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Complex Methods, and the Estimation of Operator Norms and Spectra from Real Numerical Ranges*

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Let X denote a (real or complex) Banach space, and $B(X)$ the algebra of all bounded linear operators (real-linear when X is real, complex-linear when X is complex). Suppose a semi-inner-product (s.i.p.) compatible with the norm of X has been chosen (there always exists at least one compatible s.i.p.; see [1, 8], for properties of s.i.p. spaces). Denote by $W(T)$ the numerical range of an operator $T \in B(X)$, and by $|W(T)|$ the "radius" of $W(T)$, or "numerical radius" (again see [8], [1] where the notation $v(T)$ is used for $|W(T)|$).

As is well known [1, 3, 8], there exists a constant C, such that for every complex X (and independently of the compatible s.i.p. chosen),

$$\|T\| \leqslant C\,|W(T)|, \qquad \forall T \in B(X). \tag{1}$$

One also knows that $C = e$ is the best (smallest admissible) value for C, [4]. The estimate (1) is a powerful generalization, with different constant, of the well-known fact that if X is a complex Hilbert space with inner-product (,), then for every $T \in B(X)$ one has

$$\|T\| \leqslant 2\,|W(T)|, \tag{1'}$$

where $|W(T)| = \sup\{|(Tx, x)| : x \in X,\ (x, x) = 1\}$. On the other hand, one sees immediately that no inequality such as (1) can hold in general for X real. Simply take X to be the two-dimensional real Hilbert space and T "a 90° rotation," so that $|W(T)| = 0$, $\|T\| = 1$; of course, in this example $|W(T^2)| = 1$, and in [2] Bonsall and Duncan showed that, in general, $|W(T)| = |W(T^2)| = 0$ implies

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$T = 0$. Bonsall told us privately (Edinburgh, 1968) of his conjecture that there should exist constants c_1, c_2, such that for any real X, and any compatible s.i.p. one has

$$\| T \| \leqslant c_1 \, | W(T) | + c_2 \, | W(T^2) |^{1/2}, \qquad \forall T \in B(X). \tag{2}$$

Our first result will be to prove this conjecture. For the proof we use complex methods, in particular the numerical ranges "in the complexification," and the resolvent; these methods also play an important part in the rest of the paper. What we learn about the resolvent in proving the above conjecture, we use next to obtain precise estimates for the spectral radius in terms of real numerical ranges. From (2) it is clear that one can estimate $| \operatorname{sp}(T) |$ = spectral radius of T, from $\sup(| W(T) |, | W(T^2) |^{1/2})$, though (2) would not lead to a sharp estimate of that type. We obtain below a sharp estimate, not only in the above situation, but indeed for $| \operatorname{sp}(T) |$ in terms of $\sup(| W(T) |, | W(T^n) |^{1/n})$, n being any positive even integer. These are

$$| \operatorname{sp}(T) | \leqslant \sqrt{3} \sup(| W(T) |, | W(T^2) |^{1/2}) \tag{3}$$

$$| \operatorname{sp}(T) | \leqslant \sigma_n \sup(| W(T) |, | W(T^n) |^{1/n}), \qquad n = 4, 6, 8, \ldots, \tag{3'}$$

where $\sigma_4 = \sqrt{7}$, and, in general, the σ_n are determined as a certain root of a polynomial depending on n [see (13) below].

For X a Hilbert space, we can answer completely the question of best constants for estimates of $\| T \|$ of type (2) as well as for estimates in terms of $\sup(\alpha \, | W(T) |, \beta \, | W(T^2) |^{1/2})$, and compare these two types of estimates. We also discuss some aspects of the general problem of best constants.

We give applications. In particular we introduce an "invariant" $\delta(A)$ (the "dual diameter") defined for every unital Banach algebra A, i.e., Banach algebra with unit element of norm 1 [see below for the definition of $\delta(A)$]; we show that there is a constant δ_0, $0 < \delta_0 \leqslant 1$, such that $\delta(A) < \delta_0$ implies that A must coincide with either the reals, complexes, or quaternions, and δ_0 is the largest number with that property (we note that "unique supporting hyperplane to the unit ball at $1 \in A$" is equivalent to "$\delta(A) = 0$", and for that special case we recover a result of Bonsall and Duncan [2]).

Of course, our results concerning operators on general real Banach spaces can be formulated equivalently for arbitrary real unital Banach algebras.

We refer to [5] as a general reference in functional analysis. For details on complexification, spectrum, in real Banach algebras, one may consult [9]. (Below we recall how the complexification goes, and do this with a very slight variation relative to the presentation in [9]).

THEOREM 1. *There exist constants c_1, c_2 such that for any real Banach space X, one has*

$$\| T \| \leqslant c_1 \mid W(T)| + c_2 \mid W(T^2)|^{1/2}, \qquad \forall T \in B(X).$$

Before proceeding with the proof, and since we shall be making (throughout the paper) systematic use of the complexification of real Banach spaces and algebras, let us recall a few simple facts about it. If X is a real Banach space, we can make $X' = \{x + iy : x, y \in X\}$ into a complex Banach space, by defining

$$\| x + iy \|' = \sup_{0 \leqslant \theta < 2\pi} \| x \cos \theta + y \sin \theta \|. \tag{4}$$

If A is a real unital Banach algebra, we can first consider it as a Banach space and obtain the complexified Banach space A' in the way just described. Next, we can associate to every $a' \in A'$, the operator $\tilde{a}'$ which a' determines on A' by left multiplication. Then $\tilde{A} = \{\tilde{a}' : a' \in A'\}$, normed with the operator norm

$$\| \tilde{a}' \| = \sup_{0 \neq z' \in A'} \frac{\| a'z' \|'}{\| z' \|'}$$

is a complex Banach algebra, and A is isomorphic to the subalgebra $\{\tilde{a} : a \in A\}$ of $\tilde{A}$. We shall refer to $\tilde{A}$ as the standard complexification of A. Similarly we shall call X' defined as above, the standard complexification of X.

Notice that given a real Banach space X and its standard complexification X', we can associate to any $T \in B(X)$, $T' \in B(X')$, defined by $T'(x + iy) = Tx + iTy$, and one has $\| T \| = \| T' \|$.

Finally, for lack of a better place, we point out here that we shall always use the same symbol 1 to denote the unit element of a Banach algebra, the identity operator on a Banach space, and the number one.

Proof of Theorem 1. Consider the standard complexification X' of X; consider any $T \in B(X)$, and the operator $T' \in B(X')$ defined above. Our first step will be to estimate the resolvent of T' in terms of $| W(T)|$ and $| W(T^2)|^{1/2}$. To begin, notice that

$$\begin{aligned} \sup \operatorname{Re} W(T') &= \lim_{t \to 0^+} (\| 1 + tT' \| - 1)\, t^{-1} \\ &= \lim_{t \to 0^+} (\| 1 + tT \| - 1)\, t^{-1} = \sup W(T); \end{aligned}$$

and a similar relation holds for the infs; i.e.,

$$\begin{matrix}\sup\\ \inf\end{matrix} \operatorname{Re} W(T') = \begin{matrix}\sup\\ \inf\end{matrix} W(T). \tag{5}$$

Denote by Λ the angular region in the complex plane $\{\lambda \text{ complex} : |\arg\lambda| \leqslant \pi/3 \text{ or } |\arg(-\lambda)| \leqslant \pi/3\}$. Let $\lambda \in \Lambda$. $W(1 + \lambda T') = 1 + \lambda W(T') = \{1 + \lambda w : w \in W(T')\}$, and writing $\theta = \arg\lambda$, we conclude from (5) that the distance from 0 to the convex hull of $W(1 + \lambda T')$ is not less than $\cos\theta - |\lambda|\,|W(T)|$. This implies, by well-known facts (see [8]), that

$$(1 + \lambda T')^{-1} \text{ exists for } \lambda \in \Lambda,\ |\lambda| < \frac{1}{2\,|W(T)|} \tag{6}$$

and also that, with λ as in (6),

$$\|(1 + \lambda T')^{-1}\| \leqslant \frac{1}{\cos\theta - |\lambda|\,|W(T)|} \leqslant \frac{1}{\frac{1}{2} - |\lambda|\,|W(T)|}\,. \tag{7}$$

Now using the relation $(1 + \lambda iT')(1 - \lambda iT') = 1 + \lambda^2 T'^2$, and applying (6) and (7) with λ, T', replaced by λ^2, T'^2, we conclude that whenever $\lambda i \in \complement\Lambda$ = complement of Λ in the complex plane, and $|\lambda| < 1/\sqrt{2}\,|W(T^2)|^{1/2}$, then $(1 + \lambda iT')^{-1}$ exists, and

$$\|(1 + \lambda iT')^{-1}\| \leqslant \frac{1}{\frac{1}{2} - |\lambda|^2\,|W(T^2)|}(1 + |\lambda|\,\|T\|).$$

Now set

$$\omega(T) = \inf\left(\frac{1}{2\,|W(T)|}, \frac{1}{\sqrt{2}\,|W(T^2)|^{1/2}}\right)$$

attributing (if the situation arises) the meaning ∞ to any expression $1/0$. Combining what we have shown above, we see that

$$\begin{gathered}(1 + \lambda T')^{-1} \text{ exists whenever } |\lambda| < \omega(a),\\ \|(1 + \lambda T')^{-1}\| \leqslant \sup\left(\frac{1}{\frac{1}{2} - R\,|W(T)|}, \frac{1 + R\,\|T\|}{\frac{1}{2} - R^2\,|W(T^2)|}\right)\\ \text{for } |\lambda| \leqslant R < \omega(a).\end{gathered} \tag{8}$$

Below, we shall write $M = M(T, R)$ for the bound of $\|(1 + \lambda T')^{-1}\|$ in (8), without expliciting T and R whenever there seems to be no danger of confusion.

As the next step in our proof, we shall now, loosely speaking, estimate $(1 + \lambda T')^{-1}$ from below for $\lambda > 0$, and use this in turn to establish (2). Consider the expansion

$$(1 + \lambda T')^{-1} = 1 - T'\lambda + T'^2\lambda^2 - \cdots,$$

which is valid in the largest open disc centered at 0 in which $(1 + \lambda T')^{-1}$ exists. By the usual Cauchy estimates for vector-valued holomorphic functions, one has

$$\| T^n \| \leqslant M(T, R)/R^n, \qquad n = 1, 2, 3, \ldots . \tag{9}$$

For λ real, $0 \leqslant \lambda < R$, and any $x \in X'$, $\| x \| = 1$, we have, denoting by [,] a compatible s.i.p. on X',

$$\begin{aligned} \|(1 + \lambda T')^{-1} x \| &\geqslant |[(1 + \lambda T')^{-1} x, x]| \\ &= | 1 - \lambda[T'x, x] + \lambda^2[T'^2x, x] - \cdots | \\ &\geqslant | 1 - \lambda \operatorname{Re}[T'x, x] + \lambda^2 \operatorname{Re}[T'^2x, x] - \lambda^3 \operatorname{Re}[T'^3x, x] + \cdots | \\ &\geqslant 1 - (\lambda \mid W(T)| + \lambda^2 \mid W(T^2)| + \lambda^3 \| T^3 \| + \cdots). \end{aligned}$$

Using (9) we see that

$$\lambda^3 \| T^3 \| + \lambda^4 \| T^4 \| + \cdots \leqslant \frac{\lambda^3}{R^3} M \frac{1}{1 - \lambda/R}$$

Hence,

$$\|(1 + \lambda T')^{-1} x \| \geqslant 1 - \left(\lambda \mid W(T)| + \lambda^2 \mid W(T^2)| + \frac{\lambda^3}{R^3} M \frac{1}{1 - \lambda/R}\right) \tag{10}$$

holds for $0 \leqslant \lambda < R$ and all $x \in X'$ of norm one; therefore $\| 1 + \lambda T' \| = \|((1 + \lambda T')^{-1})^{-1} \|$ is bounded by the inverse of the second member of (10) provided the latter is $\geqslant 0$, (and $0 \leqslant \lambda < R$). We now complete the proof. Set $\Omega(T) = 1/\omega(T)$ for any $T \in B(X)$; we shall prove that $\exists K$, a constant, such that $\| T \| \leqslant K\Omega(T)$ for any $T \in B(X)$, X being any real Banach space. It is immediate that it suffices to prove that $\| T \| \leqslant K\Omega(T)$ holds, whenever $\Omega(T) > 0$. Suppose no such K exists. Then we can find spaces X_n and operators $T_n \in B(X_n)$, $\| T_n \| = 1$, with $\omega(T_n) \to +\infty$ [but each $\omega(T_n)$ finite]. We now apply (10) with $R_n = \frac{1}{2}\omega(T_n)$ and λ_n of the form $\sqrt[3]{R_n}$ in lieu of R and λ. With our choice of R_n, we see readily from (8) that $M(T_n, R_n) \leqslant 4(1 + R_n)$, and can then verify without difficulty that the second member of (10) tends to 1 as $n \to +\infty$. This tells us that $\| 1 + \lambda_n T_n \| \leqslant 1 + \epsilon$ for any fixed $\epsilon > 0$ and n large enough. But this is impossible, since $\lambda_n = \| \lambda_n T_n \| \leqslant \| 1 + \lambda_n T_n \| + 1$ and $\lambda_n \to +\infty$. Q.E.D.

By the previous result, one has, in particular,

$$| \operatorname{sp}(T)| \leqslant K \sup(| W(T)|, | W(T^2)|^{1/2})$$

for some constant K. Using the techniques of the preceding proof we shall give a sharp estimate of that type, and show that a similar result holds for each positive even integer n, in terms of $\sup(| W(T)|, | W(T^n)|^{1/n})$.

THEOREM 2. *For arbitrary real Banach space X, and $T \in B(X)$, one has*

$$| \operatorname{sp}(T)| \leqslant \sqrt{3} \sup(| W(T)|, | W(T^2)|^{1/2}) \tag{11}$$

and the above estimate is sharp (i.e., if $\sqrt{3}$ is replaced by a smaller constant, the statement no longer holds in general). Moreover, for each n positive even integer, there exists a constant σ_n, such that for X and T as above,

$$| \operatorname{sp}(T)| \leqslant \sigma_n \sup(| W(T)|, | W(T^n)|^{1/n}) \tag{12}$$

and the latter estimate is sharp. $\sigma_4 = \sqrt{7}$, and in general, $1/\sigma_n$ is the smallest root of the equation

$$\lambda^n - (1 - \lambda^2)^{n/2} + \binom{n}{2} (1 - \lambda^2)^{n/2-1} \lambda^2 - \cdots + (-1)^{n/2+1} \lambda^n = 0 \tag{13}$$

satisfying $0 < 1/\sigma_n < 1$.

Remark 3. Trivially, by considering again the operator defined by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ on a two-dimensional real Hilbert space, we see that no estimate of type (12) can hold for n an odd positive integer.

Proof of Theorem 2. We prove directly the general case. Let n be a positive even integer. For any Φ, $0 < \Phi < \pi/2$, define $\Lambda_\Phi = \{\lambda \text{ complex} : | \arg \lambda | \text{ or } | \arg(-\lambda)| \leqslant \Phi\}$. Let X be any real Banach space, and $T \in B(X)$. The same reasoning that leads to (6) in the proof of Theorem 1 shows that (T' being the operator in $B(X')$ corresponding to T, as before)

$$(1 - \lambda T')^{-1} \text{ exists for } \lambda \in \Lambda_\Phi,\ | \lambda | < \frac{\cos \Phi}{| W(T)|}. \tag{14}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the n-th roots of unity, i.e., the roots of the equation $\lambda^n - 1 = 0$. We then have the algebraic identity

$$1 - \lambda^n T'^n = (1 - \lambda\alpha_1 T')(1 - \lambda\alpha_2 T') \cdots (1 - \lambda\alpha_n T'). \tag{15}$$

Now if $n = 2m$, and m is even, then for the appropriate j, $\alpha_j = i$, whereas if m is odd then for the appropriate j, $i = \alpha_j e^{(\pi/n)i}$. In the first case we use (15) with λ replaced by $\tilde{\lambda} = \lambda\alpha_j$, and in the second case with λ replaced by $\tilde{\lambda} = \lambda\alpha_j e^{(\pi/n)i}$. In either case, $\tilde{\lambda}^n = \pm\lambda^n$; so the first member of (15) reads $1 \pm \lambda^n T'^n$, and one of the factors in the second member of (15) is $(1 - \lambda i T')$. Therefore $(1 - \lambda i T')^{-1}$ exists if $(1 \pm \lambda^n T'^n)^{-1}$ exists, and we apply (14) with $\Phi = n\psi$, choosing a ψ such that $0 < n\psi < \pi/2$. This tells us that

$$(1 - \lambda i T')^{-1} \text{ exists for } \lambda \in \Lambda_\psi \qquad \text{if } \quad |\lambda| < (\cos n\psi)^{1/n}/|W(T^n)|^{1/n}. \tag{16}$$

For any ψ as above, combining (14) and (16) with $\Phi = (\pi/2) - \psi$ (so that the union of the angular regions Λ_Φ and $(\Lambda_\psi)i$ covers the whole plane) we have then

$$|\operatorname{sp}(T)| \leqslant \sup\left(\frac{|W(T)|}{\sin\psi}, \frac{|W(T^n)|^{1/n}}{(\cos n\psi)^{1/n}}\right). \tag{17}$$

Now, one sees easily that there exists exactly one ψ_n such that $0 < \psi_n < \pi/2n$ and $\sin\psi_n = (\cos n\psi_n)^{1/n}$. Set $\sigma_n = (1/\sin\psi_n)$, then we have

$$|\operatorname{sp}(T)| \leqslant \sigma_n \sup(|W(T)|, |W(T^n)|^{1/n})$$

and we shall see that this estimate is sharp. For that purpose, let X be at present a real two-dimensional Hilbert space, and U the operator on X defined by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ relative to an orthonormal basis. Set $\Phi = (\pi/2) - \psi_n$, and $T = (\cos\Phi)\,1 + (\sin\Phi)U$. $|W(T)| = \sin\psi_n$ since $|W(U)| = 0$; also, since T is a "rotation by Φ radians," we have $|W(T^n)| = |\cos n\Phi| = \cos n\psi_n$, and $|\operatorname{sp}(T)| = 1$. Hence, (12) is sharp.

Finally it follows easily from De Moivre's formula, that $\sin\psi_n$ is the smallest of the roots lying between 0 and 1 of Eq. (13). Solving that equation for $n = 2, 4$, one finds $\sigma_2 = \sqrt{3}$, $\sigma_4 = \sqrt{7}$. Q.E.D.

Next, we give several applications of Theorem 1 and the corresponding techniques. First we like to point out that a number of known facts concerning real unital Banach algebras follow almost at once from the above. For instance, COROLLARY 4. ([2]). *If A is a real unital Banach algebra whose unit ball is smooth at* 1, *then A must coincide with the reals, complexes, or quaternions.*

Proof. The hypothesis implies that $\forall a \in A$, $W(a)$ consists of one value λ_a. Using the complexification $\tilde{A}$, and (5), we have $\operatorname{Re} \operatorname{sp}(\tilde{a}) = \{\lambda_a\}$. If $0 \in \operatorname{sp}(\tilde{a})$, then $\lambda_a = 0$, so $|W(a)| = 0$; and then also $0 \in \operatorname{sp}(\tilde{a}^2)$, so $|W(a^2)| = 0$. Therefore $a = 0$. Thus, A is a division algebra and the statement follows from this, as is well-known. Q.E.D.

This corollary contains, in particular, the result [6] that any real unital Hilbert algebra must be the reals, complexes, or quaternions; and in this case the above argument yields a very short and elementary proof of that result (since we shall see below that for X a real Hilbert space Theorem 1 has an entirely elementary proof). Also most of the results on the vertex property obtained by Ingelstam in [7] can be derived similarly. As an example, (using the terminology of [7]),

COROLLARY 5 ([7], Theorem 4). *A real Banach algebra of strongly real type, with identity, has the vertex property.*

Proof. The given algebra A becomes unital under equivalent renorming. We consider $\tilde{A}$ as above. If $a \in A$, $|W(a)| = 0$, then $i\tilde{a} = h$ is a generalized hermitian, and $\operatorname{sp}(\tilde{a}^2) = -\operatorname{sp}(h^2) = -(\operatorname{sp}(h))^2$ lies on the closed left half of the real axis. Since $1 + t^2a^2$ must be invertible for every real t, we must have $\operatorname{sp}(\tilde{a}) = \{0\}$, hence $\operatorname{sp}(h) = \{0\}$ and therefore $h = 0$, $a = 0$. But the implication $|W(a)| = 0 \Rightarrow a = 0$, which we have proved, is exactly the vertex property. Q.E.D.

The result of Bonsall and Duncan stated as Corollary 4 above, tells us that if the unit ball of a real unital Banach algebra A has only one supporting hyperplane at 1, then A must be the reals, complexes, or quaternions. We shall prove below (Theorem 6), that indeed unless a real unital Banach algebra is one of the above fields, the set of supporting hyperplanes at 1 can not be "very thin". For any real unital Banach algebra A, denote by $S(A)$ the set of normalized states, i.e., $S(A) = \{f^* \in A^*$, the dual of A as a Banach space: $\|f^*\| = f^*(1) = 1\}$. Denote by $\delta(A)$ the diameter of the set $S(A)$—of course, the equation of any supporting hyperplane to the unit ball of A at 1 is of the form $f^*(a) = 1$, $f^* \in S(A)$. We shall call $\delta(A)$ the dual diameter of A.

THEOREM 6. *There exists a constant c, $0 < c \leqslant 1$, such that any real unital Banach algebra A for which $\delta(A) < c$, must coincide with* *either the reals, the complexes, or the quaternions, and no number larger than c has the same property.*

Proof. For any $a \in A$, denote by $\delta(W(a))$ the diameter of the numerical range $W(a)$ of a, where of course we suppose again that a compatible s.i.p. has been chosen on A

$$(W(a) = \{[ax, x] : x \in A, \| x \| = 1\}).$$

It is easy to check that $\delta(A) = \sup_{0 \neq a \in A} [\delta(W(a))/\| a \|]$. Now suppose $\delta(A) \leqslant k$. Consider the standard complexification $\tilde{A}$ of A, and use $\overline{\text{co}}$ to denote "closed convex hull of." If $a \in A$ is not invertible, then by (5), and the fact that $\text{sp}(a) \subset \overline{\text{co}}\ W(\tilde{a})$, it follows that $0 \in \overline{\text{co}}\ W(a)$. If a is not invertible, neither is a^2, so that also $0 \in \overline{\text{co}}\ W(a^2)$. Hence, in that case, $| W(a)| \leqslant \delta(W(a)) \leqslant k \| a \|$, and $| W(a^2)| \leqslant \delta(W(a^2)) \leqslant k \| a^2 \| \leqslant k \| a \|^2$. By Theorem 1, we conclude

$$\| a \| \leqslant (c_1 k + c_2 k^{1/2}) \| a \|$$

and if $c_1 k + c_2 k^{1/2} < 1$, this implies $a = 0$, i.e., A is a division algebra. So, we have proved that for k small enough, $\delta(A) < k$ implies that A is a division algebra. Let c be the sup of all such values of k; then c is itself one of these values k. Consider now the case in which X is a two-dimensional real Hilbert space, and A is the algebra of operators on X represented with respect to an orthonormal basis of X by the real matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}.$$

In that case, one sees immediately that

$$\sup_{0 \neq a} \frac{\delta(W(a))}{\| a \|} = \delta(A) = 1,$$

while A is not a division algebra. Hence $c \leqslant 1$. Q.E.D.

Theorem 7 below, another extension of Corollary 4, further illustrates the strong connection between geometry and algebra, in normed algebras.

THEOREM 7. *Suppose A is any real unital Banach algebra and $a \in A$. Then, if the intersections of the linear spaces generated by* 1 *and a, and by* 1 *and a^2, with unit sphere, are smooth at* 1, *a is algebraic over the reals of degree two, and invertible if nonzero.*

Proof. We consider the complexification $\tilde{A}$ as before. By (5) the hypothesis on a and a^2 implies that Re $W(\tilde{a})$ and Re $W(\tilde{a}^2)$ consist each of one number only, say α and β, respectively. Hence, $\tilde{a} - \alpha 1 = ih$, $\tilde{a}^2 - \beta 1 = ih'$, where h and h' are generalized hermitians. If $r + is \in \operatorname{sp}(\tilde{a})$, then $r = \alpha$, and by the spectral mapping theorem $r^2 - s^2 = \beta$, $s^2 = \alpha^2 - \beta$. So $\operatorname{sp}((\tilde{a} - \alpha 1)^2) = \{\beta - \alpha^2\}$, and $\operatorname{sp}(\tilde{a}^2 - 2\alpha\tilde{a} + 2\alpha^2 - \beta) = \{0\}$. But since $\tilde{a}^2 - 2\alpha\tilde{a} + 2\alpha^2 - \beta = i(-2\alpha h + h')$, we must have

$$\tilde{a}^2 - 2\alpha\tilde{a} + 2\alpha^2 - \beta = 0, \qquad a^2 - 2\alpha a + 2\alpha^2 - \beta = 0.$$

Also, if a is not invertible, we must have $\alpha = \beta = 0$, so that $\tilde{a} = ih$ and $a^2 = 0$, which implies indeed $a = 0$. Q.E.D.

We shall now discuss the question of "best constants" for estimates of type (2), as well as for estimates of $\| T \|$ in terms of "weighted sups of $| W(T)|$, $| W(T^2)|^{1/2}$". We give a complete answer for the case in which X is a real Hilbert space, obtain some indications for the general Banach space situation, and state several related open problems.

DEFINITION 8. Denote by R^+ the nonnegative reals. For X any real Banach space, define $G(X)$ as $\{(c_1, c_2) \in R^+ \times R^+ : \| T \| \leqslant c_1 | W(T)| + c_2 | W(T^2)|^{1/2}, \forall T \in B(X)\}$. Define $G = \bigcap_{\text{all} X} G(X)$; of course, $G = \{(c_1, c_2) \in R^+ \times R^+ : \| T \| \leqslant c_1 | W(T)| + c_2 | W(T^2)|^{1/2}, \forall T \in B(X), \text{ any } X\}$.

PROPOSITION 9. *$G(X)$ for any X, G, are convex and closed. The boundaries $\partial G(X)$, ∂G, have vertical and horizontal asymptotes, $x = x_0$, $y = y_0$ (interpreting the c_1 as x coordinates, and the c_2 as y coordinates). For the case of G, $x_0 \geqslant e$, $y_0 \geqslant 1$.*

Proof. The convexity is obvious. Since $(x, y) \in G(X)$ implies that $\{(x', y') : x' \geqslant x, y' \geqslant y\} \subset G(X)$, $\partial G(X)$ must have vertical and horizontal asymptotes. The same holds for G; and also clearly the $G(X)$, G, are closed. Since we can find an X, $0 \neq T \in B(X)$, such that $| W(T)| = 0$, $| W(T^2)|^{1/2} = \| T \|$, we must have for G, $y_0 \geqslant 1$. We know that there is a complex Banach space $\tilde{X}$ and $0 \neq T \in B(\tilde{X})$ for which $T^2 = 0$, $| W(T)| = (1/e) \| T \|$, [4]. It suffices to consider $\tilde{X}$ as a Banach space over the reals and use the analog of (5) to see that we have then a real-linear operator T with the same properties. It follows that $x_0 \geqslant e$.

Any (c_1, c_2) on $\partial G(X)$ not interior to any horizontal or vertical line segment in $\partial G(X)$, is a pair of "best constants" in the sense that if we diminish both of them, or one of them leaving the other fixed, the estimate no longer holds. We shall say that (c_1, c_2) are "absolute best constants" for X, if for any $(c_1', c_2') \in G(X)$, $c_1' \geqslant c_1$, $c_2' \geqslant c_2$ and $(c_1, c_2) \in G(X)$. The latter means exactly that

$$G(X) = \{(c_1', c_2') : c_1' \geqslant c_1 \geqslant 0, c_2' \geqslant c_2 \geqslant 0\}.$$

THEOREM 10. *When X is any real Hilbert space of dimension >1, $(2, 1)$ are absolute best constants for X, i.e., we have the sharp estimate*

$$\| T \| \leqslant 2 \, | W(T)| + | W(T^2)|^{1/2}, \qquad \forall T \in B(X), \tag{18}$$

and $G(X) = \{(c_1, c_2) : c_1 \geqslant 2, c_2 \geqslant 1\}$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned} 2 \, |(Tx, y) + (Ty, x)| &= |(T(x+y), x+y) - (T(x-y), x-y)| \\ &\leqslant | W(T)| \, (\| x+y \|^2 + \| x-y \|^2) \\ &= 2 \, | W(T) | (\| x \|^2 + \| y \|^2). \end{aligned}$$

Hence,

$$|(Tx, y) + (Ty, x)| \leqslant | W(T)| \, (\| x \|^2 + \| y \|^2). \tag{19}$$

We may assume $T \neq 0$, and apply (19) with $y = Tx/\| T \|$, $\| x \| \leqslant 1$, to obtain $| \, \| Tx \|^2 + (T^2x, x)| \leqslant 2 \| T \| \, | W(T)|$ for all $x \in X$, $\| x \| \leqslant 1$. It follows that $\| T \|^2 \leqslant 2 \, | W(T)| \, \| T \| + | W(T^2)|$, and therefore $\| T \|$ is $\leqslant$ the largest root of the equation

$$-\lambda^2 + 2 \, | W(T)| \, \lambda + | W(T^2)| = 0.$$

Hence,

$$\| T \| \leqslant | W(T)| + \sqrt{| W(T)|^2 + | W(T^2)|} \tag{20}$$

and since $\sqrt{t}$ is a subadditive function of $t \geqslant 0$, (18) follows immediately from (20).

Since the dimension of X is >1, we can find $T_1 \in B(X)$ such that $T_1^2 = 0$, $| W(T_1)| = \frac{1}{2}$, and $T_2 \in B(X)$ such that $| W(T_2)| = 0$, $| W(T_2^2)|^{1/2} = 1$, and therefore if $(c_1, c_2) \in G(X)$, $c_1 \geqslant 2$, $c_2 \geqslant 1$. In view of the above, we see that $(2, 1)$ are absolute best constants, and $G(X) = \{(c_1, c_2) : c_1 \geqslant 2, c_2 \geqslant 1\}$. Q.E.D.

Now let again X be any real Hilbert space of dimension >1. For any given $\alpha, \beta > 0$, there exists by the preceding theorem some constant $c > 0$ such that

$$\| T \| \leqslant c \sup(\alpha \mid W(T)|, \beta \mid W(T^2)|^{1/2}) \qquad \forall T \in B(X).$$

Denote by $c(\alpha, \beta)$ the smallest of such constants c for α, β fixed.

THEOREM 11. *For any real Hilbert space X of dimension >1, we have*

$$c(\alpha, \beta) = (1/\alpha) + \sqrt{(1/\alpha^2) + (1/\beta^2)}. \tag{21}$$

The least "maximum deviation" for an estimate of the type under consideration (see below) occurs when $\alpha = \beta$, and we have in that case the sharp estimate

$$\| T \| \leqslant (1 + \sqrt{2}) \sup(| W(T)|, | W(T^2)|^{1/2}). \tag{22}$$

Proof. Fix $\alpha, \beta > 0$, and set for simplicity $\tilde{\omega}(T) = \tilde{\omega} = \sup(\alpha \mid W(T)|, \beta \mid W(T^2)|^{1/2})$. Then, $| W(T)| \leqslant \tilde{\omega}/\alpha$, $| W(T^2)| \leqslant \tilde{\omega}^2/\beta^2$, and we have in view of the inequality (20) obtained earlier

$$\| T \| \leqslant ((1/\alpha) + \sqrt{(1/\alpha^2) + (1/\beta^2)})\tilde{\omega}$$

which shows that

$$c(\alpha, \beta) \leqslant (1/\alpha) + \sqrt{(1/\alpha^2) + (1/\beta^2)}.$$

Next, let X_0 be a two-dimensional real Hilbert subspace of X. Consider U and S in $B(X)$, defined as projection on X_0 followed by the operation defined relative to an orthonormal basis of X_0, by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, respectively, and set $T = S - \epsilon U$, where ϵ is some positive constant. Since $| W(U)| = 0$, $| W(S)| = \frac{1}{2}$, $S^2 = 0$, we verify immediately that $| W(T)| = \frac{1}{2}$, $| W(T^2)| = \epsilon(1 + \epsilon)$. We shall have $\alpha \mid W(T)| = \alpha/2 = \beta \mid W(T)|^{1/2}$ if $\epsilon^2 + \epsilon - \alpha^2/(4\beta^2) = 0$. The latter occurs when

$$\epsilon = [-1 + \sqrt{1 + (\alpha^2/\beta^2)}]/2,$$

and with that choice of ϵ, since

$$\| T \| = 1 + \epsilon = [1 + \sqrt{1 + (\alpha^2/\beta^2)}]/2,$$

we have

$$\| T \|/\sup(\alpha \mid W(T)|, \beta \mid W(T^2)|^{1/2}) = [[1 + \sqrt{1 + (\alpha^2/\beta^2)}]/2]\,(2/\alpha)$$

$$= (1/\alpha) + \sqrt{(1/\alpha^2) + (1/\beta^2)}$$

and therefore

$$c(\alpha, \beta) = (1/\alpha) + \sqrt{(1/\alpha^2) + (1/\beta^2)}.$$

For fixed $\alpha, \beta > 0$, by "maximum deviation" of the sharp estimate $\| T \| \leqslant c(\alpha, \beta) \sup(\alpha \mid W(T)|, \beta \mid W(T^2)|^{1/2})\ \forall T \in B(X)$, we mean

$$\sup_{T \neq 0} [[c(\alpha, \beta) \sup(\alpha \mid W(T)|, \beta \mid W(T^2)|^{1/2})]/\| T \|].$$

It is easy to see that this "maximum deviation" is smallest for $\alpha = \beta$, in which case the estimate is independent of α and takes the form (22).
Q.E.D.

One thing one learns from the preceding results by comparing best estimates of type (2), additive, with best estimates in terms of "weighted sups"—as in the previous theorem—for X a real Hilbert space of dimension >1, is that neither kind is overall better than the other; but the question of best constants has a simpler and uniquely determined answer (at least in the Hilbert space case) for estimates of type (2), and by and large the latter estimates deviate less for a number of standard types of operators on Hilbert space. Theorem 11 also shows that with weights of 2 and 1, we need a $c > 1.6$, and with weights of e and 1, we have still $c(e, 1) > 1.43$.

Problems 12. Determine $G(X)$ for other real Banach spaces—specially the more commonly used ones—different from Hilbert space.

Determine G (or at least determine the asymptotes of G).
Find $c(\alpha, \beta)$ in the general situation just considered.

Finally, let us mention, for arbitrary X, one limiting case in which one can give a sharp estimate of type (2) for the norm of T, quite easily, in view of Sinclair's result about (generalized) hermitians (spectral radius = norm) [10].

PROPOSITION 13. *Let X be any real Banach space. If $T \in B(X)$ and $\mid W(T)| = 0$, we have*

$$\| T \| \leqslant \mid W(T^2)|^{1/2}. \tag{23}$$

Proof. If T' is the operator corresponding to T in $B(X')$ as before, then $|W(T)| = 0$ and (5) imply $T' = iH$, where H is a hermitian. Hence by [10],

$$\|T\| = \|H\| = |\operatorname{sp}(H)| = |\operatorname{sp}(H^2)|^{1/2} = |\operatorname{sp}(T'^2)|^{1/2} \leqslant |W(T^2)|^{1/2}$$

because $\operatorname{sp}(T'^2) = -\operatorname{sp}(H^2) = -(\operatorname{sp}(H))^2$ is real. Q.E.D.

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