Artinian Rings Having a Nilpotent Group of Units

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Throughout this paper the symbol $R$ will be used to denote an associative ring with a multiplicative identity. $R_0[R^x]$ denotes the subring of $R$ which is generated over $R_0$ by $R^x$, where $R_0$ is the prime subring of $R$ and $R^x$ is the group of units of $R$. We use $J(R)$ to denote the Jacobson radical of $R$.

In Section 1 we prove that if $R$ is a finite ring and at most one simple component of the semi-simple ring $R/J(R)$ is a field of order 2, then $R^x$ is a nilpotent group iff $R$ is a direct sum of two-sided ideals that are homomorphic images of group algebras of type $SP$, where $S$ is a particular commutative finite ring, $P$ is a finite $p$-group, and $p$ is a prime number.

In Section 2 we study the artinian rings $R$ (i.e., rings with minimum condition) with $R_0$ a finite ring, $R$ finitely generated over its center and so that every block of $R$ is as in Lemma 1.1. We note that every finite-dimensional algebra over a field $F$ of characteristic $p > 0$, so that $F$ is not the finite field of order 2 is of this type. We prove that $R$ is of this type and $R^x$ is a nilpotent group iff $R$ is a direct sum of two-sided ideals that are particular homomorphic images of crossed products. Finally we present an analogous theorem when $R/J(R)$ is a commutative ring.

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1. FINITE RINGS

Let $R$ be a right (left) artinian ring. The next three results are known and we only sketch the proofs.

**Lemma 1.1.** $R = R_0[R^x]$ if and only if at most one simple component of the semi-simple ring $R/J(R)$ is a field of order 2.
Proof. Since \( J(R) \) is a nilpotent two-sided ideal we may assume that \( R \) is a semi-simple ring. The result follows from Wedderburn–Artin Structure Theorems.

**Lemma 1.2.** If \( R \) is of characteristic \( p' \), where \( p \) is a prime number and \( r \) is a positive integer, then the multiplicative group \( H = 1 + J(R) \) is a \( p \)-group of bounded period.

*Proof.* If \( J(R)^{k+1} = 0 \) and \( s \) is the highest power of \( p \) dividing \( k! \), then \( (1 + j)^{p^{s-1}} = 1 \) for every \( j \in J(R) \).

We recall that a field \( F \) is said to be absolute if and only if it is algebraic over a finite field.

**Lemma 1.3.** Suppose that \( R/J(R) \cong F_1 \oplus F_2 \oplus \cdots \oplus F_m \), where \( F_i \) are fields of characteristic \( p \). Then \( R^x \) is periodic if and only if \( F_i \) are absolute fields for \( i = 1, 2, \ldots, m \).

*Proof.* Note that every element of an absolute field belongs to a finite field and the result follows from Lemma 1.2.

Now we are looking for the finite rings \( R \) with \( R^x \) a nilpotent group. Since every artinian ring is a direct sum of blocks we can limit our attention to rings which contain no non-trivial blocks. We recall that an artinian ring \( R \) is called a completely primary ring if \( R/J(R) \) is a skewfield.

The following proposition reduces the problem to the case of completely primary rings.

**Proposition 1.4.** Let \( R \) be a right artinian ring which contains no non-trivial blocks and with \( R^x \) a periodic group. If at most one simple component of the semi-simple ring \( R/J(R) \) is a field of order 2 and \( R^x \) is nilpotent, then \( R \) is a completely primary ring and \( R/J(R) \) is an absolute field.

*Proof.* Since \( R^x \) is nilpotent it follows that \( R^x \) is solvable. Now \( R/J(R) \cong S_1 \oplus S_2 \oplus \cdots \oplus S_m \), a direct sum of simple rings and each \( S_i \) is isomorphic to \( M_n(D) \), a suitable full matrix ring over a skewfield \( D \). Thus \( D^x \) is solvable and a result of Hua [6] asserts that \( D = F \) is a field. But then \( GL_n(F) \) is solvable so, as is known (see [4]) we have either \( n = 1 \) or \( n = 2 \) and \( F = GF(2) \) or \( GF(3) \). Since the latter implies that \( S_i^x \) is not nilpotent we conclude that each \( S_i \) is a field. Since \( R^x \) is a periodic group, each \( S_i \) is an absolute field, by Lemma 1.3.

It remains to show that \( m = 1 \). Notice now that the center of \( R \) contains no non-trivial idempotents. It follows that \( R \) is of characteristic \( p' \), where \( p \) is a prime number. By Lemma 1.2 the multiplicative group \( H = 1 + J(R) \) is a \( p \)-group. Because each \( S_i \) is a field of characteristic \( p \) it follows that \( H \) is
a $p$-Sylow subgroup of $R^*$. Since $R^*$ is nilpotent each $p$-Sylow subgroup of $R^*$ is normal in $R^*$ and since $R^*$ is periodic it is the weak direct product of its Sylow subgroups [8, p. 390].

Let $\bar{e}$ be a primitive idempotent of $R/J(R)$. Notice first it will be sufficient to find an idempotent $f$ which is contained in the center of $R$ and $f$ is in the coset $\bar{e}$. Since $J(R)$ is a nilpotent two-sided ideal it is possible to choose an idempotent $f$ in the coset $\bar{e}$. By Lemma 1.1, there exist the elements $u_1, \ldots, u_r \in R^*$ and $a_1, \ldots, a_r \in R_0$ such that $f = u_1 a_1 + \cdots + u_r a_r$. Since $R^*$ is the weak direct product of its Sylow subgroups there exist the elements $v_1, \ldots, v_r \in R^*$, $b_1, \ldots, b_r \in R_0$ and $j \in J(R)$ such that each $v_i$ has the order relatively prime to $p$ and $f = v_1 b_1 + \cdots + v_r b_r + j$. Because $R^*/H$ is an abelian group it follows that each $v_i$ belongs to the center of $R^*$ and because $R = R_0[R^*]$ the element $c = v_1 b_1 + \cdots + v_r b_r$ belongs to the center of $R$. Now $f = c + j$ and let $s$ be as in the proof of Lemma 1.2. So $f = f^{p^{s+1}} = c^{p^{s+1}}$ belongs to the center of $R$. This completes the proof of the proposition.

**Corollary 1.5.** Let $R$ be a finite ring with $R^*$ a nilpotent group. Then $R^*$ is a direct product of finite $p_i$-groups and cyclic groups of order $p_i^\alpha - 1$, where $p_i$ are primes which are factors of the order of $R_0$.

**Proof.** As in the proof of Proposition 1.4 we know that $R/J(R)$ is a commutative ring. Since $R$ is a direct sum of rings of prime power characteristic the result is a consequence of Lemma 1.2 and of the Theorem of [8, p. 390].

**Remarks.** 1. Since the ring $R$ of $n \times n$ upper triangular matrices with entries from the field $GF(2)$ has $R^*$ a nilpotent group, Proposition 1.4 is not true for all artinian rings with $R^*$ a periodic group.

2. By Proposition 1.4, the ring $R$ of $n \times n$ upper triangular matrices with entries from an absolute field, not of order 2, has not $R^*$ nilpotent, when $n > 1$. So the Corollary XXI.10 of [9, p. 404] is not true.

Even in the case of completely primary rings there can exist artinian rings $R$ with $R/J(R)$ commutative and $R^*$ is not a nilpotent group.

**Example 1.6.** Let $F$ be a field of characteristic $p$ and $F \neq F_0$. If $\sigma$ is the morphism $f \mapsto f^p$, for every $f \in F$, then we consider the skew polynomial ring $F(X, \sigma)$ where $Xf = f^p X$, for every $f \in F$. Let $R = F[X, \sigma]/(X^2)$. Then $J(R) = (X)$, $R/J(R) \cong F$ and $J(R)^2 = 0$. Since $F \neq F_0$ there exist an element $f \in F$ so that $f^{p^n - 1} \neq 1$. For $P(X) \in F[X, \sigma]$ let $\overline{P}(X)$ the image of $P(X)$ in $R$. If $u_1, \ldots, u_n, u_{n+1} \in R^*$ then we write the commutator $u_1 u_2 u_1^{-1} u_2^{-1}$ as $(u_1, u_2)$ and $((u_1, u_2, \ldots, u_n), u_{n+1})$ as $(u_1, u_2, \ldots, u_{n+1})$. Then $(1 + x, f) = (1 + x f) f (1 + x) f = (1 + x f) f (1 + x f) f = 1 + (1 - f^{p^n - 1}) X \neq I$. 

**ARTINIAN RINGS**

255
Since \((u_1, \ldots, u_n, u_{n+1}) = ((u_1, \ldots, u_n), u_{n+1})\) for every \(u_i \in R^x\) we use induction on \(n\). If \(u_1 = \bar{1} + \bar{x}\) and \(u_2 = \cdots = u_{n+1} = \bar{f}\), then \((u_1, \ldots, u_n, u_{n+1}) = \bar{1} + (\bar{1} - \bar{f}^1 - \bar{f})^n \bar{x} \neq \bar{1}\) for every \(n\). So \(R^x\) is not a nilpotent group.

In order to describe the finite completely primary rings with \(R^x\) nilpotent we need the following.

**Lemma 1.7.** Let \(C\) be a commutative local artinian ring of characteristic \(p^r\), where \(p\) is a prime number. If \(P\) is a finite \(p\)-group, then the group algebra \(CP\) is a completely primary artinian ring and the field \(C/J(C) \cong CP/J(CP)\).

**Proof:** Let \(F = C/J(C)\) and \(I = J(C)CP\) the kernel of the natural morphism \(CP \to FP\). If \(J(C)^{k+1} = 0\) and the order of \(P\) is \(p^s\), then we set \(n = p^k + 1\). Since every element of \(J(C)\) is in the center of \(CP\) it follows that \(I^r = 0\). \(C\) being artinian and \(P\) being finite the group algebra \(CP\) is an artinian ring. Because \(FP\) is a completely primary ring and the elements \(\{g - 1 \mid g \in P, g \neq 1\}\) form a basis of \(J(FP)\) \([3, \text{pp. 189, 377}]\), it follows that \(CP\) is a completely primary ring, \(J(CP) = I + \sum g \in P (g - 1)CP\) and \(C/J(C) \cong CP/J(CP)\).

**Theorem 1.8.** Let \(R\) be a finite completely primary ring. Then \(R^x\) is a nilpotent group if and only if \(R\) is a homomorphic image of the group algebra \(CP\) so that the following holds:

\(P\) is a finite \(p\)-group, where \(p\) is a prime number; \(C\) is a homomorphic local image of the ring \(Z_{p^s}[X]/(X^{p^s-1}-1)\),

where \(Z\) is the ring of rational integers, \(r, s\) are positive integers, and \(Z_{p^s} = Z/(p^s)\).

**Proof:** If \(R^x\) is nilpotent, then it is the direct product of its Sylow subgroups. We set \(F = R/J(R)\) and let \(p\) be the characteristic of this finite field. Then \(F^x\) is a cyclic group of order \(p^s - 1\), where \(s\) is a positive integer. Because \(R = R_0[R^x]\), there exists a multiplicative group \(G \cong F^x\) so that every element of \(G\) is in the center of \(R\). Then \(R = C[P]\), where \(P = 1 + J(R)\) and \(C = R_0[G]\). By Lemma 1.2, \(P\) is a \(p\)-group and since \(R\) is a finite completely primary ring, \(R_0 = Z_{p^s}\). But \(G\) is a cyclic group of order \(p^s - 1\) and the center of \(R\) contains no non-trivial idempotents. Hence \(C\) is a homomorphic local image of the ring \(Z_{p^s}[X]/(X^{p^s-1}-1)\) and \(R\) is a homomorphic image of the group algebra \(CP\).

Conversely, if \(R\) is homomorphic image of the group algebra \(CP\), then \((CP)^x\) is a nilpotent group by Theorem 1 of \([7]\). By Lemma 1.7, \(CP\) is a completely primary artinian ring. Hence every ideal of \(CP\) is nilpotent and
$R^x$ is a homomorphic image of the nilpotent group $(CP)^x$. Then we conclude that $R^x$ is a nilpotent group.

**Corollary 1.9.** Let $R$ be a finite ring so that at most one simple component of the semi-simple ring $R/J(R)$ is a field of order 2. Then $R^x$ is a nilpotent group if and only if $R$ is a direct sum of rings $R_1, R_2, ..., R_m$, where $R_i$ is a finite completely primary ring as in Theorem 1.8.

**Corollary 1.10.** Let $R$ be a finite completely primary ring and $R^x$ a nilpotent group. If $J(R)^2 = 0$, then $R$ is a commutative ring.

## 2. Artinian Rings and Crossed Products

In this section we prove two theorems concerning right artinian rings finitely generated over their centers. Central role will be played by the concept of a crossed product, the definition of which we reproduce here for completeness: A crossed product $S(G, \gamma, \sigma)$ (see [2, p. 482]) is an associative ring determined by a ring $S$ with a multiplicative identity, a multiplicative group $G$, a factor set $\gamma$, and a map $\sigma: G \rightarrow \text{Aut} S$, where $\text{Aut} S$ denotes the group of automorphisms of the ring $S$. Recall that

$$\gamma(g_1, g_2, g_3) = \gamma(g_1 g_2, g_3) \gamma(g_1, g_2)^{-\gamma(g_2, g_3)};$$

$$a^{g_1 \cdot g_2 \cdot \sigma} = \gamma(g_1, g_2)^{-1} \sigma^{(g_1, g_2)} \gamma(g_1, g_2),$$

for all $a \in S$ and $g_1, g_2, g_3 \in G$.

The elements of $S(G, \gamma, \sigma)$ are the expressions $\sum_{g \in G} t_g a_g$, where $a_g \in S$ and $a_g = 0$ for all but a finite number of $g \in G$. We consider $\sum_{g \in G} t_g a_g = \sum_{g \in G} t_g b_g$ if and only if $a_g = b_g$ for all $g \in G$ and we define

$$\sum_{g \in G} t_g a_g + \sum_{g \in G} t_g b_g = \sum_{g \in G} t_g(a_g + b_g),$$

$$(\sum_{g \in G} t_g a_g) (\sum_{h \in G} t_h b_h) = \sum_{g, h \in G} t_{gh} \gamma(g, h) a_g^{\sigma} b_h.$$

We note also that the element $t_{1, (1, 1)}^{-1}$ is the multiplicative identity for $S(G, \gamma, \sigma)$. Moreover the elements $t_g$ are units and $t_g^{-1} = (\gamma(1, 1) \gamma(g^{-1}, g))^{-1} t_{g^{-1}}$ by Proposition 2 of [2].

**Proposition 2.1.** Let $R$ be a right artinian ring finitely generated over its center $C$, let $R$ contain no non-trivial blocks so that $R/J(R)$ has at most one simple component isomorphic to the field of order 2. If $R_0$ is a finite ring and
$R^x$ is a nilpotent group, then $R$ is a homomorphic image of the ring $S(G, \gamma, \sigma)/I$ so that the following four conditions hold.

(a) $S$ is a completely primary artinian ring and there exist a finite $p$-group $P$, where $p$ is a prime number, so that $S \cong CP/L$, where $L$ is a two-sided ideal of the group algebra $CP$ of the $p$-group $P$ over the local commutative artinian ring $C$ of characteristic a power of $p$. Moreover, $P^\sigma = P$ for all $g \in G$, where $P$ is the image of $P$ in $S$.

(b) $G$ is a direct product $G = (g_1) \times \cdots \times (g_r)$ of non-trivial cyclic groups.

(c) $\gamma(g', g'') \in \overline{P}$ for all $g', g'' \in G$ and $\gamma(1, 1) = 1$.

(d) $I$ is a finitely generated two-sided ideal so that

$$I = \sum_{i=1}^{r} S(G, \gamma, \sigma) x_i S(G, \gamma, \sigma) + \sum_{k=1}^{s} S(G, \gamma, \sigma) y_k S(G, \gamma, \sigma).$$

Thus every $x_i$ is of type $t_{g_i}^{n(i)} + t_{g_i}^{n(i)-1}a_{i1} + \cdots + a_{im(i)}$, where $a_{ij} \in S$ and $n(i)$ are positive integers and there exists a positive integer $m$ so that the set $(y_1, ..., y_r) = \{(z_1, ..., z_m) - t_{g_1} z_1, ..., t_{g_r} z_r\} \cup \overline{P}; \ i = 1, 2, ..., m$, where $(z_1, ..., z_m)$ are commutators as in Example 1.6.

Proof. Since $R$ is finitely generated over $C$, by Theorem 2.3 of [1], $C$ is an artinian ring. Because $R$ contains no non-trivial blocks, $C$ is a local commutative artinian ring and because $R_0$ is a finite ring, the characteristic of $R$ is a power of a prime number $p$.

By Lemma 1.1 we have $R \cong R_0[ R^x ]$. Then $R = C[ R^x ]$ and since $R$ is finitely generated over $C$ there exist elements $k_1, ..., k_n \in R^x$ so that $R = C[ K ]$, where $K$ is the multiplicative group $\langle k_1, ..., k_n \rangle$. Since $R^x$ is a nilpotent group, it follows that $K$ is a finitely generated nilpotent group and the commutator subgroup $K' \subset (R^x)'$. As in the proof of Proposition 1.4 we prove that $R^x/H$ is a commutative group, where $H = 1 + J(R)$. Then $(R^x)' \subset H$. By Lemma 1.2, $H$ is a $p$-group of bounded period and $K' \subset (R^x)' \subset H$. So $K'$ is a finitely generated nilpotent group (see [5, p. 153]), and it is a $p$-group. Then $K'$ is a finite $p$-group by Lemma 1.2 of [10, p. 470]. Let $P = K'$ and let $S = C[ P ]$. By Lemma 1.7, $S$ is a completely primary artinian ring which is a homomorphic image of the group algebra $CP$ and $\overline{P} \cong P$.

Set $G \cong K/P$. Then $G$ is a finitely generated abelian group and then there exist $g_1, ..., g_r \in G$ so that $G = \langle g_1 \rangle \times \cdots \times \langle g_r \rangle$.

Let $\{ \overline{i}_g \mid g \in G \}$ be a left transversal for $P$ in $K$, so that $\overline{i}_1 = 1$. Then $R = \sum_{g \in G} \overline{i}_g S$ and since $P$ is a normal subgroup of $K$ we set $g\sigma \in \text{Aut } S$, the inner automorphism induced by $\overline{i}_x$. Hence $\overline{i}_x^{-1} \overline{i}_g \overline{i}_x = \gamma(g', g'') \in P$ for all $g', g'' \in G$, $\gamma(1, 1) = 1$ and $R$ is a homomorphic image of the ring $S(G, \gamma, \sigma)$.
Since $R$ is finitely generated over $C$, it follows that $R/J(R)$ is a finite-dimensional algebra over the field $C/J(C)$. Then there exist the positive integers $n(i), i = 1, 2, \ldots, r$ and the elements $a_{ij} \in S$ so that

\[ i_{g_i}^{n(i)} + i_{g_i}^{n(i) - 1}a_{1i} + \cdots + a_{in(i)} = 0 \quad \text{in } R, \]

because $J(R)$ is a nilpotent ideal. Moreover, if the group $R^x$ is nilpotent of class $m - 1$, then $(x_1, \ldots, x_m) = 1$, for all $x_i \in R^x$. It follows that $R$ is a homomorphic image of the ring $S(G, \gamma, \sigma)/I$, where $I$ is a two-sided ideal as in (d). This concludes the proof of this proposition.

**Theorem 2.2.** The following properties of the ring $R$ are equivalent:

(a) $R$ is a right artinian ring finitely generated over its center and with $R_0$ a finite ring, so that every block of $R$ is an artinian ring as in Lemma 1.1 and so that $R^x$ is a nilpotent group.

(b) $R$ is a direct sum of the rings $R_1, R_2, \ldots, R_n$, where every $R_i$ is a homomorphic image of a crossed product as in Proposition 2.1.

**Proof.** Let $R$ be as in (a). Since every right artinian ring is a direct sum of blocks, we can limit our attention to rings which contain no non-trivial blocks. By Proposition 2.1, (b) holds.

Conversely, let us assume that $R$ is as in (b). Since a direct product of a finite number of nilpotent groups is a nilpotent group, we can assume that $R$ is a homomorphic image of a crossed product as in Proposition 2.1. We proceed in a series of steps.

**Step 1.** $R$ is a right artinian ring.

Since $t_{g'g^"} = t_g t_{g'} \gamma(g', g')^{-1}$ in $S(G, \gamma, \sigma)$, for all $g', g'' \in G$, by conditions of Proposition 2.1 it follows that $R$ is finitely generated over a homomorphic image of $S$. Then because $S$ is a right artinian ring we conclude that $R$ is a right artinian ring.

**Step 2.** The set $K = \{ t_g a_g \mid g \in G; a_g \in \bar{P} \}$ is a nilpotent subgroup of $R^x$, where $t_g a_g$ denotes the image of the element $t_g a_g$ in $R$.

Since $t_g a_g t_g a_g = t_g \gamma(g', g'') a_g^\sigma a_g$ and since $\gamma(g', g'') \in \bar{P}$ and $\mathbb{P}^{x, \sigma} \subseteq \bar{P}$, for all $g', g'' \in G$, it follows that $t_g a_g t_g a_g \in K$. Moreover

\[ (t_g a_g)^{-1} = a_g^{-1} t_g^{-1} = a_g^{-1} \gamma(g^{-1}, g)^{-1} t_g^{-1} = t_g^{-1} \gamma(g^{-1}, g)^{-1} a_g^{-1} \in K. \]

Thus $K$ is a subgroup of $R^x$.

At this point we observe that every commutator $(z'_1, z'_2, \ldots, z'_k) = u_1 u_2 \cdots u_k$, where $z'_1, z'_2, \ldots, z'_k \in K$, $u_i = (\tilde{z}_1, \ldots, \tilde{z}_{k(i)})$ or $u_i = (\tilde{z}_1, \ldots, \tilde{z}_{k(i)})^{-1}$ with $k(i) \geq k$ and $\tilde{z}_1, \ldots, \tilde{z}_{k(i)} \in \{ t_{g_1}, \ldots, t_{g_k} \} \cup \{ a \mid a \in \bar{P} \}$. The proof will be by induction on $k$, the statement being immediate for $k = 1$. Assume the statement true for $k - 1$. Since $(z'_1, \ldots, z'_{k - 1}, z'_k) = ((z'_1, \ldots, z'_{k - 1}), z'_k)$ and since $(z'_1 z'_2, z'_3) = ((z'_2, z'_3), z'_1)^{-1}(z'_2, z'_3)(z'_1, z'_3)$ and $(z'_1, z'_2 z'_3) = (z'_2, z'_3)^{-1}(z'_2, z'_3)(z'_1, z'_3)$.
(z', z')(z', z')⁻¹(z', z')⁻¹) this statement follows from condition (b) of Proposition 2.1.

Now, by condition (d) of Proposition 2.1, it follows that K is a nilpotent subgroup of $R^x$.

Step 3. $R^x$ is a nilpotent group.

The proof of this step is similar to that of [7], for group algebras. Notice that the two-sided ideal $V = \sum_{h \in P} (1 - h)R$ is a nilpotent ideal. This statement is a consequence of the proof of Lemma 1.7 and of conditions (a) and (c) of Proposition 2.1.

We recall that K is a nilpotent group, by Step 2.

Then let

$$K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_{f-1} \supseteq K_f = 1$$

be the lower central series of K. If $V^{r+1} = 0$ and $V^r \neq 0$, then let $d$ be a positive integer so that $d \leq e$. We set $M = \{(\alpha, \beta) | \alpha, \beta$ are nonnegative integers; $0 \leq \alpha \leq r+1; \alpha \leq \beta \leq (f-1)\alpha \}$. Then $M$ is a finite set. We introduce the lexicographic ordering in this set: that is, we say that $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$ if $\alpha_1 < \alpha_2$ or if $\alpha_1 = \alpha_2$ and $\beta_1 \leq \beta_2$.

Let $v \in V^d$, and then $v = v_1 + \cdots + v_k$, where $v_i = c_i(h_i^{(1)} - 1) \cdots (h_i^{(d)} - 1)w_i$ with $d(i) \geq d$, $c_i \in C$, $h_i^{(d)} \in P$ and $w_i \in K$. We set $\mu(h - 1) = f'$ if and only if $h \in K_f$ and $h \notin K_{f+1}$. Then we define $\mu(v_1) = \mu(h_1^{(d)} - 1) + \cdots + \mu(h_i^{(d)} - 1)$ and $\mu(v_1 + \cdots + v_k) = \min_{1 \leq i \leq k} \mu(v_i)$.

Define map $\rho: R \to M$ by setting $\rho(0) = (e+1, e+1)$; $\rho(v) = (0, 0)$ if $v \notin V$ and $\rho(v) = (d, \beta(v))$ if $v \in V^d$ and $v \notin V^{d+1}$. Here $\beta(v)$ is the maximum of the integers $\mu(v_1 + \cdots + v_k)$, for all expressions of $v$ of type $v = v_1 + \cdots + v_k$.

Note that $\rho(vu) = \rho(v)$ for all $u \in R^x$ and if $\mu(v_1 + \cdots + v_k) > (f-1)d$, then $v = v_1 + \cdots + v_k \in V^{d+1}$. Since $M$ is a finite set, we conclude that it will be sufficient to show that

$$\rho(u_1 - 1) < \rho((u_1, u_2) - 1) \quad \text{for all } u_1, u_2 \in R^x.$$
But \((w_1, h_i) h_i - 1 = (w_1, h_i)(h_i - 1) + ((w_1, h_i)(h_i - 1))\) and \(\forall w \in K_1\) it follows that \(w = h\) with \(h \in \mathcal{P}\). Then we obtain that every summand \(v_i\) of \([v, w_1]\) is of type \(v(h - 1)\) or \(\mu(v_i) > \mu(v_{ij})\), where \(v_{ij}\) is a summand of \(vw^{-1} w_1 w\). Since \(\mu(v) = \mu(vw^{-1} w_1 w)\), it follows that \(\mu([v, w_1]) > \mu(v)\), for all \(w_1 \in K\). We now find a particular element \(v = v_i\) and a particular element \(w_1 \in K\) so that \(\mu([v_1 + \cdots + v_k, u_2]) > \mu([v, w_1])\) and \(\mu(v_1 + \cdots + v_k)\). Then \(\rho([u_1 - 1, u_2]) > \rho(u_1 - 1)\) and \(R^x\) is a nilpotent group.

Step 4. \(R\) is finitely generated over its center and every block of \(R\) is as in Lemma 1.1.

It is sufficient to note that \(R\) is finitely generated over the homomorphic image of \(\bar{C}\) by the units of type \(t^{(1)}_{1} \cdots t^{(r)}_{g} w\), where \(0 \leq g(i) \leq n(i)\) and \(w \in \mathcal{P}\).

This completes the proof of the theorem.

**Proposition 2.3.** Let \(R\) be a right artinian ring finitely generated over its center \(C\), let \(R\) contain no non-trivial blocks so that \(R\) is as in Lemma 1.1. If \(R_0\) is a finite ring and \(R/J(R)\) is a commutative ring, then \(R\) is a homomorphic image of the ring \(S(G, \gamma, \sigma)/I\), so that the following four conditions hold.

(a) \(S\) is a completely primary artinian ring and there exists a finite \(p\)-group \(P\), where \(p\) is a prime number, so that \(S \cong CP/L\). Here \(L\) is a two-sided ideal of the group algebra \(CP\) of the \(p\)-group \(P\) over the local commutative artinian ring \(C\) of characteristic a power of \(p\). Moreover \(P^{g_0} \subset \mathcal{P}\) for all \(g \in G\), where \(\mathcal{P}\) is the image of \(P\) in \(S\).

(b) \(G\) is a direct product \(G = \langle g_1 \rangle \times \cdots \times \langle g_r \rangle\) of non-trivial cyclic groups.

(c) \(\gamma(1, 1) = 1\) and \(\gamma(g', g'') \in 1 + J(S)\) for all \(g', g'' \in G\).

(d) \(I\) is a finitely generated two-sided ideal so that \(I = \sum_{i=1}^{t} S(G, \gamma, \sigma) x_i S(G, \gamma, \sigma)\). Here every \(x_i\) is of type \(t^{n_{i}^{(i)}}_{g_{i}^{(i)}} + t^{n_{i}^{(i) - 1}}_{g_{i}^{(i)}} a_{i1} + \cdots + a_{im(i)}\), where \(a_{ij} \in S\) and \(n(i)\) are positive integers.

**Proof.** From the argument given in the first paragraph of the proof of Proposition 2.1, it follows that \(C\) is a local commutative artinian ring of characteristic a power of a primo \(p\). Then \(J(R)\) is finitely generated over \(C\) by the elements \(y_1, \ldots, y_m\). Let \(P\) be the subgroup of \(R^x\) generated by the units \(1 + y_i\), for \(i = 1, 2, \ldots, m\). Since \(H = 1 + J(R)\) is a nilpotent group (see [11, p. 1134]), it follows from Lemma 1.2 and from Lemma 1.2 of [10, p. 170] that \(P\) is a finite \(p\)-group. Let \(K\) be as in the proof of Proposition 2.1. Then we set \(G \cong KH/H\). Because \(K\) is a finitely generated multiplicative group and because \(R/J(R)\) is a commutative ring, it follows
that \( G = \langle g_1 \rangle \times \cdots \times \langle g_r \rangle \). Let \( S = C[P] \) and let \( \{ \hat{i}_g \mid g \in G \} \) be a left transversal for \( H \) in \( K \), so that \( \hat{i}_1 = 1 \).

Since every element \( g \in H \) belongs to \( 1 + J(S) \) it follows that \( \hat{i}_g^{-1} \hat{i}_{g'} \hat{i}_g = \gamma(g', g'') \hat{i}_g \) for all \( g', g'' \in K \) and \( R \) is a homomorphic image of the ring \( S(G, \gamma, \sigma) \). Now the proof is similar to that of Proposition 2.1.

**Theorem 2.4.** The following properties of the ring \( R \) are equivalent:

(a) \( R \) is a right artinian ring which is finitely generated over its center, with \( R_0 \) a finite ring, so that every block of \( R \) is an artinian ring as in Lemma 1.1 and so that \( R/J(R) \) is a commutative ring.

(b) \( R \) is a direct sum of the rings \( R_1, R_2, \ldots, R_n \), where every \( R_i \) is a homomorphic image of a crossed product as in Proposition 2.3.

**Proof.** Let \( R \) be as in (b). Then we can assume that \( R \) is a homomorphic image of a crossed product as in Proposition 2.3. From the arguments given in Steps 1 and 4 of the proof of Theorem 2.2, it follows that \( R \) is a right artinian ring, finitely generated over its center, so that every block of \( R \) is an artinian ring as in Lemma 1.1. Since the two-sided ideal \( V = \sum_{h \in F} (1 - h)R \) is a nilpotent ideal, it follows that \( V \subseteq J(R) \). Hence \( R/J(R) \) is a commutative ring. Now the proof is similar to that of Theorem 2.2.

**References**