JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **183**, 155–166 (1994)

Inequalities for Means

M. K. VAMANAMURTHY*

Department of Mathematics and Statistics, University of Auckland, Auckland, New Zealand

AND

M. VUORINEN†

Department of Mathematics, University of Helsinki, 00100 Helsinki, Finland

Submitted by E. R. Love

Received April 7, 1992

A monotone form of L'Hospital's rule is obtained and applied to derive inequalities between the arithmetic–geometric mean of Gauss, the logarithmic mean, and Stolarsky's identric mean. Some related inequalities are given for complete elliptic integrals. © 1994 Academic Press, Inc.

1. INTRODUCTION

For positive x and y, the *arithmetic mean*, the *geometric mean*, the *logarithmic mean*, and the *Gauss arithmetic–geometric mean* (AGM), are defined by

$$A(x, y) = \frac{x+y}{2}, \qquad G(x, y) = \sqrt{xy},$$

$$L(x, y) = \frac{x-y}{\log x - \log y}, \qquad x \neq y, \quad L(x, x) = x,$$

$$AG(x, y) = \lim x_n = \lim y_n,$$

where $x_0 = x$, $y_0 = y$ and $x_{n+1} = A(x_n, y_n)$, $y_{n+1} = G(x_n, y_n)$. We study some generalizations of these given, e.g., in [8].

A very extensive bibliography on the AGM appears in [6]. The books [23, 14] are excellent references for general properties of means.

* E-mail: vamanamu@mat.aukuni.ac.nz.

† E-mail: vuorinen@csc.fi.

0022-247X/94 $6.00

Copyright © 1994 by Academic Press, Inc.
All rights of reproduction in any form reserved.

Next, the *Stolarsky mean* is defined by

$$S_p(x, y) = \left[\frac{x^p - y^p}{p(x-y)}\right]^{1/(p-1)}, \qquad p \neq 0, 1,$$

with

$$S_0(x, y) = \lim_{p \to 0} S_p(x, y) = L(x, y)$$

and

$$S_1(x, y) = \lim_{p \to 1} S_p(x, y) = e^{-1}(x^x y^{-y})^{1/(x-y)} = I(x, y).$$

The mean $I(x, y)$ is known as the *identric mean.*

We also need the *t-modification of a mean* defined by

$$M_t(x, y) = M(x^t, y^t)^{1/t}, \qquad t \in R \setminus \{0\},$$

for $M = A$, G, AG, and L. Clearly $M_t(x, y) = M_t(y, x)$, $M_{-t}(1/x, 1/y) \cdot M_t(x, y) = 1$, and $M_1 = M$. Hence it is enough to study only the case $t > 0$. It is well known that for each $x, y > 0$, the means $A_t(x, y)$ and $S_t(x, y)$ are continuous increasing functions of t [7, 26]. We first obtain a similar result for L_t and AG_t, making use of the following variant of L'Hospital's rule, which should also be of general interest, cf. [23, p. 106].

1.1. Lemma (Monotone Form of L'Hospital's Rule). *For $a < b$, let f, g be continuous on $[a, b]$ and differentiable on (a, b) and let g' never vanish on (a, b). If f'/g' is (strictly) increasing (respectively, decreasing) on (a, b), then so are $(f(x) - f(a))/(g(x) - g(a))$ and $(f(x) - f(b))/(g(x) - g(b))$.*

1.2. Theorem. *For x, y positive and distinct,*

(1) *$L_t(x, y)$ is a continuous and strictly increasing function of t from $(0, \infty)$ onto $(\sqrt{xy}, \max\{x, y\})$,*

(2) *$AG_t(x, y)$ is a continuous and strictly increasing function of t from $(0, \infty)$ onto $(\sqrt{xy}, \max\{x, y\})$.*

There are several inequalities between these means. From the definition it is clear that $G(x, y) \leqslant AG(x, y) \leqslant A(x, y)$ and $L_t^t = S_t^{t-1} S_0$. The inequality $G(x, y) \leqslant L(x, y)$ is given in [16, 18, 19, p. 21]. Very recently, the inequality

$$L(x, y) \leqslant AG(x, y)$$

appeared in [21]. The next result gives majorants for AG in terms of L, I, and A.

1.3. THEOREM. *For x, y positive and distinct,*

(1) $AG(x, y) < L_2(x, y) = [A(x, y)\, L(x, y)]^{1/2}$.

(2) $AG(x, y) < (\pi/2)\, L(x, y)$.

(3) $AG(x, y) < I(x, y) < A(x, y)$.

(4) $AG(x, y) < A_{1/2}(x, y)$.

As a consequence we obtain, e.g., the inequalties

$$G(x, y) < L(x, y) < AG(x, y) < I(x, y) < A(x, y) \tag{1.4}$$

for all x, y positive and distinct.

We recall the *Gauss identity* [2, 8, 19, 22],

$$AG(1, r')\, \mathscr{K}(r) = \frac{\pi}{2}, \tag{1.5}$$

for r in $[0, 1)$ and $r' = \sqrt{1 - r^2}$. As usual, $\mathscr{K}$ and $\mathscr{E}$ denote the complete elliptic integrals [17] given by

$$\begin{aligned} \mathscr{K}(r) &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \qquad \mathscr{K}'(r) = \mathscr{K}(r'), \\ \mathscr{E}(r) &= \int_0^1 \sqrt{\frac{1-r^2x^2}{1-x^2}}\, dx, \qquad \mathscr{E}'(r) = \mathscr{E}(r'). \end{aligned} \tag{1.6}$$

Thus in view of (1.4) and (1.5), Theorems 1.2 and 1.3 give inequalities for these elliptic integrals. See [12, 9, 11, 28] for recent extensions of (1.5) and [15, 24] for other recent results on the AGM. Note that in [1, 17.3.1] the argument of $\mathscr{K}(r)$ is written as r^2.

2. PROOFS

2.1. *Proof of Lemma* 1.1. By the intermediate value property [5, Theorem 5.16] for derivatives, it follows that $g'(x)$ never changes its sign on (a, b). Suppose that $g'(x)$ is positive and $f'(x)/g'(x)$ is strictly increasing. By the Cauchy mean value theorem [5, Theorem 5.11], for each x in (a, b), there is a y in (a, x) such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(y)}{g'(y)} < \frac{f'(x)}{g'(x)},$$

which yields

$$\frac{d}{dx}\left[\frac{f(x) - f(a)}{g(x) - g(a)}\right] > 0.$$

The other cases are proved similarly. ∎

2.2. LEMMA. (1) $f(r) = (\mathcal{E}(r) + r'\mathcal{K}(r))/(1 + r')$ *is strictly decreasing from* $(0, 1)$ *onto* $(1, (\pi/2))$.

(2) $g(r) = (1 - r')\,\mathcal{K}(r)/\log(1/r')$ *is strictly decreasing from* $(0, 1)$ *onto* $(1, (\pi/2))$.

(3) $h(r) = (r\mathcal{K}(r))^2/\log(1/r')$ *is strictly increasing from* $(0, 1)$ *onto* $((\pi^2/2), \infty)$.

(4) $F(r) = (r \log r)/(r - 1) - 2\log(1 + \sqrt{r})$ *is strictly decreasing from* $(0, 1)$ *onto* $(\log(e/4), 0)$.

(5) $G(r) = (1 + \sqrt{r'})^2\,\mathcal{K}(r)$ *is strictly increasing from* $(0, 1)$ *onto* $(2\pi, \infty)$.

(6) $H(r) = (r \log r)/(r - 1) - \log(1 + r)$ *is strictly increasing from* $(0, 1)$ *onto* $(0, \log(e/2))$.

Proof. From [8, Theorem 1.2(d)], $f(r) = \mathcal{E}[(1 - r')/(1 + r')]$ and (1) follows. For (2), let $g_1(r)$ and $g_2(r)$ denote the numerator and denominator of $g(r)$, respectively. Then [17, 710.00] $g_1'(r)/g_2'(r) = f(r)$, and the result follows from (1) and Lemma 1.1. Next, for (3), let $h_1(r)$ and $h_2(r)$ denote the numerator and denominator of $h(r)$, respectively. Then [17, 710.00–02] $h_1'(r)/h_2'(r) = 2\mathcal{K}(r)\,\mathcal{E}(r)$, which is easily shown to be increasing from $(0, 1)$ onto $(\pi^2/2, \infty)$, so that (3) follows by Lemma 1.1. The assertion (4) follows by differentiating and using the elementary inequality $\log x < (x - 1)/\sqrt{x}$, for all $x > 1$. For (5), if we apply Landen's transformation [8, Theorem 1.2(b)] twice, we get

$$G(r) = 4\mathcal{K}\left[\left(\frac{1 - \sqrt{r'}}{1 + \sqrt{r'}}\right)^2\right];$$

hence the result follows. Finally, (6) follows by writing $H(r)$ as a quotient and applying Lemma 1.1. ∎

2.3. *Remark.* There is a slight error in [8, Exercise 2(b), p. 16]. In the identity

$$\mathcal{K}(x) = \frac{4}{(1 + \sqrt{x'})^2}\,\mathcal{K}\left[\left(\frac{1 - \sqrt[4]{1 - x^4}}{1 + \sqrt[4]{1 - x^4}}\right)^2\right],$$

the expression $\sqrt[4]{1 - x^4}$ should be corrected to $\sqrt{x'}$, where $x' = \sqrt{1 - x^2}$.

The next result is an immediate consequence of Lemma 2.2.

2.4. COROLLARY. *For* r *in* $(0, 1)$, $r' = \sqrt{1 - r^2}$,

$$1 < \frac{\mathcal{E}(r) + r'\mathcal{K}(r)}{1 + r'} < \frac{\pi}{2}, \tag{1}$$

$$1 < \frac{(1 - r')\,\mathcal{K}(r)}{\log(1/r')} < \frac{\pi}{2}, \tag{2}$$

$$\frac{\pi^2}{2} < (r\mathscr{K}(r))^2/\log(1/r'), \tag{3}$$

$$\pi e < 2\mathscr{K}(r)(r')^{r'/(r'-1)}. \tag{4}$$

2.5. *Proof of Theorem* 1.3 (1). Assuming $0<x<y$, divide by y and put $r'=x/y$. By the Gauss identity (1.5) the inequality reduces to Corollary 2.4 (3). ∎

2.6. *Proof of Theorem* 1.3 (2). Assuming that $0<x<y$, setting $x/y=r'$, $0<r<1$, and using (1.5), the inequality follows immediately from Corollary 2.4(2). ∎

2.7. *Proof of Theorem* 1.3 (3). By homogeneity we may assume that $x=1$ and $0<y<1$. Then the first inequality follows from Corollary 2.4(4) and the second inequality follows from Lemma 2.2(6). ∎

2.8. *Proof of Theorem* 1.3 (4). Assuming $0<y<x$, setting $r'=y/x$ and using (1.5), the result follows from Lemma 2.2(5). ∎

2.9. *Proof of Theorem* 1.2 (1). Continuity is obvious. Assuming $0<x<y$, put $u=y/x$ and $v=u^t$ and let $f(t)=L_t(x,y)$. Denote $h(t)=\log((v-1)/(\log v))$ and $g(t)=\log f(t)-\log x$. Then

$$g(t)=\log f(t)-\log x=\frac{h(t)}{t},$$

and

$$h'(t)=\frac{(\log u)\,F(v)}{G(v)},$$

where $F(v)=v\log v-v+1$, and $G(v)=(v-1)\log v$. Then $F(1)=F'(1)=G(1)=G'(1)=0$, and $F''(v)/G''(v)=v/(v+1)$. Hence by Lemma 1.1, $g(t)$ is increasing and so $f(t)$ is increasing. Next as t tends to 0, by L'Hospital's Rule,

$$\begin{aligned}\lim_{t\to 0} g(t)=\lim_{t\to 0} h'(t)&=\lim_{v\to 1}\frac{v\log v-v+1}{(v-1)\log v}(\log u)\\ &=(\log u)\lim_{v\to 1}\frac{v}{v+1}=\log\sqrt{u}.\end{aligned}$$

Hence

$$\lim_{t\to 0}\log f(t)=\log x+\log\sqrt{u}=\log\sqrt{xy},$$

so that

$$\lim_{t \to 0} f(t) = \sqrt{xy}.$$

Finally,

$$\lim_{t \to \infty} g(t) = \lim_{t \to \infty} h'(t) = \log(u) \lim_{v \to \infty} \frac{v}{v+1} = \log u,$$

and $\lim_{t \to \infty} f(t) = y$. ∎

2.10. *Proof of Theorem* 1.2 (2). Continuity is clear. Assuming $0 < x < y$, let $f(t) = AG_t(x, y)$, $r = x/y$ and $u' = r^t$. Then $t = (\log(1/u'))/(\log(1/r))$ so that

$$\frac{\log y - \log f(t)}{\log(1/r)} = \frac{\log(2\mathcal{K}(u)/\pi)}{\log(1/u')} = \frac{g(u)}{h(u)},$$

where $h(u) = \log(1/u')$ and $g(u) = \log(2\mathcal{K}(u)/\pi)$. (Note: If t increases from 0 to ∞, then u increases from 0 to 1). Then by [17, 710.00–02]

$$\frac{h'(u)}{g'(u)} = \frac{u^2 \mathcal{K}}{\mathcal{E} - (u')^2 \mathcal{K}} = \frac{h_1(u)}{g_1(u)},$$

where

$$\mathcal{K} = \mathcal{K}(u), \qquad \mathcal{E} = \mathcal{E}(u), \qquad h_1(u) = u^2 \mathcal{K}, \qquad g_1(u) = \mathcal{E} - (u')^2 \mathcal{K}.$$

Next by [17, 710.00–04]

$$\frac{h_1'(u)}{g_1'(u)} = 2 + \frac{\mathcal{E} - (u')^2 \mathcal{K}}{(u')^2 \mathcal{K}},$$

which is increasing, since $\mathcal{E} - (u')^2 \mathcal{K}$ is increasing and $(u')^2 \mathcal{K}$ is decreasing [17, 710.04; 3, 2.2(3)]. Hence applying Lemma 1.1 twice we see that $h(u)/g(u)$ is increasing. Now first

$$\lim_{u \to 0} \frac{h(u)}{g(u)} = \lim_{u \to 0} \frac{h'(u)}{g'(u)} = \lim_{u \to 0} \frac{h_1'(u)}{g_1'(u)} = 2.$$

Next as t goes to infinity (i.e., u goes to 1),

$$\lim_{u \to 1} \frac{h(u)}{g(u)} = \lim_{u \to 1} \frac{h'(u)}{g'(u)} = \infty.$$

Consequently, $f(t)$ is increasing and

$$\lim_{t\to 0} f(t)=\sqrt{xy}, \qquad \lim_{t\to\infty} f(t)=y. \quad \blacksquare$$

2.11. COROLLARY. *For each $x, y, t>0$, $x\neq y$, we have*

$$\sqrt{xy}<L_t(x, y)<AG_t(x, y)<\max\{x, y\}.$$

The first inequality is sharp as t tends to 0, while the last inequality is sharp as t tends to ∞.

2.12. THEOREM. *For $r\in(0, 1)$, we have*

$$\log\left(\frac{e}{r'}\right)<\mathscr{K}(r)/\mathscr{E}(r)<\log\left(\frac{4}{r'}\right).$$

Proof. Let f be defined on $[0, 1]$ by $f(0)=0$, $f(1)=\log(4/e)$, and

$$f(r)=\log(r')+\frac{\mathscr{K}(r)-\mathscr{E}(r)}{\mathscr{E}(r)}$$

for $0<r<1$. Then

$$f'(r)=\frac{(\mathscr{K}-\mathscr{E})^2}{r\mathscr{E}^2}>0$$

on $(0, 1)$. Thus f is strictly increasing on $[0, 1]$. Hence $0<f(r)<\log(4/e)$ on $(0, 1)$, and the result follows. $\blacksquare$

Since $\mathscr{E}(r)\in(1, (\pi/2))$ for all $r\in(0, 1)$ it follows by [3, 2.3] that

$$\mathscr{K}(r)\leqslant\frac{\pi}{2}\log\left(\frac{e}{r'}\right).$$

These two facts together with Theorem 2.12 show that the inequality in Theorem 2.12 is quite sharp.

2.13. THEOREM. *For all positive x, y*

$$A(x, y)\leqslant AG(x^2, y^2)/AG(x, y)\leqslant A_2(x, y),$$

with equality iff $x=y$.

Proof. The first inequality appears in [10, Proposition 2.6], but we give here an alternative proof. By symmetry and homogeneity we may assume that $x<y=1$. If we put $r'=x$ in Lemma 2.2(2) we see that the function

$$(1-x)\,\mathscr{K}'(x)/[\log(1/x)]$$

is strictly increasing from (0, 1) onto $(1, \pi/2)$. In particular since $x > x^2$, this gives

$$\mathscr{K}'(x) > [(1+x)/2]\, \mathscr{K}'(x^2)$$

as desired. Next, writing in terms of elliptic integrals (cf. (1.5)), the second inequality is equivalent to

$$\mathscr{K}'(x) \leqslant A_2(1, x)\, \mathscr{K}'(x^2)$$

which by the Landen transform [8, Theorem 1.2] can be written as

$$\mathscr{K}'(x) \leqslant \mathscr{K}'\left(\frac{2x}{1+x^2}\right)\Big/ A_2(1, x),$$

or also as

$$\sqrt{x}\, \mathscr{K}'(x) \leqslant \sqrt{\frac{2x}{1+x^2}}\, \mathscr{K}'\left(\frac{2x}{1+x^2}\right).$$

This inequality is true since the function $f(u) = \sqrt{u}\, \mathscr{K}'(u)$ is increasing on (0, 1), by [3, Theorem 2.2(3)]. ∎

3. Sharpness of Results

A natural question is whether we can sharpen the earlier inequalities by replacing a mean M by its t-modification M_t and then adjusting the parameter t optimally. First, let us point out that the inequality

$$G_t(x, y) \leqslant L_t(x, y) \leqslant AG_t(x, y) \leqslant A_t(x, y)$$

for all $t > 0$ and all $x, y > 0$ follows directly from (1.4). Note that $G_t = G$ for all $t > 0$.

3.1. Theorem. *The inequality in* 1.3(4) *is sharp in the sense that* $\frac{1}{2}$ *cannot be replaced by any smaller constant.*

Proof. Since for $t > 0$ and small $x > 0$

$$AG(1-x, 1) = 1 - \frac{x}{2} - \frac{x^2}{16} + O(x^3),$$

$$A_t(1-x, 1) = 1 - \frac{x}{2} - \frac{(1-t)x^2}{8} + O(x^3),$$

we see that the inequality $AG(1-x, 1) \leqslant A_t(1-x, 1)$ holds for small x only if $t \geqslant \frac{1}{2}$. ∎

In view of Theorem 1.3 (3) it is natural to ask if the inequality $AG \leqslant I$ can be improved to a better one of the form $AG \leqslant S_p$ with $p < 1$. Indeed since

$$S_{1/2}(x, y) = A_{1/2}(x, y), \tag{3.2}$$

it follows that $AG \leqslant S_{1/2}$ holds. Since we know that $A_0 = G \leqslant L = S_0$ we may ask if $A_p \leqslant S_p$ for $p \in (0, \frac{1}{2}]$. The next theorem provides an answer.

3.3. THEOREM. *Let x, y be positive and distinct. Then*

$$S_{p+1}(x, y) \leqslant A_p(x, y), \tag{1}$$

for $p \in [1, \infty)$ with equality iff $p = 1$,

$$S_{p+1}(x, y) > A_p(x, y), \tag{2}$$

for each $p \in (0, 1)$,

$$S_p(x, y) \leqslant A_p(x, y) \leqslant S_{p+1}(x, y), \tag{3}$$

for $p \in [\frac{1}{2}, 1]$ with equality on the left iff $p = \frac{1}{2}$ and on the right iff $p = 1$, and finally

$$A_p(x, y) < S_p(x, y), \tag{4}$$

for $p \in (0, \frac{1}{2})$.

Proof. The cases of equality are clear and we only prove strict inequalities. We may clearly assume that $y = 1$ and $x > 1$. We have

$$S_{p+1}(x, 1)^p = \frac{x^{p+1} - 1}{(p+1)(x-1)}, \qquad A_p(x, 1)^p = \frac{x^p + 1}{2}.$$

Let

$$f(x) = (p-1)(1 - x^{p+1}) + (p+1)(x^p - x).$$

Then $f(1) = 0$ and $f'(x) = (p+1)[-(p-1)x^p + px^{p-1} - 1]$. Now $f'(1) = 0$, and

$$f''(x) = p(p+1)(p-1)(1-x)\,x^{p-2}.$$

If $p > 1$ then $f''(x) < 0$, hence $f'(x) < f'(1) = 0$, so that $f(x) < f(1) = 0$ and part (1) is proved. Next if $p \in (0, 1)$, then $f''(x) > 0$, hence $f'(x) > f'(1) = 0$, so that $f(x) > f(1) = 0$ and (2) follows.

The second inequality in (3) follows from (2), since $A_1(x, y) = S_2(x, y)$. To prove the first inequality in (3) observe first that $S_{1/2}(x, y) = A_{1/2}(x, y)$ is obvious by (3.2) while

$$S_1(x, y) = I(x, y) < A(x, y)$$

was proved in Theorem 1.3(3). Assume next that $p \in (\frac{1}{2}, 1)$. Then $S_p(x, 1) < A_p(x, 1)$, iff $S_{1/p}(u, 1) < A(u, 1)$, where $u = x^p$. But this is clearly true since $1/p < 2$ and so $S_{1/p}(x, 1) < S_2(u, 1) = A(u, 1)$ by [26]. Finally to prove (4), let $p \in (0, \frac{1}{2})$ and $u = x^p$. Then $S_p(x, 1) > A_p(x, 1)$, iff $S_{1/p}(u, 1) > A(u, 1)$, which is clearly true, since $1/p > 2$, so that $S_{1/p}(u, 1) > S_2(u, 1)$ by [26]. ∎

In view of Theorem 1.3(2) we may ask if $AG \geqslant L_t$ for some $t > 1$.

3.4. THEOREM. *The inequality* $AG \geqslant L_t$ *holds if and only if* $t \in (0, 1]$. *Furthermore, for each* $t \in (1, \frac{3}{2})$ *there exists an* $x_0 \in (0, 1)$ *such that the inequalities,*

$$AG(1, x) < L_t(1, x), \qquad L_t(1, 1-x) < AG(1, 1-x),$$

hold for all $x \in (0, x_0)$. *Furthermore, for* $t > 1$, $L_t(1, x)/AG(1, x) \to \infty$ *as* $x \to 0$.

Proof. To prove the last assertion, we observe that by [19, (6.10-8) and (8.3-16)] for $t > 1$

$$\frac{L_t(x, 1)}{AG(x, 1)} \sim \frac{2}{\pi} t^{-1/t} \left(\log \frac{1}{x}\right)^{1 - 1/t} \to \infty$$

as $x \to 0$. This also proves the first assertion. For the second assertion, let $1 < t < \frac{3}{2}$. Now

$$AG(1-x, 1) = 1 - \frac{x}{2} - \frac{x^2}{16} + O(x^3),$$

$$L_t(1-x, 1) = 1 - \frac{x}{2} + \frac{t-3}{24} x^2 + O(x^3),$$

for small $x > 0$. The second assertion follows from these expansions. ∎

Further results relating S_q and A_p with $q = (p+1)/3$ were obtained by K. B. Stolarsky [27].

3.5. *Remark.* Theorem 3.4 is due to B. C. Carlson. He has kindly informed us that Theorem 3.3, except the first inequality in (3), can also be derived from Theorem 4 in [20]. Other kinds of inequalities for means occur in [13].

Finally, we give a recent result due to Borwein and Borwein [10] which provides an extremely sharp majorant for $AG(1, x)$ for x close to 1. Recall from the introduction that the inequality $L(x, y) \leqslant AG(x, y)$ appears in [21].

3.6. THEOREM [10]. *The inequality* $AG(x, y) \leqslant L_{3/2}(x, y)$ *holds for all* $x, y > 0$.

3.7. *Remark.* Lin [25] has proved that the inequality $L \leqslant A_t$ holds if and only if $t \in [\frac{1}{3}, \infty)$. Consequently, for $s, t \in (0, \infty)$, we have $L_s \leqslant A_t$ iff $0 < s \leqslant 3t$. In particular, we see that $L_{3/2} \leqslant A_{1/2}$ and hence Theorem 3.6 improves Theorem 1.3(4).

3.8. *Open Problems.* (1) Is it true that $AG_t \geqslant L$ for some $t \in (0, 1)$?

(2) Several conjectures about the behavior of $AG(1, x)$ as $x \to 0$ are given in [4, 3.22]. We now recall one of those conjectured inequalities written in terms of $\mathscr{K}(r)$ for $r \in (0, 1)$ and $r' = \sqrt{1 - r^2}$:

$$\mathscr{K}(r) < \log\left(1 + \frac{4}{r'}\right) - \left(\log 5 - \frac{\pi}{2}\right)(1 - r).$$

ACKNOWLEDGMENTS

The authors thank Professors G. D. Anderson, D. Askey, J. M. Borwein, B. C. Carlson, H. Flanders, and the referee for many helpful comments. The manuscript was completed during the first author's visit to Michigan State University on sabbatical leave from University of Auckland and the second author's visit to the University of Michigan under the support of a grant of the Academy of Finland and Professor F. W. Gehring's NSF Grant NSF-DMS-9003438. The authors also thank J. Sandor for recently drawing our attention to the paper, "Inequalities between arithmetic and logarithmic means", Univ. Beograd, *Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **680** (1980), 15–18, by A. O. Pittenger, which contains several interesting inequalities.

REFERENCES

1. M. ABRAMOWITZ AND I. A. STEGUN, "Pocketbook of Mathematical Functions," Verlag Harri Deutsch, Frankfurt am Main, 1984. [Abridged edition of Handbook of Mathematical Functions, material selected by M. Danos and J. Rafelski]
2. G. ALMKVIST AND B. BERNDT, Gauss, Landen, Ramanujan, the arithmetic–geometric mean, ellipses, pi, and the *Ladies' Diary*, *Amer. Math. Monthly* **95** (1988), 585–608.
3. G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN, Functional inequalities for complete elliptic integrals, *SIAM J. Math. Anal.* **21** (1990), 536–549.

4. G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN, Functional inequalities for hypergeometric functions and complete elliptic integrals, *SIAM J. Math. Anal.* **23** (1992), 512–524.
5. T. M. APOSTOL, "Mathematical Analysis," 2nd ed. Addison–Wesley, Reading, MA, 1975.
6. J. ARAZY, T. CLEASON, S. JANSON, AND J. PEETRE, Means and their iterations, *in* "Proceedings, 19th Nordic Congress of Mathematics, Reykjavik, 1984," pp. 191–212, Icelandic Math. Soc., 1985.
7. E. F. BECKENBACH AND R. BELLMAN, "Inequalities," Springer-Verlag, Berlin–Heidelberg–New York, 1961.
8. J. M. BORWEIN AND P. B. BORWEIN, "Pi and the AGM," Wiley, New York, 1987.
9. J. M. BORWEIN AND P. B. BORWEIN, A cubic counterpart of Jacobi's identity and the AGM, *Trans. Amer. Math. Soc.* **323** (1991), 691–701.
10. J. M. BORWEIN AND P. B. BORWEIN, Inequalities for compound mean iterations with logarithmic asymptotes, *J. Math. Anal. Appl.* **177** (1993), 572–582.
11. J. M. BORWEIN, P. B. BORWEIN, AND F. GARVAN, Hypergeometric analogues of the arithmetic–geometric mean iteration, in preparation.
12. P. B. BORWEIN, Quadratically converging rational mean iterations, *J. Math. Anal. Appl.* **154** (1991), 364–376.
13. J. L. BRENNER AND B. C. CARLSON, Homogeneous mean values, *J. Math. Anal. Appl.* **123** (1987), 265–280.
14. S. BULLEN, D. S. MITRINOVIĆ, AND P. M. VASIĆ (Eds.), "Means and Their Inequalities," Reidel, Dordrecht, 1988.
15. S. BULLETT, Dynamics of the arithmetic–geometric mean, *Topology* **30** (1991), 171–190.
16. F. BURK, Geometric, logarithmic, and arithmetic mean inequalities, *Amer. Math. Monthly* **94** (1987), 527–528.
17. P. F. BYRD AND M. D. FRIEDMAN, "Handbook of Elliptic Integrals for Engineers and Physicists," Die Grundlehren der Mathematischen Wissenschaften, Vol. 57, Springer-Verlag, Berlin/Göttingen/Heidelberg, 1954.
18. B. C. CARLSON, The logarithmic mean, *Amer. Math. Monthly* **79** (1972), 615–618.
19. B. C. CARLSON, "Special Functions of Applied Mathematics," Academic Press, New York, 1977.
20. B. C. CARLSON AND M. D. TOBEY, A property of the hypergeometric mean value, *Proc. Amer. Math. Soc.* **19** (1968), 255–262.
21. B. C. CARLSON AND M. VUORINEN, An inequality of the AGM and the logarithmic mean, *SIAM Rev.* **33** (1991), Problem 91-17, 655.
22. D. A. COX, Gauss and the arithmetic–geometric mean, *Notices Amer. Math. Soc.* **32** (1985), 147–151.
23. G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, "Inequalities," Cambridge Univ. Press, London/New York, 1959.
24. H. HARUKI, New characterisations of the arithmetic–geometric mean of Gauss and other well-known mean values, *Publ. Math. Debrecen* **38** (1991), 323–332.
25. T.-P. LIN, The power mean and the logarithmic mean, *Amer. Math. Monthly* **81** (1974), 879–883.
26. K. B. STOLARSKY, Generalizations of the logarithmic mean, *Math. Mag.* **48** (1975), 87–92.
27. K. B. STOLARSKY, The power and generalized logarithmic means, *Amer. Math. Monthly* **87** (1980), 545–548.
28. J. TODD, The Weierstrassian mean, *Numer. Math.* **57** (1990), 737–746.