Irreducibility of Unitary Group Representations and Reproducing Kernels Hilbert Spaces*

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Appendix on two point homogeneous compact ultrametric spaces in collaboration with
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Abstract: We discuss unitary representations of groups in Hilbert spaces of functions
given together with reproducing kernels, and in particular irreducibility. Our main focus is
on examples, including spherical representations of orthogonal groups, distance-transitive
finite graphs, irreducibility of induced representations, the discrete series of $SL(2, \mathbb{R})$, and
some representations of $PGL(2, \mathbb{Q}_p)$.
The appendix contains examples involving groups acting on rooted trees.

Keywords: Irreducible representations, unitary representations, reproducing kernels.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space of functions on a set $X$; the scalar product of vectors
$\psi, \chi \in \mathcal{H}$ is denoted by $\langle \psi | \chi \rangle$, is linear in $\psi$ and antilinear in $\chi$. For each $x \in X$,
we assume that the evaluation $\mathcal{H} \ni \psi \mapsto \psi(x) \in \mathbb{C}$ is a continuous linear form;
consequently, there exists a unique function $\phi_x \in \mathcal{H}$ such that $\psi(x) = \langle \psi | \phi_x \rangle$.
The kernel $\Phi : X \times X \rightarrow \mathbb{C}$ defined by $\Phi(x, y) = \langle \phi_y | \phi_x \rangle = \phi_y(x)$ is then reproducing in
the sense that
$$\psi(x) = \langle \psi(\cdot) | \Phi(\cdot, x) \rangle \quad \text{for all} \quad \psi \in \mathcal{H} \quad \text{and} \quad x \in X$$
(about choosing $\langle \phi_y | \phi_x \rangle$ rather than $\langle \phi_x | \phi_y \rangle$, see the footnote in Example 2).

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In case $\mathcal{H}$ is a closed subspace of genuine functions in the Hilbert space $L^2(X, \mu)$, for some measure $\mu$ on $X$, the latter formula takes the form

$$\psi(x) = \int_X \Phi(x, y)\psi(y)d\mu(y) \quad \text{for all} \quad \psi \in \mathcal{H} \quad \text{and} \quad x \in X,$$

possibly more familiar (we have used the identity $\Phi(x, y) = \overline{\Phi(y, x)}$, see below); we insist that any $\psi$ in such a space is a genuine function on $X$, not just a function class modulo equality almost everywhere, so that $\mathcal{H}$ is also a subspace of the vector space denoted by $L^2(X, \mu)$ in [Bourb–65].

Kernels of this type are classical objects of study. There are several foundational articles on them which seem to be independent of each other, in particular by N. Aronszajn [Arons–44], [Arons–50] and M.G. Krein [Krein–49], [Krein–50]. But Aronszajn refers to older work [Moor–16+] and Krein builds on previous work by E. Cartan [Carta–29] and H. Weyl [Weyl–34]. In the context of complex analysis, work of Bergman has to be mentioned ([Bergm–33], [Bergm–50]). Moreover, reproducing kernels can be viewed as kernels of positive type (with a slightly different emphasis in the point of view, as explained in detail in the introduction of [Arons–50]). The early history of the subject should therefore include many other contributions including Mercer [Mercer–09], Schur [Schur–11], Schoenberg [Schoe–38], [Schoe–42], Kolmogorov [Kolmo–41] and Godement [Godem–48]. The theory has been useful in domains including complex analysis, partial differential equations, dilation of linear operators, and stochastic processes (see [Alpay–98], [Chat–83a], [Chat–83b], [Hille–72], [Mesch–62]), as well as group representations (see [Carey–77], [Carey–78], [Kunze–67], and [Ville–68]).

There is a rather small number of methods for showing the irreducibility of a representation $\pi$ of a group $G$ in a space $\mathcal{H}$. One is first to decompose $\mathcal{H}$ in a direct sum of pairwise inequivalent irreducible subspaces with respect to a subgroup of $G$, and then to check that appropriate elements of $G$ interchange these subspaces; a most elementary example is that of a two-dimensional irreducible representation of a finite dihedral group. Another very standard method involves Schur’s lemma, and the proof that the commutant of $\pi(G)$ in the ring of continuous linear endomorphisms of $\mathcal{H}$ contains nothing more than the scalar multiples of the identity.

The purpose of this expositional paper is to advertise the method of reproducing kernel Hilbert spaces: for such a space $\mathcal{H}$ given together with a unitary representation $\pi$ of a group $G$ related to an appropriate action of $G$ on $X$, it is enough to check that the space of $K$-invariant functions in $\mathcal{H}$ is one-dimensional, where $K$ in $G$ is the isotropy group of some point in $X$ (Proposition 2).

Section 2, on Hilbert spaces of functions, is reduced to the definition, one proposition and two standard examples: homogeneous polynomials in $n$ variables, and Bergman spaces. Group representations appear in Section 3, where some of the most simple cases of the method are exposed, and where it is shown how any cyclic representation [respectively any representation of the discrete series] can be viewed as operating on a Hilbert space of functions [respectively a subspace of $L^2(G)$]. After a short Section 4 with further general properties, Section 5 shows for representations of class one how reproducing kernels can be expressed in terms of zonal spherical functions. Then, these basic ideas are shown to extend to cases involving cocycles in Section 6 and vector-valued functions in Section 7.
In particular, we show that the approach via reproducing kernels Hilbert spaces is relevant for

- spherical representations of orthogonal groups $SO(n)$ and $O(n)$, and of unitary groups $SU(n)$ (Examples 3 and 4);
- providing a general setting for unitary representations, in particular those in the discrete series (Examples 5 and 6);
- finite graphs which are distance-transitive for some finite group of automorphisms, and in particular some representations of symmetric groups $\text{Sym}(n)$ and related groups, as well as other finite metric spaces (Examples 7 and 8, and the appendix);
- proving results of Godement and Mackey on the irreducibility of induced representations (Example 10);
- proving irreducibility for representations of $SL(2, \mathbb{R})$ in the holomorphic discrete series (Example 11);
- proving irreducibility for some cuspidal representations of $PGL(2, \mathbb{Q}_p)$ (Example 12).

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2. Hilbert space of functions

We consider as in the introduction a set $X$ and a Hilbert space of functions $\mathcal{H}$ on $X$, such that each evaluation $\mathcal{H} \ni \psi \mapsto \psi(x) \in \mathbb{C}$ is a continuous linear form, the functions $\phi_x \in \mathcal{H}$ such that $\psi(x) = \langle \psi \mid \phi_x \rangle$, and the corresponding reproducing kernel $\Phi : X \times X \ni (x, y) \mapsto \langle \phi_y \mid \phi_x \rangle \in \mathbb{C}$.

For $y \in X$, the function $\phi_y \in \mathcal{H}$ will also be denoted by $\Phi(\cdot, y)$.

1. Proposition. Let the notation be as above.

   (i) The kernel $\Phi$ is of positive type. In particular, its diagonal values $\Phi(x, x)$ are real positive, and $|\Phi(x, y)|^2 \leq \Phi(x, x)\Phi(y, y)$ for all $x, y \in X$.
   (ii) The family $(\phi_x)_{x \in X}$ generates $\mathcal{H}$.
   (iii) If $\mathcal{H} \neq 0$, then $\Phi(x, x) \neq 0$ for some $x \in X$.
   (iv) If moreover $\mathcal{H}$ is of the form $\mathcal{H}_\mathbb{R} \otimes \mathbb{R} \mathbb{C}$ where $\mathcal{H}_\mathbb{R}$ is a real Hilbert space of real-valued functions on $X$, then $\Phi(x, y) \in \mathbb{R}$ for all $x, y \in X$.

Proof. (i) By definition, the kernel $\Phi$ is of positive type if

\[ (*) \quad \sum_{j,k=1}^n \lambda_j \lambda_k \Phi(x_j, x_k) \geq 0 \]

for all integers $n$, complex numbers $\lambda_1, \ldots, \lambda_n$ and points $x_1, \ldots, x_n$ in $X$. This is clear here, since the left-hand term $\sum_{j,k=1}^n \lambda_j \lambda_k \langle \phi_{x_k} \mid \phi_{x_j} \rangle$ of (*) is equal to the square of the Hilbert-space norm of the sum $\sum_{k=1}^n \lambda_k \phi_{x_k}$. The last claim of (i) follows by the Cauchy-Schwarz inequality.

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\[ 1 \] This condition is of course automatic when the dimension of $\mathcal{H}$ is finite.
(ii) Observe that, for $\psi \in \mathcal{H}$, the condition $\langle \psi \mid \phi_x \rangle = 0$ for all $x \in X$ implies that $\psi(x) = 0$ for all $x \in X$.

(iii) Assume that $\Phi(x, x) = 0$ for all $x \in X$. Then, by (i), $\Phi(x, y) = 0$ for all $x, y \in X$, and it follows from (ii) that $\mathcal{H} = 0$.

Claim (iv) is obvious. \[\square\]

The conclusions of Proposition 1 are readily verified in the simplest cases. For example in the case of a one-dimensional Hilbert space generated by one complex-valued function $\psi$ on $X$ of Hilbert norm 1, for which the corresponding kernel is given by $\Phi(x, y) = \psi(x)\overline{\psi(y)}$. Another example is the Sobolev space $\mathcal{H}^1(\mathbb{R}_+)$ of those absolutely continuous functions $\psi : \mathbb{R}_+ \to \mathbb{C}$ such that $\psi(0) = 0$ and $\int_0^\infty |\psi'(x)|^2 \, dx < \infty$, corresponding to functions $\phi_x$ defined by $\phi_x(t) = \min\{t, x\}$, and therefore to the kernel defined by $\Phi(x, y) = \min\{x, y\}$. There is an abundance of classical examples related to orthogonal polynomials; see e.g. [Szegö-75], in particular pages 38–44.

Here are two other standard examples.

**Example 1.** A reproducing Hilbert space of homogeneous polynomial functions.

Consider integers $n \geq 1$, $k \geq 0$, and the space $\mathcal{P}^{(k)}(\mathbb{R}^n)$ of complex polynomial functions $\mathbb{R}^n \to \mathbb{C}$ which are homogeneous of degree $k$. For any $\alpha \in \mathbb{R}^n$, define $\phi^{k}_\alpha \in \mathcal{P}^{(k)}(\mathbb{R}^n)$ by

$$
\phi^{k}_\alpha(x) = \langle \alpha \mid x \rangle^k = \sum_{|j|=k} \frac{k!}{j!} \alpha^j x^j,
$$

where $\langle \alpha \mid x \rangle$ denotes the canonical scalar product of the vectors $\alpha$ and $x$ in $\mathbb{R}^n$. (We use standard notation for a multi-index $j \in \mathbb{N}^n$, namely $|j| = j_1 + \cdots + j_n$, and $j! = j_1! \cdots j_n!$, and $\alpha^j = \alpha_1^{j_1} \cdots \alpha_n^{j_n}$ if $\alpha \in \mathbb{R}^n$. Also, our set $\mathbb{N}$ of natural numbers is frenchlike: it contains 0.)

Let $\psi, \chi \in \mathcal{P}^{(k)}(\mathbb{R}^n)$; write $\psi(x) = \sum_{|j|=k} \psi_j x^j$ and define

$$
\psi \left( \frac{\partial}{\partial x} \right) = \sum_{|j|=k} \psi_j \frac{\partial |j|}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}};
$$

write similarly $\chi(x) = \sum_{|j|=k} \chi_j x^j$. Observe that $\psi \left( \frac{\partial}{\partial x} \right) \chi(x)$ is a constant function and define a scalar product on $\mathcal{P}^{(k)}(\mathbb{R}^n)$ by

$$
\langle \psi \mid \chi \rangle = \psi \left( \frac{\partial}{\partial x} \right) \chi(x) = \sum_{|j|=k} j! \psi_j \chi_j
$$

(the so-called “Fischer scalar product”). Then

$$
\langle \phi^{k}_\alpha \mid \chi \rangle = k! \sum_{|j|=k} \alpha^j \chi_j = k! \chi(\overline{\alpha})
$$

for all $\alpha \in \mathbb{R}^n$.

It follows that $\mathcal{P}^{(k)}(\mathbb{R}^n)$ is a Hilbert space of functions on $\mathbb{R}^n$ with reproducing kernel

$$
\mathbb{R}^n \times \mathbb{R}^n \ni (\alpha, x) \mapsto \frac{1}{k!} \phi^{k}_\alpha(x) = \frac{1}{k!} \langle \alpha \mid x \rangle^k \in \mathbb{R}.
$$
Example 2. **Bergman kernels.**

Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and let \( \mathcal{H} \) be the space of those holomorphic functions \( \psi \) on \( \Omega \) such that \( \int_{\Omega} |\psi(z)|^2 \, dx_1 dy_1 \cdots dx_n dy_n < \infty \). The corresponding reproducing kernel is the **Bergman kernel** \( \Phi : \Omega \times \Omega \rightarrow \mathbb{C} \) of \( \Omega \), which is holomorphic in the first variable and antiholomorphic in the second variable\(^2\).

For example, if \( \Omega \) is a simply connected bounded domain in \( \mathbb{C} \), its Bergman kernel is related to uniformizing mappings of \( \Omega \) onto the open unit disc \( \mathbb{D}_1 \). More precisely, if \( f : \Omega \rightarrow \mathbb{D}_1 \) is a holomorphic diffeomorphism, if \( w = f^{-1}(0) \in \Omega \), and if \( f'(w) > 0 \), then \( f'(z) = \frac{\pi}{\Phi(w,w)} \); see e.g. Section V.10 in [Nehar–52]. In the special case of \( \Omega = \mathbb{D}_1 \), we have \( \Phi(z, w) = \frac{1}{\pi} \frac{1}{1 - \overline{w}z} \).

If a relatively compact domain \( \Omega \subset \mathbb{C}^n \) has Bergman kernel \( \Phi \), the **Bergman metric** on \( \Omega \) is the Hermitian metric defined by \( g_{j,k}(z) = \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \ln \Phi(z, z) \). Bergman kernels are often useful in the theory and rarely computable explicitly. Some of them are known, however. For example, the Bergman kernel of the open unit ball in \( \mathbb{C}^n \) is given by

\[
\Phi(z, w) = \frac{n!}{\pi^n} \frac{1}{(1 - \overline{w}_1 z_1 - \cdots - \overline{w}_n z_n)^{n+1}}
\]

and that of the polydisc \( \{ z \in \mathbb{C}^n \mid |z_1| < 1, \ldots, |z_n| < 1 \} \) by

\[
\Phi(z, w) = \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{1}{(1 - \overline{w}_j z_j)^2}
\]

(see e.g. [Krant–82]).

3. **Unitary group representations on \( G \)-Hilbert spaces of functions**

Let \( \mathcal{H} \) be, as in Section 2, a Hilbert space of functions on a set \( X \), and let moreover \( G \) be a group acting on \( X \). There is a canonical action \( \pi \) of \( G \) on the space \( \mathbb{C}^X \) of functions on \( X \), according to the formula

\[
(\pi(g)\psi)(y) = \psi(g^{-1}y),
\]

where \( g \in G, y \in X, \) and \( \psi : X \rightarrow \mathbb{C} \). Define a **\( G \)-Hilbert space of functions on \( X \)** to be a Hilbert space of functions as above, say \( \mathcal{H} \), such that \( \pi(g)\psi \in \mathcal{H} \) and \( \|\pi(g)\psi\| = \|\psi\| \) for all \( g \in G \) and \( \psi \in \mathcal{H} \), namely such that \( \pi \) is a unitary representation of \( G \) in \( \mathcal{H} \). For a subgroup \( K \) of \( G \), denote by \( \mathcal{H}^K \) the subspace of \( \mathcal{H} \) consisting of \( K \)-invariant functions, that is of functions \( \psi \) such that \( \pi(k)\psi = \psi \) for all \( k \in K \).

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\(^2\)This is because reproducing kernels are defined by \( \Phi(x, y) = \langle \phi_y | \phi_x \rangle = \phi_y(x) \). The opposite choice \( \Phi(x, y) = \langle \phi_x | \phi_y \rangle = \phi_x(y) \) would give rise to Bergman kernels \( \Phi(x, w) \) antiholomorphic in the first variable, whereas Hilbert space scalar products \( \langle \xi | \eta \rangle \) are here antilinear in the second variable.
2. Proposition. With the notation above,

(i) \( \pi(g)\phi_x = \phi_{gx} \) and \( \Phi(gx, gy) = \Phi(x, y) \) for all \( g \in G \) and \( x, y \in X \).

Assume moreover that the action of \( G \) on \( X \) is transitive; let \( \omega \) be some point in \( X \) and let \( K \) be its isotropy group \( \{ k \in G \mid kw = \omega \} \).

(ii) If \( \mathcal{H} \neq \{0\} \), then \( \mathcal{H}^K \neq \{0\} \).

(iii) If \( \dim_C(\mathcal{H}^K) = 1 \), then the representation \( \pi \) of \( G \) in \( \mathcal{H} \) is irreducible.

Proof. (i) For \( g \in G \) and \( x \in X \), we have by unitarity of \( \pi \)

\[ \langle \psi | \pi(g)\phi_x \rangle = \langle \pi(g^{-1}) \psi | \phi_x \rangle = \psi(gx) = \langle \psi | \phi_{gx} \rangle \]

for all \( \psi \in \mathcal{H} \), so that \( \pi(g)\phi_x = \phi_{gx} \). Also \( \Phi(gx, gy) = \langle \phi_{gy} | \phi_{gx} \rangle = \langle \phi_y | \phi_x \rangle = \Phi(x, y) \).

(ii) If \( G \) acts transitively on \( X \), it follows that there exists a constant \( d \geq 0 \) such that \( \Phi(x, x) = d \) for all \( x \in X \). Moreover, if \( \mathcal{H} \neq \{0\} \), then \( d \neq 0 \) by Proposition 1.iii. In particular, the function \( X \ni x \mapsto \Phi(x, \omega) \in \mathbb{C} \) is not zero; as this function is clearly \( K \)-invariant, this shows that \( \mathcal{H}^K \neq \{0\} \).

(iii) Let now \( \mathcal{H}_0 \) be a non-zero \( G \)-invariant closed subspace of \( \mathcal{H} \), and denote by \( \Phi_0 \) its reproducing kernel. The function \( X \ni x \mapsto \Phi_0(x, \omega) \in \mathbb{C} \) is in \( \mathcal{H}_0^K \), and a fortiori in \( \mathcal{H}^K \). If \( \dim_C(\mathcal{H}^K) = 1 \), there exists a constant \( c \) such that \( \Phi_0(x, \omega) = c\Phi(x, \omega) \) for all \( x \in X \), indeed (using the invariance of \( \Phi \) and the transitivity of \( G \) on \( X \)) such that \( \Phi_0(x, y) = c\Phi(x, y) \) for all \( x, y \in X \). Choose now \( \chi \in \mathcal{H}_0, \chi \neq 0 \), and \( y \in X \) such that \( \chi(y) \neq 0 \). Then

\[ \chi(y) = \langle \chi(\cdot) | \Phi_0(\cdot, y) \rangle = \overline{c} \chi(\cdot, y) = \overline{c} \chi(y) \]

so that \( c = 1 \) and \( \Phi_0 = \Phi \). Proposition 1.ii shows now that \( \mathcal{H}_0 = \mathcal{H} \), namely that \( \mathcal{H} \) is irreducible as a \( G \)-space. \( \square \)

Remarks. (i) An irreducible representation \( \pi : G \rightarrow U(\mathcal{H}) \) is said to be of class one with respect to a subgroup \( K \) if \( \dim_C(\mathcal{H}^K) = 1 \).

(ii) In Proposition 2, there is no converse to (iii). Indeed, consider a finite group \( G \neq \{e\} \), a subgroup \( A \), a character \( \sigma : A \rightarrow \{z \in \mathbb{C} \mid |z| = 1\} \), and the Hilbert space \( \mathcal{H} \) of functions \( \xi : G \rightarrow \mathbb{C}^* \) such that \( \xi(xa) = \xi(x)\sigma(a) \) for all \( x \in G \) and \( a \in A \). The scalar product in \( \mathcal{H} \) is given by \( \langle \xi | \eta \rangle = \sum_{t \in G/A} \xi(t)\overline{\eta(t)} \) where \( \sum_{t \in G/A} \) indicates a summation over a set of representatives of \( G/A \) in \( G \), and \( \pi \) is the representation induced by \( \sigma \) from \( A \) to \( G \). We have

\[ \phi_x(z) = \begin{cases} \sigma(x^{-1}z) & \text{if } x^{-1}z \in A \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \langle \phi_x | \phi_y \rangle = \begin{cases} \sigma(x^{-1}y) & \text{if } x^{-1}y \in A \\ 0 & \text{otherwise} \end{cases} \]

For any choice of \( \omega \) in \( G \), the isotropy group \( K \) is reduced to \( \{1\} \) and \( \mathcal{H}^K = \mathcal{H} \); in particular, \( \dim_C(\mathcal{H}^K) > 1 \) as soon as \( A \) is a proper subgroup of \( G \). There are standard cases for which \( \pi \) is irreducible, for example that with \( G \) a non-abelian group of order 6, with \( A \) a subgroup of order 3, and with \( \sigma \) a character of \( A \) distinct from the unit character.
(iii) Let $G$ be a locally compact group and $K$ a compact subgroup. Assume that $(G,K)$ is a Gelfand pair, namely that the convolution algebra of $K$-biinvariant continuous functions with compact support on $G$ is commutative. Assume also that $\mathcal{H}^K$ is cyclic for $\pi$, namely that the closed linear span of $\pi(G) (\mathcal{H}^K)$ is the whole of $\mathcal{H}$. Then the converse of (iii) holds: the irreducibility of $\pi$ implies the equality $\dim_C (\mathcal{H}^K) = 1$; see Theorem 2 in Chapter IV of [Lang-75]. For an introduction to Gelfand pairs, see [Farau-80] or [Dieud-75]; the terminology refers to a paper by Gelfand [Gelfa-50].


Notation being as in Example 1, assume now that $n \geq 2$ and denote by

$$\mathcal{H}^{(k)}(\mathbb{R}^n) = \left\{ \psi \in \mathcal{P}^{(k)}(\mathbb{R}^n) \mid \Delta \psi = 0 \right\}$$

the space of harmonic homogeneous polynomials of degree $k$, where $\Delta$ denotes the usual Laplacian $\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. We identify any function $\psi \in \mathcal{H}^{(k)}(\mathbb{R}^n)$ with its restriction to the unit sphere $S^{n-1} \subset \mathbb{R}^n$; we define a scalar product on $L^2(S^{n-1}, \mu)$, and in particular on $\mathcal{H}^{(k)}(\mathbb{R}^n)$, by

$$\langle \psi \mid \chi \rangle = \int_{S^{n-1}} \psi(x) \overline{\chi(x)} d\mu(x)$$

where $\mu$ denotes the probability measure on the sphere which is invariant by the group $O(n)$ of orthogonal transformations of $\mathbb{R}^n$. (The scalar products $[\psi \mid \chi]$ and $\langle \psi \mid \chi \rangle$ should not be confused. For example, if $n = 2$, $\psi(x) = x_1^2$ and $\chi(x) = x_2^2$, observe that $[\psi \mid \chi] = 0 \neq \langle \psi \mid \chi \rangle$.)

The space $\mathcal{H}^{(k)}(\mathbb{R}^n)$ is a $O(n)$-Hilbert space of functions on $S^{n-1}$. With reference to the notation of Proposition 2, we consider now the north pole $\omega = (0, \ldots, 0, 1)$ in $S^{n-1}$ and its isotropy group $K = O(n-1)$. It can be shown that

- elements in $\mathcal{P}^{(k)}(\mathbb{R}^n)$ which are invariant by $K$ are functions of the form $x \mapsto \sum_{j=0}^{[k/2]} (-1)^j c_j \langle x \mid x \rangle^j \langle x \mid \omega \rangle^{k-2j}$ where $c_0, c_1, \ldots$ are constants,
- such a function is harmonic if and only if some linear recurrence holds for the $c_j$, so that $\dim_C (\mathcal{H}^{(k)}(\mathbb{R}^n)^K) = 1$,
- the representation of $O(n)$ in $\mathcal{H}^{(k)}(\mathbb{R}^n)$ is consequently irreducible.

There exists a polynomial of degree $k$ in one indeterminate

$$P^{(k)}(T) = \sum_{j=0}^{[k/2]} (-1)^j c_j^{(k)} T^{k-2j}$$

with $c_0^{(k)}, c_1^{(k)}, \ldots > 0$, which is some form of a Gegenbauer polynomial (see e.g. § IX.3 in [Vilen-68]), such that the reproducing kernel $\Phi^{(k)}$ of $\mathcal{H}^{(k)}(\mathbb{R}^n)$ is given by

$$\Phi^{(k)}(x, y) = \sum_{j=0}^{[k/2]} (-1)^j c_j^{(k)} \langle x \mid y \rangle^{k-2j}$$
for all \( x, y \in S^{n-1} \) and such that the unique \( \psi \in \mathcal{H}^{(k)}(\mathbb{R}^n)^K \) with \( \psi(\omega) = \dim_{\mathbb{C}} \mathcal{H}^{(k)}(\mathbb{R}^n) \) is given by

\[
\psi(x) = \sum_{j=0}^{[k/2]} (-1)^j c_j^{(k)}(x) \langle x \mid x \rangle^j (x \mid \omega)^{k-2j}
\]

for all \( x \in \mathbb{R}^n \) (in particular \( \psi(x) = \Phi^{(k)}(x, \omega) \) for all \( x \in S^{n-1} \)).

The Laplacian \( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \) has an expression in spherical coordinates of the form

\[
\frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} + \Delta_S
\]

where \( \Delta_S \) denotes the Laplacian on the sphere \( S^{n-1} \). Then

\[
\Delta_S \psi = -k(n + k - 2)\psi
\]

for all \( k \geq 0 \) and \( \psi \in \mathcal{H}^{(k)}(\mathbb{R}^n) \), and we have an orthogonal sum \( \bigoplus_{k=0}^{\infty} \mathcal{H}^{(k)}(\mathbb{R}^n) = L^2(S^{n-1}, \mu) \). In other words, the subspaces \( \mathcal{H}^{(k)}(\mathbb{R}^n) \) of \( L^2(S^{n-1}, \mu) \) are precisely the eigenspaces of the self-adjoint operator \( \Delta_S \) on \( L^2(S^{n-1}, \mu) \).

Moreover, if \( n \geq 3 \), similar arguments show that the representation of the special orthogonal group \( \text{SO}(n) \) in \( \mathcal{H}^{(k)}(\mathbb{R}^n) \) is irreducible. For all this, see for example [SteWe-71] or [Vilen-68]; for the action of \( O(n) \) on the space of all polynomial functions \( \mathbb{R}^n \rightarrow \mathbb{C} \), see also No 5.2.3 in [GooWa-98].

**Example 4. Unitary variation.**

For integers \( n \geq 2 \) and \( p, q \geq 0 \), denote by \( \mathcal{P}^{(p,q)}(\mathbb{C}^n) \) the subspace of \( \mathcal{P}^{(p+q)}(\mathbb{R}^{2n}) \) of complex-valued polynomial functions on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) which are homogeneous of degree \( p \) in \( z_1, \ldots, z_n \) and homogeneous of degree \( q \) in \( \bar{z}_1, \ldots, \bar{z}_n \). Recall that the Laplacian \( \Delta \) is given by \( \frac{1}{4} \left( \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \cdots + \frac{\partial^2}{\partial z_n \partial \bar{z}_n} \right) \); denote by \( \mathcal{H}^{(p,q)}(\mathbb{C}^n) \) the space \( \mathcal{P}^{(p,q)}(\mathbb{C}^n) \cap \mathcal{H}^{(p+q)}(\mathbb{R}^{2n}) \) of harmonic polynomials which are bi-homogeneous of degree \( (p, q) \), and view it again as a subspace of the appropriate space \( L^2(S^{2n-1}, \mu) \) of functions on the sphere; it is easy to check that \( \mathcal{H}^{(k)}(\mathbb{R}^{2n}) \) is the direct sum \( \bigoplus_{p+q=k} \mathcal{H}^{(p,q)}(\mathbb{C}^n) \).

Then \( \mathcal{H}^{(p,q)}(\mathbb{C}^n) \) is a \( U(n) \)-Hilbert space of functions on the \( (2n-1) \)-sphere and the subspace \( \mathcal{H}^{(p,q)}(\mathbb{C}^n)_{U(n-1)} \) is of dimension 1, so that the corresponding unitary representation of \( U(n) \) in \( \mathcal{H}^{(p,q)}(\mathbb{C}^n) \) is irreducible. These representations remain irreducible when they are restricted to \( SU(n) \). Details appear in, for example, Chapter 11 of [VilKl-93] (in particular pages 296–298).

**Example 5. Cyclic representations.**

Let \( G \) be a topological group, let \( \pi : G \rightarrow U(\mathcal{K}) \) be a continuous unitary representation of \( G \) in a Hilbert space \( \mathcal{K} \), and let \( \eta_0 \in \mathcal{K} \). Let \( \imath \) denote the linear mapping from \( \mathcal{K} \) to the space of continuous functions on \( G \) defined by

\[
(\imath(\eta))(x) = c(\eta \mid x) \eta_0 \quad \text{for all} \quad \eta \in \mathcal{K} \quad \text{and} \quad x \in G
\]
where $c > 0$ is an appropriate constant (the reason for the introduction of $c$ appears in Example 6). Let $\mathcal{H}$ denote the image $\iota(\mathcal{K})$. Then $\iota$ is $G$-equivariant, namely

$$\left(\iota(\pi(g)\eta)\right)(x) = \left(\iota(\eta)\right)(g^{-1}x) \quad \text{for all} \quad \eta \in \mathcal{K} \quad \text{and} \quad g, x \in G.$$ 

Assume moreover that the vector $\eta_0$ is cyclic for $\pi$, namely that the closed linear span of $\{\pi(g)\eta_0\}_{g \in G}$ is the whole of $\mathcal{K}$. Then $\iota : \mathcal{K} \rightarrow \mathcal{H}$ is a $G$-equivariant linear isomorphism; it follows that the space $\mathcal{H}$ (a priori just a vector space of continuous functions on $G$) can be given the structure of a Hilbert space isometric to $\mathcal{K}$. In particular, up to equivalence,

any cyclic continuous unitary representation of $G$ occurs in a $G$-Hilbert space of continuous functions on $G$ itself.

**Example 6. The discrete series.**

Let $G$ be a locally compact group and let $\pi : G \rightarrow \mathcal{U}(\mathcal{K})$ be a continuous unitary representation which is irreducible and in the discrete series. For simplicity, let us assume first that $G$ is unimodular; then, $\pi$ in the discrete series means that the functions $g \mapsto \langle \eta | \pi(g)\eta' \rangle$ are in $L^2(G)$ for all $\eta, \eta' \in \mathcal{K}$. Let $d_\pi$ denote the formal dimension of $\pi$, and consider the isomorphism $\iota : \mathcal{K} \rightarrow \mathcal{H}$ of the previous example with $c = \sqrt{d_\pi}$. Then, for $\eta_1, \eta_2 \in \mathcal{K}$, we have

$$\langle \iota(\eta_1) | \iota(\eta_2) \rangle_{L^2(G)} = d_\pi \int_G \langle \eta_1 | \pi(x)\eta_0 \rangle_{\mathcal{K}} \overline{\langle \eta_2 | \pi(x)\eta_0 \rangle_{\mathcal{K}}} \, dx = \langle \eta_1 | \eta_2 \rangle_{\mathcal{K}}$$

by Schur's orthogonality relations. (For these relations as well as for the definition of "formal dimension", see Theorem 14.3.3 in [DixC*-69].) In particular, up to equivalence,

for a unimodular group $G$, any unitary representation in the discrete series occurs in a closed subspace of $L^2(G)$ which is also a $G$-Hilbert space of continuous functions on $G$.

This carries over to a locally compact group $G$ which is not necessarily unimodular [DufMo-76]. Recall that an irreducible representation $\pi : G \rightarrow \mathcal{U}(\mathcal{K})$ is defined to be in the discrete series if there exists a vector $\eta \neq 0$ in $\mathcal{K}$ such that the function $g \mapsto \langle \eta | \pi(g)\eta \rangle$ is in $L^2(G)$. To $\eta$ is associated a (a priori unbounded) closed operator $D_\eta : \mathcal{K} \rightarrow L^2(G)$, with domain $\{\xi \in \mathcal{K} | \langle \xi | \pi(\cdot)\eta \rangle \in L^2(G)\}$, defined by $D_\eta(\xi) = \langle \xi | \pi(\cdot)\eta \rangle$. It can then be shown that there exists a constant $c$ such that $cD_\eta$ extends to an isometry from $\mathcal{K}$ onto a closed subspace of $L^2(G)$.

4. Further properties of kernels of positive type

Whenever a Hilbert space of functions is given together with an orthonormal basis, the following proposition shows one way to compute the corresponding reproducing kernel.
3. Proposition. Let $\mathcal{H}$ be a Hilbert space of functions on a set $X$, with reproducing kernel $\Phi$, and let $(e_j)_{j \in J}$ be an orthonormal basis of $\mathcal{H}$. Then

(i) \[ \Phi(x, y) = \sum_{j \in J} e_j(x)\overline{e_j(y)} \]

for all $x, y \in X$.

Suppose moreover that there exists a positive finite measure $\mu$ on $X$ such that $\mathcal{H}$ is a closed subspace of $L^2(X, \mu)$, and that $\int_{X \times X} |\Phi(x, y)|^2 \, d\mu(x) \, d\mu(y) < \infty$. Then

(ii) \[ \dim_{\mathbb{C}}(\mathcal{H}) = \int_X \Phi(x, x) \, d\mu(x) < \infty. \]

In particular, if there exists moreover a group $G$ which acts transitively on $X$ and preserves $\mu$, and if $\mathcal{H}$ is both a closed subspace of $L^2(X, \mu)$ and a $G$-Hilbert space of functions on $X$, then

(iii) \[ \Phi(x, x) = \frac{\dim_{\mathbb{C}}(\mathcal{H})}{\mu(X)} \]

for all $x \in X$.

Proof. For (i), evaluate the Fourier expansion $\phi_x = \sum_{j \in J} \langle \phi_x | e_j \rangle e_j$ at the point $x$, and obtain $\Phi(x, y) = \sum_{j \in J} \langle \phi_x | e_j \rangle \langle e_j | \phi_y \rangle = \sum_{j \in J} e_j(x)\overline{e_j(y)}$.

For (ii), denote by $K_\Phi : L^2(X, \mu) \to L^2(X, \mu)$ the linear operator defined by the kernel $\Phi$, namely $\int_X \Phi(x, y) \psi(y) \, d\mu(y)$ for all $\psi \in \mathcal{H}$ and $x \in X$. On the one hand, $K_\Phi$ is a Hilbert-Schmidt operator, because of the $L^2$-condition on $\Phi$; on the other hand, $K_\Phi$ is the identity, since the kernel $\Phi$ is reproducing. It follows that the dimension of $\mathcal{H}$ is finite. Moreover, we have

\[ \int_X \Phi(x, x) \, d\mu(x) = \sum_{j \in J} \int_X |e_j(x)|^2 \, d\mu(x) = \sum_{j \in J} \|e_j\|^2 = \dim_{\mathbb{C}}(\mathcal{H}) \]

by (i).

The equality in (iii) follows now from the transitivity of $G$ on $X$, which implies that $\Phi(x, x)$ is constant (Proposition 2.1). \qed

Remarks. (i) If $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt norm, we have

\[ \int_{X \times X} |\Phi(x, y)|^2 \, d\mu(x) \, d\mu(y) = \|K_\Phi\|^2_{HS} = \text{trace}(K_\Phi^* K_\Phi) = \sum_{j \in J} |\langle K_\Phi e_j | e_j \rangle|^2 = \dim_{\mathbb{C}}(\mathcal{H}) \]

where the last equality follows from $K_\Phi = id$. 
(ii) Suppose that all the hypotheses of the proposition are satisfied as in (iii), that $X$ is a finite set, and that $\mu$ is the counting measure. Then $\Phi(x, x) \in \mathbb{Q}$. We don’t know under which conditions (if any) $\Phi(x, y) \in \mathbb{Q}$ for all $x, y \in X$.

Let us now show how any kernel of positive type arises as in Proposition 1. This is often referred to as a GNS construction, in honour of Gelfand, Naimark and Segal. (To whom we could add Kolmogorov, since Lemma 2 of [Kolmo-41] shows that a matrix $(c_{m,n})_{m,n \geq 1}$ is of positive type if and only if there exists a sequence of vectors $\xi_n$ in some Hilbert space such that $c_{m,n} = \langle \xi_m, \xi_n \rangle$.) The following proposition is sometimes called the “Moore-Aronszajn theorem”.

4. Proposition. Let $\Phi : X \times X \rightarrow \mathbb{C}$ be a kernel of positive type on a set $X$.

(i) There exists a Hilbert space of functions $\mathcal{H} \subset \mathbb{C}^X$ such that $\Phi$ is the reproducing kernel of $\mathcal{H}$.

Suppose moreover that $X$ is a topological space and that the kernel $\Phi$ is continuous.

(ii) The embedding $X \ni x \mapsto \phi_x \in \mathcal{H}$ is continuous.

(iii) Functions in $\mathcal{H}$ are continuous on $X$.

Proof. Let $\mathcal{H}_{pre}$ denote the subspace of $\mathbb{C}^X$ generated by the functions $\phi_y : X \ni x \mapsto \Phi(x, y) \in \mathbb{C}$. For two functions $\psi = \sum_{j=1}^{m} c_j \phi_{x_j}, \chi = \sum_{k=1}^{n} d_k \phi_{y_k}$ in $\mathcal{H}_{pre}$, observe that

$$
\sum_{1 \leq j \leq m} c_j \overline{d_k} \Phi(y_k, x_j) = \sum_{1 \leq k \leq n} \overline{d_k} \sum_{1 \leq j \leq m} c_j \phi_{x_j}(y_k) = \sum_{1 \leq k \leq n} \overline{d_k} \psi(y_k)
$$

$$
= \sum_{1 \leq j \leq m} c_j \sum_{1 \leq k \leq n} d_k \phi_{y_k}(x_j) = \sum_{1 \leq j \leq m} \overline{c_j} \chi(x_j)
$$

is independent on the chosen representations for $\psi$ and $\chi$. We may thus define

$$
\langle \psi | \chi \rangle = \sum_{1 \leq j \leq m} c_j \overline{d_k} \Phi(y_k, x_j).
$$

Now, we have

$$
(*) \quad |\psi(x)|^2 = |\langle \psi | \phi_x \rangle|^2 \leq \langle \phi_x | \phi_x \rangle |\psi | \psi \rangle = \Phi(x, x) \|\psi\|^2
$$

for all $\psi \in \mathcal{H}_{pre}$ and $x \in X$. The first consequence of this is that $\langle \psi | \psi \rangle = 0$ implies $\psi(x) = 0$ for all $x \in X$, namely $\psi = 0$; thus $\mathcal{H}_{pre}$ together with $\langle \cdot | \cdot \rangle$ is a prehilbert space. Let $\mathcal{H}$ denote its completion.

The second consequence of $(*)$ is that, for $(\psi_n)_{n \geq 1}$ a Cauchy sequence in $\mathcal{H}_{pre}$ converging towards some $\psi \in \mathcal{H}$, the sequence of numbers $(\psi_n(x))_{n \geq 1}$ converges for all $x \in X$, so that $\mathcal{H}$ is indeed a Hilbert space of functions on $X$, with reproducing kernel $\Phi$.

Claim (ii) is a straightforward consequence of the continuity of $\Phi$.

If the kernel $\Phi$ is continuous, the functions $\phi_y = \Phi(\cdot, y)$ are continuous, so that any function is $\mathcal{H}_{pre}$ is clearly continuous. If $(\psi_n)_{n \geq 1}$ is a Cauchy sequence in $\mathcal{H}_{pre}$
converging towards \( \psi \in \mathcal{H} \) as above, then the numerical sequences \( (\psi_n(x))_{n \geq 1} \) converge locally uniformly towards \( \psi(x) \), and it follows that \( \psi \) is continuous. This shows Claim (iii). \( \square \)

**Remarks.** (i) The previous proposition shows in particular that “starting with \( \Phi \)” and “starting with \( \mathcal{H} \)” correspond to dual points of view, as explained in the introduction of [Arons-50].

(ii) Assume that \( X \) is a locally compact space, and let the space \( C(X) \) of complex-valued continuous functions on \( X \) be endowed with the topology of uniform convergence on compact subspaces. For \( \Phi \) a kernel of positive type on \( X \) and \( \mathcal{H} \) the corresponding Hilbert space of functions, the two following conditions are equivalent:

\[ \mathcal{H} \text{ is a subset of } C(X) \text{ and the inclusion is continuous,} \]
\[ \Phi \text{ is separately continuous and locally bounded.} \]

(Proposition 24 of [Schwa-64].) Further regularity conditions on \( \Phi \) imply corresponding regularity conditions on functions in \( \mathcal{H} \); see Proposition 25 of [Schwa-64].

(iii) For a systematic exposition of the theory of Hilbert spaces of functions, see [Mesch-62].

5. Reproducing kernels and spherical functions

In this section, we want to show how to express reproducing kernels in terms of spherical functions. For simplicity, we restrict our attention to the case of a compact group \( G \) given together with a closed subgroup \( K \). Set \( X = G/K \), let \( \mu \) be a positive finite \( G \)-invariant measure on \( X \), and let \( \pi \) denote the natural unitary representation of \( G \) on the Hilbert space \( L^2(X, \mu) \). We choose a \( \pi(G) \)-invariant irreducible closed subspace \( \mathcal{H} \) of \( L^2(X, \mu) \) and we set \( \mathcal{H}^K = \{ \psi \in \mathcal{H} \mid \pi(k)\psi = \psi \text{ for all } k \in K \} \). It follows from Peter-Weyl theory that \( \mathcal{H} \) is a finite-dimensional \( G \)-Hilbert space of continuous functions on \( X \). We assume moreover that

\[ \dim_{C}(\mathcal{H}^K) = 1, \]

namely that the restriction of \( \pi \) to \( \mathcal{H} \) is a class one representation of \( G \) with respect to \( K \).

Choose a unit vector \( \xi \in \mathcal{H}^K \). To \( \xi \) corresponds the zonal spherical function \( f : G \rightarrow \mathbb{C} \) defined by

\[ f(g) = \langle \pi(g)\xi \mid \xi \rangle; \]

observe that \( f \) is a \( K \)-bi-invariant function. Let \( \Phi : X \times X \rightarrow \mathbb{C} \) be the reproducing kernel of the \( G \)-Hilbert space of function \( \mathcal{H} \). Denote by \( \omega \in X \) the canonical base point (the class \( K \) in \( G/K \)), and let \( \phi_\omega : X \rightarrow \mathbb{C} \) be defined as in Section 1 by \( \phi_\omega(x) = \Phi(x, \omega) \). Both \( \xi \) and \( \phi_\omega \) are in \( \mathcal{H}^K \), so that there exists a constant \( c \in \mathbb{C}^* \) such that \( \phi_\omega = c\xi \). Since \( \phi_\omega(\omega) > 0 \) (see Propositions 1 and 2), we can further normalize \( \xi \) by \( \xi(\omega) > 0 \); then \( c \) is real positive.
5. Proposition. The notation being as above, we have

\[ \Phi(gw, g'w) = \frac{\dimc(\mathcal{H})}{\mu(X)} f(g^{-1}g') \quad \text{and} \quad \xi(gw) = \sqrt{\frac{\dimc(\mathcal{H})}{\mu(X)}} f(g^{-1}) \]

for all \( g, g' \in G \).

Proof. We have

\[ \Phi(gw, g'w) = \Phi((g^{-1}g')^{-1}w, w) = \phi_w((g^{-1}g')^{-1}w) = \langle \pi(g^{-1}g')\phi_w | \phi_w \rangle = c^2 f(g^{-1}g'). \]

Together with Equality (iii) in Proposition 3, this implies

\[ \dimc(\mathcal{H}) = \mu(X)\Phi(\omega, \omega) = \mu(X)c^2 f(1) = \mu(X)c^2, \]

so that the first equality holds. Then

\[ \xi(gw) = c^{-1}\phi_w(gw) = c^{-1}\Phi(gw, \omega) = cf(g^{-1}) \]

for all \( g \in G \). \( \square \)

Remarks (i) It may be convenient to normalize \( \xi \) by \( \xi(\omega) = 1 \), instead of \( \|\xi\| = 1 \) and \( \xi(\omega) > 0 \).

(ii) In the situation of Proposition 5, the terminology for \( \Phi, \xi, \) and \( f \) is not uniform. For example, in Number 9.2.3 of [ConSl-99], it is the kernel \( J(x, y) = \dimc(\mathcal{H})\Phi(x, y) \) which is the zonal spherical function.

(iii) Suppose that \((G, K)\) is a compact Gelfand pair (see Remark (iii) following Proposition 2). Then, for every irreducible unitary representations \( \pi : G \rightarrow \mathcal{U}(\mathcal{H}) \), we have \( \dimc(H^K) \leq 1 \); the proof involves showing that the representation of \( \mathcal{C}(K \setminus G/K) \) on \( \mathcal{H}^K \) is irreducible, where \( \mathcal{C}(K \setminus G/K) \) denotes the abelian convolution algebra of complex-valued functions on \( G \) which are \( K \)-biinvariant (see Section IV.2 in [Lang-75]). Moreover \( \dimc(\mathcal{H}^K) = 1 \) if and only if \( \pi \) is a subrepresentation of the quasi-regular representation on \( L^2(X, \mu) \).

(iv) If \( G = SO(3) > K = SO(2) \), the function \( f \) "is" a Legendre polynomial. See e.g. Section III.4.10 in [Vilen-68].

Example 7. Distance-transitive graphs.

Let \( X \) be a connected finite graph (here a simple graph, namely a graph without loops or multiple edges). We view its vertex set \( X \) as a metric space for the usual combinatorial distance, written \( d \). Let \( \ell^2(X) \) be the finite-dimensional Hilbert space of functions \( X \rightarrow \mathbb{C} \), for the scalar product defined by \( \langle \psi | \chi \rangle = \sum_{x \in X} \psi(x)\chi(x) \). With the notation of Proposition 1, the function \( \phi_x \) is the characteristic function of \( x \in X \) and the kernel \( \Phi \) is the characteristic function of the diagonal in \( X \times X \).

Let \( A \) be the adjacency matrix of \( X \), viewed as a self-adjoint operator on \( \ell^2(X) \); equivalently \( (A\xi)(x) = \sum_{y \sim x} \xi(y) \) for all \( \xi \in \ell^2(X) \) and \( x \in X \), where \( \sum_{y \sim x} \) indicates
a summation over those vertices \( y \in X \) which are connected to \( x \) by an edge. For each eigenvalue \( \lambda \) of \( A \), let \( \mathcal{H}_\lambda \) denote the corresponding eigenspace of \( A \).

Assume that \( X \) is \textit{distance-transitive}, namely that there exists a group \( G \) of automorphisms of \( X \) which acts transitively on pairs of equidistant vertices; equivalently, that \( G \) is transitive on \( X \) and that the isotropy subgroup \( K \) of some vertex \( \omega \in X \) is transitive on spheres in \( X \) centred at \( \omega \). Then \( \ell^2(X) \) is a \( G \)-Hilbert space of functions on \( X \).

As \( A \) commutes with \( \pi(G) \), each eigenspace \( \mathcal{H}_\lambda \) is \( G \)-invariant. Any function \( \psi \) in \( \ell^2(X)^K \) depends on a vertex \( x \) only via the distance between \( x \) and the basis vertex \( \omega \). It follows that the space \( \mathcal{H}_\lambda^K \) of \( K \)-invariant solutions of the difference equation \( A\psi = \lambda \psi \) is of dimension one; thus, by Proposition 2, the canonical representation of \( G \) on \( \mathcal{H}_\lambda \) is irreducible.

Let \( (\delta_x)_{x \in X} \) denote the canonical orthonormal basis of \( \ell^2(X) \), so that an operator \( T \) in the algebra \( \mathcal{L}(\ell^2(X)) \) of linear transformations of \( \ell^2(X) \) can be described by its matrix \( (T_{x,y})_{x,y \in X} \). Such a \( T \) commutes with \( G \) if and only if \( T_{x,y} = T_{x',y'} \) for any \( x, y, x', y' \in X \) with \( d(x, y) = d(x', y') \). If \( D \) denotes the diameter of \( X \), the commutant algebra \( \pi(G)' \) of \( G \) in \( \mathcal{L}(\ell^2(X)) \) is therefore of dimension \( D + 1 \). Define for \( j \in \{0, \ldots, D\} \) the matrix \( A_j \) by \( A_j(x, y) = 1 \) if \( d(x, y) = j \) and \( A_j(x, y) = 0 \) otherwise \((x, y \in X)\). On the one hand, \( \{A_0 = I, A_1, \ldots, A_D\} \) is a linear basis of \( \pi(G)' \). On the other hand,

\[
AA_j = b_{j-1}A_{j-1} + a_jA_j + c_{j+1}A_{j+1}
\]

for appropriate integers \( a_j, b_j, c_j \). Consequently, \( A_j = p_j(A) \) for the sequence of polynomials defined by \( p_0(t) = 1, p_1(t) = t \), and

\[
 tp_j(t) = b_{j-1}p_{j-1}(t) + a_jp_j(t) + c_{j+1}p_{j+1}(t)
\]

for \( j \in \{2, \ldots, D - 1\} \). In particular, \( \pi(G)' \) is generated by \( A \) and is abelian, so that \((G, K)\) is a Gelfand pair. Also, the degree of the minimal polynomial of \( A \) is \( D + 1 \) and \( A \) has exactly \( D + 1 \) pairwise distinct eigenvalues. It follows that the representations of \( G \) in the spaces \( \mathcal{H}_\lambda \) are pairwise inequivalent.

The convolution algebra of \( K \)-invariant functions on \( X \) (of \( K \)-biinvariant functions on \( G \)) coincides with the algebra of operators on \( \ell^2(X) \) generated by \( A \). It has various names, such as the \textit{adjacency algebra} in [Biggs–93], the appropriate \textit{Hecke algebra} in [CurRe–62], and the \textit{centraliser ring} in [Neuma–77].

Let \( \lambda \) be an eigenvalue of \( A \). The function \( \xi_\lambda \in (\mathcal{H}_\lambda)^K \), now normalised by \( \xi_\lambda(\omega) = 1 \), is given by

\[
\xi_\lambda(x) = \frac{1}{|B_k(\omega)|} p_k(\lambda)
\]

for \( x \in X \) with \( d(\omega, x) = k \), where \( |B_k(\omega)| \) denotes the number of vertices of \( X \) at distance \( k \) from \( \omega \). See e.g. Section 13.1.5 of [VilKl–93].

\textbf{Example 8.} \textit{Johnson graphs} and \textit{Hamming graphs}.

As a first particular case of the previous example, consider two integers \( n, D \) with \( 1 \leq D \leq \frac{n}{2} \) and the corresponding \textit{Johnson graph} \( J(n, D) \): its vertices are the subsets of
size \( D \) in \( \{1, \ldots, n\} \), and two such subsets \( x, y \) are joined by an edge if \( |x \cap y| = D - 1 \).

The symmetric group on \( n \) objects \( G = \text{Sym}(n) \) acts transitively on the vertex set of \( J(n, D) \), with isotropy groups isomorphic to \( K = \text{Sym}(D) \times \text{Sym}(n-D) \). The diameter of the graph is \( D \), and the eigenvalues are

\[
\lambda_j = (D - j)(n - D - j) - j \quad \text{of multiplicity} \quad \binom{n}{j} - \binom{n}{j-1}
\]

for \( j \in \{0, 1, \ldots, D\} \) (see e.g. [BrCoN-89], page 255).

Spherical functions can be written in terms of the so-called Hahn polynomials. This, and some description of the eigenspaces of the adjacency matrix \( A \), can be found in [VilKl-93] (pages 487–488); see also [Dunkl-76] and [Delsa-78].

There exists a classification of all Gelfand pairs of the form \((\text{Sym}(n), K)\) for \( n > 18 \) (the hypothesis \( n > 18 \) is not strictly necessary, but the analysis of smaller \( n \) involves many “sporadic groups”). See [Saxl-81].

As a second particular case of Example 7, consider again two integers \( n, D \geq 1 \) and the corresponding Hamming graph \( H(D, n) \): its vertex set is \( \{1, \ldots, n\}^D \), and two such vertices \( x, y \) are joined by an edge if they differ in just one coordinate. The group of automorphisms of \( H(D, n) \) is a wreath product \( \text{Sym}(n) \wr \text{Sym}(D) \), namely a semi-direct product \( \left( \bigoplus_{k=1}^{D} S_k \right) \rtimes \text{Sym}(D) \) where each \( S_k \) holds for a copy of the symmetric group \( \text{Sym}(n) \). The diameter of the graph is \( D \) and the eigenvalues are

\[
\lambda_j = D(n-1) - nj \quad \text{of multiplicity} \quad \binom{D}{j}(n-1)^j
\]

for \( j \in \{0, 1, \ldots, D\} \) (see [BrCoN-89], page 261). Spherical functions can be written in terms of the so-called Krawtchouk polynomials (see e.g. Chapter 13 in [VilKl-93]).

There are many other examples of distance-transitive graphs in [BrCoN-89]. However, there are also finiteness results. Indeed, the diameter \( D \) of a distance-transitive graph of degree \( k \geq 3 \) is bounded above by \( (k^6)12^2k \); there are known better bounds: 8 for \( k = 3 \), and \( 2k - 1 \) for all known distance-transitive graphs of degree \( k > 3 \). This is a result of Cameron, Praeger, Saxl & Seitz, and Weiss: see Corollary 7.3.2, page 220 in [BrCoN-89].

For examples of finite metric spaces which are not underlying finite graphs, see the appendix below.

6. Reproducing kernels for a \( G \)-space with a cocycle

Consider again a set \( X \), a Hilbert space \( \mathcal{H} \) of functions on \( X \) such that each evaluation \( \psi \mapsto \psi(x) \) is continuous, and the corresponding reproducing kernel \( \Phi \). Consider moreover a group \( G \) acting on \( X \) and a cocycle \( \alpha : G \times X \to \mathbb{C}^* \), the cocycle condition being

\[
\alpha(g_1g_2, x) = \alpha(g_1, g_2x)\alpha(g_2, x) \quad \text{for all} \quad g_1, g_2 \in G \quad \text{and} \quad x \in X.
\]

There is an action \( \pi_\alpha \) of \( G \) on the space \( \mathbb{C}^X \) of functions on \( X \), according to the formula

\[
(\pi_\alpha(g)\psi)(x) = \alpha(g^{-1}, x)\psi(g^{-1}x).
\]
Example 9. Cocycles corresponding to induced characters.

Let the notation be as in the remark following Proposition 2, about the induced representation \( \pi = \text{Ind}_A^G(\sigma) \) of a unitary character

\[
\sigma : A \rightarrow \{ z \in \mathbb{C} \mid |z| = 1 \}.
\]

Recall that the representation \( \pi \) acts in the Hilbert space \( \mathcal{H} \) of functions \( \xi : G \rightarrow \mathbb{C}^* \) such that \( \xi(xa) = \xi(x)\sigma(a) \) for all \( x \in G \) and \( a \in A \), and such that \( \sum_{x \in G/A} |\xi(x)|^2 < \infty \).

Let \( T \subset G \) be a set of representatives of \( G/A \). For each class \( b \in G/A \), let \( t_b \in T \) be its representative. Since \( G \) acts on \( T \) by \( (g, t_b) \mapsto t_{gb} \), there is a natural mapping \( \tilde{\alpha} : G \times T \rightarrow A \) defined by \( g t_b = t_{gb} \tilde{\alpha}(g, t_b) \), as well as a mapping \( \alpha : G \times T \rightarrow \{ z \in \mathbb{C} \mid |z| = 1 \} \) defined by \( \alpha(g, t_b) = \sigma(\tilde{\alpha}(g, t_b)) \). For \( g_1, g_2 \in G \) and \( b \in B \), we have

\[
g_1g_2t_b = t_{g_1g_2b} \tilde{\alpha}(g_1g_2, t_b) = t_{g_1t_b} \tilde{\alpha}(g_2, t_b) = t_{g_1g_2b} (t_{g_2b} \tilde{\alpha}(g_2, t_b)),
\]

so that \( \alpha \) is a \( \{ z \in \mathbb{C} \mid |z| = 1 \} \)-valued cocycle. Observe that, if \( t_A = 1 \) is chosen to represent the class \( A \) in \( G/A \), then \( \tilde{\alpha}(a, t_A) = a \), and thus \( \alpha(a, t_A) = \sigma(a) \) for all \( a \in A \).

There is a natural isometry from the Hilbert space \( \mathcal{H} \) onto \( \mathcal{H}' = \ell^2(T) \), mapping a function \( \xi \in \mathcal{H} \) to its restriction \( \xi' = \xi|T \). The representation \( \pi \) is accordingly equivalent to the representation \( \pi' \) on \( \mathcal{H}' \) defined by

\[
(\pi'(g)\xi')(t_b) = \xi(t_{g^{-1}b} \tilde{\alpha}(g^{-1}, t_b)) = \sigma(\tilde{\alpha}(g^{-1}, t_b)) \xi'(t_{g^{-1}b}).
\]

This shows that, when expressed as \( \pi' \), the induced representation \( \text{Ind}_A^G(\sigma) \) is a particular case of a representation \( \pi_\alpha \). □

Given a \( G \)-set \( X \) and a cocycle \( \alpha : G \times X \rightarrow \mathbb{C}^* \), define a \( (G, \alpha) \)-Hilbert space of functions on \( X \) to be a Hilbert space \( \mathcal{H} \) of functions on \( X \) such that \( \pi_\alpha(g)\psi \in \mathcal{H} \) and \( \|\pi_\alpha(g)\psi\| = \|\psi\| \) for all \( g \in G \) and \( \psi \in \mathcal{H} \), namely such that \( \pi_\alpha \) is a unitary representation of \( G \) in \( \mathcal{H} \). For \( \omega \in X \) and \( K = \{ k \in G \mid k\omega = \omega \} \) its isotropy subgroup, observe that \( K \ni k \mapsto \alpha(k, \omega) \in \mathbb{C}^* \) is a group homomorphism. Set

\[
\mathcal{H}^{K, \alpha} = \{ \psi \in \mathcal{H} \mid \pi_\alpha(k)\psi = \alpha(k^{-1}, \omega)\psi \text{ for all } k \in K \}.
\]

The following statements generalize those of Proposition 1 to the present setting.

6. Proposition. With the notation above,

(i) \( \pi_\alpha(g)\varphi_x = \overline{\alpha(g, x)} \varphi_{gx} \) and \( \Phi(gx, gy) = \alpha(g, x)^{-1} \overline{\alpha(g, y)}^{-1} \Phi(x, y) \) for all \( g \in G \) and \( x, y \in X \). In particular, \( |\alpha(g, x)| = 1 \) for all \( g \in G \) and \( x \in X \) such that \( \Phi(x, x) \neq 0 \).

Assume moreover that the action of \( G \) on \( X \) is transitive; let \( \omega \) be some point in \( X \) and let \( K \) be its isotropy group.

(ii) If \( \mathcal{H} \neq \{0\} \), then \( \mathcal{H}^{K, \alpha} \neq \{0\} \).

(iii) If \( \dim_{\mathbb{C}}(\mathcal{H}^{K, \alpha}) = 1 \), then the representation \( \pi_\alpha \) of \( G \) in \( \mathcal{H} \) is irreducible.
Irreducibility and Reproducing Kernels

NB: kernels which have the property of (i) are \textit{projectively invariant under }G, in the terminology of [ParSc-72].

\textbf{Proof.} (i) It is straightforward to check these claims; see the proof of Proposition 2.

(ii) If \( \mathcal{H} \neq \{0\} \), then \( \Phi(x, x) \neq 0 \) for some \( x \) in \( X \) (by Proposition 1) and thus \( \Phi(x, x) \neq 0 \) for all \( x \in X \) by (i). In particular, \( |\alpha(g, x)| = 1 \) for all \( g \in K \) and \( x \in X \).

The function \( \phi_\omega = \Phi(\cdot, \omega) \) is not zero, and we have

\[ \pi_\alpha(k)\phi_\omega = \alpha(k, \omega)\phi_\omega = \alpha(k^{-1}, \omega)\phi_\omega \]

for all \( k \in K \) by (i). This shows Claim (ii).

The proof of (iii) goes as for Proposition 2. \( \square \)

\textbf{Example 10. On the irreducibility of induced representations (results of Godement and Mackey).}

Consider a group \( G \), a subgroup \( K \), the quotient space \( X = G/K \) with its standard base point \( \omega = K \), a cocycle \( \alpha: G \times X \to \{z \in \mathbb{C} \mid |z| = 1\} \), the Hilbert space \( \ell^2(X) \), and the corresponding unitary representation \( \pi_\alpha \) of \( G \) in \( \mathcal{H} \).

With the same notation as in Proposition 1, we have \( \phi_x = \delta_x \) (the Dirac “measure” at \( x \)) for all \( x \in X \), so that the reproducing kernel of \( \ell^2(X) \) is given by the Kronecker symbol: \( \Phi(x, y) = \delta_{x,y} \). The isotropy subgroup of \( \omega \) is of course \( K \); the subspace

\[ \ell^2(X)^{K,\alpha} = \{ \psi \in \mathcal{H} \mid \pi_\alpha(k)\psi = \alpha(k^{-1}, \omega)\psi \ \text{for all} \ k \in K \} \]

of \( \mathcal{H} \) is contained in the subspace

\[ \ell^2(X)^{K,\text{abs}} = \{ \psi \in \ell^2(X) \mid \text{the absolute value of } \psi \text{ is constant on each } K\text{-orbit} \} \]

It follows that

\textit{if all the orbits of } K \text{ on the complement of } \{\omega\} \text{ in } X \text{ are infinite, then the representation } \pi_\alpha \text{ is irreducible.}

This is the result of Appendix A in [Godem-48].

For an appropriate cocycle \( \alpha \) (see Example 9), the representation \( \pi_\alpha \) above is an induced representation, the result is a particular case of a result of Mackey, and the converse does hold: see e.g. Sections 3.3 and 3.4 in [Macke-76] (pages 146 and 158), or [Curti-02].

To express differently the result for \( \text{Ind}^G_K(1_K) \), define for a subgroup \( K \) of a group \( G \) the function \( L: G \to \mathbb{N} \cup \{\infty\} \) by \( L(g) = [K : K \cap gKg^{-1}] \), this index being also the size of the \( K \)-orbit of \( gK \in G/K \), and the \textit{commensurator} (other authors write “quasi-normaliser”)

\[ \text{Comm}_G(K) = \{ g \in G \mid L(g) < \infty \ \text{and} \ L(g^{-1}) < \infty \} \]

It follows from the definition that \( \text{Comm}_G(K) = K \) if and only if all orbits of \( K \) on \( G/K \) distinct from \( \{\omega\} \) are infinite. (Caveat: in general, \( L(g) < \infty \), or even \( L(g) = 1 \), \textit{does not} imply \( L(g^{-1}) < \infty \); however, if \( \{ g \in G \mid L(g) < \infty \} = K \), then \( \text{Comm}_G(K) = K \).) Hence:

\textit{The quasi-regular representation } \text{Ind}^G_K(1_K) \text{ of } G \text{ on } \ell^2(G/K) \text{ is irreducible if and only if } \text{Comm}_G(K) = K.

Further work related to Example 10 appear in [BurHa-97] and [BekCu].
Example 11. Unitary representations of the holomorphic discrete series of the group $PSU(1,1)$.

We follow essentially Chapter 17 in [Rober–83], with a slight amount of translation in the reproducing kernel language. See also, e.g. Chapter V in [Vesen–84].

The special unitary group

$$G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \left| |a|^2 - |b|^2 = 1 \right. \right\} \cong SL(2,\mathbb{R})$$

acts transitively on the open unit disc $D_1 = \{ z \in \mathbb{C} \mid |z| < 1 \}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} z = (az + b)(\bar{b}z + \bar{a})^{-1}.$$

The isotropy subgroup of the origin $\omega = 0 \in D_1$ is

$$K = \left\{ g_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in [0, 2\pi] \right\} \cong U(1).$$

As $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts as the identity, one could replace $G$ by $PSU(1,1)$, which is the quotient of $G$ by its centre of order 2. As $\frac{d}{dz} \left( \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} z \right) = (\bar{b}z + \bar{a})^{-2}$, Leibniz’ rule implies that the mapping

$$\alpha_k : G \times D_1 \ni \left( \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, z \right) \longmapsto (\bar{b}z + \bar{a})^{-k} \in \mathbb{C}^*$$

is a cocycle for any $k \in \mathbb{Z}$.

Choose an integer $k$. Let $\mathcal{O}(D_1)$ denote the space of holomorphic functions on $D_1$, and define a Hilbert space

$$H_k = \left\{ \psi \in \mathcal{O}(D_1) \mid \int_{D_1} |\psi(z)|^2 \left( 1 - |z|^2 \right)^{k-2} \frac{1}{2i} dz \wedge d\bar{z} < \infty \right\}.$$

For $k \geq 2$, it can be checked that any polynomial in $z$ is in $H_k$. Using the $G$-invariance of the 2-form $(1 - |z|^2)^{-2} \frac{1}{2i} dz \wedge d\bar{z}$ and the identity $1 - |g z|^2 = |\alpha_{-2}(g,z)|(1 - |z|^2)$ for $g \in G$ and $z \in D_1$, we can check that the function $z \mapsto \alpha_k(g^{-1}, z)\psi(g^{-1}z)$ is in $H_k$ and has the same norm as $\psi$ for all $\psi \in H_k$. Thus $H_k$ is a infinite dimensional $(G, \alpha_k)$-Hilbert space of functions on $D_1$. The corresponding unitary representation $\pi_k$ of $G$ in $H_k$ is a holomorphic discrete series representation$^3$ of $G$.

Let us show that $\pi_k$ is irreducible, using Proposition 5.iii. For $\theta \in [0, 2\pi]$, observe that $\alpha_k(g_\theta, z) = e^{ik\theta}$ for all $z \in D_1$, so that

$$(\pi_k(g_\theta)\psi)(z) = e^{-ik\theta} \psi(e^{-2i\theta} z) = \alpha_k(g_\theta^{-1}, 0)\psi(e^{-2i\theta} z)$$

$^3$Let $\sigma_k$ denote the unitary character $g_\theta \mapsto e^{ik\theta}$ of $K$. There is a natural realisation $\tilde{\pi}_k$ of the representation $\text{Ind}_K^G(\sigma_k)$ in $L^2_k(D_1, (1 - |z|^2)^{-2} \frac{1}{2i} dz \wedge d\bar{z})$, and $\pi_k$ is then the subrepresentation of $\tilde{\pi}_k$ corresponding to the closed subspace $\mathcal{O}(D_1) \cap L^2_k$ of $L^2_k$.\]
for all \( \psi \in \mathcal{H}_k \). In particular, if \( \psi_0 \in \mathcal{H}_k \) denotes the constant function of value 1, then \( \psi_0 \in \mathcal{H}_{k,\alpha_k} \). Let now \( \psi \in \mathcal{H}_{k,\alpha_k} \); then the meromorphic function \( \frac{\psi}{\psi_0} \) is \( \pi_k(K) \)-invariant, namely is constant on circles centred at the origin in \( \mathbb{D}_1 \). Since such a function is constant, \( \psi \) is a constant multiple of \( \psi_0 \). Hence \( \dim_{\mathbb{C}} (\mathcal{H}_{k,\alpha_k}) = 1 \), and this ends the proof of the irreducibility of \( \pi_k \).

### 7. Hilbert spaces of vector-valued functions

We consider a set \( X \), a Hilbert space \( V \), and a Hilbert space \( \mathcal{H} \) of \( V \)-valued functions on \( X \) such that each evaluation \( \mathcal{H} \ni \psi \mapsto \psi(x) \in V \) is a continuous linear operator.

Denote by \( \mathcal{L}(V) \) the C*-algebra of all continuous linear operators on \( V \). For each \( x \in X \) and \( u \in V \), there exists a unique element \( \phi_{x,u} \in \mathcal{H} \) such that

\[
\langle \psi \mid \phi_{x,u} \rangle_{\mathcal{H}} = \langle \psi(x) \mid u \rangle_V.
\]

The vector \( \phi_{x,u} \) depends on \( u \) in a way which is linear (obvious), and bounded, since

\[
\left\| \phi_{x,u} \right\|_{\mathcal{H}} = \sup_{\| \psi \| \leq 1} \left| \langle \psi \mid \phi_{x,u} \rangle \right| = \sup_{\| \psi \| \leq 1} \left( \| \psi(x) \| \| u \| \right) \leq c_x \| u \|
\]

where \( c_x \) is the norm of the linear mapping \( \mathcal{H} \ni \psi \mapsto \psi(x) \in V \). It follows that there exists a linear operator \( \phi_x : V \to \mathcal{H} \) such that \( \phi_{x,u} = \phi_x(u) \).

Define the reproducing kernel of \( \mathcal{H} \) to be

\[
\Phi : X \times X \ni (x,y) \mapsto \langle \phi_x \rangle^* \phi_y \in \mathcal{L}(V).
\]

Thus, for \( x, y \in X \) and \( u, v \in V \), we have

\[
\langle \Phi(x,y)u \mid v \rangle_V = \langle \phi_{y,v} \mid \phi_{x,u} \rangle_{\mathcal{H}} = \langle \phi_{y,v}(x) \mid u \rangle_V
\]

\[
\phi_{y,v}(\cdot) = \Phi(\cdot,y)v \in \mathcal{H} \subset V^X.
\]

The following statements generalize those of Proposition 1 to the present setting.

**7. Proposition.** Let the notation be as above.

(i) The kernel \( \Phi \) is of positive type: \( \sum_{j,k=1}^{n} \lambda_j \lambda_k \Phi(x_j, x_k) \) is a positive operator in \( \mathcal{L}(V) \) for all integer \( n \), complex numbers \( \lambda_1, ..., \lambda_n \), and points \( x_1, ..., x_n \) in \( X \).

(ii) The family \( \left\{ \phi_{x,u} \right\}_{x \in X, u \in V} = \left\{ \Phi(\cdot, x)u \right\}_{x \in X, u \in V} \) generates \( \mathcal{H} \).

(iii) If \( \mathcal{H} \neq 0 \), then \( \Phi(x,x) \neq 0 \) for some \( x \in X \).

**Proof.** (i) follows from a straightforward computation.

To show (ii), let \( \psi \in \mathcal{H} \) be such that \( \langle \psi \mid \phi_{x,u} \rangle_{\mathcal{H}} = 0 \) for all \( x \in X \) and \( u \in V \). Then \( \langle \psi(x) \mid u \rangle_V = 0 \) for all \( x \in X \) and \( u \in V \), that is, \( \psi = 0 \).

(iii) Assume that \( \Phi(x,x) = 0 \) for all \( x \in X \). Then

\[
\| \phi_x(u) \|^2 = \langle (\phi_x)^* \phi_x(u) \mid u \rangle_V = 0
\]
and hence $\phi_x(u) = \tilde{\phi}_{x,u} = 0$ for all $x \in X$ and $u \in V$. Therefore, $\mathcal{H} = 0$, by (ii).

Consider moreover a group $G$ acting on $X$ and a cocycle $\alpha : G \times X \rightarrow GL(V)$, the cocycle condition being

$$\alpha(g_1g_2, x) = \alpha(g_2, x)\alpha(g_1, g_2x) \quad \text{for all} \quad g_1, g_2 \in G \quad \text{and} \quad x \in X$$

(mind the order of the two operators $\alpha(\ldots)$). There is an action $\pi_\alpha$ of $G$ on the space $V^X$ of $V$-valued functions on $X$, according to the formula

$$(\pi_\alpha(g)\psi)(x) = \alpha(g^{-1}, x)\psi(g^{-1}x).$$

Define a $(G, \alpha)$-Hilbert space of $V$-valued functions on $X$ to be a Hilbert space $\mathcal{H}$ as above such that $\pi_\alpha(g)\psi \in \mathcal{H}$ and $\|\pi_\alpha(g)\psi\| = \|\psi\|$ for all $g \in G$ and $\psi \in \mathcal{H}$, namely such that $\pi_\alpha$ is a unitary representation of $G$ in $\mathcal{H}$.

Propositions 8 and 9 generalize Propositions 2 and 6 to the present setting. (For a generalization of the GNS construction of Proposition 4, we refer to [Kunze–67]; for more on operator-valued positive definite kernels, see [Chat–83a] and references given there, as well as Section 4.3 of [EvaKa–98].)

8. Proposition. With the notation above, we have $\pi_\alpha(g)\phi_x = \phi_{g^2}\alpha(g, x)^*$ and

$$\alpha(g, x)\Phi(gx, gy)\alpha(g, y)^* = \Phi(x, y)$$

for all $g \in G$ and $x, y \in X$.

Proof. For all $\psi \in \mathcal{H}$, $g \in G$, $x \in X$, and $u \in V$, we have

$$\langle \psi \mid \pi_\alpha(g)\tilde{\phi}_{x,u} \rangle = \langle \pi_\alpha(g^{-1})\psi \mid \tilde{\phi}_{x,u} \rangle = \langle \alpha(g, x)\psi(gx) \mid u \rangle_V$$

and consequently

$$\pi_\alpha(g)\phi_x = \phi_{g^2}\alpha(g, x)^*.$$

Let $x, y \in X$ and $g \in G$. Since $\pi_\alpha(g)$ is a unitary, it follows from the above equality that

$$\alpha(g, x)\Phi(gx, gy)\alpha(g, y)^* = \alpha(g, x)(\phi_{g^2})^*\phi_{gy}\alpha(g, y)^*$$

$$= \alpha(g, x)\alpha(g, x)^{-1}(\phi_x)^*\pi_\alpha(g^{-1})\phi_y\alpha(g, y)^{-1}\alpha(g, y)^*$$

$$= \Phi(x, y).$$

\[\square\]

Let $\omega$ be some point in $X$ and $K = \{k \in G \mid k\omega = \omega\}$ its isotropy subgroup. Observe that

$$K \ni k \mapsto \alpha(k, \omega) \in GL(V)$$

is a group homomorphism. In the sequel, we shall assume that $\alpha(k, \omega)$ is a unitary operator on $V$ for all $k \in K$, that is, $k \mapsto \alpha(k, \omega)$ is a unitary representation of $K$.

Let $F$ be the vector space of all mappings $\Psi : X \rightarrow L(V)$ such that

$$\Psi(kx) = \alpha(k, x)^{-1}\Psi(x)\alpha(k, \omega)$$

for all $k \in K$, $x \in X$ and such that $\Psi(\cdot)u \in \mathcal{H}$ for all $u \in V$.

Observe that, by the previous proposition, the mapping $x \mapsto \Phi(x, \omega)$ is an element from $F$. 

\[\square\]
9. Proposition. Assume that the action of $G$ on $X$ is transitive.

(i) If $H \neq \{0\}$, then $F \neq \{0\}$.

(ii) If $\dim_{C}(F) = 1$, then the representation $\pi_{\alpha}$ of $G$ in $H$ is irreducible.

Proof. (i) If $H \neq \{0\}$, then $\Phi(x, x) \neq 0$ for some $x \in X$ by Proposition 6. By transitivity of the action of $G$, it follows that $\Phi(x, x) \neq 0$ for all $x \in X$. The mapping $x \mapsto \Phi(x, \omega)$ is in $F$ and is non zero.

(ii) Let $H_{0}$ be a non-zero $G$-invariant closed subspace of $H$, and denote by $\Phi_{0}$ its reproducing kernel. The mapping $x \mapsto \Phi_{0}(x, \omega)$ is in $F$. If $\dim_{C}F = 1$, there exists a constant $c \neq 0$ such that $\Phi_{0}(x, \omega) = c\Phi(x, \omega)$ for all $x \in X$. Hence, using the transitivity of $G$ on $X$ and the equivariance property of $\Phi_{0}$ and $\Phi$ from the previous proposition, it follows that $\Phi_{0}(x, y) = c\Phi(x, y)$ for all $x, y \in X$. Since $(\Phi(\cdot, y)_{u})_{y \in X, u \in V}$ and $(\Phi_{0}(\cdot, y)_{u})_{y \in X, u \in V}$ generate $H$ and $H_{0}$, respectively, we obtain that $H = H_{0}$. \]

Example 12. Irreducibility of a discrete series representation of $PGL(2, \mathbb{Q}_{p})$.

Fix a prime integer $p$. Let $\mathbb{Q}_{p}$ be the field of the $p$-adic numbers and $\mathbb{Z}_{p}$ the compact and open subring consisting of the $p$-adic integers. Let $G = PGL(2, \mathbb{Q}_{p})$, the quotient of the general linear group $GL(2, \mathbb{Q}_{p})$ by its centre. The subgroup $K = PGL(2, \mathbb{Z}_{p})$ is compact and open in $G$. We will show that for certain irreducible unitary representations $\sigma$ of $K$, the induced representation $Ind_{K}^{G}\sigma$ is irreducible and has matrix coefficients with compact support. Such representations were first constructed by Mautner (see Section 9 in [Maut-64]).

As is well known, the quotient $X = G/K$ can be equipped with the structure of an infinite regular tree of degree $p + 1$ on which $G$ acts (in the natural way) by isometries (see Chapter II in [Serre-83]). Let $\omega$ be the coset $K$. For each integer $n \geq 1$, denote by $S_{n}$ the sphere of radius $n$ with centre $\omega$ and by $A_{n}$ the matrix

$$
A_{n} = \begin{pmatrix} p^{n} & 0 \\ 0 & 1 \end{pmatrix} \in PGL(2, \mathbb{Q}_{p})
$$

(we make no notational distinction between a matrix in $GL(2, \mathbb{Q}_{p})$ and its image in $PGL(2, \mathbb{Q}_{p})$). The vertex $A_{n}\omega$ is at distance $n$ from $\omega$ and we choose $(p + 1)p^{n-1}$ elements $k_{j}^{(n)} \in K$ with $k_{1}^{(n)} = I$ such that

$$
S_{n} = \left\{ k_{j}^{(n)}A_{n}\omega \mid j = 1, \ldots, (p + 1)p^{n-1} \right\}.
$$

The family

$$
T = \left\{ k_{j}^{(n)}A_{n} \mid n \in \mathbb{N}, j = 1, \ldots, (p + 1)p^{n-1} \right\}
$$

is a set of representatives of $G/K$. Let $\tilde{\alpha} : G \times X \longrightarrow K$ be the corresponding cocycle as in Example 9. Let $\sigma$ be unitary irreducible representation of $K$ on a Hilbert space $V$, and let $\alpha : G \times X \longrightarrow GL(V)$ be the cocycle defined by $\alpha(g, x) = \sigma(\tilde{\alpha}(g, x))$. The Hilbert space $H$ of all mappings $\psi : X \longrightarrow V$ such that

$$
\sum_{x \in X} \|\psi(x)\|^{2} < \infty
$$
is a \((G, \alpha)\)-Hilbert space of \(V\)-valued functions on \(X\). It is easily shown (as in Example 10) that the unitary representation \(\pi_\alpha\) of \(G\) on \(\mathcal{H}\) is equivalent to the induced representation \(\text{Ind}_K^G \sigma\).

Let \(K(p)\) be the closed normal subgroup of \(K\) consisting of all matrices \(k \in K\) such that \(k \equiv 1 \pmod{p\mathbb{Z}_p}\). Choose an irreducible unitary representation \(\sigma\) in a finite dimensional Hilbert space \(V\) of the quotient group \(K/K(p)\) with the property that the restriction of \(\sigma\) to the subgroup \(\mathbb{Z}_p\)

\[
N = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}
\]

has no non-zero invariant vectors. We view \(\sigma\) as representation of \(K\). The corresponding representation \(\pi_\alpha\) of \(G\) is irreducible. To show this, we apply the criterion from the previous proposition. Let \(\Psi : X \rightarrow GL(V)\) be a mapping such that

\[
\Psi(kx) = \alpha(k, x)^{-1}\Psi(x)\alpha(k, \omega)
\]

for all \(k \in K\) and all \(x \in X\). We claim that \(\Psi(x) = 0\) for all \(x \neq \omega\). Indeed, let \(x_n = A_n \omega\). For

\[
k = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \in N,
\]

we have, for \(n \geq 1\),

\[
\tilde{\alpha}(k, x_n) = A_n^{-1}kA_n = \begin{pmatrix} 1 & 0 \\ p^n a & 1 \end{pmatrix} \in K(p)
\]

and hence \(\alpha(k, x_n) = I\). It follows from this that \(kx_n = x_n\) and

\[
\Psi(x_n) = \Psi(kx_n) = \Psi(x_n)\alpha(k, \omega) = \Psi(x_n)\sigma(k).
\]

Hence, every column of the matrix \(\Psi(x_n)\) is invariant under \(\sigma(A)\). By our assumption on \(\sigma\), it follows that \(\Psi(x_n) = 0\) for all \(n \geq 1\).

As \(X = \bigcup_{n \in \mathbb{N}} Kx_n\), this implies that \(\Psi(x) = 0\) for all \(x \in X\), \(x \neq \omega\). Observe that, since \(\Psi(\omega) = \Psi(k\omega) = \sigma(k)^{-1}\Psi(\omega)\sigma(k)\), the matrix \(\Psi(\omega)\) commutes with \(\sigma(k)\) for all \(k \in K\). Hence, by Schur's lemma, \(\Psi(\omega)\) is a multiple of the identity \(I\) on \(V\). It follows that the space \(\mathcal{F}\) from the previous proposition is one-dimensional, proving the irreducibility of \(\pi_\alpha\).

We claim that \(\pi_\alpha\) is square integrable. In fact, \(\pi_\alpha\) is a so-called cuspidal representation, that is, \(\pi_\alpha\) has a non-zero matrix coefficient with compact support. Indeed, fix a unit vector \(v \in V\). Let \(\psi : X \rightarrow V\) be the element in \(\mathcal{H}\) defined by \(\psi(\omega) = v\) and \(\psi(x) = 0\) otherwise. Then \(g \rightarrow \langle \pi_\alpha(g)\psi \mid \psi \rangle\) has compact support. Normalizing the Haar measure \(\mu\) on \(G\) by \(\mu(K) = 1\), we have by Schur's orthogonality relations

\[
\int_G |\langle \pi_\alpha(g)\psi \mid \psi \rangle|^2d\mu(g) = \int_K |\langle \pi_\alpha(k)\psi \mid \psi \rangle|^2d\mu(k)
\]

\[
= \int_K |\langle \sigma(k^{-1})v \mid v \rangle|^2d\mu(k) = 1/ \dim V,
\]
showing that the formal dimension of $\pi_\alpha$ is equal to $\dim V$.

In order to construct a representation $\sigma$ with the required properties, observe that $K/K(p)$ is isomorphic to the finite group $PGL(2, \mathbb{Z}/p\mathbb{Z})$. Choose now an irreducible representation of $PGL(2, \mathbb{Z}/p\mathbb{Z})$ such that its restriction to the subgroup of strictly lower triangular matrices has no non-zero invariant vectors. For example, in the case $p = 2$, we may take as $\sigma$ the signature representation of $PGL(2, \mathbb{Z}/2\mathbb{Z}) \cong \text{Sym}(3)$.

Appendix – in collaboration with Rostislav Grigorchuk

A1. Two point homogeneous compact ultrametric spaces

The spaces.

Let $\mathbf{m} = (m_n)_{n \geq 1}$ be an infinite sequence of integers, all at least 2. The corresponding spherically symmetric rooted tree $T(\mathbf{m})$ is defined as follows. The vertices of $T(\mathbf{m})$ are the finite sequences $\mathbf{j} = (j_1, \ldots, j_n)$ with $n \geq 0$ and $j_k \in \{0, 1, \ldots, m_k - 1\}$ for $k \in \{1, \ldots, n\}$; the index $n$ is the level of the vertex $\mathbf{j}$. There is exactly one vertex of level 0 (the empty sequence) which is the root of $T(\mathbf{m})$. Ancestors of a vertex $(j_1, \ldots, j_n)$ are the vertices $(j_1, \ldots, j_k)$ with $0 \leq k \leq n$. The edges of $T(\mathbf{m})$ connect pairs $((j, j'))$ where $j$ is a vertex of level $n \geq 1$ and $j'$ its ancestor of level $n - 1$.

For each $n \geq 0$, we denote by $T_n(\mathbf{m})$ the finite subtree of $T(\mathbf{m})$ containing vertices of level at most $n$, and by $L_n(\mathbf{m})$ the set of vertices of level $n$ in $T(\mathbf{m})$, namely the set of leaves of $T_n(\mathbf{m})$. Observe that $|L_n(\mathbf{m})| = \prod_{k=1}^{n} m_k$. The integers $m_k$ are called the branch indices of $T(\mathbf{m})$ and of the $T_n(\mathbf{m})$'s.

The distances and their isometries.

Let $\lambda = (\lambda_n)_{n \geq 0}$ be a strictly decreasing sequence of positive real numbers such that $\lim_{n \to \infty} \lambda_n = 0$. Define a mapping

$$d_\lambda : L_n(\mathbf{m}) \times L_n(\mathbf{m}) \to \mathbb{R}_+$$

by $d_\lambda(x, y) = 0$ if $x = y$ and $d_\lambda(x, y) = \lambda_k$ if $x \neq y$, where $k$ is the largest level of a common ancestor of $x$ and $y$. Thus $d_\lambda$ is an ultrametric distance, and makes $L_n(\mathbf{m})$ an ultrametric space of diameter $\lambda_0$. The group of isometries of this space is canonically isomorphic to the group of automorphisms of the finite graph $T_n(\mathbf{m})$. In particular, this group

$$\text{Is}(L_n(\mathbf{m}), d_\lambda) = \text{Aut}(T_n(\mathbf{m}))$$

is independent of the choice of $\lambda$. Indeed, it is an iterated permutational wreath product

$$\text{Aut}(T_n(\mathbf{m})) = \left( \cdots \left( \text{Sym}(m_n) \rtimes L_{n-1} \text{Sym}(m_{n-1}) \right) \rtimes L_{n-2} \cdots \right) \rtimes L_1 \text{Sym}(m_1)$$

where $H \rtimes_k \text{Sym}(m_k)$ indicates the semidirect product $(H \oplus \cdots \oplus H) \times \text{Sym}(m_k)$ defined by the natural action of the symmetric group of $\{0, 1, \ldots, m_k - 1\}$ on the direct sum of $m_k$ copies of a group $H$. 

Similarly, the space \( \partial T(m) \) of ends of the infinite tree \( T(m) \) is a compact ultrametric space of diameter \( \lambda_0 \) for a distance \( d_\lambda \) defined by the same formulas as above. The isometry group of this space is isomorphic to the group of root-preserving\(^4\) automorphisms of the graph \( T(m) \), and also the appropriate inverse limit of the groups \( \text{Aut}(T_n(m)) \).

The action of the group \( G_n = \text{Is}(L_n(m), d_\lambda) \) on \( X = L_n(m) \) is two point homogeneous: this means that, given \( x, y, x', y' \in X \) with \( d_\lambda(x, y) = d_\lambda(x', y') \), there exists \( g \in G_n \) such that \( gx = x' \) and \( gy = y' \). Similarly, the action of \( G = \text{Is}(\partial T(m), d_\lambda) \) on \( \partial T(m) \) is two point homogeneous. We have the following characterization.

A compact ultrametric space with a transitive group of isometries is isometric to one of the spaces \((L_n(m), d_\lambda)\) or \((\partial T(m), d_\lambda)\) defined above. Moreover the action of the full isometry group on such a space is two point homogeneous. See Proposition 6.2 in [GrNeS-00] or Section 3 in [Fig-01]; see also Proposition 6.3 of the first reference for an extension to the case of totally disconnected metric spaces.

Whenever there is no ambiguity, we write \( T, \partial T, T_n, L, \sim \) instead of \( T(m), \partial T(m), T_n(m), L_n(m) \).

The Hilbert spaces.

Let \( m \) be fixed. Consider an integer \( n \geq 0 \), the level \( L_n \), and the Hilbert space \( \ell^2(L_n) \) of complex-valued functions on \( L_n \), with the scalar product defined by \( \langle \xi | \eta \rangle = \sum_{x \in L_n} \xi(x) \overline{\eta(x)} \); it is a complex Hilbert space of dimension \( \prod_{k=1}^{n} m_k \). Our next target is to describe a natural orthogonal decomposition of this space.

For each \( k \in \{0, 1, \ldots, n\} \), let \( \sim^k \) denote the equivalence relation on \( L_n \) defined by \( x \sim^k y \) if the largest level of a common ancestor of \( x \) and \( y \) is at least \( k \) (namely if \( d_\lambda(x, y) \leq \lambda_k \)). For \( k \in \{1, \ldots, n\} \), let \( \mathcal{H}^k_n \) denote the subspace of \( \ell^2(L_n) \) of functions \( \xi \) such that \( \xi(x) = \xi(y) \) whenever \( x \sim^k y \), and \( \sum_{x \sim^k y} \xi(x) = 0 \) for each \( x \in L_n \). Then

\[
\dimc (\mathcal{H}^k_n) = m_1 m_2 \cdots m_{k-1} (m_k - 1)
\]

and we have an orthogonal decomposition

\[
\ell^2(L_n) = \bigoplus_{k=0}^{n} \mathcal{H}^k_n
\]

where \( \mathcal{H}^0_n \) denotes the space of constant functions on \( L_n \).

There is a natural projection from \( \ell^2(L_n) = \bigoplus_{k=0}^{n} \mathcal{H}^k_n \) onto \( \ell^2(L_{n-1}) = \bigoplus_{k=0}^{n-1} \mathcal{H}^k_{n-1} \) which identifies all factors but the last one of the \( n \)th level to the corresponding factors of the \((n - 1)\)st level; the kernel of this projection is \( \mathcal{H}^n_n \). Consequently, there are canonical isomorphisms

\[
\mathcal{H}^k_n \approx \mathcal{H}^k_{n-1} \approx \cdots \approx \mathcal{H}^k_k
\]

whenever \( k \leq n \); we may view each of these spaces as a space \( \mathcal{H}^k \) of functions defined on \( L_n \), for some \( n \geq k \) depending on the context, as well as on \( \partial T \). If \( \mu \) denotes the

\(^4\)This condition is not always empty, as shown by the case with \( m_1 = 3 \) and \( m_n = 2 \) for \( n \geq 2 \).
usual $\text{Aut}(T)$-invariant probability measure on $\partial T$ (which can be viewed as the infinite product of the uniform probability measures on the finite sets $\{0, 1, \ldots, m_k - 1\}$), we have an orthogonal decomposition

$$L^2(\partial T, \mu) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k.$$

The representations.

Let $\pi_n$ denote the natural unitary representation of the group $G_n = \text{Aut}(T_n)$ on $\ell^2(L_n)$. It is clear that the above orthogonal decomposition of $\ell^2(L_n)$ is $\pi_n(G_n)$-invariant. Denote by $\omega_n$ the vertex $(m_1 - 1, m_2 - 1, \ldots, m_n - 1) \in L_n$, and by $P_n$ its isotropy group in $G_n$. The two point homogeneous property discussed above implies that $(G_n, P_n)$ is a Gelfand pair.

Each $\pi_n$ can also be viewed as a representation of the group $G = \text{Aut}(T)$. The transitive action of $G$ on $\partial T$ leaves the probability measure $\mu$ invariant, and provides a representation $\pi$ of $G$ on $L^2(\partial T, \mu)$. The direct sum $L^2(\partial T, \mu) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k$ is a decomposition of $\pi$ into irreducible representations, and is multiplicity free. If $P$ denotes the isotropy subgroup of the point $\omega = (m_k - 1)_{k \geq 1} \in \partial T$ in the group $G = \text{Aut}(T)$, then $(G, P)$ is again a Gelfand pair (now with $G$ a profinite group and $P$ a closed subgroup).

For simplicity, we restrict for some time the discussion to the Gelfand pair $(G_n, P_n)$ and the decomposition $\ell^2(L_n) = \bigoplus_{k=0}^{n} \mathcal{H}_n^k$, for some given $n \geq 0$.

Let $x \in L_n$; the set of distances $d_\Delta(x, y)$ with $y \in L_n$ takes exactly $n + 1$ values: $0, \lambda_{n-1}, \lambda_{n-2}, \ldots, \lambda_0$. An argument already used in Example 7 shows that the dimension of the commutant $\pi_n(G_n)'$ is $n+1$, and the first claim of the following proposition follows. The other claims record computations which are now straightforward.

10. Proposition. We keep the notation above.

(i) The representation $\pi_n$ of $G_n$ splits as a direct sum

$$\ell^2(L_n) = \bigoplus_{k=0}^{n} \mathcal{H}_n^k$$

of $n + 1$ pairwise inequivalent irreducible subrepresentations.

(ii) If $\xi_n^k$ is the function in $(\mathcal{H}_n^k)^{P_n}$ such that $\xi_n^k(\omega_n) = 1$, we have

$$\xi_n^k(j_1, \ldots, j_n) = \begin{cases} 
1 & \text{if } (j_1, \ldots, j_k) = (m_1 - 1, \ldots, m_k - 1) \\
-1/m_k - 1 & \text{if } (j_1, \ldots, j_{k-1}) = (m_1 - 1, \ldots, m_{k-1} - 1) \text{ and } j_k \neq m_k - 1 \\
0 & \text{otherwise}
\end{cases}$$

for $k \in \{1, \ldots, n\}$, and $\xi_n^0$ is the constant function of value 1 on $L_n$. 

(iii) If \( f_n^k \) is the zonal spherical function on \( G_n \) corresponding to \( \zeta_n^k \), we have

\[
\begin{align*}
  f_n^k(g) &= \begin{cases} 
    1 & \text{if } g \in \mathcal{P}_k \\
    \frac{-1}{m_k - 1} & \text{if } g \in \mathcal{P}_{k-1} \text{ and } g \not\in \mathcal{P}_k \\
    0 & \text{otherwise}
  \end{cases}
\end{align*}
\]

for \( k \in \{1, \ldots, n\} \), and \( f_n^0 \) is the constant function of value 1 on \( G_n \).

Remarks. (i) The reproducing kernel of the \( G_n \)-Hilbert space of functions \( \mathcal{H}_n^k \) is simply given in terms of \( f_n^k \) as in Proposition 5.

(ii) The functions \( \zeta_n^k \) are in some sense classical: they appear in Section 12 of [Letac-82].

The averaging operators.

The analogue here of the operator \( A \) and its powers in Example 7 can be described as follows. Define for each \( k \in \{0, 1, \ldots, n\} \) an operator \( A_k \) on \( \ell^2(L_n) \) by

\[
(A_k \psi)(x) = \left( \prod_{j=1}^{k} m_j \right)^{-1} \sum_{y \in x} \psi(y).
\]

Then \( A_k \) is the orthogonal projection of \( \ell^2(L_n) \) on \( \bigoplus_{j=0}^{k} \mathcal{H}_j^k \).

Its eigenvalues are 1, with multiplicity \( \prod_{j=1}^{k} m_j \), and 0. In particular, \( A_n \) is the identity operator, \( A_0 \) is of rank one, and \( \mathcal{H}_n^k \) is the common eigenspace of those \( \psi \in \ell^2(L_n) \) for which \( A_j \psi = \psi \) when \( j \leq k \) and \( A_j \psi = 0 \) when \( j > k \).

The algebra of operators on \( \ell^2(L_n) \) which commute with \( \pi_n(G_n) \) is abelian of dimension \( n + 1 \), and \( \{A_0, A_1, \ldots, A_n\} \) is a linear basis. This algebra is naturally isomorphic to the convolution algebra (also called Hecke algebra) of \( \mathcal{P}_n \)-biinvariant functions on \( G_n \), with \( \{f_n^0, f_n^1, \ldots, f_n^n\} \) as linear basis. One could express elements of one of these basis in terms of the other; the formulas would be in some sense analogous to change of basis formulas for the space of central functions on a finite group, the two natural basis being given there by irreducible characters on the one hand and characteristic functions of conjugacy classes on the other hand.

A2. A Finitely generated example on the binary tree

The group \( G = \text{Aut}(T(m)) \) of the previous section is not finitely generated, even in the topological sense: it is a compact group in which no finitely generated subgroup is dense. In this section we want to show that there are finitely generated subgroups \( G \) of \( G \) which are two point homogeneous on each level \( L_n(m) \), thus providing sequences \( (G_n, P_n)_{n \geq 0} \) of Gelfand pairs, with the \( G_n \) appropriate finite quotients of \( G \).

The group \( G \) and the pairs \( (G_n, P_n) \).

Let \( m \) be the sequence with \( m_n = 2 \) for all \( n \geq 1 \), so that \( T = T(m) \) is the binary tree. Let \( a \in \text{Aut}(T) \) be defined on the vertices of \( T \) by

\[
a(j_1, j_2, \ldots, j_n) = (\overline{j_1}, j_2, \ldots, j_n),
\]

where \( \overline{0} = 1 \) and \( \overline{1} = 0 \); thus \( a \) "exchanges the two halves of \( T \)". An automorphism \( g \) of \( T \) which fixes the vertices 0 and 1 is usually written \( (g_0, g_1) \), with \( g_0, g_1 \in \text{Aut}(T) \) describing the action of \( g \) in the two halves of \( T \). Define recursively \( b, c, d \in \text{Aut}(T) \) by

\[
b = (a, c) \quad c = (a, d) \quad d = (1, b).
\]
For example, \( b(0) = 0, b(1) = 1, \) and
\[
\begin{align*}
\{ b(0, j_2, j_3, \ldots, j_n) &= (0, j_2, j_3, \ldots, j_n) \\
\{ b(1, j_2, j_3, \ldots, j_n) &= (1, c(j_2, j_3, \ldots, j_n))
\end{align*}
\]
for \( n \geq 2. \) It is easy to check that \( a^2 = b^2 = c^2 = d^2 = 1. \) Let \( G \) be the subgroup of \( G = \text{Aut}(T) \) generated by \( \{a, b, c, d\}. \) This is the “first Grigorchuk group” from [Grigo-80]; for other references and an exposition, see [Harpe-00]. For each \( n \geq 0, \) we denote by \( G_n \) the finite quotient of \( G \) which acts naturally on the level \( L_n, \) and by \( P_n \) the isotropy subgroup of the vertex \( \omega_n = (1, \ldots, 1) \in L_n. \)

The group \( G_n \) is two point homogenous on \( L_n \) for any metric \( d_3 \) of the previous section. This has been shown in Theorem 9.17 of [BarGr-02], but we want to give now a different argument.

**Three properties of the action of \( G \) on \( T. \)**

**Property (i).** For each vertex \( u \) of \( T, \) let \( T_u \) denote the binary tree of those vertices \( x \) of \( T \) which have \( u \) as ancestor. On the one hand, we denote by \( St_G(u)|T_u \) the restriction to \( T_u \) of the isotropy subgroup \( \{ g \in G \mid g(u) = u \}. \) On the other hand, there is a canonical isomorphism \( T \approx T_u \) by which we identify \( \text{Aut}(T) \) and its subgroups with groups of automorphisms of \( T_u. \)

The action of \( G \) on \( T \) is fractal, which means that
\[
St_G(u)|T_u = G
\]
for all vertices \( u \in T. \) This is easily checked by induction on the level of \( u. \) For example, it holds for \( u = 0 \) in \( L_1, \) since the first coordinates of \( b = (a, c), ada = (b, 1), aba = (c, a), \) and \( aca = (d, a) \) generate \( G, \) and similarly for \( u = 1 \) in \( L_1, \) since the second coordinates of \( aba, d, b, c \) also generate \( G. \)

**Property (ii).** The action of \( G \) on the level \( L_n \) is transitive, for all \( n \geq 0. \) This is obvious for \( n = 1 \) (since \( a \in G \)), and then follows from Property (i) by induction on \( n. \)

**Property (iii).** Let \( K = \langle (ab)^2 \rangle^G \) be the smallest normal subgroup of \( G \) containing \( (ab)^2. \) Then \( K \) is a subgroup of \( G \) which fixes the two vertices of \( L_1. \) The group \( K \) contains the subgroup \( K \times K \) of \( G, \) where the first factor of the product \( K \times K \) holds for the group \( K \) itself acting on “the first half” \( T_0 \) of \( T, \) and similarly for the second factor and \( T_1. \) (It is known that \( K \) is of index 16 in \( G \) and that \( K \times K, \) viewed as above, is a subgroup of index 4 in \( K. \)) Moreover, \( K \) is generated by the elements
\[
\begin{align*}
(ab)^2 &= (ca, ac) \\
(bada)^2 &= ((ab)^2, 1) \\
(abad)^2 &= (1, (ab)^2)
\end{align*}
\]
of \( G. \) For this, we refer to Proposition VIII.30 in [Harpe-00].
11. **Lemma.** For each \( n \geq 1 \), the group \( K \) has exactly two orbits on \( L_n \), one being \( \{(j_1, \ldots, j_n) \in L_n \mid j_1 = 0\} \) and the other \( \{(j_1, \ldots, j_n) \in L_n \mid j_1 = 1\} \).

**Proof.** Let \( M \) be the subgroup \( St_K(0)/T_0 \) of \( G \). From \((*)\) above, it follows that \( M \) is generated by \( ca \) and \( (ab)^2 \). Similarly, the subgroup \( M' = St_K(1)/T_1 \) of \( G \) is generated by \( ac \) and \( (ab)^2 \). Thus \( M = M' \), since \( ac = (ca)^{-1} \). To prove the lemma, we have to show that \( M \) acts transitively on \( L_n \) for each \( n \geq 0 \).

The stabilizer \( St_M(0) \) of the vertex 0 in \( L_1 \) contains clearly the three elements

\[
(ab)^2 = (ca, ac) \\
(ca)(ab)^2ac = (ca, bad) \\
(ac)^2 = (ad, da)
\]

which generate in \( M \) a subgroup of index 2. Hence they generate \( St_M(0) \). If \( N = St_M(0)/T_0 \) and \( N' = St_M(1)/T_1 \), then

\[
N = \langle ca, ad \rangle = \langle ca, b \rangle \\
N' = \langle ac, bad, da \rangle = N.
\]

Moreover, \( St_N(0) \) contains

\[
(ca)^2 = (ad, da) \\
(bacab) = (aca, dad) \\
b = (a, c)
\]

and we have

\[
St_N(0)/T_0 = \langle ad,aca,a \rangle = G \\
St_N(1)/T_1 = \langle da,dad,c \rangle = G.
\]

Now \( M \) acts transitively on \( L_1 \), since \( M \) contains \( ca \), which exchanges the two vertices of \( L_1 \). Similarly, \( M \) acts transitively on \( L_2 \), since \( N \) contains \( ca \). Finally, \( M \) acts transitively on \( L_n \) for \( n \geq 3 \), since \( St_N(0)/T_0 = St_N(1)/T_1 = G \) acts transitively on each level. \( \square \)

12. **Proposition.** The action of the group \( G \) defined above on each level \( L_n \) of the binary tree \( T \) is two point homogeneous (for some metric \( d_A \), or equivalently for any metric \( d_A \)). In particular, \((G_n, P_n)\) is a Gelfand pair for each \( n \geq 1 \).

With the notation of Proposition 10.i, the direct summand \( H^K_n \) of \( \ell^2(L_n) \) remains irreducible for the restriction of \( \pi_n \) to \( G_n \), the functions \( \xi^K_n \) of Proposition 10.ii are in \((H^K_n)^{P_n}\), and the zonal spherical functions for the pairs \((G_n, P_n)\) are the restrictions to \( G_n \) of the functions \( f^K_n \) of Proposition 10.iii.

**Proof.** Recall that, for each \( n \geq 0 \), we denote by \( \omega_n \) the vertex \((1, \ldots, 1) \in L_n \); let \( P_n \) be the corresponding isotropy subgroup in \( G \) (or in \( G_n \)). Similarly, we have \( \omega = 1^\infty \in \partial T \) and \( P = \{ g \in G \mid g\omega = \omega \} = \cap_{n=0}^\infty P_n \). Set moreover \( u_j = 1^{j-1}0 \in L_j \) for all \( j \geq 1 \).
It follows from Property (i) of the action of \( G \) on \( T \) that there exists for each \( j \geq 0 \) an element \( g_j \in P \) such that \( g_j|_{u_j} = b \); observe that \( g_j(u_{j+1}) = u_{j+1} \) and that \( g_j|_{u_{j+1}} = a \). It follows then from Property (iii) that \( P \) contains for each \( j \geq 1 \) a subgroup \( K(j) \) such that \( K(j)|_{u_j} = K \), and which fixes any vertex of \( T \) of which \( u_j \) is not an ancestor. (The group \( P \) contains the infinite direct sum \( \bigoplus_{j \geq 1} K(j) \), but we will not use this fact here.) As a consequence, the subgroup \( \langle K(j), g_{j-1} \rangle \) acts on \( T_{u_j} \) as \( \langle K, a \rangle \), and it follows from Lemma 11 that this action is transitive on each level of \( T_{u_j} \).

Consider now an integer \( n \geq 0 \), and the action of \( P \) on \( L_n \). For any metric \( \delta \), there are exactly \( n + 1 \) spheres around \( u_n \) in \( L_n \), of radii \( 0, \ldots, n-1, n-2, \ldots, 0 \). The sphere of radius \( n-1 \) consists precisely of the vertices in \( L_n \) which have \( u_j \) as ancestor \((1 \leq j \leq n)\). Thus, the transitivity of \( \langle K(j), g_{j-1} \rangle \) on \( T_{u_j} \) can be reformulated as the transitivity of \( P \) on the sphere of radius \( n-1 \) around \( u_n \) in \( L_n \).

This shows that \( G \) is two point homogeneous on each level of \( T \). The other claims of the proposition are now straightforward. \( \square \)

**Remark.** Inside the automorphism group \( G \) of the binary tree, there is a group \( \tilde{G} \) generated by four automorphisms \( \{a, b, c, d\} \), which contains strictly the group \( G = \langle a, b, c, d \rangle \) discussed above (see [BarGr-02], page 65). Proposition 12 holds for \( \tilde{G} \) and the actions of the appropriate finite quotients \( \tilde{G}_n \) on the levels \( L_n \).

### A3. Examples on regular rooted trees of higher degrees

Choose an integer \( d \geq 3 \). Let \( m \) be the sequence with \( m_n = d \) for all \( n \geq 1 \), so that \( T = T(m) \) is the \( d \)-ary tree. We consider now a subgroup \( G \) of the automorphism group \( G = \text{Aut}(T) \). For each \( n \geq 0 \), let \( G_n \) denote again the finite quotient of \( G \) which acts naturally on the level \( L_n \) of \( T \), and let \( P_n \) denote the isotropy subgroup of \( \omega_n \in L_n \). For any of the metrics \( \delta \) introduced in Section A1, the level \( L_n \) splits as the disjoint union of \( n + 1 \) spheres \( S_n(0) = \{\omega_n\}, S_n(1), \ldots, S_n(n) \) of increasing radii around \( \omega_n \); the group \( P_n \) acts naturally on each of these spheres. We consider moreover a permutation \( a \) of the first level \( L_1 = \{0, 1, \ldots, d-1\} \) which is a \( d \)-cycle.

### Groups which cannot be two point homogeneous on the levels.

Assume now that, for each vertex \( u \in T \), the stabilizer \( St_G(u)|_{T_u} \) acts on the first level \( \{u_0, u_1, \ldots, u(d-1)\} \) of \( T_u \) as a power of \( a \). Then, for each \( n \geq 1 \) and for each index \( k \in \{1, \ldots, n\} \), the action of \( P_n \) on \( S_n(k) \) has at least \( d-1 \) orbits. This is obvious if \( n = 1 \), since \( P_1 \) is reduced to one element. Similarly, for \( n \geq 2 \) and for each \( k \in \{1, \ldots, n\} \), the group \( P_n \) fixes the vertex \( \omega_{n-k} \in L_{n-k} \); it follows that \( P_n \) fixes also the \( d-1 \) vertices \( (d-1)^{n-k} j \in L_{n-k+1} \), with \( j \in \{0, 1, \ldots, d-2\} \). Thus \( P_n \) has at least \( d-1 \) orbits in \( S_n(k), 1 \leq k \leq n \), and \( (d-1)n + 1 \) orbits on \( L_n \).

The previous argument shows that a subgroup \( G \) of \( \text{Aut}(T) \) with two point homogeneous actions on all levels \( L_n \) cannot have its first quotient group \( G_1 \) reduced to some powers of \( a \). P. Neumann has found an example of a perfect group \( G \) which is residually finite and isomorphic to the wreath product \( G \wr \text{Alt}(6) \), namely to the semi-direct product \( G^6 \rtimes \text{Alt}(6) \) corresponding to the natural action of the alternate group \( \text{Alt}(6) \) on the direct sum of 6 copies of \( G \) (see [Neuma-86], in particular pages 308-310, where our
$G$ is written $C$). This group is fractal; its action on each level of the 6-ary tree $T$ (with $d = 6$) is transitive and two point homogeneous on each level of the tree. Neumann's example carries over to any integer $d \geq 6$. More generally, it is a natural project to construct other classes of finitely generated automorphism groups of $d$-ary trees which are two point homogeneous on all levels.

Groups acting on the ternary tree.

Our last purpose is to show examples of groups $G$ of automorphisms of the ternary tree with the following properties: the group $G_1$ is the group of powers of $a$, so that the groups $G_n$ are not two point homogeneous on the corresponding levels, but the pairs $(G_n, P_n)$ are nevertheless Gelfand pairs for all $n \geq 1$. In each of our examples, $P_n$ has $2n + 1$ orbits on $L_n$ for all $n \geq 0$. To prove that the commuting algebra $\pi_n(G_n)'$ is abelian (by induction on $n$), the crucial observation is that the kernel of the natural projection $p_{n-1}^n : \pi_n(G_n)' \rightarrow \pi_{n-1}(G_{n-1})'$ is a two-sided ideal of dimension two in a semisimple algebra, and is therefore an abelian direct summand.

We set from now on $d = 3$, and we use for the groups the same notation as in [BarGr–02]. Let $a \in \text{Aut}(T)$ be defined on the vertices of $T$ by

$$a(j_1, j_2, \ldots, j_n) = (j_1 + 1, j_2, \ldots, j_n)$$

where $j_1 + 1 \in \{0, 1, 2\}$ is understood modulo 3. Each of the groups below has two generators of order 3, one being $a$ and the other being defined recursively by an equality of the form $b = (a, *, b)$, where $*$ denotes a suitable element depending on the example; compare with the definition of $G$ in Section A2.

The Fabrikowski-Gupta group $\Gamma$.

Let $\Gamma$ be the subgroup of $\text{Aut}(T)$ generated by $a$ and $t = (a, 1, t)$ [FabGu–91]. Let $K$ denote the subgroup of commutators in $\Gamma$.

Our first claim is that

$$K \times K \times K \leq K. \tag{*}$$

Indeed, we have $ata^{-1} = (t, a, 1)$ and therefore

$$[t, ata^{-1}] = ([a, t], 1, 1).$$

It follows that $K \times \{1\} \times \{1\} \leq K$. Similar arguments show that $\{1\} \times K \times \{1\} \leq K$ and $\{1\} \times \{1\} \times K \leq K$ are subgroups of $K$.

Our second claim is that $K$ has exactly 3 orbits on $L_2$. Indeed, on the one hand it is obvious that $K$ has at least 3 orbits on $L_2$ since $K$ acts as the identity on $L_1$. On the other hand

$$t^{-1}ata^{-1} = (a^{-1}t, a, t^{-1})$$

acts transitively on the three vertices 00, 01, 02 of $L_2$ below 0, and also on the three vertices 10, 11, 12 below 1, whereas

$$t^{-1}a^{-1}ta = (a^{-1}, t, t^{-1}a)$$
acts transitively on the three vertices 20, 21, 22 below 2.

Our third claim is that $\Gamma$ is fractal. Indeed, from

\[
\begin{align*}
\alpha^{-1}t^{-1}at &= \alpha^{-1}(t^{-1}ata^{-1})a = (a, t^{-1}, a^{-1}t) \\
\alpha^{-1}t^{-1}a^{-1}ta^2 &= (t, t^{-1}a, a^{-1})
\end{align*}
\]

we see that the stabilizer of the vertex $0 \in L_1$ coincides with $(a, t) = \Gamma$. More generally $St_\Gamma(u)/T_u = \Gamma$, first for any $u \in L_1$ by a similar argument, and then for any vertex $u$ of the tree $T$ by induction on the level of $u$.

As $\Gamma$ is a fractal group which acts transitively on the first level, it acts also transitively on any level of the tree. (This is a general and easy fact: see e.g. Lemma 2.3 in [BarGr-02].)

As a consequence of this and of the second claim, the group $K$ has exactly 3 orbits on $L_n$ for each $n \geq 1$.

Consider now the base point $w = 2^\infty \in \partial T$ and the isotropy subgroup $P$ of $w$ in $\Gamma$. On the one hand, $P$ contains $t = (a, 1, t)$ which acts as a 3-cycle on the three vertices below 0, and $ata^{-1} = (t, a, 1)$ which acts as a 3-cycle on the three vertices below 1. On the other hand, by (*), we have inside $P$ a subgroup of the form

\[
(K \times K) \times (K \times K) \times \cdots \times (K \times K)
\]

(n pairs of factors),

where, for $j = 1, \ldots, n$ and the vertices $u = 2^{i-1}0, v = 2^{i-1}1$ in $L_j$, the $j$th pair $K \times K$ acts by the product action on $T_u \times T_v$ (observe that the $(n - 1)$st and the $n$th pairs act trivially on vertices of $L_n$). It follows that $P$ has exactly $2n + 1$ orbits on $L_n$.

We denote by $\Gamma_n, P_n$ the finite quotients of $\Gamma, P$ which act naturally on $L_n$.

13. Proposition. For each $n \geq 1$, the pair $(\Gamma_n, P_n)$ defined by the Fabrikowski-Gupta group $\Gamma$ acting on the $n$th level of the ternary tree is a Gelfand pair.

The representation $\pi_n$ of $\Gamma_n$ on $\mathbb{C}^{L_n}$ splits as the direct sum of $2n + 1$ pairwise inequivalent irreducible representations.

Proof. The group $\Gamma_n$ acts diagonally on $L_n \times L_n$ with the same number of orbits, namely $2n + 1$, as the number of orbits of $P_n$ acting on $L_n$. If $\pi_n$ denotes the natural permutation representation of $\Gamma$ on the vector space $\mathbb{C}^{L_n}$, it follows that the commutant algebra $(\pi_n(\Gamma_n))'$ is a subalgebra of $\text{End}(\mathbb{C}^{L_n})$ of dimension $2n + 1$.

If $n \geq 2$, the space of functions on $L_{n-1}$ embeds naturally inside the space of functions on $L_n$, and this induces an embedding $j^n_{n-1} : (\pi_{n-1}(\Gamma_{n-1}))' \rightarrow (\pi_n(\Gamma_n))'$. Moreover, if $A$ denotes the kernel of the natural projection $\pi^1_{n-1} : \pi_n(\Gamma_n)' \twoheadrightarrow \pi_{n-1}(\Gamma_{n-1})'$, we have a direct sum decomposition

\[
(\pi_n(\Gamma_n))' = j^n_{n-1}\left((\pi_{n-1}(\Gamma_{n-1}))'\right) \oplus A
\]

and the two-sided ideal $A$ is abelian, since it is of dimension $(2n + 1) - (2n - 1) = 2$.

It follows by induction on $n$ that the algebra $(\pi_n(\Gamma_n))'$ is abelian for all $n \geq 0$. \[\square\]
The Gupta-Sidki group $\Gamma$ and the Bartholdi-Grigorchuk group $\bar{\Gamma}$.

Let $\Gamma$ be the subgroup of $\text{Aut}(T)$ generated by $a$ and

$$w = (a, a^{-1}, w)$$

([GuSi-83a], [GuSi-83b]). Let $K$ denote now the subgroup of commutators in $\bar{\Gamma}$.

The previous considerations apply to $\Gamma$ with minor changes only. For example, setting $w^\gamma = \gamma w \gamma^{-1}$ and $w^{-\gamma} = \gamma^{-1} w \gamma$ for $\gamma \in \bar{\Gamma}$ ($\gamma \neq 1$), we have

$$\left[ w^{-a^2} w^{-a}, w^{-a} w^{-1} \right] = \left( [w, a^{-1}], 1, 1 \right)$$

and we use this to show that $K \times K \times K \leq K$.

It follows that Proposition 13 holds for the Gupta-Sidki group $\bar{\Gamma}$.

Proposition 13 holds also for a third example, the Bartholdi-Grigorchuk group $\bar{\Gamma}$, generated by $a$ and

$$v = (a, a, v)$$

[BarGr-02]. Arguments for $\bar{\Gamma}$ are slightly different than for $\Gamma$ and $\bar{\Gamma}$, but we will not give details here.

REFERENCES


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