Sufficient Conditions for Stability of Linear Neutral Systems with a Single Delay

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Abstract—This paper deals with the asymptotic stability of linear neutral systems with a single delay. Simple delay-independent stability criteria are derived in terms of the measure and norm of the corresponding matrices. The significance of the main criterion is that it takes into consideration the structure information of the system matrices A, B, and C, thus reducing the conservatism found in the existing results. Numerical examples are given to demonstrate the validity of our main criteria.

Keywords—Neutral systems, Algebraic stability criteria, Delay independent.

1. INTRODUCTION

Asymptotic stability of neutral delay-differential systems is playing an increasingly important role in many disciplines such as engineering, science, and mathematics. A number of stability criteria based on the characteristic equation approach, involving the determination of eigenvalues, measures and norms of matrices, or matrix conditions in terms of Hurwitz matrices, have been presented by Stroinski [1], Hale et al. [2], Li [3], Hale and Verduyn Lunel [4], Hu and Hu [5], Bellen et al. [6], Park and Won [7], and Hu et al. [8]. Some stability criteria (delay-independent and/or delay-dependent) are given in terms of the Lyapunov function and matrix inequalities (see, for example, [9-14]).

This paper investigates the problem of asymptotic stability of linear neutral systems with a single time delay. Scalar inequalities involving the measures and norms of the corresponding matrices constitute the mathematical foundations of our approach. Based on the characteristic equation of the system, simple delay-independent stability criteria are derived. Involving

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the structure information of the coefficient matrices $A$, $B$, and $C$, the new criteria can significantly reduce the conservation of the results in the literature. Numerical examples are given to demonstrate the validity of our main criteria and to compare them with the existing ones.

2. NOTATIONS AND PRELIMINARIES

Let $\mathbb{R}^n(\mathbb{C}^n)$ denote the $n$-dimensional real (complex) space and $\mathbb{R}^{n \times n}(\mathbb{C}^{n \times n})$ denote the set of all real (complex) $n \times n$ matrices. $I$ denotes the unit matrix of appropriate order. $\lambda_j(A)$ and $\rho(A)$ denote the $j$th eigenvalue and the spectral radius of $A$, respectively. $|A|$ denotes the modulus matrix of $A$; $A \leq B$ represents that the elements of $A$ and $B$ satisfy the inequality $a_{ij} \leq b_{ij}$ for all $i$ and $j$. $\|A\| := \sqrt{\max(A^*A)}$ and $\mu(A) := (1/2)\max(A + A^*)$ denote the spectral norm and the matrix measure of $A$, respectively.

Consider the following linear neutral delay-differential system:

$$\dot{x}(t) = Ax(t) + [Bz(t - \tau) + Ck(t - \tau)],$$

where $x(t) \in \mathbb{C}^{n \times 1}$ is the state vector, the constant parameter $\tau \geq 0$ represents the delay argument, $A$, $B$, and $C \in \mathbb{C}^{n \times n}$. The system matrix $A$ is assumed to be a Hurwitz matrix, that is, all the eigenvalues of $A$ have negative real parts.

The following two lemmas are cited and will be used in the proof of our main results.

**LEMMA 2.1.** (See Theorem 1 in [5].) The neutral system (1) is asymptotically stable if $A$ is a Hurwitz matrix, $\rho(C) < 1$ and

$$\Re \lambda_1 [(I - \xi C)^{-1}(A + \xi B)] < 0, \quad \forall \xi \in \mathbb{C} \text{ such that } |\xi| \leq 1.$$

**LEMMA 2.2.** (See [15].) Let $R \in \mathbb{C}^{n \times n}$. If $\|R\| < 1$, then $(I - R)^{-1}$ exists, and $(I - R)^{-1} = I + R + R^2 + \cdots$.

3. MAIN RESULTS

**THEOREM 3.1.** Assume that $A$ is a Hurwitz matrix. Then, system (1) is asymptotically stable if $\|C\| < 1$ and there exists an invertible matrix $T$ such that

$$\varepsilon_1 T = \mu(T^{-1}AT) + \|T^{-1}(CA + B)T\| + \frac{\|T^{-1}\|\|C(CA + B)T\|}{1 - \|C\|} < 0.$$

**PROOF.** Taking notice of $\|\xi C\| \leq \|\xi\|\|C\|$, we have $||\xi C|| < 1$ for $|\xi| \leq 1$. Using Lemma 2.2, we have the following inequality:

$$\|T^{-1}\| = 1 + \|C\| + \|C^2\| + \cdots \leq (1 - \|C\|)^{-1}, \quad \text{for } |\xi| \leq 1.$$

According to Lemma 2.1, (1) is asymptotically stable if

$$\Re \lambda_1 [(I - \xi C)^{-1}(A + \xi B)] < 0, \quad \forall \xi \in \mathbb{C} \text{ such that } |\xi| \leq 1.$$

By making use of the relation

$$(I - \xi C)^{-1} = I + \xi(I - \xi C)^{-1}C,$$

we can obtain

$$(I - \xi C)^{-1}(A + \xi B) = (I + \xi(I - \xi C)^{-1}C)A + \xi(I - \xi C)^{-1}B$$

$$= A + \xi(I - \xi C)^{-1}(CA + B)$$

$$= A + \xi(CA + B) + \xi^2(I - \xi C)^{-1}C(CA + B).$$
Therefore, (5) is equivalent to

\[ \Re \lambda_i [A + \xi(CA + B) + \xi^2(I - \xi C)^{-1}C(CA + B)] < 0, \quad \forall \xi \in \mathbb{C} \text{ such that } |\xi| \leq 1. \]  

(8)

In view of the similarity invariants and the properties of the measure of matrix, we have, for $|\xi| \leq 1$,

\[ \Re \lambda_i [A + \xi(CA + B) + \xi^2(I - \xi C)^{-1}C(CA + B)] = \Re \lambda_i \left[ T^{-1}AT + \xi T^{-1}(CA + B)T + \xi^2 T^{-1}(I - \xi C)^{-1}C(CA + B)T \right] \]
\[ \leq \mu \left( T^{-1}AT \right) + \mu \left( \xi T^{-1}(CA + B)T \right) + \mu \left( \xi^2 T^{-1}(I - \xi C)^{-1}C(CA + B)T \right) \]
\[ \leq \mu \left( T^{-1}AT \right) + \left\| T^{-1}(CA + B)T \right\| + \left\| T^{-1}(I - \xi C)^{-1}C(CA + B)T \right\|. \]  

(9)

In terms of inequality (4), this leads to

\[ \Re \lambda_i [A + \xi(CA + B) + \xi^2(I - \xi C)^{-1}C(CA + B)] \]
\[ \leq \mu \left( T^{-1}AT \right) + \left\| T^{-1}(CA + B)T \right\| + \frac{\left\| T^{-1}\right\| \left\| C(CA + B)T \right\|}{1 - \|C\|}. \]  

(10)

Therefore, condition (3) implies that (5) holds. The proof is completed.

If $\mu(A) < 0$, taking $T = I$, one can directly obtain the following corollary.

**COROLLARY 3.1.** Assume that $A$ is a Hurwitz matrix. Then, system (1) is asymptotically stable if $\|C\| < 1$ and

\[ \varepsilon_{1T} \triangleq \mu(A) + \|CA + B\| + \frac{\|C(CA + B)\|}{1 - \|C\|} < 0. \]  

(11)

In analogy to that of Theorem 4 of [5], we define

\[ \beta_T(q) \triangleq \sum_{j=1}^{q} \left\| T^{-1}C^j(CA + B)T \right\| + \frac{T^{-1}\|C^q+1(CA + B)T\|}{1 - \|C\|}. \]  

(12)

**THEOREM 3.2.** Assume that $A$ is a Hurwitz matrix. Then, system (1) is asymptotically stable if $\|C\| < 1$ and there exists an invertible matrix $T$ such that

\[ \varepsilon_{2T} \triangleq \mu \left( T^{-1}AT \right) + \left\| T^{-1}(CA + B)T \right\| + \beta_T(q) < 0, \]  

(13)

where $\beta_T(q)$ is defined by (12).

**PROOF.** Following the process in the proof of Theorem 3.1, we only need to prove that (5) holds. Since $\|\xi C\| \leq \|C\| < 1$, it follows from Lemma 2.2 that

\[ (I - \xi C)^{-1}(A + \xi B) = A + \xi(I - \xi C)^{-1}(CA + B) \]
\[ = A + \xi \left( I + \xi C + \xi^2 C^2 + \cdots \right) (CA + B) \]
\[ = A + \sum_{j=0}^{q} \xi^{j+1} C^j(CA + B) \]
\[ + \xi^{q+2} \left( I + \xi C + \xi^2 C^2 + \cdots \right) C^{q+1}(CA + B) \]
\[ = A + \sum_{j=0}^{q} \xi^{j+1} C^j(CA + B) + \xi^{q+2}(I - \xi C)^{-1}C^{q+1}(CA + B). \]  

(14)
Therefore, in view of the similarity invariants and the properties of the measure of matrix, we have, for $|\xi| \leq 1$,
\[
\Re \lambda_i \left[ (I - \xi C)^{-1}(A + \xi B) \right] = \Re \lambda_i \left[ T^{-1}(I - \xi C)^{-1}(A + \xi B)T \right]
\]
\[
= -\Re \lambda_i \left[ T^{-1} \left( A + \sum_{j=0}^{q} \xi^{j+1} C^j (CA + B) + \xi^{q+2} (I - \xi C)^{-1} C^{q+1} (CA + B) \right) T \right]
\]
\[
\leq \mu (T^{-1} A T) + \|T^{-1} (CA + B)T\| + \beta_T(q),
\]
where $\beta_T(q)$ is defined by (12). Therefore, condition (13) implies that (5) holds. The proof is completed.

**Remark 3.1.** Since for integer $q \geq 1$
\[
\beta_T(q + 1) - \beta_T(q) = \|T^{-1} C^{q+1} (CA + B)T\| + \|T^{-1}\| C^{q+2} (CA + B)\| 1 - \|C\|^{-1}
\]
\[
- \|T^{-1}\| C^{q+1} (CA + B)\| 1 - \|C\|^{-1}
\]
\[
\leq \|T^{-1}\| \left( 1 + \|C\|^{-1} \right) \|C^{q+1} (CA + B)\| = 0,
\]
we have $\beta_T(q + 1) \leq \beta_T(q)$. Moreover, one can easily prove that Theorem 3.2 is sharper than Theorem 3.1.

If $\mu(A) < 0$, taking $T = I$, one can directly obtain the following corollary.

**Corollary 3.2.** Assume that $A$ is a Hurwitz matrix. Then, system (1) is asymptotically stable if $\|C\| < 1$ and
\[
\varepsilon_{21} \triangleq \mu(A) + \|CA + B\| + \beta_1(q) < 0,
\]
where
\[
\beta_1(q) \triangleq \sum_{j=1}^{q} \|C^j (CA + B)\| + \frac{\|C^{q+1} (CA + B)\|}{1 - \|C\|}.
\]

**4. ILLUSTRATIVE EXAMPLES**

We will compare our new criteria with the following criteria for asymptotic stability of system (1) in the case of $\|C\| < 1$:

**Criterion 1** [5]:
\[
k_1 \triangleq \mu(A) + \|B\| + \frac{\|CA\| + \|CB\|}{1 - \|C\|} < 0,
\]

**Criterion 2** [3]:
\[
k_2 \triangleq \mu(A) + \|B\| + \frac{\|CA\| + \|C\| \|B\|}{1 - \|C\|} < 0,
\]

**Criterion 3** [6]:
\[
k_3 \triangleq \mu(A) + \frac{\|AC + B\|}{1 - \|C\|} < 0,
\]

**Criterion 4** [8, Theorem 3.3]:
\[
k_4 \triangleq \rho(G_0) < 1 \text{ and } \rho(\|N\|) < 1,
\]

**Criterion 5** [8, Theorem 3.4]:
\[
k_5 \triangleq \rho(G_q) < 1 \text{ for some integer } q > 1 \text{ and } \rho(\|N\|) < 1,
\]
where
\[
G_0 = |L| + |M| + (I - |N|)^{-1}(|NL| + |NM|),
\]
\[
G_q = |L| + |M| + \sum_{j=1}^{q} \left( |N^j L| + |N^j M| \right) + (I - |N|)^{-1} \left( |N^{q+1} L| + |N^{q+1} M| \right),
\]
and
\[ L = (I - A)^{-1}(B + C), \quad M = (I - A)^{-1}(B - C), \quad N = (I - A)^{-1}(I + A). \]

**Example 4.1.** Consider system (1) with
\[
A = \begin{bmatrix} -29 & 1 \\ -2 & -30 \end{bmatrix}, \quad B = \begin{bmatrix} 21.1 & -3.7 \\ -16.8 & 10.6 \end{bmatrix}, \quad C = \begin{bmatrix} 0.7 & -0.1 \\ -0.6 & 0.3 \end{bmatrix}.
\]

It is evident that the system matrix \( A \) is Hurwitz. With simple computation, we have \( ||C|| = 0.9621, \epsilon_1 = -2.3868 < 0, \) and \( \epsilon_2 = -21.8534 < 0 \) for integer \( q = 20 \). According to Corollary 3.1 (or Corollary 3.2), system (1) is asymptotically stable. Since \( k_1 = 1392.9348 > 0, k_2 = 1496.5837 > 0, k_3 = 20.9667 > 0, \) and \( k_4 = 26.0148 > 1, \) we cannot determine whether system (1) is stable using Criterion 1–Criterion 4. Moreover, one can calculate that \( \rho(||L|| + |M||) = 1.6444 > 1. \) Thus, for all \( q \geq 1, k_5 = \rho(G_q) > 1. \) This shows all the cited criteria are not applicable in this example.

**Example 4.2.** Consider system (1) with
\[
A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1.3 & 0.6 \\ -0.7 & 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & 0.1 \\ -0.2 & 0.1 \end{bmatrix}.
\]

Since \( \mu(A) = 0.0811 > 0, \) Corollary 3.1 and Corollary 3.2 are not applicable. Take a similarity transformation by the following similarity matrix:
\[
T = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.
\]

Then, with simple computation, we have \( ||C|| = 0.3618, \epsilon_{IT} = 1.1229 > 0, \) and \( \epsilon_{IT} = -0.0194 < 0 \) for integer \( q = 8. \) Since \( \epsilon_{IT} > 0, \) the conditions of Theorem 3.1 are not satisfied. However, according to Theorem 3.2, system (1) is asymptotically stable. This shows Theorem 3.2 is sharper than Theorem 3.1. Making the similarity transformation (19), we can calculate \( k_1 = 5.4564 > 0, k_2 = 6.1858 > 0, k_3 = 1.3514 > 0, \) and \( k_4 - k_5 - 1.0742 > 1. \) Thus, all the cited criteria are not applicable in this example.

**REFERENCES**