ d\lambda_G(s) \right) \ d\mu(t) \\
= \int_G \left( \int_G \pi(ut^{-1}) \otimes \tau(u) x \ d\lambda_G(u) \right) \ d\mu(t) \\
= \int_G \pi^\beta(x) \cdot \pi(t^{-1}) \ d\mu(t) = \pi^\beta(x) \cdot \pi(\bar{\mu})
\]
and
\[
\mu \star \pi^\beta(\zeta) = \int_G \tau(t)(\pi^\beta(\zeta)) \, d\mu(t)
\]
\[
= \int_G \tau(t) \left( \int_G \pi(s) \cdot \tau(s) \zeta \, d\lambda_G(s) \right) \, d\mu(t)
\]
\[
= \int_G \left( \int_G \pi(s) \cdot \tau(t)(\tau(s) \zeta) \, d\lambda_G(s) \right) \, d\mu(t)
\]
\[
= \int_G \left( \int_G \pi(t^{-1}u) \cdot \tau(u) \zeta \, d\lambda_G(u) \right) \, d\mu(t)
\]
\[
= \int_G \left( \tau(t^{-1}) \cdot \int_G \pi(u) \cdot \tau(u) \zeta \, du \right) \, d\mu(t)
\]
\[
= \int_G \pi(t^{-1}) \cdot \pi^\beta(\zeta) \, d\mu(t) = \pi(\mu^* \cdot \pi^\beta(\zeta)).
\]

**Remark 2.3.** On account of Lemma 2.3 and Remark 2.2, we have
\[
\pi_{ij} \star \tau(t)x = \sum_{k=1}^n \overline{\pi_{jk}(t)} \pi_{ik} \star x
\]
and
\[
\tau(t)(\pi_{ij} \star x) = \sum_{k=1}^n \overline{\pi_{ki}(t)} \pi_{kj} \star x
\]
for all \(x \in X\) and \(t \in G\).

**Lemma 2.4.** The following assertions hold:

(i) If \([\pi], [\varpi] \in \hat{G}\) and \(U : H_\pi \to H_\varpi\) is a unitary operator such that \(U \pi(t) = \varpi(t)U\) for each \(t \in G\), then \(U \cdot \pi^\beta(x) = \varpi^\beta(x) \cdot U\) for each \(x \in X\);

(ii) If \(x \in X\) is such that \(\pi^\beta(x) = 0\) for each \([\pi] \in \hat{G}\), then \(x = 0\).

**Proof.** (i) For every \(x \in X\), we have
\[
U \cdot \pi^\beta(x) = \int_G U \cdot (\pi(t) \otimes \tau(t)x) \, d\lambda_G(t)
\]
\[
= \int_G (U \pi(t)) \otimes \tau(t)x \, d\lambda_G(t) = \int_G (\varpi(t)U) \otimes \tau(t)x \, d\lambda_G(t)
\]
\[
= \int_G (\varpi(t) \otimes \tau(t)x) \cdot U \, d\lambda_G(t) = \varpi^\beta(x) \cdot U.
\]
(ii) From the continuity of the functionals $T \otimes x \mapsto \langle Tu, v \rangle \tilde{\zeta}(x)$ for all $u, v \in H_\pi$ and $\tilde{\zeta} \in X_\ast$ we deduce that
\[
\int_G \langle \pi(t)u, v \rangle \tilde{\zeta}(\tau(t)x) \, d\lambda_G(t) = 0
\]
for all $\pi \in \hat{G}$, $u, v \in H_\pi$, and $\tilde{\zeta} \in X_\ast$. Consequently,
\[
\int_G p(t)\tilde{\zeta}(\tau(t)x) \, d\lambda_G(t) = 0
\]
for each trigonometric polynomial $p$ and $\tilde{\zeta} \in X_\ast$. Since trigonometric polynomials are dense in $C(G)$ with the uniform norm [9, Theorem 27.39], it may be concluded that $\int_G f(t)\tilde{\zeta}(\tau(t)x) \, d\lambda_G(t) = 0$ for all $f \in C(G)$ and $\tilde{\zeta} \in X_\ast$. This yields $\tilde{\zeta}(\tau(t)x) = 0$ for all $t \in G$ and $\tilde{\zeta} \in X_\ast$, which implies that $\tau(t)x = 0$ for each $t \in G$ and so $x = 0$.

In order to measure the size of a subset $M$ of $X$ we shall consider the subset $\Delta(M)$ of $\hat{G}$ defined by
\[
\Delta(M) = \{ [\pi] \in \hat{G} : \pi^\ast(M) \neq \{0\} \}.
\]
This definition makes sense because, on account of Lemma 2.4(i), if $\varpi \in [\pi]$ and $U : H_\pi \to H_\varpi$ is a unitary operator such that $U\pi(t) = \varpi(t)U$ for each $t \in G$, then $\pi^\ast(M) = \{0\}$ if and only if $\varpi^\ast(M) = \{0\}$. On the other hand, on account of Lemma 2.4(ii), $M = \{0\}$ if and only if $\Delta(M) = \emptyset$.

For a given $[\pi] \in \hat{G}$, we call an element $\zeta \in L(H_\pi) \hat{\otimes} X$ $\pi$-invariant if it is invariant with respect to the representation $\tau^\pi$, which is equivalent to the property
\[
\tau(t)\zeta = \pi(t^{-1}) \cdot \zeta
\]
for each $t \in G$. Thus, the $\pi$-invariant elements can be viewed as a nonabelian generalization of the scalar $G$-submodules of [12]. On the other hand, the operator $\pi^\ast$ gives a projection from $L(H_\pi) \hat{\otimes} X$ on the $\pi$-invariant elements of $L(H_\pi) \hat{\otimes} X$.

Remark 2.4. When we fix an orthonormal basis on $H_\pi$, then with respect to this basis the $\pi$-invariant elements are the matrices $(x_{ij})$ whose entries satisfy
\[
\tau(t)x_{ij} = \sum_{k=1}^n \pi_{kj}(t)x_{kj}
\]
for all $i,j \in \{1, \ldots, n\}$ and $t \in G$. 
3. Nonuniqueness of invariant norms

It turned out in [5,6] that the uniqueness of invariant norms on the spaces $L^p(G)$ is intimately connected with the classical problem whether or not every invariant functional on $L^p(G)$ is automatically continuous. However a special feature of replacing the spaces $L^p(G)$ with an arbitrary Banach space $X$ is the fact that the class of invariant functionals of $X$ is far too restrictive, and we now become involved with a somewhat more general class of invariant functionals.

A linear functional $\phi$ on $X$ is said to be invariant if

$$\phi(\tau(t)x) = \phi(x)$$

for all $t \in G$ and $x \in X$. Let $[\pi] \in \hat{G}$. A linear functional $\phi: L(H_{\pi}) \otimes X \mapsto \mathbb{C}$ is said to be $\pi$-invariant if it is invariant with respect to the representation $\tau^\pi$. This is equivalent to

$$\phi(\tau(t)\zeta) = \phi(\pi(t^{-1}) \cdot \zeta)$$

for all $\zeta \in L(H_{\pi}) \otimes X$ and $t \in G$.

**Remark 3.1.** Let $[\pi] \in \hat{G}$ and let $(e_i)_{i=1}^n$ an orthonormal basis of $H_{\pi}$. Let $E_{ij}$ the linear operator from $H_{\pi}$ into itself defined by $E_{ij}(e_k) = \delta_{jk} e_i$ for $i,j,k = 1, \ldots, n$. It is clear that every linear functional $\phi$ on $L(H_{\pi}) \otimes X$ can be thought of as a matrix $(\phi_{ij})$ of linear functionals on $X$, where $\phi_{ij}(x) = \phi(E_{ij} \otimes x)$ for all $i,j \in \{1, \ldots, n\}$ and $x \in X$. Furthermore, $\phi$ is a $\pi$-invariant functional if and only if

$$\phi_{ij}(\tau(t)x) = \sum_{k=1}^n \pi_{jk}(t) \phi_{kj}(x)$$

for all $i,j \in \{1, \ldots, n\}$, $t \in G$, and $x \in X$.

**Example 3.1.** Let $G$ be a compact abelian group, let $\gamma \in \hat{G} \setminus \{1\}$, and let $X$ be a Banach space. Define $\tau(t)x = \overline{\gamma(t)}x$ for all $t \in G$ and $x \in X$. Then a linear functional $\phi$ on $X$ is invariant if and only if $\phi = 0$. Indeed, if $t \in G$ is such that $\gamma(t) \neq 1$ and $x \in X$, then $\phi(x) = \phi(\tau(t)x) = \phi(\overline{\gamma(t)}x) = \overline{\gamma(t)} \phi(x)$ and thus $\phi(x) = 0$. On the other hand, it is clear that every linear functional on $X$ is $\gamma$-invariant.

The preceding example shows that even in the case when every invariant functional on $X$ is automatically continuous, $X$ may carry discontinuous $\pi$-invariant functionals for some $[\pi] \in \hat{G}$.

**Theorem 3.1.** Let $G$ be a compact group, let $(X,|| \cdot ||)$ be a Banach space, and let $\tau$ be a $\sigma(X,X_*)$-continuous representation of $G$ on $X$ (where either $X_* = X^*$ or $X$ is a dual Banach space and $X_*$ is a predual of $X$). Suppose that there exist a non-zero $\pi$-invariant element $a$ and a discontinuous $\pi$-invariant functional $\phi$ for some $[\pi] \in \hat{G}$. Then there exists a...
complete norm $|\cdot|$ on $X$ which is not equivalent to $||\cdot||$ and $\tau$ is a representation of $G$ on $(X, |\cdot|)$ with $|\tau(t)| \leq ||\tau(t)||$ for each $t \in G$.

Proof. Since there is a $\pi$-invariant element, it follows that $\pi^\circ (L(H_x) \hat{\otimes} X) \neq \{0\}$. Since $\pi^\circ (T \otimes x) = \pi^\circ (x) \cdot T$ for all $T \in L(H_x)$ and $x \in X$, we conclude that $\pi^\circ (X) \neq \{0\}$. Let $u \in X$ such that $\pi^\circ (u) \neq 0$ and let $i, j \in \{1, \ldots, n\}$ be such that $\pi_{ij} \star u \neq 0$.

Let $\phi$ be a discontinuous $\pi$-invariant functional.

We claim that, for every $l \in \{1, \ldots, n\}$, the map $T_l : X \to X$ defined by

$$T_l(x) = \sum_{k=1}^n \phi(E_{kl} \otimes x) \pi_{kj} \star u$$

for each $x \in X$ commutes with all the translation operators. On account of Remarks 2.3 and 3.1, we have

$$T_l(\tau(t)x) = \sum_{k=1}^n \phi(E_{kl} \otimes \tau(t)x) \pi_{kj} \star u$$

$$= \sum_{k=1}^n \left( \sum_{m=1}^n \pi_{km}(t) \phi(E_{ml} \otimes x) \right) \pi_{kj} \star u$$

$$= \sum_{m=1}^n \left( \phi(E_{ml} \otimes x) \sum_{k=1}^n \pi_{km}(t) \pi_{kj} \star u \right)$$

$$= \sum_{m=1}^n \phi(E_{ml} \otimes x) \tau(t)(\pi_{mj} \star u) = \tau(t) T_l(x).$$

Our next objective is to show that there exists $l \in \{1, \ldots, n\}$ such that the map $T_l$ is discontinuous. To obtain a contradiction, suppose that $T_l$ is continuous for each $l \in \{1, \ldots, n\}$. Then the map $x \mapsto \pi_{ij} \star T_l(x)$ from $X$ into itself is continuous for each $l \in \{1, \ldots, n\}$. According to [9, Theorem 27.20(iii)], we have $\pi_{ij} \star \pi_{kj} = \frac{1}{n} \delta_{jk} \pi_{ij}$ for each $k \in \{1, \ldots, n\}$ and so $\pi_{ij} \star T_l(x) = \frac{1}{n} \phi(E_{jl} \otimes x) \pi_{ij} \star u$. Since $\pi_{ij} \star u \neq 0$, it follows that the functional $x \mapsto \phi(E_{jl} \otimes x)$ is continuous for each $l \in \{1, \ldots, n\}$. Since the linear subspace of $H_x$ given by

$$\left\{ \sum_{m=1}^N \alpha_m \pi(t_m) e_j : N \in \mathbb{N}, \; \alpha_1, \ldots, \alpha_N \in \mathbb{C}, \; t_1, \ldots, t_N \in G \right\}$$

is non-zero and $\pi$-invariant, we conclude that for every $k \in \{1, \ldots, n\}$ there exist $N_k \in \mathbb{N}, \; \alpha_{1k}, \ldots, \alpha_{N_k} \in \mathbb{C}$, and $t_{1k}, \ldots, t_{N_k} \in G$ such that

$$\sum_{m=1}^{N_k} \alpha_{mk} \pi(t_{mk}) e_j = e_k.$$
On the other hand, we have

$$\phi(E_{jl} \otimes \tau(t_{mk}^{-1})x) = \sum_{p=1}^{n} \pi_{jl}(t_{mk}^{-1})\phi(E_{pl} \otimes x) = \sum_{p=1}^{n} \langle e_j, \pi(t_{mk}^{-1})e_p \rangle \phi(E_{pl} \otimes x)$$

$$= \sum_{p=1}^{n} \langle \pi(t_{mk})e_j, e_p \rangle \phi(E_{pl} \otimes x)$$

and we thus get

$$\sum_{m=1}^{N_k} \alpha_m \phi(E_{jl} \otimes \tau(t_{mk}^{-1})x) = \sum_{p=1}^{n} \left( \sum_{m=1}^{N_k} \alpha_m \pi(t_{mk})e_j, e_p \right) \phi(E_{pl} \otimes x)$$

$$= \sum_{p=1}^{n} \langle e_k, e_p \rangle \phi(E_{pl} \otimes x) = \phi(E_{kl} \otimes x)$$

for all $k,l \in \{1, \ldots, n\}$ and $x \in X$. Since the functionals $x \mapsto \phi(E_{jl} \otimes \tau(t_{mk}^{-1})x)$ are continuous for all $k,l \in \{1, \ldots, n\}$ and $m \in \{1, \ldots, N_k\}$, we see that the functional $\phi(E_{kl} \otimes x)$ is continuous for all $k,l \in \{1, \ldots, n\}$, which clearly implies the continuity of the functional $\phi$, contrary to our assumption.

Pick $l \in \{1, \ldots, n\}$ such that $T_l$ is discontinuous. Let $\alpha \in \mathbb{C}\setminus\{0\}$ with $\alpha \notin \text{sp}(T_{lT_l(x)})$, where $\text{sp}(T_{lT_l(x)})$ stands for the spectrum of the restriction of $T_l$ to $T_l(X)$. We claim that the map $S: X \to X$ defined by $S(x) = \alpha x - T_l(x)$ is bijective. We first prove the injectivity. Assume that $\alpha x - T_l(x) = 0$. If $T_l(x) = 0$, then $\alpha x = 0$ and so $x = 0$. Since $\alpha T_l(x) = T_l(T_l(x))$, if $T_l(x) \neq 0$, then $\alpha \in \sigma(T_{lT_l(x)})$, a contradiction. Our next concern is the surjectivity. Let $y \in X$ and take

$$x = \frac{1}{\alpha} \left[ y + (\alpha I_{T_l(x)} - T_{lT_l(x)})^{-1}(T_l(y)) \right].$$

Since $I_{T_l(x)} + (\alpha I_{T_l(x)} - T_{lT_l(x)})^{-1}T_{lT_l(x)} = \alpha (\alpha I_{T_l(x)} - T_{lT_l(x)})^{-1}$, we have

$$T_l(x) = \frac{1}{\alpha} \left[ T_l(y) + (\alpha I_{T_l(x)} - T_{lT_l(x)})^{-1}T_l(T_l(y)) \right]$$

$$= \frac{1}{\alpha} [I_{T_l(x)} + (\alpha I_{T_l(x)} - T_{lT_l(x)})^{-1}T_{lT_l(x)}](T_l(y))$$

$$= \frac{1}{\alpha} [\alpha (\alpha I_{T_l(x)} - T_{lT_l(x)})^{-1}](T_l(y))$$

$$= (\alpha I_{T_l(x)} - T_{lT_l(x)})^{-1}(T_l(y)).$$
On the other hand, we have
\[ \mathcal{A}x = y + (\mathcal{A}I_{|T_i(x)|} - T_{|T_i(x)|})^{-1}(T_i(y)), \]
which together with the last identity above gives \( \mathcal{A}x - T_i(x) = y \), as required.

Consequently, the map \( | \cdot | \) defined on \( X \) by \( |x| = ||S(x)|| \) for each \( x \in X \) is a norm on \( X \) which is not equivalent to \( || \cdot || \). On the other hand, we have
\[ ||\tau(t)|| = ||S(\tau(t)x)|| = ||\tau(t)S(x)|| = ||\tau(t)|| ||S(x)|| = ||\tau(t)|| |x| \]
for all \( t \in G \) and \( x \in X \), which shows that \( \tau \) is a representation of \( G \) on \( (X, | \cdot |) \) and that \( ||\tau(t)|| \leq ||\tau(t)|| |x| \) for each \( t \in G \).

**Remark 3.2.** Let assumptions of Theorem 3.1 hold and let \( S \) be the map obtained in the proof. Then the dual space of \( (X, | \cdot |) \) is \( X^\sigma = \{ \xi \circ S^{-1} : \xi \in X^* \} \) and \( X_2 = \{ \xi \circ S^{-1} : \xi \in X_2 \} \) is a predual of \( (X, | \cdot |) \), in the case when \( (X, || \cdot ||) \) is a dual Banach space. On the other hand, we have \( (\xi \circ S^{-1})(\tau(t)x) = \xi(\tau(t)(S^{-1}(x))) \) for all \( \xi \in X_2 \), \( x \in X \), and \( t \in G \), which clearly shows that \( \tau \) is \( (X, X_2) \)-continuous. Furthermore, since
\[ ||\tau(t)x - \tau(s)x|| = ||S(\tau(t)x - \tau(s)x)|| = ||\tau(t)(S(x)) - \tau(s)(S(x))|| \]
for all \( s, t \in G \) and \( x \in X \), it follows that \( \tau \) is a strongly continuous representation of \( G \) on \( (X, | \cdot |) \) provided that \( \tau \) is a strongly continuous representation of \( G \) on \( (X, || \cdot ||) \).

Consequently, the existence of discontinuous \( \pi \)-invariant functionals does not allow the uniqueness of invariant norms even when we restrict the sense of the uniqueness to the narrower context of strongly continuous representations.

4. Uniqueness of invariant norms

4.1. Gliding hump sequences

Let \( \mathfrak{U} \) denote the subalgebra of \( M(G) \) generated by all the unit point mass measures \( \delta_t \) with \( t \in G \). It is worth pointing out that \( \mathfrak{U} \) is the \( * \)-subalgebra of \( M(G) \) consisting of discrete measures with finite support and that every unitary representation \( \pi \) of \( G \) lifts to an algebra \( * \)-homomorphism, \( \pi^{\mathfrak{U}} \), from \( \mathfrak{U} \) into \( L(H_\pi) \), which is nothing but the restriction of \( \pi^{M(G)} \) to \( \mathfrak{U} \). If \( \pi \) is an irreducible unitary representation of \( G \), then \( \pi^{\mathfrak{U}} \) is an algebraically irreducible representation of the algebra \( \mathfrak{U} \) on the space \( H_\pi \). Furthermore, on account of [9, Theorem 27.13], if \( \pi \) and \( \varpi \) are non-equivalent irreducible unitary representations of \( G \), then the representations \( \pi^{\mathfrak{U}} \) and \( \varpi^{\mathfrak{U}} \) of \( \mathfrak{U} \) are not equivalent. We shall write \( \pi \) for \( \pi^{\mathfrak{U}} \) when no confusion can arise.
Lemma 4.1. Let \( \pi_1, \ldots, \pi_n \) be pairwise non-equivalent irreducible unitary representations of \( G \). Then the map \( \mu \mapsto (\pi_1(\mu), \ldots, \pi_n(\mu)) \) from \( \mathcal{U} \) into \( L(H_{\pi_1}) \oplus \cdots \oplus L(H_{\pi_n}) \) is an epimorphism.

Proof. \( \pi_1, \ldots, \pi_n \) are pairwise non-equivalent algebraically irreducible representations of the algebra \( \mathcal{U} \) on the spaces \( H_{\pi_1}, \ldots, H_{\pi_n} \), respectively. Thus the result is a consequence of [3, Corollary 1.4.39].

Lemma 4.2. Let \( \Sigma \) be an infinite set of pairwise non-equivalent irreducible unitary representations of \( G \). Then there exist sequences \( (\pi_n) \) in \( \Sigma \) and \( (\mu_n) \) in \( \mathcal{U} \) for which one of the following assertions holds:

(i) \( \pi_n(\mu_k \star \cdots \star \mu_1) \neq 0 \) and \( \pi_n(\mu_{n+1} \star \cdots \star \mu_1) = 0 \) for each \( n \in \mathbb{N} \).

(ii) \( \pi_n(\mu_k \star \mu_n) \neq 0 \) and \( \pi(\mu_m \star \mu_n) = 0 \) for all \( \pi \in \Sigma \) and \( m, n \in \mathbb{N} \) with \( m \neq n \).

Proof. Let \( \Pi \) be the set of those \( \pi \in \Sigma \) for which there exists \( v_\pi \in \mathcal{U} \) with the property that \( \pi(v_\pi) \neq 0 \) and \( \pi'(v_\pi) = 0 \) for each \( \pi' \in \Sigma \setminus \{\pi\} \).

We first assume that \( \Pi \) is infinite. Let \( \pi_n \) be a sequence of pairwise different elements of \( \Pi \) and let us write \( v_\pi = v_{\pi_n} \) for each \( n \in \mathbb{N} \). For every \( n \in \mathbb{N} \) let \( x_n \in H_{\pi_n} \) be such that \( \pi_n(v_{\pi_n})(x_n) \neq 0 \) and we apply Lemma 4.1 to get \( \varrho_n \in \mathcal{U} \) such that \( \pi(\varrho_n)(\pi_n(v_{\pi_n})x_n) = x_n \). Set \( \mu_n = \varrho_n \star v_n \) for each \( n \in \mathbb{N} \). Then it is easily seen that \( \pi_n(\mu_k \star \mu_n)x_n = x_n \neq 0 \) and \( \pi(\mu_n \star \mu_n) = 0 \) for all \( \pi \in \Sigma \) and \( m, n \in \mathbb{N} \) with \( m \neq n \).

We now assume that \( \Pi \) is finite. Set \( \pi_1 \in \Sigma \setminus \Pi \) and \( \mu_1 = \delta_{\pi_1} \). Then \( \pi_1(\mu_1) \neq 0 \). Assume that \( \pi_1, \ldots, \pi_n \in \Sigma \setminus \Pi \) and \( \mu_1, \ldots, \mu_n \in \mathcal{U} \) have been chosen so that \( \pi_k(\mu_k \star \cdots \star \mu_1) \neq 0 \) if \( k \leq n \) and \( \pi_k(\mu_{k+1} \star \cdots \star \mu_1) = 0 \) if \( k < n \). We claim that \( \pi(\mu_n \star \cdots \star \mu_1) \neq 0 \) for some \( \pi \in \Sigma \setminus (\Pi \cup \{\pi_n\}) \). Suppose, contrary to our claim, that \( \pi(\mu_n \star \cdots \star \mu_1) = 0 \) for each \( \pi \in \Sigma \setminus (\Pi \cup \{\pi_n\}) \). From Lemma 4.1 it follows that there exists \( \mu \in \mathcal{U} \) such that \( \pi(\mu) = 0 \) for each \( \pi \in \Pi \) and \( \pi(\mu) = I_n \), where \( I_n \) stands for the identity operator on \( H_{\pi_n} \). We thus get \( \pi(\mu \star \mu_n \star \cdots \star \mu_1) = 0 \) for each \( \pi \in \Sigma \setminus \{\pi_n\} \) and \( \pi_n(\mu \star \mu_n \star \cdots \star \mu_1) \neq 0 \), which implies \( \pi_n \in \Pi \), a contradiction. Pick \( \pi_{n+1} \in \Sigma \setminus (\Pi \cup \{\pi_n\}) \) such that \( \pi_{n+1}(\mu_n \star \cdots \star \mu_1) \neq 0 \). By Lemma 4.1 there exists \( \mu_{n+1} \in \mathcal{U} \) such that \( \pi_n(\mu_{n+1}) = 0 \) and \( \pi_{n+1}(\mu_{n+1}) = I_{n+1} \). The sequences \( (\pi_n) \) and \( (\mu_n) \) constructed in this way satisfy the requirements of the first assertion.

4.2. The basic principle

Throughout this section we consider the case when \( X \) is equipped with another norm \( | \cdot | \) for which \( \tau \) becomes a representation of \( G \) on \( (X, | \cdot |) \). In order to study whether \( | \cdot | \) is necessarily equivalent to \( \| | \cdot | \| \) we are required to bring a number of tools from the automatic continuity theory.

A particularly important notion in automatic continuity is that of the separating space \( \mathcal{E}(\Phi) \) of a linear map \( \Phi \) from a Banach space \( X \) into a Banach space \( Y \) which is
defined as follows:

\[ \Xi(\Phi) = \{ y \in \mathcal{Y} : \text{there exists } (x_n) \to 0 \text{ in } \mathcal{X} \text{ with } (\Phi(x_n)) \to y \}. \]

The separating space is a closed subspace of \( \mathcal{Y} \). Moreover, it is an immediate restatement of the closed graph theorem that \( \Phi \) is continuous if and only if \( \Xi(\Phi) = \{0\} \). Another standard fact we shall use is that \( \Psi \Phi \) is continuous if and only if \( \Psi(\Xi(\Phi)) = \{0\} \) whenever \( \Psi \) is any continuous linear operator from \( \mathcal{Y} \) into another Banach space \( \mathcal{Z} \). We refer the reader to [3], where the basic properties of the separating space are explored. Our method involves the so-called \textit{gliding hump technique}. The next result, which is a slight modification of [3, Theorem 5.2.6], formalizes this method.

**Lemma 4.3.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach spaces and let \( \Phi : \mathcal{X} \to \mathcal{Y} \) be a linear map. Suppose that there exist \( T_n \in L(\mathcal{X}) \) and continuous linear maps \( S_n \) from \( \mathcal{Y} \) into Banach spaces \( \mathcal{Y}_n \) for \( n \in \mathbb{N} \) with the property that each map \( S_n \Phi T_1 \cdots T_m \) is continuous for \( m > n \), then \( S_n \Phi T_1 \cdots T_n \) is continuous for sufficiently large \( n \).

We shall denote by \( \Phi \) the identity operator from \( (X, \| \cdot \|) \) into \( (X, \| \cdot \|) \) and we shall write its separating space simply \( \Xi \). It is important to note here that all the convolutions and \( \ast \)-operations we shall use in the sequel are considered with respect to the norm \( \| \cdot \| \). Every \( \mu \in \mathfrak{A} \) can be expressed in the form \( \mu = \sum_{k=1}^{n} \alpha_k \delta_{t_k} \) with \( n \in \mathbb{N} \), \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \), and \( t_1, \ldots, t_n \in G \), so that \( \mu \star x \) is nothing but \( \sum_{k=1}^{n} \alpha_k \tau(t_k)x \). In order to check whether or not \( \Xi = \{0\} \) we shall consider the subset \( A(\Xi) \) of \( \hat{G} \) defined in Section 2 as

\[ A(\Xi) = \{ [\pi] \in \hat{G} : \pi^\ast(\Xi) \neq \{0\} \}. \]

We shall write \( L(H_\pi) \otimes (X, \| \cdot \|) \) and \( L(H_\pi) \otimes (X, \| \cdot \|) \) as \( L(H_\pi) \otimes \| \cdot \| \) and \( L(H_\pi) \otimes \| \cdot \| \), respectively. Recall that the underlying spaces of both \( L(H_\pi) \otimes \| \cdot \| \) and \( L(H_\pi) \otimes \| \cdot \| \) are nothing but \( L(H_\pi) \otimes X \) and we shall write the corresponding norms simply by \( \| \cdot \| \) and \( \| \cdot \| \), respectively.

**Lemma 4.4.** The following assertions hold:

(i) \( \Xi(\Phi^{-1}) = \Xi \) and therefore \( \Xi \) is closed in both \( (X, \| \cdot \|) \) and \( (X, \| \cdot \|) \);

(ii) \( \Xi \) is an invariant subspace of \( X \);

(iii) \( \| \cdot \| \) and \( \| \cdot \| \) are equivalent if and only if \( A(\Xi) = \emptyset \).

**Proof.** (i) The proof is just a straightforward verification.

(ii) Since \( \Phi \tau(t) = \tau(t) \Phi \) for each \( t \in G \), this is an easy consequence of [3, Proposition 5.2.2(iii)].

(iii) It is known that \( \| \cdot \| \) and \( \| \cdot \| \) are equivalent if and only if \( \Phi \) is continuous and we already know that this is equivalent to the condition \( \Xi(\Phi) = \{0\} \). On account of Lemma 2.4(ii), this latter condition is equivalent to \( A(\Xi) = \emptyset \). \( \square \)
Lemma 4.5. One of the following assertions hold:

(i) \( \Delta(\Xi) \) is finite.

(ii) There exists \([\pi] \in \Delta(\Xi)\) such that the map \( x \mapsto \pi^\#: x \mapsto \pi^\#: \Xi, [\cdot] \Rightarrow L(H_{\pi}) \otimes ||\cdot||X \) is continuous.

Proof. Suppose that \( \Delta(\Xi) \) is infinite. Let \( \Sigma \) be a set of pairwise non-equivalent irreducible unitary representation of \( G \) such that \{\([\pi] \in \Sigma \}\} = \Delta(\Xi)\). Let \( (\pi_n) \) and \( (\mu_n) \) be the sequences given in Lemma 4.2.

We first examine the case when the sequences satisfy the first assertion of Lemma 4.2. For every \( n \in \mathbb{N} \) let \( T_n \) be the linear operator from \( X \) into itself given by \( T_n(x) = \pi_n \star x \) for each \( x \in X \). Since \( T_n \) is in the linear span of the translation operators on \( X \), it follows that it is a continuous linear operator from \( (X, ||\cdot||) \) into itself for each \( n \in \mathbb{N} \). We now define \( S_n : X \rightarrow H_{\pi_n} \otimes ||\cdot||X \) by \( S_n(x) = \pi_n^\#: x \) for all \( x \in X \) and \( n \in \mathbb{N} \). It is clear that \( S_n \) is a continuous linear operator from \( (X, ||\cdot||) \) into \( L(H_{\pi_n}) \otimes ||\cdot||X \). On the other hand, we have

\[
(S_n \Phi T_1 \cdots T_m)(x) = \pi_n^\#: (\pi_1 \star \cdots \star \pi_m \star x)
\]

for all \( x \in X \) and \( m, n \in \mathbb{N} \). Since \( \pi_n(\mu_m \star \cdots \star \mu_1) = 0 \) if \( m > n \), Lemma 4.3 now yields \( \langle \Xi, [t]_v \rangle = 0 \) for each \( t \in G \). On the other hand, on account of the translation operators on \( X \), we have \( \pi_n^\#: \Xi, [t]_v \mapsto \pi_n^\#: (\pi_n(\mu_n \star \cdots \star \mu_1)) \) for all \( t \in G \). Since \( \Xi(\Phi) \) is invariant (Lemma 4.4(ii)), for all \( x \in \Xi \) and \( t \in G \) we have

\[
0 = \pi_n^\#: (\delta_{t^{-1}} \star x) \cdot T_0 = \pi_n^\#: (x) \cdot (\pi_n^\#: (t) T_0)
\]

and so \( \pi_n^\#: (\Xi), (\Xi, [\cdot]_v) \mapsto \{0\} \) for each \( \mu \in \mathfrak{M} \). On the other hand, on account of Lemma 4.1, we have \( \pi_n^\#: (\mathfrak{M}) = L(H_{\pi_n}) \) and so \( \pi_n^\#: (\Xi), (\Xi, [\cdot]_v) \mapsto \{0\} \). For every continuous linear functional \( \xi \) on \( (X, ||\cdot||) \) we consider the continuous linear functional \( \pi_n^\#: (\Xi), (\Xi, [\cdot]_v) \mapsto R\xi(T \otimes x) = \xi(x) T \) for all \( T \in L(H_{\pi_n}) \) and \( x \in X \). Note that \( R\xi(T \otimes x) = R\xi(T) \) for all \( \xi \in L(H_{\pi_n}) \otimes ||\cdot||X \) and \( T \in L(H_{\pi_n}) \). From what has already been proved, it follows that

\[
0 = R\xi(\pi_n^\#: \Xi), (\Xi, [\cdot]_v) \mapsto R\xi(\pi_n^\#: \Xi), (\Xi, [\cdot]_v) \mapsto T T_0
\]

for all \( T \in L(H_{\pi_n}) \). Since \( T_0 \neq 0 \), we conclude that \( R\xi(\pi_n^\#: \Xi), (\Xi, [\cdot]_v) \mapsto \{0\} \) for arbitrary \( \xi \), which entails that \( \pi_n^\#: (\Xi) \mapsto \{0\} \). Since this contradicts our choosing \( \pi_n^\#: \Xi \) in \( \Delta(\Xi) \), it follows that this case does not arise.

We now turn to the case when the sequences \( (\pi_n) \) and \( (\mu_n) \) satisfy the second assertion of Lemma 4.2. Let us observe that if \( m \neq n \), then \( \pi_n^\#: (\mu_m \star \pi_n \star \Xi) = \{0\} \) for each \( \pi \in \mathfrak{G} \) and therefore Lemma 2.4(ii) yields \( \overline{\mu_m \star \pi_n \star \Xi} = \{0\} \). We now
claim that there exists $N \in \mathbb{N}$ such that the map

$$x \mapsto \overline{\mu}_N^* \star \overline{\mu}_N^* \star x$$

from $(\mathcal{E}, \cdot)$ into $(\mathcal{E}, \|\cdot\|)$ is continuous. To this end it is of interest to recall that, on account of Lemma 4.4(i), $\mathcal{E}$ is closed in $(X, \cdot)$. Suppose, contrary to our claim, that all those maps are discontinuous. Then there exists a sequence $(x_n)$ in $\mathcal{E}$ such that

$$|x_n| < 2^{-n}(1 + z_n)^{-1}$$

and

$$||\overline{\mu}_n^* \star \overline{\mu}_n^* \star x_n|| > n(1 + \beta_n),$$

where $z_n$ and $\beta_n$ stand for the norm of the operator $x \mapsto \overline{\mu}_n^* \star x$ from $(X, \cdot)$ into itself and from $(X, \|\cdot\|)$ into itself, respectively, for each $n \in \mathbb{N}$. Define

$$x_0 = \sum_{k=1}^{\infty} \overline{\mu}_k^* \star x_k,$$

the series being absolutely convergent in $(\mathcal{E}, \cdot)$. On account of the continuity of the map $x \mapsto \overline{\mu}_n^* \star x$ from $(X, \cdot)$ into itself together with the fact that $\overline{\mu}_m^* \star \overline{\mu}_m^* \star \mathcal{E} = \{0\}$, we have

$$\overline{\mu}_n^* \star x_0 = \sum_{k=1}^{\infty} \overline{\mu}_n^* \star \overline{\mu}_k^* \star x_k = \overline{\mu}_n^* \star \overline{\mu}_n^* \star x_n$$

and thus

$$(1 + \beta_n)n < ||\overline{\mu}_n^* \star \overline{\mu}_n^* \star x_n|| = ||\overline{\mu}_n^* \star x_0|| \leq \beta_n||x_0||$$

for every $n \in \mathbb{N}$, which gives a contradiction. Pick $N \in \mathbb{N}$ such that the map $x \mapsto \overline{\mu}_N^* \star \overline{\mu}_N^* \star x$ from $(\mathcal{E}, \cdot)$ into $(\mathcal{E}, \|\cdot\|)$ is continuous. Since this map is nothing but the composition of the map $\Phi$ when restricted to $\mathcal{E}$ with the continuous linear operator $x \mapsto \overline{\mu}_N^* \star \overline{\mu}_N^* \star x$ from $(\mathcal{E}, \|\cdot\|)$ into itself, it may be concluded $\overline{\mu}_N^* \star \overline{\mu}_N^* \star \mathcal{E} = \{0\}$, where $\mathcal{E}$ denotes the separating space of the map $\Phi$ when restricted to $\mathcal{E}$. Accordingly,

$$\{0\} = \pi_N^p(\overline{\mu}_N^* \star \overline{\mu}_N^* \star \mathcal{E}) = \pi_N^p(\mathcal{E}) \cdot \pi_N^p(\mu_N \star \mu_N).$$

Since $\mathcal{E}$ is invariant, we see at once that $\mathcal{E}$ is invariant and then we can proceed as in the preceding case with $T_0$ replaced with $\pi_N(\mu_N \star \mu_N)$ in order show that $\pi_N^p(\mathcal{E}) = \{0\}$. Since $\pi_N^p(\mathcal{E}) = \{0\}$ and the map $\pi_N^p$ is a continuous linear operator from $(X, \|\cdot\|)$ into $L(H_{\pi_N}) \otimes \|\cdot\|X$, it follows that the map $x \mapsto \pi_N^p(x)$ from $(\mathcal{E}, \cdot)$ into $L(H_{\pi_N}) \otimes \|\cdot\|X$ is continuous, as required.  \[\square\]
We are now in a position to prove the basic principle that we shall use in the sequel.

**Theorem 4.1.** Let $G$ be a compact group, let $(X, ||\cdot||)$ be a Banach space, let $\tau$ be a $\sigma(X, X_\ast)$-continuous representation of $G$ on $X$ (where either $X_\ast = X^\ast$ or $X$ is a dual Banach space and $X_\ast$ is a predual of $X$), and let $||\cdot||$ be a norm on $X$ such that $\tau$ becomes a representation of $G$ on $(X, ||\cdot||)$. Suppose that for every $[\pi] \in \hat{G}$ with $\pi^\#(X) \neq \{0\}$ the following properties hold:

1. $\dim \pi^\#(X) < \infty$;
2. There exist $A_\pi, K_\pi \subseteq G$ such that each $\zeta \in L(H_\pi) \otimes X$ has a representation of the form
   \[
   \zeta = \pi^\#(\zeta) + \sum_{k=1}^{N_\pi} [\pi(t_k^{-1}) \cdot \zeta_k - \tau(t_k)\zeta_k],
   \]
   where $t_k \in K_\pi$, $\zeta_k \in L(H_\pi) \otimes X$, and $||\zeta_k|| \leqslant A_\pi||\zeta||$ for $k = 1, \ldots, N_\pi$;
3. There exists a constant $B_\pi$ such that
   \[
   ||\tau(t)x|| \leqslant B_\pi|x|
   \]
   for all $x \in X$ and $t \in K_\pi$.

Then $||\cdot||$ and $||\cdot||$ are equivalent.

**Proof.** Assume towards a contradiction that the result is false so that $A(\mathfrak{S}) \neq \emptyset$.

We begin by proving that there exists $[\pi] \in A(\mathfrak{S})$ such that the map $x \mapsto \pi^\#(x)$ from $(\mathfrak{S}, ||\cdot||)$ into $L(H_\pi) \otimes ||\cdot|| X$ is continuous. On account of Lemma 4.5, we are reduced to proving the claim for $A(\mathfrak{S})$ finite. Suppose that $A(\mathfrak{S}) = \{[\pi_1], \ldots, [\pi_N]\}$. If $x \in \mathfrak{S}$ is such that $\pi_k^\#(x) = 0$ for each $k \in \{1, \ldots, N\}$, then $\pi^\#(x) = 0$ for each $[\pi] \in \hat{G}$ and Lemma 2.4(ii) now shows that $x = 0$. Consequently, the map $x \mapsto (\pi_1^\#(x), \ldots, \pi_N^\#(x))$ from $\mathfrak{S}$ into $\pi_1^\#(X) \times \cdots \times \pi_N^\#(X)$ is injective. Since $\dim \pi_k^\#(X) < \infty$ for each $k = 1, \ldots, N$, we conclude that $\mathfrak{S} \cong \mathbb{R}^N$. Hence for any $[\pi] \in A(\mathfrak{S})$ the map $x \mapsto \pi^\#(x)$ from $(\mathfrak{S}, ||\cdot||)$ into $L(H_\pi) \otimes ||\cdot|| X$ is continuous.

We now pick $[\pi] \in A(\mathfrak{S})$ such that the map $x \mapsto \pi^\#(x)$ from $(\mathfrak{S}, ||\cdot||)$ into $L(H_\pi) \otimes ||\cdot|| X$ is continuous. This clearly forces that the linear space $M = \{\zeta \in L(H_\pi) \otimes \mathfrak{S}; \pi^\#(\zeta) = 0\}$ is closed in $L(H_\pi) \otimes ||\cdot|| X$. Let $Y$ be the quotient Banach space $(L(H_\pi) \otimes ||\cdot|| X)/M$. Let us denote by $||\cdot||_Y$ the quotient norm on $Y$ and let us denote by $Q$ the quotient map from $L(H_\pi) \otimes ||\cdot|| X$ onto $Y$. Let $\Psi$ be the identity map from $L(H_\pi) \otimes ||\cdot|| X$ onto $L(H_\pi) \otimes ||\cdot|| X$ and note that $\mathfrak{S}(\Psi) = L(H_\pi) \otimes \mathfrak{S}$.

For every $t \in G$ we define $\Psi_t : L(H_\pi) \otimes ||\cdot|| X \to L(H_\pi) \otimes ||\cdot|| X$ by

\[
\Psi_t(\zeta) = \pi(t^{-1}) \cdot \zeta - \tau(t)\zeta
\]
for each \( \zeta \in L(H_{\pi}) \otimes_X X \). It is clear that the operator \( \Psi_t \) is continuous with \( \|\Psi_t\| \leq \|\pi(t^{-1})\| + \|\pi(t)\| \leq 1 + B_{\pi} \) for each \( t \in K_{\pi} \). We also check at once that \( \Psi_t(L(H_{\pi}) \otimes \mathcal{Z}) \subset L(H_{\pi}) \otimes \mathcal{Z} \) for each \( t \in G \). On the other hand, we have

\[
\pi^\beta[\pi(t^{-1}) \cdot (T \otimes x) - \pi(t)(T \otimes x)] \\
= \pi^\beta(x) \cdot (\pi(t^{-1})T) - \pi^\beta(\delta_t \star x)) \cdot T \\
= \pi^\beta(x) \cdot (\pi(t^{-1})T) - (\pi^\beta(x) \cdot \pi(t^{-1})) \cdot T = 0
\]

for all \( T \in L(H_{\pi}), x \in X, \text{ and } t \in G \). Consequently, \( \Psi_t(\mathcal{Z}(\Psi)) \subset M \) and so \( Q^\Psi_t \Psi \) is continuous for each \( t \in G \). On account of [24, Lemma 1.3], there is a constant \( C_{\pi} \) (independent of \( t \)) such that \( \|Q^\Psi_t \Psi\| \leq C_{\pi} \|Q^\Psi_t\| \leq C_{\pi}(1 + B_{\pi}) \) for each \( t \in G \).

On the other hand, since \( \dim \pi^\beta(X) < \infty \) and \( \pi^\beta(L(H_{\pi}) \otimes X) \) is the linear span of \( \pi^\beta(X) \cdot L(H_{\pi}) \) (because \( \pi^\beta(T \otimes x) = \pi^\beta(x) \cdot T \) for all \( T \in L(H_{\pi}) \) and \( x \in X \)), it may be concluded that \( \dim \pi^\beta(L(H_{\pi}) \otimes X) < \infty \) and therefore that there exists a constant \( D_{\pi} \) such that \( \|\pi^\beta(\zeta)\| \leq D_{\pi}\|\pi^\beta(\zeta)\| \) for each \( \zeta \in L(H_{\pi}) \otimes X \).

On account of (ii), for every \( \zeta \in L(H_{\pi}) \otimes X \), we have

\[
\zeta = \pi^\beta(\zeta) + \sum_{k=1}^{N_{\pi}} [\pi(\tau_k^{-1}) \cdot \zeta_k - \tau(\tau_k)\zeta_k],
\]

where \( t_k \in K_{\pi}, \zeta_k \in L(H_{\pi}) \otimes X \), and \( \|\zeta_k\| \leq A_{\pi}\|\zeta\| \) for \( k = 1, \ldots, N \). Hence

\[
Q^\Psi(\zeta) = Q(\pi^\beta(\zeta)) + \sum_{k=1}^{N_{\pi}} (Q^\Psi_t \Psi)(\zeta_k)
\]

and

\[
|Q^\Psi(\zeta)|_Y \leq |Q(\pi^\beta(\zeta))|_Y + \sum_{k=1}^{N_{\pi}} |(Q^\Psi_t \Psi)(\zeta_k)|_Y \\
\leq D_{\pi}\|\pi^\beta(\zeta)\| + \sum_{k=1}^{N_{\pi}} C_{\pi}(1 + B_{\pi})\|\zeta_k\| \\
\leq D_{\pi}\|\pi^\beta\|\|\zeta\| + \sum_{k=1}^{N_{\pi}} C_{\pi}(1 + B_{\pi})A_{\pi}\|\zeta\| \\
= [D_{\pi}\|\pi^\beta\| + N_{\pi}C_{\pi}(1 + B_{\pi})A_{\pi}]\|\zeta\|.
\]

This clearly forces that \( Q^\Psi \) is continuous and therefore that \( L(H_{\pi}) \otimes \mathcal{Z} \subset M \). In particular \( \pi^\beta(\mathcal{Z}) = \{0\} \), which contradicts the choice of \( \pi \). \( \square \)
4.3. Invariant functionals: a new insight

Let $\mathcal{T}(X)$ denote the linear span of the set \( \{x - \tau(t)x : x \in X, t \in G\} \). It is important to note here that a linear functional $\phi$ on $X$ is invariant if and only if $\phi(\mathcal{T}(X)) = \{0\}$.

**Lemma 4.6.** There exists a discontinuous invariant functional on $X$ if and only if at least one of the following assertions holds:

(i) $\mathcal{T}(X)$ is not closed in $X$.
(ii) $\mathcal{T}(X)$ has infinite codimension in $X$.

**Proof.** Suppose $\mathcal{T}(X)$ is not closed in $X$. Set $u \in \overline{\mathcal{T}(X)} \backslash \mathcal{T}(X)$. We define $\phi$ to be $1$ at $u$ and $0$ on $\mathcal{T}(X)$, and then we extend $\phi$ to $X$ by linearity.

We now suppose that $\dim X/\mathcal{T}(X) = \infty$. Set a sequence $(x_n)$ in $X$ such that $(x_n + \mathcal{T}(X) : n \in \mathbb{N})$ is linearly independent in $X$. We define $\phi(x_n) = n||x_n||$ for each $n \in \mathbb{N}$ and $\phi(\mathcal{T}(X)) = \{0\}$, and then we extend $\phi$ to $X$ by linearity.

Finally, suppose that $\mathcal{T}(X)$ is closed in $X$ and that $\dim X/\mathcal{T}(X) < \infty$. Hence there exists a finite-dimensional subspace $M$ of $X$ such that $X$ can be expressed as the topological sum $X = \mathcal{T}(X) \oplus M$. If $\phi$ is an invariant functional on $X$, then $\phi(\mathcal{T}(X)) = \{0\}$ and thus $\phi$ is continuous on $\mathcal{T}(X)$. Of course, $\phi$ is continuous on $M$ and therefore $\phi$ is continuous on $X$. $\square$

Our next goal is to explore the consequences of $\mathcal{T}(X)$ being closed in $X$.

**Lemma 4.7.** Let $Y$ be a dual Banach space, let $Y_s$ be a predual of $Y$, and let $\tau$ be a $\sigma(Y, Y_s)$-continuous representation of $G$ on $Y$ such that $\tau(G)$ consists of $\sigma(Y, Y_s)$-continuous operators. If $X$ is an invariant closed subspace of $Y$ such that $\mathcal{T}(X)$ is closed in $X$, then there exist $n \in \mathbb{N}$ and $C > 0$ with the property that for all $y \in \mathcal{P}^{\sigma(Y, Y_s)}$ and $\mu \in \mathcal{P}(G)$ (the regular Borel probability measures on $G$) the element $y - \mu \star y \in Y$ has a representation of the form

$$y - \mu \star y = \sum_{k=1}^{n} [y_k - \tau(t_k)y_k],$$

where $t_k \in G$, $y_k \in Y$, and $||y_k|| \leq C||y||$ for each $k \in \{1, \ldots, n\}$.

**Proof.** Our proof follows the pattern established in [23, Lemmas 1 and 2]. We shall denote by $B_Y$, $B_X$, and $B_{\mathcal{T}(X)}$ the closed unit balls of $Y$, $X$, and $\mathcal{T}(X)$, respectively.

On account of Lemma 2.2, $\tau$ is the adjoint of a strongly continuous representation $\tau_s$ of $G$ on $Y_s$.

For every $n \in \mathbb{N}$ we define

$$V_n = \left\{ \sum_{k=1}^{n} [y_k - \tau(t_k)y_k] : t_k \in G, \ y_k \in nB_Y, \ k = 1, \ldots, n \right\}.$$
Our first objective is to prove that $V_n$ is a $\sigma(Y, Y_*)$-compact subset of $Y$. To this end we first check that the map $(t, y) \rightarrow y - \tau(t)y$ from $G \times BY$ into $Y$ is continuous when both $BY$ and $Y$ are equipped with the $\sigma(Y, Y_*)$-topology. Let $(t_i, y_i)$ be a convergent net in $G \times BY$ with $\lim(t_i, y_i) = (t, y)$. If $\xi \in Y_*$, then we have

$$|(y_i - \tau(t_i)y_i)(\xi) - (y - \tau(t)y)(\xi)|$$

$$\leq |y_i(\xi) - y(\xi)| + |(\tau(t_i)y_i)(\xi) - (\tau(t)y_i)(\xi)| + |(\tau(t)y_i)(\xi) - (\tau(t)y)(\xi)|$$

$$\leq |y_i(\xi) - y(\xi)| + ||\tau(t_i^{-1})\xi - \tau(t^{-1})\xi|| + |y_i(\tau(t_i^{-1})\xi) - y(\tau(t^{-1})\xi)|$$

and the latter line clearly converges to zero. By Banach–Alaoglu theorem $BY$ is $\sigma(Y, Y_*)$-compact which, together with the compactness of $G$ and the continuity just checked, entails that the set $W = \{y - \tau(t)y : t \in G, \ y \in BY\}$ is $\sigma(Y, Y_*)$-compact. We now observe that $V_n$ is the $n$-fold sum of $nW$.

Since $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} (\mathcal{F}(X) \cap V_n)$, Baire category theorem now shows that at least one of the sets $\mathcal{F}(X) \cap V_n$ has a non-empty interior in $\mathcal{F}(X)$. This implies that there exist $m \in \mathbb{N}$, $x \in \mathcal{F}(X) \cap V_m$, and $r > 0$ such that $x + rBY \subset V_m$, which gives

$$B_{\mathcal{F}(X)} \subset r^{-1}(V_m - x) \subset r^{-1}(V_m - V_m) \subset r^{-1}V_{2m}.$$ 

We now consider the set

$$\Theta = \left\{ \mu \in M(G) : y - \mu \mathbf{1} y \in 2\overline{B_{\mathcal{F}(X)}}_{\sigma(Y, Y_*)}, \forall y \in \overline{B_X}_{\sigma(Y, Y_*)} \right\}.$$ 

The task is to prove that $\Theta$ is $\sigma(M(G), C(G))$-closed. Of course we are reduced to proving that the map $\mu \mapsto \mu \mathbf{1} y$ from $M(G)$ into $Y$ is $\sigma(M(G), C(G)) - \sigma(Y, Y_*)$-continuous for each $y \in Y$. Let $(\mu_i)$ a net in $M(G)$ with $\lim \mu_i = \mu$ with respect to the $\sigma(M(G), C(G))$-topology. Then, for every $\xi \in Y_*$, we have

$$(\mu_i \mathbf{1} y)(\xi) = \int_G (\tau(t)y)(\xi) \, d\mu_i(t) = \int_G y(\tau(t_i^{-1})\xi) \, d\mu_i(t)$$

$$\rightarrow \int_G y(\tau(t_i^{-1})\xi) \, d\mu_i(t) = (\mu \mathbf{1} y)(\xi).$$

Our next claim is that $\delta_t \in \Theta$ for each $t \in G$ and that $\Theta$ is convex. Let $t \in G$ and $x \in B_X$. Then $x - \tau(t)x \in 2B_{\mathcal{F}(X)}$. The $\sigma(Y, Y_*)$-continuity of $\tau(t)$ on $Y$ now yields

$$y - \tau(t)y \in 2\overline{B_{\mathcal{F}(X)}}_{\sigma(Y, Y_*)}$$

for each $y \in \overline{B_X}_{\sigma(Y, Y_*)}$. We thus get $\delta_t \in \Theta$. Let $\mu, v \in \Theta$, $0 \leq \alpha \leq 1$, and $y \in \overline{B_X}_{\sigma(Y, Y_*)}$. Then

$$y - (\alpha \mu + (1 - \alpha)v) \mathbf{1} y = \alpha(y - \mu \mathbf{1} y) + (1 - \alpha)(y - v \mathbf{1} y) \in 2\overline{B_{\mathcal{F}(X)}}_{\sigma(Y, Y_*)},$$
which gives \( x\mu + (1-x)v \in \Theta \). On the other hand, \( P(G) = \{ \mu \in M(G) : \mu \geq 0 \text{ and } \mu(G) = 1 \} \). So \( P(G) \) is a \( \sigma(M(G), C(G)) \)-compact convex subset of \( M(G) \). By Kreĭn–Mil'man theorem \( P(G) \) is the \( \sigma(M(G), C(G)) \)-closure of the convex hull of its extreme points, which are nothing but the point mass measures \( \delta_t \) with \( t \in G \) [2, Theorem 8.4]. Consequently, \( P(G) \subseteq \Theta \). Since \( V_{2m} \) is \( \sigma(Y, Y_0) \)-closed in \( Y \) and \( B_{\mathcal{F}(X)} \subset r^{-1}V_{2m} \), it may be concluded that \( 2B_{\mathcal{F}(X)} = 2r^{-1}V_{2m} \) and therefore that \( y - \mu \star y^* V_{2m} \) for all \( y \in B_{\mathcal{F}(X)} \) and \( \mu \in P(G) \). If \( y \in \mathcal{F}(X) \setminus \{0\} \), then \( ||y||^{-1} y \in B_{\mathcal{F}(X)} \) and so \( ||y||^{-1}(y - \mu \star y) \in 2r^{-1}V_{2m} \). Therefore there exist \( t_1, \ldots, t_{2m} \in G \) and \( y_1, \ldots, y_{2m} \in Y \) such that \( ||y_k|| \leq 2m \) for each \( k \in \{1, \ldots, 2m\} \) and

\[
||y||^{-1}(y - \mu \star y) = 2r^{-1} \sum_{k=1}^{2m} (y_k - \tau(t_k)y_k).
\]

This gives

\[
y - \mu \star y = \sum_{k=1}^{2m} (z_k - \tau(t_k)z_k),
\]

where \( z_k = ||y||2r^{-1}y_k \) and \( ||z_k|| \leq 2r^{-1}2m||y|| \) for each \( k \in \{1, \ldots, 2m\} \), which completes the proof. \( \Box \)

**Theorem 4.2.** Let \( X \) be a dual Banach space, let \( X_\ast \) be a predual of \( X \), and let \( \tau \) be a \( \sigma(X, X_\ast) \)-continuous representation of a compact group \( G \) on \( X \) such that \( \tau(G) \) consists of \( \sigma(X, X_\ast) \)-continuous operators. Then the following assertions are equivalent:

(i) Every invariant functional on \( X \) is continuous.

(ii) The subspace consisting of the invariant elements of \( X \) is finite-dimensional and there exist \( N \in \mathbb{N} \) and a constant \( A \) such that each \( x \in X \) has a representation of the form

\[
x = \lambda_G \star x + \sum_{k=1}^{N} [x_k - \tau(t_k)x_k],
\]

where \( t_k \in G, x_k \in X \), and \( ||x_k|| \leq A ||x|| \) for \( k = 1, \ldots, N \).

**Proof.** We first suppose that assertion (i) holds. On account of Lemma 4.3, \( \mathcal{F}(X) \) is closed in \( X \). Then we apply Lemma 4.7 with \( Y = X \) to get \( N \in \mathbb{N} \) and a constant \( A \) such that every \( x \in X \) can be expressed in the form given in (ii) (take into account that \( \lambda_G \) has been normalized to be a probability). We now consider the case when the space of invariant elements of \( X \), which is nothing but \( \lambda_G \star X \), is infinite-dimensional. Therefore there exists a discontinuous linear functional \( \psi \) on \( \lambda_G \star X \). We define a linear functional \( \phi \) on \( X \) by \( \phi(x) = \psi(\lambda_G \star x) \) for each \( x \in X \). For all \( x \in X \) and \( t \in G \), we have

\[
\phi(\tau(t)x) = \psi(\lambda_G \star (\tau(t)x)) = \psi(\lambda_G \star (\delta_t \star x)) = \psi((\lambda_G \star \delta_t) \star x) = \psi(\lambda_G \star x) = \phi(x)
\]
and so \( \phi \) is an invariant functional. Since \( \phi(x) = \psi(x) \) for each \( x \in \lambda G \otimes X \), it follows that \( \phi \) is discontinuous. From what has already been proved, it follows that (i) implies (ii).

We finally suppose that assertion (ii) holds and that \( \phi \) is an invariant functional on \( X \). Then \( \phi(x) = \phi(\lambda G \otimes x) \) for each \( x \in X \). Since \( \dim \lambda G \otimes X < \infty \), it follows that \( \phi \) is continuous. □

**Remark 4.1.** It is evidently of interest to know that if \( \tau \), does not weakly contain the trivial representation, then the zero functional is the only invariant functional on \( X \) [28]. Let us recall that a representation \( \pi \) of a group \( H \) on a Banach space is said to contain the trivial representation of \( H \) weakly if there exists a net \( (x_i) \) in \( X \) with \( \|x_i\| = 1 \) and \( \lim \|x_i - \pi(t)x_i\| = 0 \) for each \( t \in H \).

We are thus led to the following characterization of the uniqueness of invariant norms.

**Theorem 4.3.** Let \( X \) be a dual Banach space, let \( X_\sigma \) be a predual of \( X \), and let \( \tau \) be a \( \sigma(X, X_\sigma) \)-continuous representation of a compact group \( G \) on \( X \) such that \( \tau(G) \) consists of \( \sigma(X, X_\sigma) \)-continuous operators. Then the following assertions are equivalent:

(i) \( X \) carries a unique invariant norm.

(ii) Every \( \pi \)-invariant functional is continuous whenever \( [\pi] \in \hat{G} \) is such that there exists a non-zero \( \pi \)-invariant element.

**Proof.** Theorem 3.1 clearly entails that (i) implies (ii).

We now suppose that (ii) holds. It is a simple matter to check that \( \tau^\pi(G) \) consists of \( \sigma(L(H_\pi) \hat{\otimes} X, (L(H_\pi) \hat{\otimes} X)_\sigma) \)-continuous operators. Since the \( \pi \)-invariant functionals on \( L(H_\pi) \hat{\otimes} X \) are exactly the invariant functionals on \( L(H_\pi) \hat{\otimes} X \) with respect to the representation \( \tau^\pi \), Theorem 4.2 now yields \( \dim \tau^\pi(L(H_\pi) \hat{\otimes} X) < \infty \) and there exist \( N_\pi \in \mathbb{N} \) and a constant \( A_\pi \) such that for every \( \zeta \in L(H_\pi) \hat{\otimes} X \) the element \( \zeta - \pi^\pi(\zeta) \) has a representation of the form \( \sum_{k=1}^{N_\pi} (v_k - \tau^\pi(t_k)v_k) \), where \( t_k \in G \), \( v_k \in L(H_\pi) \hat{\otimes} X \), and \( \|v_k\| \leq A_\pi \|\zeta\| \) for \( k = 1, \ldots, N_\pi \). We thus get \( \dim \pi^\beta(X) < \infty \) and

\[
\zeta = \pi^\beta(\zeta) + \sum_{k=1}^{N_\pi} (v_k - \tau^\pi(t_k)v_k) = \pi^\beta(\zeta) + \sum_{k=1}^{N_\pi} (\pi(t_k^{-1}) \cdot \zeta_k - \tau(t_k)\zeta_k),
\]

where \( \zeta_k = \pi(t_k) \cdot v_k \) and so \( \|\zeta_k\| \leq \|\pi(t_k)\| \|v_k\| = \|v_k\| \leq A_\pi \|\zeta\| \) for \( k = 1, \ldots, N_\pi \). Consequently, conditions (i) and (ii) of Theorem 4.1 are satisfied. On the other hand, if \( \cdot \cdot \) is any invariant norm on \( X \), then assertion (iii) of Theorem 4.1 is also satisfied and therefore \( \cdot \cdot \) is equivalent to \( \| \cdot \| \). □

In the next sections we explore the consequences of the results obtained in Sections 3 and 4 for the spaces of Example 2.3.
5. Actions on locally compact spaces

5.1. Non-transitive actions

Throughout this section \( \Omega \) stands for a locally compact Hausdorff space on which \( G \) acts and that is endowed with an invariant positive Radon measure \( \lambda_\Omega \). We shall denote by \( \Omega/G \) the quotient space and by \( \vartheta \) the canonical quotient map.

**Theorem 5.1.** Let \( G \) be a compact group acting on a locally compact Hausdorff space \( \Omega \) which is endowed with an invariant positive Radon measure \( \lambda_\Omega \). If the quotient space \( \Omega/G \) is infinite, then \( L^p(\Omega) \) with \( 1 \leq p < \infty \) does not carry a unique invariant norm.

**Proof.** We shall put into action [1, VII, Section 2.1-3]. Since \( G \) is compact, it follows that \( G \) acts properly on \( \Omega \), which means that for all compact subsets \( K_1 \) and \( K_2 \) of \( \Omega \), the set \( \{ t \in G : tK_1 \cap K_2 \neq \emptyset \} \) is compact.

Let \( P: L^p(\Omega) \to L^p(\Omega) \) the operator of convolution with \( \lambda_G \). Then \( P \) is a projection on \( L^p(\Omega) \). \( \mathcal{F}(L^p(\Omega)) \) is contained in \( \ker P \) and the subspace of \( C_c(\Omega) \) consisting of functions constant on \( G \)-orbits is contained in \( P(L^p(\Omega)) \). Since this subspace is isomorphic to \( C_c(\Omega/G) \), which is infinite-dimensional, it follows that \( \mathcal{F}(L^p(\Omega)) \) has infinite codimension. \( \Box \)

**Remark 5.1.** On account of the preceding theorem, one may expect the space \( L^p(\Omega) \) to carry a unique invariant norm for some \( 1 \leq p < \infty \) only in the case when \( G \) acts **almost transitively** on \( \Omega \) in the sense that \( \Omega/G \) is finite. Suppose that \( G \) acts almost transitively on \( \Omega \) so that \( \Omega \) is the disjoint union of the sets \( G\omega_1, \ldots, G\omega_N \) for a suitable choice of \( \omega_1, \ldots, \omega_N \in \Omega \). Then we may identify \( L^p(\Omega) \) with the Banach space \( \ell_p - \bigoplus_{k=1}^N L^p(\Omega_{G_k}) \). Note that \( G \) acts transitively on each of the spaces \( G\omega_k \) and that, on account of [7, Proposition 2.44], all the spaces \( G\omega_k \) become **homogeneous spaces**. This means that there exist closed subgroups \( H_1, \ldots, H_N \) of \( G \) such that \( G\omega_k \) is homeomorphic to the quotient space \( G/H_k \) of left cosets of \( H_k \) for \( k = 1, \ldots, N \).

5.2. Transitive actions

In this section we shall study whether \( L^p(G/H) \) carries a unique invariant norm in the case when \( H \) is a closed subgroup of \( G \) and \( G/H \) stands for the quotient space of left cosets of \( H \). In the sequel, \( \lambda_H \) denotes the Haar measure on \( H \) normalized so that \( \lambda_H(H) = 1 \) and \( \vartheta \) denotes the canonical quotient map from \( G \) onto the homogeneous space \( G/H \). According to [1, Theorem 2 in Chapter VII, Section 2.5], \( G/H \) carries an invariant positive Radon measure \( \lambda_{G/H} \) which is uniquely determined up to a constant factor. This measure is the unique Radon measure on \( G/H \) satisfying

\[
\int_G f(t) \, d\lambda_G(t) = \int_{G/H} \left( \int_H f(ts) \, d\lambda_H(s) \right) \, d\lambda_{G/H}(tH)
\]

for each \( f \in C(G) \).
We begin by showing that \( L^1(\Omega) \) never carries a unique invariant norm provided that \( \Omega \) is infinite.

**Theorem 5.2.** Let \( G \) be a compact group acting on an infinite locally compact Hausdorff space \( \Omega \) which is endowed with an invariant positive Radon measure \( \lambda_\Omega \). Then \( L^1(\Omega) \) does not carry a unique invariant norm.

**Proof.** On account of Theorem 5.1, we are reduced to proving the theorem for \( \Omega/G \) finite.

We proceed to prove that there exists a discontinuous invariant functional on \( L^1(\Omega) \). We first consider the case when \( \Omega = G/H \), where \( H \) is a closed subgroup of \( G \). The proof consists in proving that \( \mathcal{T}(L^1(\Omega)) \) is not closed in \( L^1(\Omega) \). To obtain a contradiction, suppose that \( \mathcal{T}(L^1(\Omega)) = \mathcal{T}(L^1(\Omega)) \). We can apply Lemma 4.7 to the Banach spaces \( X = L^1(\Omega) \) and \( Y = M(\Omega) \). Observe that \( L^1(\Omega) \) is \( \sigma(M(\Omega), C(\Omega)) \)-dense in \( M(\Omega) \). Consequently, Lemma 4.7 gives \( N \in \mathbb{N} \) and \( C > 0 \) such that, for every \( \mu \in M(\Omega) \), the element \( \mu - \lambda_G \star \mu \) has a representation of the form

\[
\mu - \lambda_G \star \mu = \sum_{k=1}^{N} (\mu_k - \tau(t_k)\mu_k),
\]

where \( t_k \in G \) and \( \mu_k \in M(\Omega) \) with \( ||\mu_k|| \leq C||\mu|| \) for each \( k \in \{1, \ldots, N\} \). We now pick \( \omega \in \Omega \) and consider the unit point mass measure \( \delta_\omega \) at \( \omega \). There exist \( \mu_1, \ldots, \mu_N \in M(\Omega) \) and \( t_1, \ldots, t_N \in G \) with

\[
\delta_\omega - \lambda_G \star \delta_\omega = \sum_{k=1}^{N} (\mu_k - \tau(t_k)\mu_k).
\]

Our next claim is that \( \lambda_G \star \delta_\omega \) is a continuous measure. Write \( \omega = xH \) with \( x \in G \) and set \( s \in G \). Since \( G \) is compact, \( G \) is unimodular and therefore we have

\[
(\lambda_G \star \delta_\omega)((\{sH\}) = \lambda_G(\{t \in G : txH = sH\}) = \lambda_G(sHx^{-1}) = \lambda_G(H).
\]

What is left is to show that \( \lambda_G(H) = 0 \). Since \( \Omega \) is infinite, there exists a sequence \( (s_n) \) in \( G \) with \( s_n H \cap s_m H = \emptyset \) if \( n \neq m \). Hence

\[
\sum_{n=1}^{\infty} \lambda_G(s_n H) = \lambda_G \left( \bigcup_{n=1}^{\infty} s_n H \right) \leq \lambda_G(G) < \infty,
\]

which gives \( \lambda_G(H) = 0 \), since \( \lambda_G(H) = \lambda_G(s_n H) \) for each \( n \in \mathbb{N} \). We are now in a position to complete the proof. To this end we consider the discrete parts in the identity \( \delta_\omega - \lambda_G \star \delta_\omega = \sum_{k=1}^{N} (\mu_k - \tau(t_k)\mu_k) \) to obtain

\[
\delta_\omega = \sum_{k=1}^{N} (\mu_k - \tau(t_k)\mu_k),
\]
Lemma 5.1. The following assertions hold:

(i) For every \( f \in C(G) \), the function \( Q(f) : G/H \to \mathbb{C} \) defined by

\[
Q(f)(\tau(t)) = \int_H f(ts) \, d\lambda_H(s)
\]

for each \( t \in G \) is in \( C(G/H) \) and the map \( Q \) is a continuous linear operator from \( C(G) \) onto \( C(G/H) \) with \( \tau(t) \circ Q = Q \circ \tau(t) \) for each \( t \in G \);

(ii) For every \( g \in C(G/H) \), the function \( J(g) : G \to \mathbb{C} \) defined by \( J(g) = g \circ \delta \) is in \( C(G) \) and the map \( J \) is a linear isometry from \( C(G) \) into \( C(\Omega) \) with \( \tau(t) \circ J = J \circ \tau(t) \) for each \( t \in G \) and \( Q(J(g)) = g \) for each \( g \in C(G/H) \);

(iii) There exists a closed invariant subspace \( M \) of \( C(G) \) such that \( C(G) \) splits as the topological sum \( C(G) = J(C(G/H)) \oplus M \).

Accordingly, \( C(G/H) \) can be thought of as an invariant-complemented subspace of \( C(G) \).

Proof. (i) It is shown in [1, Remark VII, Section 2.1] that \( Q \) is a continuous linear operator from \( C(G) \) onto \( C(G/H) \) and it is easily seen that \( Q \) commutes with translations.

(ii) It is straightforward to check that \( J \) is a linear isometry from \( C(G) \) into \( C(G/H) \) which commutes with translations. On the other hand, if \( g \in C(G/H) \) and \( t \in G \), then we have

\[
Q(J(g))(\tau(t)) = \int_H J(g)(ts) \, d\lambda_H(s) = \int_H g(\delta(ts)) \, d\lambda_H(s) = \int_H g(\delta(t)) \, d\lambda_H(s) = g(\delta(t)).
\]

(iii) Define \( P = J \circ Q \). By (i) and (ii), \( P \) is a continuous linear operator from \( C(G) \) into itself such that \( P^2 = P \). Consequently, \( P(C(G)) \) and \( M = \ker(P) \) are closed subspaces of \( C(G) \) and \( C(G) = P(C(G)) \oplus M \). Since \( Q(C(G)) = C(G/H) \), we see
that $P(C(G)) = J(Q(C(G))) = J(C(G/H))$. On the other hand, $\tau(t) \circ P = P \circ \tau(t)$ for each $t \in G$ and therefore $M$ is invariant. \hfill \Box

**Lemma 5.2.** Let $1 \leq p < \infty$. Then the following assertions hold:

(i) For every $f \in L^p(G)$, the function $s \mapsto f(ts)$ is in $L^1(H)$ for each $tH \in G/H$ outside a null subset of $G/H$, the function $Q_p(f):G/H \to \mathbb{C}$ defined almost everywhere on $G/H$ by

$$Q_p(f)(\vartheta(t)) = \int_H f(ts) \, d\lambda_H(s)$$

is in $L^p(G)$, and the map $Q_p$ is a continuous linear operator from $L^p(G)$ onto $L^p(G/H)$ with $\tau(t) \circ Q_p = Q_p \circ \tau(t)$ for each $t \in G$;

(ii) For every $g \in L^p(G/H)$, the function $J_p(g):G \to \mathbb{C}$ defined by $J_p(g) = g \circ \vartheta$ is in $L^p(G)$ and the map $J_p$ is a linear isometry from $L^p(G/H)$ into $L^p(G)$ with $\tau(t) \circ J_p = J_p \circ \tau(t)$ for each $t \in G$ and $Q_p(J_p(g)) = g$ for each $g \in L^p(G/H)$;

(iii) There exists a closed invariant subspace $M_p$ of $L^p(G)$ such that $L^p(G) \cong J_p(L^p(G/H)) \oplus M_p$.

Accordingly, $L^p(G/H)$ can be thought of as an invariant-complemented subspace of $L^p(G)$.

**Proof.** (ii) Let $g \in L^p(G/H)$. From [1, Proposition VII, Section 2.5] we see that

$$\int_G |g(\vartheta(t))|^p \, d\lambda_G(t) = \int_{G/H} \left( \int_H |g(\vartheta(ts))|^p \, d\lambda_H(s) \right) \, d\lambda_{G/H}(tH)$$

$$= \int_{G/H} \left( \int_H |g(tH)|^p \, d\lambda_H(s) \right) \, d\lambda_{G/H}(tH)$$

$$= \int_{G/H} |g(tH)|^p \, d\lambda_{G/H}(tH).$$

Hence $J_p(g) \in L^p(G)$ and $\|J_p(g)\|_p = \|g\|_p$.

(i) Let us first consider the case $p = 1$. On account of [1, Proposition VII, Section 2.5], for every $f \in L^1(G)$, we have

$$\|Q_1(f)\|_1 \leq \int_{G/H} \left( \int_H |f(ts)| \, d\lambda_H(s) \right) \, d\lambda_{G/H}(tH) = \int_G |f(t)| \, d\lambda_G(t) = \|f\|_1.$$  

This yields the continuity of $Q_1$.  

We now turn to the case $p > 1$. Let $q$ with $\frac{1}{p} + \frac{1}{q} = 1$. Set $f \in L^p(G)$ and $g \in L^q(G/H)$. According to [1, Proposition VII, Section 2.5], we have

$$
\int_{G/H} |Q_p(f)(tH)g(tH)| \, d\lambda_{G/H}(tH) \leqslant \int_{G/H} \left( \int_H |f(ts)| \, d\lambda_H(s) \right) |g(tH)| \, d\lambda_{G/H}(tH)
= \int_{G/H} \left( \int_H |f(ts)||g(\vartheta(ts))| \, d\lambda_H(s) \right) \, d\lambda_{G/H}(tH)
= \int_G |f(t)||g(\vartheta(t))| \, d\lambda_G(t) \leqslant \|f\|_p \|g\|_q.
$$

Thus $Q_p(f) \in L^p(G/H)$ and $\|Q_p(f)\|_p \leqslant \|f\|_p$. It is easily seen that $Q_p$ commutes with translations. The property that $Q_p(J_q(g)) = g$ for each $g \in L^p(G/H)$ follows by the same direct computation as in the proof of Lemma 5.1(ii). It should be noted that it remains to prove that $Q_p$ maps onto $L^p(G/H)$.

(iii) This follows by the same method as in the proof of Lemma 5.1(iii). Define $P_p = J_p \circ Q_p$ and apply (i) and (ii) to get that $P_p$ is a continuous linear operator from $L^p(G)$ into itself such that $P_p^2 = P_p$. Therefore $P_p(L^p(G))$ and $M_p = \ker P_p$ are closed invariant subspaces of $L^p(G)$ with $L^p(G) = P_p(L^p(G)) \oplus M_p$. Since $C(G/H)$ is dense in $L^p(G/H)$ and $C(\Omega) = Q_p(C(G))$, it follows that

$$
J_p(L^p(G/H)) = J_p(C(G/H)) \subseteq J_p(C(G/H)) \subseteq J_p(Q_p(L^p(G)))
= P_p(L^p(G)) = P_p(L^p(G)).
$$

This implies that $P_p(L^p(G)) = J_p(L^p(G/H))$ and hence that $Q_p(L^p(G)) = L^p(G/H)$. □

**Theorem 5.3.** Let $G$ be a compact group and let $1 < p < \infty$. Then the following assertions are equivalent:

(i) For every compact Hausdorff space $\Omega$ on which $G$ acts with $\Omega/G$ finite, $L^p(\Omega)$ carries a unique invariant norm.

(ii) $L^p(G)$ carries a unique invariant norm.

(iii) Every invariant functional on $L^p(G)$ is continuous.

(iv) Every invariant functional on $L^p(G)$ is a constant multiple of the Haar integral.

**Proof.** (i) $\Rightarrow$ (ii) Take $\Omega = G$.

(ii) $\Rightarrow$ (i) Let $\Omega$ be a compact Hausdorff space on which $G$ acts with $\Omega/G$ finite. On account of Remark 5.1, there exist closed subgroups $H_1, \ldots, H_N$ of $G$ such that $L^p(\Omega) \cong L^p(G/H_1) \oplus \cdots \oplus L^p(G/H_N)$. According to Lemma 5.2(iii), there exist closed invariant subspaces $M_1, \ldots, M_N$ of $L^p(G)$ such that $L^p(G) \cong L^p(G/H_k) \oplus M_k$ for $k = 1, \ldots, N$. Then $M = M_1 \oplus \cdots \oplus M_N$ is a closed invariant subspace of $L^p(G)$ with the property that $L^p(\Omega) \oplus M \cong L^p(G)^N$. Let $[\pi] \in \hat{G}$ and
let $\phi$ be a $\pi$-invariant functional on $L(H_n) \otimes L^p(\Omega)$. We extend $\phi$ to a $\pi$-invariant functional $\psi$ on $L(H_n) \otimes L^p(G)^N$ by defining $\psi$ to be 0 on $L(H_n) \otimes M$. Our objective is to prove that $\psi$ is continuous. From Theorem 3.1 it follows that $\psi$ is continuous when restricted to any of the summands $L(H_n) \otimes L^p(G)$ and thus $\psi$ is continuous. Hence $\phi$ is continuous and Theorem 4.3 now shows that $L^p(\Omega)$ carries a unique invariant norm.

(ii) $\Rightarrow$ (iii) Theorem 3.1.

(iii) $\Rightarrow$ (iv) Theorem 4.2.

(iv) $\Rightarrow$ (ii) Let $[\pi] \in \hat{G}$ and let $\phi$ be a $\pi$-invariant functional on $L(H_n) \otimes L^p(G)$. We fix an orthonormal basis $(e_i)_{i=1}^n$ in $H_n$ so that $L(H_n) \otimes L^p(G)$ can be viewed as the matrix space $M_n(L^p(G))$ and $\pi$ can be thought of as an element in $M_n(C(G))$. Thus we can define a linear functional $\psi$ on $M_n(L^p(G))$ by $\psi(\zeta) = \phi(\pi\zeta)$ for each $\zeta \in M_n(L^p(G))$. It is straightforward to check that $\tau(t)(\pi\zeta) = \pi(t^{-1})\tau(t)\zeta$ for all $\zeta \in M_n(L^p(G))$ and $t \in G$. On account of the $\pi$-invariance of $\phi$, we have

$$
\psi(\tau(t)\zeta) = \phi(\pi\tau(t)\zeta) = \phi(\pi(t)\tau(t)(\pi\zeta)) = \phi(\pi\zeta)
$$

for all $\zeta \in M_n(L^p(G))$ and $t \in G$, which shows that $\psi$ is invariant. This entails that all the functionals $f \mapsto \psi(E_{ij} \otimes f)$ ($i,j \in \{1, \ldots, n\}$) on $L^p(G)$ are invariant. According to our assumption, each of them is a constant multiple of the Haar integral and so there exist $(x_{ij}) \in M_n(C)$ such that $\psi(\zeta) = \sum_{i,j} x_{ij} \int_G \zeta_{ij}(t)\,d\lambda_G(t)$ for each $\zeta \in M_n(L^p(G))$. Since $\phi(\zeta) = \psi(\pi^*\zeta)$ for each $\zeta \in M_n(L^p(G))$, we conclude that $\phi$ is continuous. Finally, Theorem 4.3 shows that $L^p(G)$ carries a unique invariant norm.

The group $G$ is said to have the mean-zero weak containment property if for all $t_1, \ldots, t_n \in G$ and $\varepsilon > 0$, there exists $f \in L_0^2(G) = \{f \in L^2(G) : \int_G f(t)\,d\lambda_G(t) = 0\}$ such that $\|f\|_2 = 1$ and $\|f - \tau(t_k)f\|_2 < \varepsilon$ for $k = 1, \ldots, n$. This means that the trivial representation of $G$ is weakly contained in the left regular representation of $G$ with the discrete topology on $L_0^2(G)$.

It is evidently of interest to know that the mean-zero weak containment property is closely related to Kazhdan’s property T. A discrete group $H$ is said to have Kazhdan’s property T if any unitary representation of $H$ which weakly contains the trivial representation has a one-dimensional invariant subspace. If $G$ contains a dense discrete subgroup with Kazhdan’s property, then the mean-zero weak containment property fails to hold (we refer the reader to [20] for more information). This is the case of the groups $SO(n)$ with $n \geq 5$ [13,25]. On the other hand, the failure of the mean-zero weak containment property is shown to imply the automatic continuity of the invariant functionals [19,28]. In fact, it is shown in [20] that for countable groups the failure of the mean-zero weak containment property is equivalent to the automatic continuity of the invariant functionals. We refer the reader to [4,13,17–19,25] for some examples of groups for which the mean-zero weak containment fails to hold.
**Theorem 5.4.** Let $G$ be a compact group for which the mean-zero weak containment fails to hold. Then for every compact Hausdorff space $\Omega$ on which $G$ acts with $\Omega/G$ finite, all the spaces $L^p(\Omega)$ for $1 < p < \infty$ and $C(\Omega)$ carry a unique topologically invariant norm.

**Proof.** Let $\Omega$ be a compact Hausdorff space on which $G$ acts with $\Omega/G$ finite and let $X$ and $Y$ denote either $L^p(\Omega)$ and $L^p(G)$ for $1 < p < \infty$, respectively, or $C(\Omega)$ and $C(G)$, respectively. Let us write the norm of $X$ and $Y$ by $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively.

As in the proof of Theorem 5.3, there exist $N \in \mathbb{N}$ and a closed invariant subspace $M$ of $Y$ such that $Y^N = X \oplus M$. We shall denote by $\| \cdot \|_{Y^N}$ the norm on $Y^N$ derived from $\| \cdot \|_Y$.

It is shown in [6, Lemma 5.1] (which was derived from [19] and [28]) that there exist $N \in \mathbb{N}$, $t_1, \ldots, t_N \in G$, and a constant $C$ such that every $f \in Y$ has a representation of the form 

$f = \lambda_G f + \sum_{k=1}^N (f_k - \tau(t_k) f_k),$

where $f_k \in Y$ and $\|f_k\|_Y \leq C \|f\|_Y$ for $k = 1, \ldots, N$. Set $[\pi] \in \hat{G}$. If $\xi \in L(H_\pi) \otimes Y$, then there exist $T_1, \ldots, T_N \in L(H_\pi)$ and $f_1, \ldots, f_N \in Y$ such that $\xi = \sum_{i=1}^N T_i \otimes f_i$ and $\sum_{i=1}^N ||T_i|| \|f_i\|_Y \leq 2||\xi||$. For every $i \in \{1, \ldots, m\}$ let $f_i, \ldots, f_{iN} \in Y$ such that

$\xi_k = \lambda_G f_i + \sum_{k=1}^N [f_i - \tau(t_k) f_k]$

with $f_k \in Y$ and $\|f_k\|_Y \leq C \|f\|_Y$ for $k = 1, \ldots, N$. We define $\xi_k = \sum_{i=1}^m T_i \otimes f_{ik}$ for $k = 1, \ldots, N$. Then $\xi = \lambda_G (\xi + \sum_{k=1}^N [\xi_k - \tau(t_k) \xi_k])$ and $||\xi_k||_Y \leq C ||\xi||_Y$ for $k = 1, \ldots, N$. We now fix an orthonormal basis $(e_i)_{i=1}^n$ in $H_\pi$ and we think of $L(H_\pi) \otimes Y$ as being $M_n(Y)$ and $\pi \in M_n(C(G))$. For every $\xi \in M_n(Y)$, we have $\pi^* \xi \in M_n(Y)$ and therefore $\pi^* \xi = \lambda_G (\pi^* \xi + \sum_{k=1}^N [\eta_k - \tau(t_k) \eta_k]$, where $\eta_k \in M_n(Y)$ and $||\eta_k|| \leq 2C ||\pi^* \xi||$ for $k = 1, \ldots, N$. On the other hand, it is easily seen that $\lambda_G (\pi^* \xi) = \pi^* \pi^* (\xi)$. We now define $\xi_k = \pi(t_k) \eta_k$ for $k = 1, \ldots, N$. We check at once that $\xi = \pi^* (\xi + \sum_{k=1}^N [\pi(t_k) \xi_k - \tau(t_k) \xi_k])$ and that $||\xi_k||_Y \leq A \|\xi||$ for $k = 1, \ldots, N$ for a suitable choice of $A$. It is obvious that the same decomposition holds if we replace $Y$ with $Y^N$.

Note that, for every $f \in Y$, $\pi^* (f) = \pi \int G \pi(t^{-1}) f(t) d\lambda_G(t)$ and therefore $\pi^* (Y)$ as well as $\pi^* (Y^N)$ are finite-dimensional.

If $| \cdot |$ is a norm on $Y^N$ such that $\tau$ is a representation of $G$ on $(Y^N, | \cdot |)$, then we can apply Theorem 4.1 with $B_\pi = \max\{||\tau(t_1)||, \ldots, ||\tau(t_N)||\}$ to get that $\| \cdot \|_{Y^N}$ and $\| \cdot \|$ are equivalent.

We are now in a position to prove the uniqueness of the norm on $X$. Let $| \cdot |$ a norm on $X$ such that $\tau$ becomes a representation of $G$ on $(X, | \cdot |)$. Then we can define a norm $|| \cdot ||$ on $Y^N$ by $||y|| = ||y_X|| + ||y_M||_{Y^N}$ for each $y \in Y^N$, where $y = y_X + y_M$ with $y_X \in X$ and $y_M \in M$. It is a simple matter to show that $\tau$ becomes a representation of $G$ on $(Y^N, || \cdot ||)$ and from what has previously been proved, it may be concluded that $|| \cdot ||_{Y^N}$ and $|| \cdot ||$ are equivalent. This clearly entails that $|| \cdot ||_X$ and $\| \cdot \|$ are equivalent. $\square$

**Remark 5.2.** It is worth pointing out that Theorems 5.3 and 5.4 generalize [6, Theorems 5.1 and 5.2]. In fact, it is important to note that in [6] we were required
to consider two-sided invariant norms while in Theorems 5.3 and 5.4 we show that just the left invariance suffices to characterize the norm.

It has been shown by Jarosz in [10] that both $L^1(S^1)$ and $C(S^1)$ do not carry a unique topologically rotation-invariant norm and that $L^p(S^1)$ with $1 < p < \infty$ carries a unique topologically rotation-invariant norm. We can now extend this result to an arbitrary $n$-dimensional sphere $S^n$. Indeed, when solving the Banach–Ruzewicz problem for the spheres, it has been shown that the mean-zero weak containment fails to hold for the groups $SO(n)$ with $n \geq 3$ (cases $n = 3, 4$ in [4] and case $n \geq 5$ in [13, 25]) and, on the other hand, the map

$$\sigma \mapsto \sigma(\text{north pole}), \ SO(n+1) \to S^n$$

turns $S^n$ into a homogeneous space (observe that $S^n = SO(n+1)/SO(n)$). Therefore Theorem 5.4 now yields the following.

**Corollary 5.1.** If $n > 1$, then all the spaces $L^p(S^n)$ with $1 < p < \infty$ and $C(S^n)$ carry a unique topologically rotation invariant norm.

**Remark 5.3.** (i) It is worth pointing out that Corollary 5.1 holds true with $S^n$ replaced by any compact Hausdorff space equipped with a positive Radon measure on which $SO(n+1)$ (with $n > 1$) acts transitively as a group of continuous invertible measure-preserving transformations, such as the $n$-dimensional projective space $\mathbb{R}P^n$ with $n > 1$ (observe that $\mathbb{R}P^n = SO(n+1)/O(n)$).

(ii) Let $M$ be a connected compact Riemannian manifold. According to [15], the group $\text{Iso}(M)$ of the isometries of $M$ is a compact Lie group when endowed with the pointwise convergence topology. Every isometry $\sigma$ of $M$ yields an isometry of the Banach space $L^p(M)$ with $1 \leq p \leq \infty$ in the obvious way $f \mapsto f \circ \sigma$. The question we wish to address is whether the classical norms $\| \cdot \|_p$ are the unique norms which are so well-behaved with respect to $\text{Iso}(M)$. On account of Theorems 5.1 and 5.2, if $L^p(M)$ carries a unique invariant norm for some $1 \leq p < \infty$, then $M/\text{Iso}(M)$ is finite and $p \neq 1$. Being $M$ connected this entails that $\text{Iso}(M)$ acts transitively on $M$. By Theorem 5.4, if the mean-zero weak containment fails to hold for $\text{Iso}(M)$ then all the spaces $L^p(M)$ with $1 < p < \infty$ and $C(M)$ carry a unique norm invariant under $\text{Iso}(M)$.

**References**