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Inverse spectral problems for Sturm–Liouville operators with singular potentials. Part III: Reconstruction by three spectra ☆

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Abstract

We solve the inverse spectral problem of recovering the singular potential from $W_2^{-1}(0,1)$ of a Sturm–Liouville operator by its spectra on the three intervals $[0,1]$, $[0,a]$, and $[a,1]$ for some $a \in (0,1)$. Necessary and sufficient conditions on the spectral data are derived, and uniqueness of the solution is analyzed.

Keywords: Inverse spectral problems; Sturm–Liouville operators; Singular potentials

1. Introduction

Suppose that a function σ belongs to $L_2(0,1)$, that c and d, $c < d$, are real numbers from the interval $[0,1]$, and assume that h_0 and h_1 are arbitrary elements of the extended complex plane $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. We denote by $T = T([c,d], \sigma, h_0, h_1)$ an operator in $L_2(c,d)$ that acts according to the formula

$$Tu = l_\sigma(u) := -(u' - \sigma u)' - \sigma u' \tag{1.1}$$

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on the domain $\operatorname{dom} T$ consisting of functions $u \in W_1^1[c,d]$ with absolutely continuous *quasi-derivative* $u^{[1]} := u' - \sigma u$, for which $l_\sigma(u) \in L_2(c,d)$ and which satisfy the boundary conditions

$$u^{[1]}(c) = h_0 u(c), \qquad u^{[1]}(d) = h_1 u(d). \tag{1.2}$$

If $h_0 = \infty$ (respectively, $h_1 = \infty$), then the corresponding boundary condition is regarded as a Dirichlet one, i.e., as $u(c) = 0$ (respectively, as $u(d) = 0$).

It is known [20] that so defined operator T is closed and has nonempty resolvent set. Moreover, since $l_\sigma(u) = -u'' + \sigma' u$ in the sense of distributions, T is a Sturm–Liouville operator with singular potential $q = \sigma' \in W_2^{-1}(0,1)$. We note that among the singular potentials that can be treated by this *regularization* method are, e.g., the Dirac δ-potentials and the Coulomb $1/x$-like potentials that have been widely used in quantum mechanics and mathematical physics; see [1,2] for particulars and detailed reference lists.

In the following the function σ will always be real-valued and the numbers h_0, h_1 will always belong to $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. Then the operator $T([c,d],\sigma,h_0,h_1)$ is selfadjoint, bounded below, and has simple discrete spectrum accumulating at $+\infty$ [20]. The *inverse spectral problem* is to reconstruct the operator $T([c,d],\sigma,h_0,h_1)$ based on its spectral data. Classical results of the inverse spectral theory [13,14,16] imply that in regular situations (i.e., for locally integrable potentials) knowing only the spectrum is not sufficient: there are many Sturm–Liouville operators with the given spectrum. The same conclusion was drawn in [10] for the class of Sturm–Liouville operators (1.1)–(1.2) with singular potentials from $W_2^{-1}(c,d)$. In the regular case the data allowing unique reconstruction of the potential are: the spectrum and the so-called norming constants [13,16], or two spectra (corresponding to Sturm–Liouville operators with the same potential but different boundary conditions) [13,14], or three spectra (one for the whole interval and the others for two parts of it) [6,15], or the spectrum and half of the potential [3,5,9,18], or the spectral function [13], etc. In [10,11] the first two settings of the inverse spectral problem were completely investigated in the class of Sturm–Liouville operators (1.1)–(1.2) with singular potentials from $W_2^{-1}(c,d)$. Our aim here is to study the third of the above-mentioned settings, i.e., reconstruction by three spectra.

More exactly, we fix an arbitrary real-valued function $\sigma \in L_2(0,1)$, an arbitrary number $a \in (c,d)$, and a triple $\mathbf{h} = (h,h_0,h_1) \in \bar{\mathbb{R}}^3$ and denote by $(\lambda_n^2)_{n\in\mathbb{N}}$, $(\lambda_{0,n}^2)_{n\in\mathbb{N}}$, and $(\lambda_{1,n}^2)_{n\in\mathbb{N}}$ the eigenvalues (in increasing order) of the operators $T([c,d],\sigma,h_0,h_1)$, $T([c,a],\sigma,h_0,h)$, and $T([a,d],\sigma,h,h_1)$, respectively. Then we have a mapping

$$(\sigma,a,\mathbf{h}) \mapsto \Lambda := \bigl((\lambda_n^2),(\lambda_{0,n}^2),(\lambda_{1,n}^2)\bigr), \tag{1.3}$$

and a natural question arises whether this mapping is injective. Also, what is its range when a and $\mathbf{h}$ are fixed and σ runs through $L_2(0,1)$? Does there exist an efficient algorithm of recovering σ, a, and $\mathbf{h}$ from Λ?

Our interest in these questions is motivated by papers [6,15], were similar problems were addressed. In particular, Gesztesy and Simon [6] considered the case $\sigma \in W_1^1[0,1]$ and found sufficient conditions on the three spectra in Λ guaranteeing uniqueness of the potential σ'. Pivovarchik [15] presented an algorithm reconstructing the potential in the case where σ belongs to $W_2^2[0,1]$, $a = 1/2$, $h = h_1 = h_2 = \infty$, and all eigenvalues are pairwise distinct.

Note that this inverse spectral problem admits the following mechanical interpretation. Consider a vibrating string of unit length and suppose that λ_n are its eigenfrequencies. Now we clamp the string at the point $x = a$ and determine the eigenfrequencies $\lambda_{0,n}$ and $\lambda_{1,n}$ of two parts. The problem is to determine the structure of the string (e.g., its mass density) from the available data.

To avoid technical complications, we restrict ourselves to the most interesting (from our point of view) case where $c = 0$, $d = 1$, and $h = h_1 = h_2 = \infty$; see [6] for discussion of other boundary conditions in the regular case. We give an explicit description of the range of mapping (1.3) (i.e., solve the direct spectral problem), find the preimage of an arbitrary point Λ in the range (i.e., solve the inverse spectral problem), and give necessary and sufficient conditions under which this preimage consists of a single point (i.e., when the inverse spectral problem admits a unique solution).

The paper is organized as follows. In Section 2 we give the necessary definitions and formulate the main results. Some preliminaries are derived in Sections 3 and 4. In Section 5 we solve the direct spectral problem, and in the last section solve the inverse spectral problem.

2. Formulation of the main results

Throughout the paper $\mathcal{H}$ will stand for the Hilbert space $L_2(0,1)$ and σ will be an arbitrary real-valued function from $\mathcal{H}$. As was mentioned in introduction, we shall only consider the case of Dirichlet boundary conditions at the points 0, 1, and a, i.e., the case where $h = h_0 = h_1 = \infty$. Respectively, the above three operators are specified as

$$\begin{aligned} T &= T(\sigma) := T\big([0,1], \sigma, \infty, \infty\big), \\ T_0 &= T_0(\sigma) := T\big([0,a], \sigma, \infty, \infty\big), \\ T_1 &= T_1(\sigma) := T\big([a,1], \sigma, \infty, \infty\big). \end{aligned} \tag{2.1}$$

Observe that $T(\sigma + c) = T(\sigma)$ and $T_j(\sigma + c) = T_j(\sigma)$ for any $c \in \mathbb{R}$, so that without loss of generality we shall impose the restriction that $\int_0^1 \sigma = 0$. The operators $T(\sigma)$ and $T_j(\sigma)$, $j = 0, 1$, are selfadjoint and bounded below and hence become positive after adding a suitable constant to the potential $q := \sigma'$ (i.e., after adding to σ a suitable multiple of $(x - 1/2)$). Since under such a transformation the spectra in Λ shift elementwise by the same constant, we may only concentrate on the case where all three operators are positive.

Denote by Σ_0^+ the set of all real-valued functions σ in $\mathcal{H}$ with zero mean value, for which the operator $T(\sigma)$ is positive. (Observe that the operators $T_j(\sigma)$, $j = 0, 1$, are positive as soon as such is $T(\sigma)$; this easily follows from the variation principle [17, Proposition XIII.15.4].) In what follows, σ will always stand for a generic element of Σ_0^+.

Suppose therefore that $\sigma \in \Sigma_0^+$ and denote by $(\lambda_n^2(\sigma))_{n\in\mathbb{N}}$ and $(\lambda_{j,n}^2(\sigma))_{n\in\mathbb{N}}$, $j = 0, 1$, the eigenvalues of the operators $T(\sigma)$ and $T_j(\sigma)$, $j = 0, 1$. According to the definition of Σ_0^+, all these eigenvalues (and the square roots of them $\lambda_n(\sigma)$ and $\lambda_{j,n}(\sigma)$) are positive; they also are pairwise distinct within each of the sequences. Therefore we can (and always shall) arrange them in strictly increasing order. We shall often omit dependence on σ, especially when no ambiguity arises.

It is known (see, e.g., [10,19,20]) that the eigenvalues $\lambda_n^2 = \lambda_n^2(\sigma)$ of the operator T obey the asymptotics

$$\lambda_n = \pi n + a_n, \quad n \in \mathbb{N}, \tag{2.2}$$

where $(a_n) \in \ell_2$. It is easily seen that the operators $T_0(\sigma)$ and $T_1(\sigma)$ are similar to the operators $a^{-2}T(\sigma_a^-)$ and $(1-a)^{-2}T(\sigma_a^+)$, respectively, where

$$\sigma_a^-(x) := a\sigma(ax), \quad x \in [0,1],$$
$$\sigma_a^+(x) := (1-a)\sigma\big(a + (1-a)x\big), \quad x \in [0,1].$$

This observation and (2.2) yield the following asymptotics of the eigenvalues $\lambda_{j,n}^2$:

$$\lambda_{0,n} = \frac{\pi n}{a} + a_{0,n}, \quad \lambda_{1,n} = \frac{\pi n}{1-a} + a_{1,n}, \quad n \in \mathbb{N}, \tag{2.3}$$

where $(a_{j,n})$, $j = 0, 1$, are some ℓ_2-sequences. In particular, the number $a \in (0,1)$ is uniquely determined by the spectrum of T_0 or T_1. There is no loss of generality in supposing a known a priori, so that mapping (1.3) should more correctly be defined as follows:

$$\Sigma_0^+ \ni \sigma \mapsto \mathfrak{l}(\sigma) := \Lambda = \big((\lambda_n^2), (\lambda_{0,n}^2), (\lambda_{1,n}^2)\big).$$

As we have shown, any element Λ from the range of $\mathfrak{l}$ must obey asymptotics (2.2) and (2.3). However, there are other conditions that any Λ in $\mathfrak{l}(\Sigma_0^+)$ must verify.

To begin with, if a number λ^2 belongs to two spectra in Λ, then λ^2 belongs to the third spectrum as well. This is best seen by inspection of the corresponding eigenfunctions.

Secondly, the spectra in Λ possess some interlacing property. To explain it, we denote by $(\mu_n^2)_{n\in\mathbb{N}}$ the sequence obtained by combining the sequences $(\lambda_{0,n}^2)$ and $(\lambda_{1,n}^2)$ into one and rearranging the union in increasing order; moreover, we repeat twice in (μ_n^2) each common element of $(\lambda_{0,n}^2)$ and $(\lambda_{1,n}^2)$. Symbolically, we shall denote this operation by $\amalg$ and write $(\mu_n^2) = (\lambda_{0,n}^2) \amalg (\lambda_{1,n}^2)$. Observe that (μ_n^2) is the sequence of eigenvalues (repeated according to multiplicity and arranged in increasing order) of the operator $T_0 \oplus T_1$. Then [4, Sections IV.8.1, IV.10] the sequences (λ_n^2) and (μ_n^2) interlace, i.e., $\lambda_1 < \mu_1$ and $\mu_n \leqslant \lambda_{n+1} \leqslant \mu_{n+1}$ for all $n \in \mathbb{N}$.

For any triple of strictly increasing real sequences $\Lambda = ((\lambda_n^2), (\lambda_{0,n}^2), (\lambda_{1,n}^2))$, we put $(\mu_n^2) := (\lambda_{0,n}^2) \amalg (\lambda_{1,n}^2)$, denote by A_Λ the set of all $n \in \mathbb{N}$ such that $\mu_n = \mu_{n+1}$, and set $B_\Lambda := \mathbb{N} \setminus A_\Lambda$. Then the combination of the above intersection and interlacing properties implies that, for each Λ in the range of $\mathfrak{l}$, we have $\mu_n = \lambda_{n+1} = \mu_{n+1}$ if $n \in A_\Lambda$ and $\mu_n < \lambda_{n+1} < \mu_{n+1}$ if $n \in B_\Lambda$.

The third, and last, restriction on Λ is somewhat technical, and we illustrate it first for the simplest case $a = 1/2$ under the assumption that the three spectra in Λ are pairwise disjoint. Then the eigenvalues λ_{2n}^2, $\lambda_{0,n}^2$, and $\lambda_{1,n}^2$ are asymptotically (i.e., for large n) close and λ_{2n}^2 lies between the other two, see Fig. 1. The third restriction states that the distances $|\lambda_{2n} - \lambda_{0,n}|$ and $|\lambda_{1,n} - \lambda_{2n}|$ must be asymptotically equal. More exactly, the requirement is that the sequence

$$\frac{\lambda_{2n} - \lambda_{0,n}}{\lambda_{1,n} - \lambda_{2n}} - 1 \tag{2.4}$$

belongs to ℓ_2.

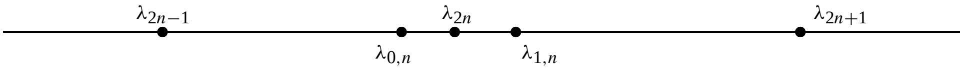

Fig. 1.

For an arbitrary $a \in (0,1)$ this can be reformulated as follows. If a is rational, i.e., if $a = r/s$ for relatively prime naturals r and s, then we put $M(a) = s\mathbb{N}$ and for every $k \in M(a)$ set $k' = k'(k,a) = ak$ and $k'' = k''(a,k) = (1-a)k$. If a is irrational, then we put

$$M(a) := \bigl\{k \in \mathbb{N} \mid \exists m \in \mathbb{N} \text{ s.t. } |ak - m| < a(1-a)/2\bigr\}$$

and for every $k \in M(a)$ we denote by $k' = k'(k,a)$ and $k'' = k''(k,a)$ unique natural numbers, for which

$$|ak - k'| < a(1-a)/2, \qquad \bigl|(1-a)k - k''\bigr| < a(1-a)/2.$$

It is easily seen that, in both cases, $M(a) = M(1-a)$ and $k' + k'' = k$.

Fix $k \in M(a)$ large enough. It follows from definition that the numbers $\lambda_{0,k'}$ and $\lambda_{1,k''}$ are close to λ_k and contain λ_k in between; cf. the above special case $a = 1/2$, where $M(a) = 2\mathbb{N}$ and $(2n)' = (2n)'' = n$ for every $n \in \mathbb{N}$. Thus it is natural to expect that condition like (2.4) should concern only $k \in M(a) \cap B_\Lambda$.

After these preparations, we are in a position to give the following definition.

Definition 2.1. We denote by $\mathfrak{L}$ the set of all triples

$$\bigl((\lambda_n^2), (\lambda_{0,n}^2), (\lambda_{1,n}^2)\bigr) =: \Lambda$$

of strictly increasing sequences of positive numbers, which satisfy the following conditions:

(1) The numbers λ_n^2 and $\lambda_{j,n}^2$, $j = 0,1$, obey the asymptotics

$$\lambda_n = \pi n + a_n, \quad \lambda_{0,n} = \frac{\pi n}{a} + a_{0,n}, \quad \lambda_{1,n} = \frac{\pi n}{1-a} + a_{1,n}, \quad n \in \mathbb{N},$$

where the sequences (a_n), $(a_{0,n})$, and $(a_{1,n})$ belong to ℓ_2;

(2) With $(\mu_n^2) := (\lambda_{0,1}^2) \amalg (\lambda_{1,n}^2)$ we have $\lambda_1 < \mu_1$ and

$$\mu_n = \lambda_{n+1} = \mu_{n+1}, \quad n \in A_\Lambda,$$
$$\mu_n < \lambda_{n+1} < \mu_{n+1}, \quad n \in B_\Lambda;$$

(3) The sequence $(\delta_k)_{k \in B'_\Lambda}$, where $B'_\Lambda := B_\Lambda \cap M(a)$ and

$$\delta_k = \delta_k(\Lambda) := \frac{a(\lambda_k - \lambda_{0,k'})}{(1-a)(\lambda_{1,k''} - \lambda_k)} - 1, \quad k \in B'_\Lambda,$$

belongs to $\ell_2(B'_\Lambda)$.

Our first result is as follows.

Theorem 2.2. *The range $\mathfrak{l}(\Sigma_0^+)$ of the mapping $\mathfrak{l}$ coincides with the set $\mathfrak{L}$. A point $\Lambda \in \mathfrak{L}$ has a unique preimage $\sigma = \mathfrak{l}^{-1}(\Lambda) \in \Sigma_0^+$ if and only if $A_\Lambda = \emptyset$.*

We note that for the case of a regular potential $q = \sigma' \in W_2^1(0,1)$ and $a = 1/2$ the asymptotics of the spectra in $\Lambda = \mathfrak{l}(\sigma)$ refines to

$$\lambda_n = \pi n + \frac{\tilde{a}}{n} + \frac{b_n}{n^2}, \quad \lambda_{0,n} = 2\pi n + \frac{\tilde{a}_0}{n} + \frac{b_{0,n}}{n^2}, \quad \lambda_{1,n} = 2\pi n + \frac{\tilde{a}_1}{n} + \frac{b_{1,n}}{n^2}, \quad n \in \mathbb{N},$$

where $\tilde{a}$ and $\tilde{a}_j$, $j = 0, 1$, are real numbers such that $\tilde{a}_0 + \tilde{a}_1 = \tilde{a}$ and $(b_n), (b_{j,n})$ are some ℓ_2-sequences. It easily follows that condition (3) of Definition 2.1 holds in this case automatically as soon as $\tilde{a}_0 \neq \tilde{a}_1$. Pivovarchik proved in [15] that, for an arbitrary triple Λ satisfying the above refined asymptotics with $\tilde{a}_0 \neq \tilde{a}_1$, assumption (2) of Definition 2.1, and the condition $A_\Lambda = \emptyset$, the inverse spectral problem has a unique solution, i.e., there exists a unique potential $q = \sigma' \in L_2[0,1]$ such that $\mathfrak{l}(\sigma) = \Lambda$. Since such a Λ necessarily falls into $\mathfrak{L}$ as explained above, Theorem 2.2 extends the results of [15] to the class of singular potentials from $W_2^{-1}(0,1)$. Observe also that there is an inconsistency in [15] between the required asymptotics of the spectral data and the declared smoothness of the restored potential q, while Theorem 2.2 gives explicit necessary and sufficient conditions for a triple Λ to be the spectral data for some $q \in W_2^{-1}(0,1)$.

An example with $A(\Lambda) \neq \emptyset$ where the uniqueness of solution to the inverse spectral problem by three spectra fails was constructed by Gesztesy and Simon in [6]. The authors conjectured therein that nonuniqueness should take place whenever $A(\Lambda) \neq \emptyset$. Theorem 2.2 justifies this conjecture even in a more general setting.

We mention that an efficient algorithm was suggested in [15] for recovering the potential q from three spectra. Basically the approach consisted in reducing the problem to recovering the potential from the Dirichlet–Dirichlet and Dirichlet–Neumann spectra and then using the classical results of Marchenko [14].

Here, we also give an efficient reconstruction algorithm, but take a slightly different approach. Namely, we reduce the inverse spectral problem to recovering the potential by the Dirichlet–Dirichlet spectrum and the sequence of so-called norming constants. The latter problem for the case of Sturm–Liouville operators with singular potentials from $W_2^{-1}(0,1)$ has been completely solved in our paper [10].

To formulate the corresponding result from [10] we need some definitions. For an arbitrary $\sigma \in \Sigma_0^+$ and nonzero $\lambda \in \mathbb{C}$, we denote by $u(\cdot, \lambda, \sigma)$ a solution to equation $l_\sigma u = \lambda^2 u$ satisfying the initial conditions $u(0) = 0$ and $u^{[1]}(0) = \sqrt{2}\lambda$. Then $\phi_n := u(\cdot, \lambda_n, \sigma)$ is an eigenfunction of the operator $T(\sigma)$ corresponding to the eigenvalue λ_n^2, and we put

$$\alpha_n = \alpha_n(\sigma) := \int_0^1 \left|\phi_n(x)\right|^2 dx.$$

We denote by $\mathfrak{A}$ the set of pairs $\{(\lambda_n^2)_{n\in\mathbb{N}}, (\alpha_n)_{n\in\mathbb{N}}\}$, in which the numbers λ_n are positive, strictly increase, and obey asymptotics (2.2), and α_n are positive and $\tilde{\alpha}_n := \alpha_n - 1$ form an ℓ_2-sequence. Any element of $\mathfrak{A}$ is naturally associated with ℓ_2-sequences (a_n) of (2.2) and $(\tilde\alpha_n)$. In this way the set $\mathfrak{A}$ is identified with a subset of the Hilbert space $\ell_2 \times \ell_2$ and thus becomes a topological space. The results of [10] easily imply the following statement.

Proposition 2.3. *The mapping $\mathfrak{a}: \Sigma_0^+ \to \mathfrak{A}$, which maps a function $\sigma \in \Sigma_0^+$ into the pair $\{(\lambda_n^2), (\alpha_n)\}$, with (λ_n^2) being the spectrum of $T(\sigma)$ and (α_n) the corresponding sequence of norming constants, is homeomorphic.*

We shall show that the three spectra in $\Lambda \in \mathfrak{L}$ determine the norming coefficients α_k for $k \in B_\Lambda$, while those for $k \in A_\Lambda$ remain undefined. This observation explains why nonempty A_Λ leads to nonuniqueness of the solution of the inverse spectral problem. More exactly, we establish the following result.

Theorem 2.4. *Suppose that $\Lambda \in \mathfrak{L}$ and $A_\Lambda \neq \emptyset$. For any sequence $(\theta_k)_{k\in A_\Lambda}$ belonging to $\ell_2(A_\Lambda)$ and satisfying the condition $1+\theta_k > 0$, $k \in A_\Lambda$, there exists a unique $\sigma \in \Sigma_0^+$ such that $\mathfrak{l}(\sigma) = \Lambda$ and $\alpha_k(\sigma) = 1+\theta_k$ for all $k \in A_\Lambda$.*

As a corollary, we get the following description of the set $\mathfrak{l}^{-1}(\Lambda)$.

Corollary 2.5. *Suppose that $\sigma \in \Sigma_0^+$ and put $\Lambda := \mathfrak{l}(\sigma)$. Then*

$$\mathfrak{l}^{-1}(\Lambda) = \bigl\{\tilde\sigma \in \Sigma_0^+ \mid \forall n \in \mathbb{N},\ \lambda_n^2(\tilde\sigma) = \lambda_n^2(\sigma) \text{ and } \forall k \in B_\Lambda,\ \alpha_k(\tilde\sigma) = \alpha_k(\sigma)\bigr\}. \tag{2.5}$$

Throughout the paper, u' and $\dot u$ will denote derivatives of a function u with respect to x and λ, respectively. Also, for a fixed function $\sigma \in \mathcal{H}$, the notation $u^{[1]}$ means the quasi-derivative $u' - \sigma u$ of a function u.

3. Preliminary results

In this section we shall recall some known facts and prove statements to be used later on.

Suppose that $\sigma \in \Sigma_0^+$; we denote by $\hat T_\sigma := T([0,1],\sigma,\infty)$ a Sturm–Liouville operator

$$-\frac{d^2}{dx^2} + \sigma'$$

with the Dirichlet boundary condition at the point $x = 0$. More exactly, $\hat T_\sigma$ acts according to the formula

$$\hat T_\sigma u = l_\sigma(u) := -(u' - \sigma u)' - \sigma u'$$

on the domain

$$\operatorname{dom} \hat T_\sigma := \bigl\{u \in W_1^1[0,1] \mid u^{[1]} \in W_1^1[0,1],\ l_\sigma(u) \in \mathcal{H},\ u(0) = 0\bigr\}.$$

In particular, $T(\sigma)$ is a restriction of $\hat T_\sigma$ imposing the Dirichlet boundary condition at the point $x = 1$.

One of the key results of [12] states that the pair $\hat{T}_\sigma$ and $\hat{T}_0$ possesses the *transformation operator* $I + K_\sigma$ that performs similarity of $\hat{T}_\sigma$ and $\hat{T}_0$ and enjoys some nice properties. Namely, K_σ is a Hilbert–Schmidt integral operator of Volterra type, i.e.,

$$(K_\sigma u)(x) = \int_0^x k_\sigma(x,t)u(t)\,dt.$$

The kernel k_σ of K_σ possesses the property that its cross-sections $f_x(\cdot) := k_\sigma(x,\cdot)$ belong to $\mathcal{H}$ for every $x \in [0,1]$ and, moreover, the mapping $x \mapsto f_x$ is continuous from $[0,1]$ into $\mathcal{H}$.

The relation $\hat{T}_\sigma(I + K_\sigma) = (I + K_\sigma)\hat{T}_0$ shows that any element of $\operatorname{dom}\hat{T}_\sigma$ has the form $(I + K_\sigma)v$, where v is an arbitrary function from $W_2^2[0,1]$ with $v(0) = 0$. Some additional properties of K_σ imply that, for $u = (I + K_\sigma)v$ with v as above, $u^{[1]}(0) = v'(0)$.

For an arbitrary nonzero $\lambda \in \mathbb{C}$, we denote by $s_-(\cdot,\lambda) = s_-(\cdot,\lambda,\sigma)$ ($s_+(\cdot,\lambda) = s_+(\cdot,\lambda,\sigma)$) a solution to the equation $l_\sigma(u) = \lambda^2 u$ satisfying the initial conditions $u(0) = 0$ and $u^{[1]}(0) = 1$ (respectively, satisfying the terminal conditions $u(1) = 0$ and $u^{[1]}(1) = 1$). The above properties of K_σ imply that

$$s_-(x,\lambda) = \frac{\sin\lambda x}{\lambda} + \int_0^x k_\sigma(x,t)\frac{\sin\lambda t}{\lambda}\,dt. \tag{3.1}$$

Changing the variable x to $1-x$, we show existence of the transformation operator $I + \hat{K}_\sigma$ for $\hat{T}_\sigma$ and $\hat{T}_0$ connected with the point $x = 1$ and possessing similar properties; in particular, the kernel $\hat{k}_\sigma$ of $\hat{K}_\sigma$ has the property that $\hat{k}_\sigma(x,\cdot) \in \mathcal{H}$ for every $x \in [0,1]$ and

$$s_+(x,\lambda) = \frac{\sin\lambda(x-1)}{\lambda} + \int_x^1 \hat{k}_\sigma(x,t)\frac{\sin\lambda(t-1)}{\lambda}\,dt. \tag{3.2}$$

According to the definition of $s_\pm$, we have $s_-(1,\lambda_n) = s_+(0,\lambda_n) = 0$ and $s_-(a,\lambda_{0,n}) = s_+(a,\lambda_{1,n}) = 0$. Moreover, for each fixed $x \in [0,1]$ the functions $s_-(x,\lambda)$ and $s_+(x,\lambda)$ are even entire functions of λ of order x and $1-x$, respectively, and thus are uniquely determined by their zeros. The corresponding representations can be derived from the following modification of Lemma 3.4.2 from [14], whose proof can be found in [11].

Proposition 3.1. *For an entire function f to admit the representation*

$$f(\lambda) = \frac{\sin\lambda}{\lambda} + \int_0^1 \hat{f}(t)\frac{\sin\lambda t}{\lambda}\,dt \tag{3.3}$$

with some function $\hat{f} \in \mathcal{H}$, it is necessary and sufficient that

$$f(\lambda) = \prod_{k=1}^{\infty}\frac{f_k^2 - \lambda^2}{(\pi k)^2}, \tag{3.4}$$

where $f_k = \pi k + \tilde{f}_k$, the sequence $(\tilde{f}_k)$ belongs to ℓ_2, and $\pm f_k$ are all the zeros of f.

Lemma 3.2. *The following formulae hold*:

$$s_-(1,\lambda)=\prod_{k=1}^{\infty}\frac{\lambda_k^2-\lambda^2}{(\pi k)^2},\qquad s_-(a,\lambda)=a\prod_{k=1}^{\infty}\frac{\lambda_{0,k}^2-\lambda^2}{(\pi k/a)^2},$$
$$s_+(a,\lambda)=-(1-a)\prod_{k=1}^{\infty}\frac{\lambda_{1,k}^2-\lambda^2}{[\pi k/(1-a)]^2}.$$

Proof. The formula for $s_-(1,\lambda)$ follows directly from (3.1) and Proposition 3.1. Putting $\hat g(t):=ak_\sigma(a,at)\in\mathcal{H}$, we find that

$$\frac{s_-(a,\lambda)}{a}=\frac{\sin a\lambda}{a\lambda}+\int_0^1\hat g(t)\frac{\sin a\lambda t}{a\lambda}\,dt=:g(a\lambda).$$

Therefore by Proposition 3.1 we have

$$g(\mu)=\prod_{k=1}^{\infty}\frac{g_k^2-\mu^2}{(\pi k)^2},$$

where $\pm g_k$ have the required asymptotics and are all the zeros of g. It is easily seen that $g_k=a\lambda_{0,k}$, so that the representation for $s_-(a,\lambda)$ follows.

In the same manner we derive the formula for $s_+(a,\lambda)$, and the lemma is proved. □

At various points we shall also use the following well-known result (see, e.g., [8]).

Proposition 3.3. *Assume that ν_k, $k\in\mathbb{Z}_+$, are pairwise distinct positive numbers such that $\nu_k-\pi k\to 0$ as $k\to\infty$. Then the systems $\{\sin\nu_k x\}_{k=1}^{\infty}$ and $\{\cos\nu_k x\}_{k=0}^{\infty}$ form Riesz bases of $L_2(0,1)$.*

We denote by $\mathcal{S}$ the set of all functions u of the form (3.3) (or, which is the same due to Proposition 3.1, of the form (3.4)). The following simple statement will play an important role for establishing uniqueness of solutions to the inverse spectral problem, so we give its proof here.

Lemma 3.4. *Suppose that $a\in(0,1)$ and that a sequence (μ_n) of positive pairwise distinct numbers obeys the asymptotics $\mu_n=\pi n+o(1)$ as $n\to\infty$. If the functions v_j, $j=1,\dots,4$, belong to $\mathcal{S}$ and*

$$v_1(a\mu_n)v_2\big((1-a)\mu_n\big)=v_3(a\mu_n)v_4\big((1-a)\mu_n\big)$$

for all $n\in\mathbb{N}$, then

$$v_1(a\lambda)v_2\big((1-a)\lambda\big)\equiv v_3(a\lambda)v_4\big((1-a)\lambda\big).$$

Proof. Put $v(\lambda):=v_1(a\lambda)v_2((1-a)\lambda)-v_3(a\lambda)v_4((1-a)\lambda)$. Integration by parts in integral representations (3.3) for the functions v_j, $j=1,\dots,4$, and subsequent simple transformations yield the representations

$$v_j(\lambda) = \int_{-1}^{1} \exp(-i\lambda t) f_j(t)\,dt$$

with $f_j = \chi_{[-1,1]}/2 + g_j$, where $\chi_{[-1,1]}$ is the indicator of the interval $[-1,1]$ and g_j are even functions from the Sobolev space $W_2^1(\mathbb{R})$ such that $\operatorname{supp} g_j \subset [-1,1]$. We define

$$f_{j,b}(x) := (1/b) f_j(x/b)$$

for $b \in (0,1)$ and $x \in \mathbb{R}$ and put

$$f := f_{1,a} * f_{2,1-a} - f_{3,a} * f_{4,1-a},$$

where $(\phi * \psi)(x) := \int_{\mathbb{R}} \phi(x-t)\psi(t)\,dt$ is the convolution of functions ϕ and ψ. Then the function f is even, belongs to $W_2^1(\mathbb{R})$, $\operatorname{supp} f \subset [-1,1]$, and

$$v(\lambda) = \int_{-1}^{1} \exp(-i\lambda t) f(t)\,dt = -2\int_0^1 \frac{\sin\lambda t}{\lambda} f'(t)\,dt.$$

By assumption,

$$\int_0^1 \sin(\mu_n t) f'(t)\,dt = -\frac{1}{2}\mu_n v(\mu_n) = 0$$

for all $n \in \mathbb{N}$; since by Proposition 3.3 the system $\{\sin \mu_n t\}_{n\in\mathbb{N}}$ is complete in $\mathcal{H}$ and $f' \in \mathcal{H}$, we get $f' = 0$. Thus $v \equiv 0$, and the lemma is proved. □

4. Properties of the functions u_Λ, $u_{0,\Lambda}$, and $u_{1,\Lambda}$

With an arbitrary $\Lambda = ((\lambda_n^2), (\lambda_{0,n}^2), (\lambda_{1,n}^2)) \in \mathfrak{L}$, we shall associate entire functions u_Λ, $u_{0,\Lambda}$, and $u_{1,\Lambda}$ given by

$$\begin{aligned} u_\Lambda(\lambda) &:= \prod_{k=1}^{\infty} \frac{\lambda_k^2 - \lambda^2}{(\pi k)^2}, \qquad u_{0,\Lambda}(\lambda) := \prod_{k=1}^{\infty} \frac{\lambda_{0,k}^2 - \lambda^2}{(\pi k/a)^2}, \\ u_{1,\Lambda}(\lambda) &:= \prod_{k=1}^{\infty} \frac{\lambda_{1,k}^2 - \lambda^2}{[\pi k/(1-a)]^2}. \end{aligned} \tag{4.1}$$

Observe that if $\sigma \in \Sigma_0^+$ is such that $\mathfrak{l}(\sigma) = \Lambda$, then these functions are related to solutions $s_\pm(\cdot,\lambda,\sigma)$ of the equation $l_\sigma(u) = \lambda u$ in the following way (see Lemma 3.2):

$$\begin{aligned} &s_-(1,\lambda,\sigma) = u_\Lambda(\lambda), \qquad s_-(a,\lambda,\sigma) = a u_{0,\Lambda}(\lambda), \\ &s_+(a,\lambda,\sigma) = -(1-a) u_{1,\Lambda}(\lambda). \end{aligned} \tag{4.2}$$

In this section, we shall establish some important properties of the functions u_Λ and $u_{j,\Lambda}$, $j = 0, 1$, that will essentially be used in Section 5. We start with the following two lemmata.

Lemma 4.1. *Suppose that $\Lambda \in \mathfrak{L}$; then we have*

$$(-1)^n \lambda_n \dot{u}_\Lambda(\lambda_n) = 1 + d_n > 0, \quad n \in \mathbb{N}, \tag{4.3}$$

where $(d_n) \in \ell_2$.

Proof. The fact that the left-hand side of (4.3) is positive follows from (4.1). Proposition 3.1 and formula (3.3) imply that

$$\lambda_n \dot{u}_\Lambda(\lambda_n) = \frac{d}{d\lambda}\bigl(\lambda u_\Lambda(\lambda)\bigr)\Big|_{\lambda=\lambda_n} = \cos\lambda_n + \int_0^1 t\hat{f}(t)\cos\lambda_n t\,dt$$

for some $\hat{f} \in \mathcal{H}$. Now the statement easily follows from the asymptotics of the numbers λ_n and the fact that by Proposition 3.3 the system $\{\cos\lambda_n t\}_{n\in\mathbb{N}}$ forms a Riesz basis of its closed linear span in $\mathcal{H}$. □

Lemma 4.2. *Suppose that $\Lambda \in \mathfrak{L}$ and that sequences $(\xi_{j,k})_{k\in\mathbb{N}}$, $j = 0, 1$, of positive pairwise distinct numbers are such that*

$$\xi_{0,k} = \frac{\pi k}{a} + \eta_{0,k}, \quad \xi_{1,k} = \frac{\pi k}{1-a} + \eta_{1,k}, \quad k \in \mathbb{N},$$

where $(\eta_{j,k}) \in \ell_2$, $j = 0, 1$. Then there exist ℓ_2-sequences $(d_{j,k})$, $j = 0, 1$, such that

$$(-1)^k \xi_{0,k}\dot{u}_{0,\Lambda}(\xi_{0,k}) = 1 + d_{0,k}, \qquad (-1)^k \xi_{1,k}\dot{u}_{1,\Lambda}(\xi_{1,k}) = 1 + d_{1,k}$$

for all $k \in \mathbb{N}$.

Proof of this statement is similar to that of Lemma 4.1 and thus is omitted.

The principal result of this section is as follows.

Theorem 4.3. *Suppose that $a \in (0, 1)$ is irrational and $\Lambda \in \mathfrak{L}$. Then for all $k \in M(a)$ (see Definition 2.1) we have*

$$\frac{a\pi k\, u_{0,\Lambda}(\lambda_k)}{\sin(a\pi k)} = \left(\frac{a}{\pi}\right)^2 \frac{\lambda_{0,k'}^2 - \lambda_k^2}{(k')^2 - (ak)^2}(1 + c_{0,k}), \tag{4.4}$$

$$\frac{(1-a)\pi k\, u_{1,\Lambda}(\lambda_k)}{\sin((1-a)\pi k)} = \left(\frac{1-a}{\pi}\right)^2 \frac{\lambda_{1,k''}^2 - \lambda_k^2}{(k'')^2 - (1-a)^2k^2}(1 + c_{1,k}), \tag{4.5}$$

where $(c_{j,k})$, $j = 0, 1$, are some sequences from $\ell_2(M(a))$.

The proof of this theorem is based on the next three lemmata.

Lemma 4.4. *Assume that $x = (x_n)_{n=1}^\infty$ is a sequence of complex numbers from ℓ_2 such that $|x_n| \leqslant 1/2$ for all $n \in \mathbb{N}$ and the series $\sum_{n=1}^\infty x_n$ converges. Put $G(t) := t\exp t$; then the following inequality holds*:

$$\left|1 - \prod_{n=1}^\infty (1 + x_n)\right| \leqslant G\left(\left|\sum_{n=1}^\infty x_n\right| + \sum_{n=1}^\infty |x_n|^2\right).$$

Proof. Taking the principal branch of the logarithm and using the inequality

$$\left|z - \log(1+z)\right| \leqslant |z|^2, \quad |z| \leqslant \frac{1}{2},$$

we find that

$$\left|\sum_{n=1}^{\infty} x_n - \sum_{n=1}^{\infty} \log(1+x_n)\right| \leqslant \sum_{n=1}^{\infty} |x_n|^2,$$

and hence that

$$\left|\log \prod_{n=1}^{\infty}(1+x_n)\right| \leqslant \left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=1}^{\infty} |x_n|^2.$$

Now, taking into account that $|1 - \exp z| \leqslant |z| \exp(|z|) = G(|z|)$ for all complex z, we conclude that

$$\left|1 - \prod_{n=1}^{\infty}(1+x_n)\right| \leqslant G\left(\left|\log \prod_{n=1}^{\infty}(1+x_n)\right|\right) \leqslant G\left(\left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=1}^{\infty} |x_n|^2\right)$$

as claimed. □

Lemma 4.5. *Suppose that $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers with $|r_n - n| < 1/2$ for all $n \in \mathbb{N}$. Then linear operators $H^{\pm}$ in ℓ_2 that act according to the formula*

$$(H^{\pm} f)(k) = \sum_{n\neq k} \frac{f(n)}{n \pm r_k}, \quad k \in \mathbb{N},$$

are continuous.

Proof. We denote by $H_0^{\pm}$ the operators corresponding to the particular case $r_n = n$ for every $n \in \mathbb{N}$; then continuity of the Hilbert operators [7, Chapter 5] implies that $H_0^{\pm}$ are continuous in ℓ_2. Now we find that

$$\left|\left(H^{\pm} f - H_0^{\pm} f\right)(k)\right| \leqslant \sum_{n\neq k} \frac{|f(n)|}{(n\pm k)^2} \leqslant \left(2\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n\neq k} \frac{|f(n)|^2}{(n-k)^2}\right)^{1/2},$$

where we have used the inequality $|y/(x+y)| \leqslant |1/x|$ holding for all real x and y with $|x| \geqslant 1$ and $|y| \leqslant 1/2$, the Cauchy–Schwarz–Bunyakowski inequality and the estimate $\sum_{n\neq k} 1/(n\pm k)^2 \leqslant 2\sum 1/n^2$. Henceforth,

$$\begin{aligned}\left\|\left(H^{\pm} f - H_0^{\pm} f\right)\right\|^2 &\leqslant \left(2\sum_{n=1}^{\infty} \frac{1}{n^2}\right) \sum_{k=1}^{\infty} \sum_{n\neq k} \frac{|f(n)|^2}{(n-k)^2} \\ &= \left(2\sum_{n=1}^{\infty} \frac{1}{n^2}\right) \sum_{n=1}^{\infty} \left|f(n)\right|^2 \sum_{k\neq n} \frac{1}{(n-k)^2} \leqslant \left(2\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^2 \|f\|^2 = \frac{\pi^4}{9} \|f\|^2,\end{aligned}$$

so that $\|H^{\pm} - H_0^{\pm}\| \leqslant \pi^2/3$, and the operators $H^{\pm}$ are continuous in ℓ_2. □

Lemma 4.6. *Suppose that $a \in (0,1)$ and (ω_n), (ω'_n) are ℓ_2-sequences. We put for each $k \in M(a)$ (see Section 2 for definition),*

$$b(n,k) := \begin{cases} \frac{(n+\omega_n)^2-(ak+\omega'_k)^2}{n^2-(ak)^2} - 1 & \text{if } n \in \mathbb{N} \setminus \{k'\}, \\ 0 & \text{if } n = k'. \end{cases}$$

Then

$$\prod_{n=1}^{\infty}\bigl(1+b(n,k)\bigr) = 1 + c_k,$$

where $(c_k) \in \ell_2(M(a))$.

Proof. The lemma will be proved as soon as we establish the following properties of the numbers $b(n,k)$:

(a) There exists $k_0 \in \mathbb{N}$ such that $|b(n,k)| \leqslant 1/2$ for all $n \in \mathbb{N}$ and all $k \in M(a)$ larger than k_0;
(b) $\sum_{k\in M(a)} |\sum_{n=1}^{\infty} b(n,k)|^2 < \infty$;
(c) $\sum_{k\in M(a)} \sum_{n=1}^{\infty} |b(n,k)|^2 < \infty$.

In fact, if (a)–(c) are satisfied and C_1 and C_2 denote the sums in (b) and (c), respectively, then, by Lemma 4.4, we have for $k \geqslant k_0$,

$$|c_k| \leqslant \left(\left|\sum_{n=1}^{\infty} b(n,k)\right| + \sum_{n=1}^{\infty}\bigl|b(n,k)\bigr|^2\right) \exp(\sqrt{C_1} + C_2),$$

so that

$$\begin{aligned}\sum_{\substack{k\in M(a)\\ k\geqslant k_0}} |c_k|^2 &\leqslant 2\left[\sum_{k\in M(a)}\left|\sum_{n=1}^{\infty} b(n,k)\right|^2 + \sum_{k\in M(a)}\left(\sum_{n=1}^{\infty}\bigl|b(n,k)\bigr|^2\right)^2\right] \\ &\quad \times \exp(2\sqrt{C_1} + 2C_2) \\ &\leqslant 2\bigl(C_1 + C_2^2\bigr)\exp(2\sqrt{C_1} + 2C_2).\end{aligned}$$

For every $k \in M(a)$ and $n \in \mathbb{N} \setminus \{k'\}$, we put

$$b_1(n,k) := \frac{2n\omega_n}{n^2-(ak)^2}, \qquad b_2(n,k) := -\frac{2ak\omega'_k}{n^2-(ak)^2},$$

$$b_3(n,k) := \frac{\omega_n^2}{n^2-(ak)^2}, \qquad b_4(n,k) := -\frac{(\omega'_k)^2}{n^2-(ak)^2},$$

and let $b_j(k',k) = 0$, $j = 1,2,3,4$. It is easily seen that

$$b(n,k) = b_1(n,k) + b_2(n,k) + b_3(n,k) + b_4(n,k), \quad n \in \mathbb{N},\ k \in M(a),$$

and therefore it suffices to establish the above properties (a)–(c) for the numbers $b_j(n,k)$, $j = 1,2,3,4$ (with the bound $1/2$ in (a) replaced with $1/8$). Actually, instead of (a) we shall establish a stronger property

(a′) $\sup_{n\in\mathbb{N}} |b_j(n,k)| \to 0$ as $k \in M(a)$ tends to ∞.

To begin with, we recall that, by definition, $|n - ak| \geqslant 1/s$ if $n \neq k'$ and $a = r/s$ with relatively prime r and s and that $|n - ak| \geqslant a(1-a)/2$ if $n \neq k'$ and a is irrational; henceforth,

$$\sup_{k\in M(a)} \sum_{n\neq k'} \frac{1}{|n^2 - (ak)^2|} < \infty, \qquad \sup_{n\in\mathbb{N}} \sum_{\substack{k\in M(a)\\ k'\neq n}} \frac{1}{|n^2 - (ak)^2|} < \infty.$$

For $j = 3$ and $j = 4$ this observation implies that $\sum_{k\in M(a)} \sum_{n\in\mathbb{N}} |b_j(n,k)| < \infty$, and thus (a′), (b), and (c) hold.

For $j = 1$, we easily derive (a′) from the representation

$$b_1(n,k) = \frac{\omega_n}{n - ak} + \frac{\omega_n}{n + ak}, \quad n \in \mathbb{N} \setminus \{k'\}.$$

Also notice that $(ak)_{k\in M(a)}$ is a subsequence of some sequence $(r_m)_{m\in\mathbb{N}}$ satisfying the assumptions of Lemma 4.5. Therefore by Lemma 4.5 we conclude that the sequences $(d_k^{\pm})$, with

$$d_k^{\pm} := \sum_{n\neq k'} \frac{\omega_n}{n \pm ak},$$

belong to $\ell_2(M(a))$ so that (b) is satisfied. Property (c) follows from the fact that

$$\sum_{k\in M(a)} \sum_{n\neq k'} \frac{|\omega_n|^2}{|n \pm ak|^2} \leqslant \left(\max_{n\in\mathbb{N}} \sum_{\substack{k\in M(a)\\ k'\neq n}} \frac{1}{|n \pm ak|^2} \right) \sum_{n\in\mathbb{N}} |\omega_n|^2 < \infty.$$

Finally, for $j = 2$ the inequality $|b_2(n,k)| \leqslant 2|\omega'_k|/|n - ak|$ justifies property (a′). In a similar way we show that

$$\sum_{n\neq k'} \left| \frac{2ak}{n^2 - (ak)^2} \right|^2 \leqslant \sum_{n\neq k'} \frac{4}{(n - ak)^2} \leqslant C$$

for some $C > 0$ independent of k, and thus (c) holds. Property (b) follows from the inequality below, in which, for $k \in M(a)$, we have put $s := |k' - ak|$ and used the bounds $s < 1/8$ and $ak > 7/8$,

$$\left| \sum_{n\neq k'} \frac{2ak}{n^2 - (ak)^2} \right| \leqslant \sum_{n\geqslant 2ak} \frac{2ak}{n^2 - (ak)^2} + \sum_{\substack{n<2ak\\ n\neq k'}} \frac{1}{n + ak} + \left| \sum_{\substack{n<2ak\\ n\neq k'}} \frac{1}{n - ak} \right|$$
$$< \sum_{n\geqslant 2ak} \frac{8ak}{3n^2} + 2 + \left[\sum_{n=1}^{k'-1} \left(\frac{1}{n - s} - \frac{1}{n + s} \right) + \frac{1}{k' - s} \right]$$

$$< \frac{8ak}{3(2ak-1)} + 2 + \frac{1}{1-s} < \frac{28}{9} + 2 + \frac{8}{7} < 7.$$

The lemma is proved. □

Proof of Theorem 4.3. Due to the symmetry between $M(a)$ and $M(1-a)$ (see Section 2), it suffices to prove only relation (4.4). Taking into account the definition of the function $u_{0,\Lambda}$ and the equality

$$\frac{\sin(a\pi k)}{a\pi k} = \prod_{n=1}^{\infty} \frac{(\pi n)^2 - (a\pi k)^2}{(\pi n)^2},$$

we find that, for all $k \in M(a)$,

$$\frac{(a\pi k)u_{0,\Lambda}(\lambda_k)}{\sin(a\pi k)} = \prod_{n=1}^{\infty} \left(\frac{a}{\pi}\right)^2 \frac{\lambda_{0,n}^2 - \lambda_k^2}{n^2 - (ak)^2}.$$

Hence the theorem will be proved as soon as we establish the relation

$$\prod_{n \neq k'} \left(\frac{a}{\pi}\right)^2 \frac{\lambda_{0,n}^2 - \lambda_k^2}{n^2 - (ak)^2} = 1 + c_{0,k}, \quad k \in M(a), \tag{4.6}$$

for some sequence $(c_{0,k}) \in \ell_2(M(a))$.

Observe that, according to (2.2) and (2.3), we have

$$\left(\frac{a}{\pi}\right)^2 \left(\lambda_{0,n}^2 - \lambda_k^2\right) = (n + \omega_n)^2 - (ak + \omega_k')^2,$$

where (ω_n) and (ω_n') are some ℓ_2-sequences. Thus representation (4.6) follows from Lemma 4.6, and the proof is complete. □

5. The direct spectral problem

In this section we shall study the direct spectral problem, i.e., shall study the spectral properties of the operators $T(\sigma)$ and $T_j(\sigma)$, $j = 0, 1$, for $\sigma \in \Sigma_0^+$. Our aim is to prove the next statement.

Theorem 5.1. *Suppose that $\sigma \in \Sigma_0^+$; then the triple $\Lambda := \mathfrak{l}(\sigma)$ belongs to the set $\mathfrak{L}$.*

In order to prove the theorem, we need only show that Λ verifies conditions (1)–(3) of Definition 2.1.

As we have already mentioned, the eigenvalue asymptotics for the Sturm–Liouville operators with singular potentials from $W_2^{-1}(0,1)$ is established in several papers (see, e.g., [19,20] and also [10]). Thus the sequences (λ_n^2) and $(\lambda_{j,n}^2)$ of Λ satisfy condition (1).

The interlacing properties of the sequences (λ_n^2) and $(\mu_n^2) := (\lambda_{0,1}^2) \sqcup (\lambda_{1,n}^2)$ (i.e., the inequalities $\lambda_1^2 < \mu_1^2$ and $\mu_n^2 \leqslant \lambda_{n+1}^2 \leqslant \mu_{n+1}^2$ for all $n \in \mathbb{N}$) is proved, e.g., in [4, Section IV.8.1]. Also, among each triple μ_n^2, λ_{n+1}^2, and μ_{n+1}^2, $n \in \mathbb{N}$, all numbers are either pairwise distinct or all equal. Indeed, if, e.g., $\mu_n^2 = \lambda_{n+1}^2$ and u_{n+1} is the corresponding eigenfunction of T, then $u_{n+1}(a) = 0$ and the restrictions of u_{n+1} onto $[0, a]$ and $[a, 1]$ give eigenfunctions of T_0 and T_1; thus also $\mu_{n+1}^2 = \lambda_{n+1}^2$. These reasonings establish (2).

It remains to verify condition (3). We shall do this in the following way: first derive a representation of the norming constants $\alpha_n(\sigma)$ for the operator $T(\sigma)$ via Λ and then use this representation and the asymptotic behaviour of $\alpha_n(\sigma)$ to establish (3).

Lemma 5.2. *Suppose that* $\sigma \in \Sigma_0^+$ *and put* $\alpha_n := \alpha_n(\sigma)$ *and* $s_\pm(\cdot, \lambda) := s_\pm(\cdot, \lambda, \sigma)$; *then for all* $n \in \mathbb{N}$ *we have*

$$\alpha_n s_+(\cdot, \lambda_n) = \lambda_n \dot{s}_-(1, \lambda_n) s_-(\cdot, \lambda_n). \tag{5.1}$$

Proof. Recall that $\phi_n := \sqrt{2}\lambda_n s_-(\cdot, \lambda_n)$ is an eigenfunction of the operator $T = T(\sigma)$ corresponding to the eigenvalue λ_n^2 and that $\alpha_n := \|\phi_n\|^2$. Therefore the Green's function of the operator T equals

$$G(x, y, \lambda^2) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\alpha_n(\lambda_n^2 - \lambda^2)}, \quad x, y \in [0, 1].$$

On the other hand, we have

$$G(x, y, \lambda^2) = \frac{1}{W(\lambda)} \begin{cases} s_-(x, \lambda)s_+(y, \lambda) & \text{if } 0 \leqslant x \leqslant y \leqslant 1, \\ s_+(x, \lambda)s_-(y, \lambda) & \text{if } 0 \leqslant y \leqslant x \leqslant 1, \end{cases}$$

where $W(\lambda) := s_-^{[1]}(x, \lambda)s_+(x, \lambda) - s_+^{[1]}(x, \lambda)s_-(x, \lambda)$ is the Wronskian of the solutions $s_+(\cdot, \lambda)$ and $s_-(\cdot, \lambda)$. The value of $W(\lambda)$ is independent of x; in particular, taking $x = 1$ we get $W(\lambda) = -s_-(1, \lambda)$. Combining the above equalities and putting $x = y$, we arrive at the identity

$$\sum_{n=1}^{\infty} \frac{2\lambda_n^2 s_-^2(y, \lambda_n)}{\alpha_n(\lambda_n^2 - \lambda^2)} \equiv -\frac{s_-(y, \lambda)s_+(y, \lambda)}{s_-(1, \lambda)}.$$

Formula (5.1) follows now after equating the residues of both sides of this identity at the poles $\lambda = \lambda_n$. □

If $n \in B_\Lambda$, then $s_\pm(a, \lambda_n) \neq 0$, so that we can take $x = a$ in equality (5.1), divide both its sides by $s_+(a, \lambda_n)$, and then express $s_\pm(a, \lambda_n)$ in terms of $u_{j,\Lambda}$ as in (4.2). This results in the following statement.

Corollary 5.3. *Suppose that* $\sigma \in \Sigma_0^+$ *and* $\Lambda = \mathfrak{l}(\sigma)$; *then for* $n \in B_\Lambda$ *it holds*

$$\alpha_n(\sigma) = -\lambda_n \dot{u}_\Lambda(\lambda_n) \frac{a u_{0,\Lambda}(\lambda_n)}{(1-a)u_{1,\Lambda}(\lambda_n)}. \tag{5.2}$$

It is proved in [10] that the norming constants (α_n) have the representation $\alpha_n = 1 + \tilde{\alpha}_n$ with $(\tilde{\alpha}_n) \in \ell_2$. Thus the right-hand side of (5.2) obeys the same asymptotics, and we shall show next that this asymptotics implies condition (3) of Definition 2.1. The crucial observation is contained in the next statement.

Lemma 5.4. *Condition* (3) *in Definition* 2.1 *of the set* $\mathfrak{L}$ *is equivalent to the following one*:

(3′) *The sequence* $(\gamma_n)_{n\in B_\Lambda}$, *where*

$$\gamma_n = \gamma_n(\Lambda) := (-1)^{n+1}\frac{au_{0,\Lambda}(\lambda_n)}{(1-a)u_{1,\Lambda}(\lambda_n)} - 1, \quad n \in B_\Lambda,$$

belongs to $\ell_2(B_\Lambda)$.

Assuming for the time being that Lemma 5.4 is already proved, we argue as follows. By Lemma 4.1 we have the relation

$$(-1)^n\lambda_n\dot{u}_\Lambda(\lambda_n) = 1 + d_n$$

with some ℓ_2-sequence (d_n); henceforth equality (5.2) for all $n \in B_\Lambda$ can be recast as

$$1 + \tilde{\alpha}_n = (1 + d_n)(1 + \gamma_n), \tag{5.3}$$

and this representation easily yields the inclusion $(\gamma_n)_{n\in B_\Lambda} \in \ell_2(B_\Lambda)$. Thus condition (3′) holds, and by Lemma 5.4 also condition (3) takes place.

To sum up, taking for granted Lemma 5.4, we have shown that $\Lambda \in \mathfrak{L}$ and thus finished the proof of Theorem 5.1.

It remains to establish Lemma 5.4.

Proof of Lemma 5.4. Suppose that a triple $\Lambda = ((\lambda_n^2), (\lambda_{0,n}^2), (\lambda_{1,n}^2))$ of strictly increasing sequences of positive numbers satisfies conditions (1) and (2) of Definition 2.1. We shall show that Λ satisfies condition (3) of this definition if and only if (3′) above is verified.

We recall that by definition $B'_\Lambda = B_\Lambda \cap M(a)$ and put $B''_\Lambda := B_\Lambda \setminus M(a)$. Our first step is to show that the sequence $(\gamma_n)_{n\in B''_\Lambda}$ always belongs to $\ell_2(B''_\Lambda)$.

Observe that, for any $k \in B''_\Lambda$ we have

$$1 + \gamma_k = (-1)^{k+1}\frac{au_{0,\Lambda}(\lambda_k)}{(1-a)u_{1,\Lambda}(\lambda_k)} = \frac{a\lambda_k u_{0,\Lambda}(\lambda_k)}{\sin(a\pi k)}\frac{\sin((1-a)\pi k)}{(1-a)\lambda_k u_{1,\Lambda}(\lambda_k)}. \tag{5.4}$$

It follows from (3.1), (3.2), and (4.2) that

$$\frac{a\lambda_k u_{0,\Lambda}(\lambda_k)}{\sin(a\pi k)} = \frac{\sin(a\lambda_k)}{\sin(a\pi k)} + \frac{1}{\sin(a\pi k)}\int_0^a f_0(t)\sin(\lambda_k t)\,dt,$$

$$\frac{(1-a)\lambda_k u_{1,\Lambda}(\lambda_k)}{\sin((1-a)\pi k)} = \frac{\sin((1-a)\lambda_k)}{\sin((1-a)\pi k)} + \frac{1}{\sin((1-a)\pi k)}\int_0^{1-a} f_1(t)\sin(\lambda_k t)\,dt,$$

where $f_0, f_1 \in \mathcal{H}$. Since by Proposition 3.3 the system $\{\sin\lambda_k t\}_{k\in\mathbb{N}}$ forms a Riesz basis of $\mathcal{H}$, the sequences $(c_{0,k})$ and $(c_{1,k})$ with

$$c_{0,k} := \int_0^a f_0(t)\sin\lambda_k t\,dt, \qquad c_{1,k} := \int_0^{1-a} f_1(t)\sin\lambda_k t\,dt$$

belong to ℓ_2. Also, by the definition of the set $M(a)$,

$$\sup_{k\in\mathbb{N}\setminus M(a)} 1\big/\big|\sin(a\pi k)\big| = \sup_{k\in\mathbb{N}\setminus M(a)} 1\big/\big|\sin\big((1-a)\pi k\big)\big| < \infty,$$

so that

$$\frac{\sin(a\lambda_k)}{\sin(a\pi k)} = 1 + O(a_k) \quad \text{and} \quad \frac{\sin((1-a)\lambda_k)}{\sin((1-a)\pi k)} = 1 + O(a_k), \quad k \in \mathbb{N}\setminus M(a),$$

with $(a_k) \in \ell_2$ being the sequence of (2.2). Thus we have shown that

$$1 + \gamma_k = \frac{1 + c'_k}{1 + c''_k}, \quad k \in B''_\Lambda,$$

where $(c'_k), (c''_k) \in \ell_2(B''_\Lambda)$, and the claim follows.

The second step is to show that, under the assumptions imposed on Λ, conditions $(\gamma_k) \in \ell_2(B'_\Lambda)$ and $(\delta_k) \in \ell_2(B'_\Lambda)$ are equivalent. To this end it suffices to prove that there exists a sequence $(\epsilon_k) \in \ell_2(B'_\Lambda)$ such that the following equality holds for all $k \in B'_\Lambda$:

$$1 + \gamma_k = (1 + \delta_k)(1 + \epsilon_k). \tag{5.5}$$

Suppose that a is irrational. Then, due to equality (5.4) and Theorem 4.3 we have

$$1 + \gamma_k = \left(\frac{a}{1-a}\right)^2 \frac{\lambda_{0,k'}^2 - \lambda_k^2}{(k')^2 - (ak)^2} \frac{(k'')^2 - (1-a)^2 k^2}{\lambda_{1,k''}^2 - \lambda_k^2}(1 + c_k), \quad k \in B'_\Lambda, \tag{5.6}$$

where $(c_k) \in \ell_2(B'_\Lambda)$. Asymptotics of the sequences (λ_k), $(\lambda_{0,k'})$, and $(\lambda_{1,k''})$ implies that

$$\frac{\lambda_{0,k'} + \lambda_k}{\lambda_{1,k''} + \lambda_k} = 1 + O(1/k), \quad k \in M(a).$$

Moreover, $k' - ak = -k'' + (1-a)k$ and

$$\frac{a(k'' + (1-a)k)}{(1-a)(k' + ak)} = 1 + O(1/k), \quad k \in M(a).$$

Combining this relation into (5.6), we easily derive (5.5) for some $(\epsilon_k) \in \ell_2(B'_\Lambda)$.

Assume now that a is rational. Then for any $k \in B'_\Lambda$ we find that

$$1 + \gamma_k = (-1)^{k+1} \frac{a u_{0,\Lambda}(\lambda_k)}{(1-a) u_{1,\Lambda}(\lambda_k)} = (-1)^{k+1} \frac{a \dot{u}_{0,\Lambda}(\tilde{\lambda}_{0,k'})(\lambda_k - \lambda_{0,k'})}{(1-a)\dot{u}_{1,\Lambda}(\tilde{\lambda}_{1,k''})(\lambda_k - \lambda_{1,k''})},$$

where $\tilde{\lambda}_{0,k'}$ (respectively, $\tilde{\lambda}_{1,k''}$) are some numbers between λ_k and $\lambda_{0,k'}$ (respectively, between λ_k and $\lambda_{1,k''}$). In particular, if the set B'_Λ is infinite, then the numbers $\tilde{\lambda}_{0,k'}$ and $\tilde{\lambda}_{1,k''}$ have the asymptotics

$$\tilde{\lambda}_{0,k'} = \pi k'/a + \eta_{0,k}, \quad \tilde{\lambda}_{1,k''} = \pi k''/(1-a) + \eta_{1,k}, \quad k \in B'_\Lambda,$$

for some $\ell_2(B'_\Lambda)$-sequences $(\eta_{j,k})$, $j = 0, 1$. Since for $k \in M(a)$ we have

$$\pi k = \pi k'/a = \pi k''/(1-a) \quad \text{and} \quad k' + k'' = k,$$

representation (5.5) easily follows from Lemma 4.2. The lemma is proved. □

Using formula (5.3), Lemmata 4.1 and 5.4, we can easily prove the following statement.

Corollary 5.5. *Assume that $\Lambda \in \mathfrak{L}$ is arbitrary and construct the numbers α_n for $n \in B_\Lambda$ according to* (5.2); *then the sequence $(\alpha_n - 1)_{n\in B_\Lambda}$ belongs to $\ell_2(B_\Lambda)$.*

6. The inverse spectral problem: Proof of the main results

In this section, we shall solve the inverse spectral problem. In other words, we shall show how a potential $\sigma \in \Sigma_0^+$ can be recovered via the spectral data $\Lambda = \mathfrak{l}(\sigma)$ and prove that any triple Λ in $\mathfrak{L}$ is the spectral data corresponding to some $\sigma \in \Sigma_0^+$, i.e., that $\mathfrak{L} = \mathfrak{l}(\Sigma_0^+)$. Since the procedure is the same for both parts, we shall treat only the second one, which is more general.

Assume therefore that $\Lambda = ((\lambda_n^2), (\lambda_{0,n}^2), (\lambda_{1,n}^2)) \in \mathfrak{L}$. We construct the corresponding functions u_Λ and $u_{j,\Lambda}$, $j = 0, 1$, as explained in Section 4 and, for every $k \in B_\Lambda$, we define a number α_k through formula (5.2). According to the results of the previous section (see Corollary 5.5), the sequence $(\alpha_k - 1)_{k\in B_\Lambda}$ belongs to $\ell_2(B_\Lambda)$. If $A_\Lambda \neq \emptyset$, then we take an arbitrary sequence $(\theta_k)_{k\in A_\Lambda}$ from $\ell_2(A_\Lambda)$ obeying the inequality $\theta_k > -1$ for all $k \in A_\Lambda$ and put $\alpha_k := 1 + \theta_k$, $k \in A_\Lambda$.

The sequence $(\alpha_n)_{n\in\mathbb{N}}$ so constructed consists of positive numbers and satisfies the asymptotic relation $\alpha_n = 1 + \tilde{\alpha}_n$ with an ℓ_2-sequence $(\tilde{\alpha}_n)$. The pair $S := \{(\lambda_k^2), (\alpha_k)\}$ belongs to $\mathfrak{A}$ and thus we can use Proposition 2.3 and the reconstruction algorithm of [10] to find a unique $\sigma \in \Sigma_0^+$ such that $\mathfrak{a}(\sigma) = S$, i.e., such that $\lambda_n^2(\sigma) = \lambda_n^2$ and $\alpha_n(\sigma) = \alpha_n$ for all $n \in \mathbb{N}$.

It remains to prove that the function σ so defined satisfies the equality $\mathfrak{l}(\sigma) = \Lambda$, i.e., that $(\lambda_{j,n}^2)$ are the spectra of the operators $T_j(\sigma)$, $j = 0, 1$.

To this end we write $\tilde{\Lambda} := \mathfrak{l}(\sigma)$ and denote by $\tilde{\lambda}_{j,n}^2 := \lambda_{j,n}^2(\sigma)$, $j = 0, 1$, the spectra of $T_j(\sigma)$ and by $u_{j,\tilde{\Lambda}}$ the corresponding entire functions. It follows from Lemma 5.2 and formula (4.2) that

$$\alpha_n(\sigma)(1-a)u_{1,\tilde{\Lambda}}(\lambda_n) = -\lambda_n \dot{u}_\Lambda(\lambda_n) a u_{0,\tilde{\Lambda}}(\lambda_n);$$

recalling the definition of α_n, we derive from this the equalities

$$u_{0,\Lambda}(\lambda_n)u_{1,\tilde{\Lambda}}(\lambda_n) = u_{0,\tilde{\Lambda}}(\lambda_n)u_{1,\Lambda}(\lambda_n) \tag{6.1}$$

for all $n \in B_\Lambda$. Since $u_{0,\Lambda}(\lambda_n) = u_{1,\Lambda}(\lambda_n) = 0$ for all $n \in A_\Lambda$, equalities (6.1) hold for all $n \in \mathbb{N}$. We apply now Lemma 3.4 and conclude that

$$u_{0,\Lambda}u_{1,\tilde{\Lambda}} \equiv u_{0,\tilde{\Lambda}}u_{1,\Lambda}.$$

The above identity implies the following property:

(∗) The functions $u_{j,\Lambda}$ and $u_{j,\tilde{\Lambda}}$, $j = 0, 1$, have the same zeros outside the set $\{\lambda_n\}_{n\in\mathbb{N}}$.

Observe that the triples Λ and $\tilde{\Lambda}$ belong to $\mathfrak{L}$ (the former by assumption and the latter by Theorem 5.1) and thus they enjoy property (2) of Definition 2.1. Put $(\mu_n^2) := (\lambda_{0,n}^2) \amalg (\lambda_{1,n}^2)$ and $(\tilde{\mu}_n^2) := (\tilde{\lambda}_{0,n}^2) \amalg (\tilde{\lambda}_{1,n}^2)$; then, in view of (∗) and property (2) of Definition 2.1, the equality $\Lambda = \tilde{\Lambda}$ holds if and only if the sequences (μ_n^2) and $(\tilde{\mu}_n^2)$ coincide, i.e., if and only if identically equal are the counting functions

$$r(t) := \#\{n \in \mathbb{N} \mid \mu_n \leqslant t\}, \quad \tilde{r}(t) := \#\{n \in \mathbb{N} \mid \tilde{\mu}_n \leqslant t\}, \quad t \in \mathbb{R}.$$

We obviously have $r(t) = \tilde{r}(t) = 0$ for $t \in (-\infty, \lambda_1]$. Assume that the equality $r(t) = \tilde{r}(t)$ is already proved for $t \in (-\infty, \lambda_n]$ for some $n \in \mathbb{N}$. If $r(\lambda_n) = \tilde{r}(\lambda_n) = n$, then property (2) of Definition 2.1 yields the relations $\mu_n = \tilde{\mu}_n = \lambda_n$, $\mu_{n+1} > \lambda_{n+1}$, $\tilde{\mu}_{n+1} > \lambda_{n+1}$ and hence $r(t) = \tilde{r}(t) = n$ for all $t \in (\lambda_n, \lambda_{n+1}]$. Otherwise $r(\lambda_n) = \tilde{r}(\lambda_n) = n - 1$, and then either (i) $\mu_n \in (\lambda_n, \lambda_{n+1})$ or (ii) $\mu_n = \lambda_{n+1}$. In case (i) we see that $\tilde{\mu}_n = \mu_n$ due to property $(*)$ and $\mu_{n+1} > \lambda_{n+1}$ and $\tilde{\mu}_{n+1} > \lambda_{n+1}$ due to property (2) of Definition 2.1; as a result, $r(t) = \tilde{r}(t)$ for $t \in (\lambda_n, \lambda_{n+1}]$. In case (ii), again by $(*)$ and property (2) of Definition 2.1, we have $\tilde{\mu}_n = \mu_n = \lambda_{n+1} = \mu_{n+1} = \tilde{\mu}_{n+1}$, and thus $r(t) = \tilde{r}(t)$ for all $t \in (\lambda_n, \lambda_{n+1}]$. The induction in n establishes now the identity $r \equiv \tilde{r}$; henceforth we have shown that $\tilde{\Lambda} = \Lambda$, and the reconstruction procedure is complete.

We are now in a position to prove Theorems 2.2 and 2.4 and Corollary 2.5.

Proof of Theorems 2.2, 2.4, and Corollary 2.5. The inclusion $\mathfrak{l}(\Sigma_0^+) \subset \mathfrak{L}$ is justified by Theorem 5.1, while the above reconstruction algorithm states that $\mathfrak{L} \subset \mathfrak{l}(\Sigma_0^+)$. Therefore the mapping $\mathfrak{l} : \Sigma_0^+ \to \mathfrak{L}$ is surjective as claimed by Theorem 2.2.

Suppose that $\sigma \in \Sigma_0^+$ and $\tilde{\sigma} \in \Sigma_0^+$ satisfy the relations $\mathfrak{l}(\sigma) = \mathfrak{l}(\tilde{\sigma}) =: \Lambda$ and $\alpha_k(\sigma) = \alpha_k(\tilde{\sigma})$ for $k \in A_\Lambda$. Then in view of Corollary 5.3 we also have $\alpha_k(\sigma) = \alpha_k(\tilde{\sigma})$ for $k \in B_\Lambda$. Thus $\mathfrak{a}(\sigma) = \mathfrak{a}(\tilde{\sigma})$ and hence $\sigma = \tilde{\sigma}$ by Proposition 2.3. This proves uniqueness statements of Theorems 2.2 and 2.4, while existence is demonstrated by the above reconstruction algorithm.

To prove Corollary 2.5, we denote the right-hand side of (2.5) by $\Sigma(\Lambda)$. For any $\tilde{\sigma} \in \mathfrak{l}^{-1}(\Lambda)$ the values $\alpha_k(\tilde{\sigma})$ of the norming constants for $k \in B_\Lambda$ are prescribed by Λ by Corollary 5.3 (in particular, $\alpha_k(\tilde{\sigma}) = \alpha_k(\sigma)$), so that the inclusion $\mathfrak{l}^{-1}(\Lambda) \subset \Sigma(\Lambda)$ holds.

To justify the reverse inclusion, we take an arbitrary $\tilde{\sigma} \in \Sigma(\Lambda)$; then the sequence $(\alpha_k(\tilde{\sigma}))_{k \in A_\Lambda}$ has the form as required in Theorem 2.4 and hence there exists a unique $\hat{\sigma} \in \Sigma_0^+$ such that $\mathfrak{l}(\hat{\sigma}) = \Lambda$ and $\alpha_k(\hat{\sigma}) = \alpha_k(\tilde{\sigma})$ for $k \in A_\Lambda$. Again by Corollary 5.3 we have that $\alpha_k(\hat{\sigma}) = \alpha_k(\sigma)$ for $k \in B_\Lambda$ and thus also $\alpha_k(\hat{\sigma}) = \alpha_k(\tilde{\sigma})$ for $k \in B_\Lambda$. Proposition 2.3 now implies that $\hat{\sigma} = \tilde{\sigma}$; henceforth $\tilde{\sigma} \in \mathfrak{l}^{-1}(\Lambda)$, and the proof is complete. □

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