Little Rip, ΛCDM and singular dark energy cosmology from Born–Infeld-$f(R)$ gravity

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A R T I C L E I N F O

A B S T R A C T

We study late-time cosmic accelerating dynamics from Born–Infeld-$f(R)$ gravity in a simplified conformal approach. We find that a variety of cosmic effects such as Little Rip, ΛCDM universe and dark energy cosmology with finite-time future singularities may occur. Unlike the convenient Born–Infeld gravity where in the absence of matter only de Sitter expansion may emerge, apparently any FRW cosmology maybe reconstructed from this conformal version of the Born–Infeld-$f(R)$ theory. Despite the fact that the explicit form of $f(R)$ is fixed by the conformal ansatz, the relation between the two metrics in this approach may be changed so as to bring out any desired FRW cosmology.

Modified gravity provides a very natural approach for the description of the evolution of the universe by generating early-time as well as late-time cosmic acceleration via the modification of standard General Relativity (for recent reviews on modified gravity, see [1]). It is remarkable that within this approach a unified description of the early-time inflation with late-time dark energy is possible, as first proved in Ref. [2].

An interesting model of modified gravity extensively considered recently is the so-called Born–Infeld (BI) theory [3], whose Palatini formulation avoids the appearance of ghosts. Some cosmological properties of BI gravity have been discussed in a number of works [4]. The peculiarities of the Palatini formulation imply that the connection is compatible with an auxiliary metric algebraically related with the space-time metric via the matter sources. This type of algebraic relations makes it quite difficult to get consistent generalizations of the original BI model. However, we have recently demonstrated [5] that a non-perturbative and consistent generalization of BI gravity is possible when an $f(R)$ term is added to the original theory (hence dubbed BI-$f(R)$ theory). In the present letter we consider the BI-$f(R)$ theory using a simplified conformal ansatz which, despite not being the most general approach (see Ref. [5] for full details), it may be used to discuss a number of interesting situations. In fact, we explicitly construct Little Rip, ΛCDM and ΛCDM-like cosmologies with finite-time future

singularities. The possibility to reconstruct eventually arbitrary cosmologies like in $f(R)$ gravity [6] is briefly mentioned.

Let us briefly review the standard BI theory [3,7]. The action for this theory is given by

$$S_{\text{EiBI}} = \frac{2}{\kappa} \int d^4 x \big[\sqrt{\big| \det \big(g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma) \big) \big|} - \lambda \sqrt{|g|} \big] + S_M[g, \Psi], \tag{1}$$

where $g_{\mu\nu}$ is the metric, $R_{\mu\nu}(\Gamma) = R^\alpha{}_{\mu\alpha\nu}$ is the Ricci tensor, where

$$R^\alpha{}_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\beta\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\beta} \tag{2}$$

is the Riemann tensor of the connection $\Gamma^\lambda_{\mu\nu}$, which is *a priori* independent of the metric (Palatini formalism), and λ is a dimensionless constant. The matter action depends on the matter fields, denoted generically by Ψ, and the metric $g_{\mu\nu}$ but not on the connection. The theory is considered under the Palatini formalism, i.e., $\Gamma^\alpha_{\mu\nu}$ is not assumed *a priori* to be the Levi-Civita connection of the metric $g_{\mu\nu}$. Additionally, we assume that the Ricci tensor is symmetric and that there is no torsion.

Varying the action (1) with respect to $g_{\mu\nu}$ gives

$$\sqrt{q}(q^{-1})^{\mu\nu} - \lambda\sqrt{g}g^{\mu\nu} = -\kappa\sqrt{g}T^{\mu\nu}. \tag{3}$$

Here $T^{\mu\nu}$ is the standard energy–momentum tensor with indices raised with the metric $g_{\mu\nu}$, $q = \det q_{\mu\nu}$,

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$$q_{\mu\nu} \equiv g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma). \tag{4}$$

Varying the action (1) with respect to the connection we obtain

$$\nabla_\alpha\left[\sqrt{q}(q^{-1})^{\mu\nu}\right] = 0, \tag{5}$$

where the covariant derivative is taken with respect to the independent connection. As is well-known, the covariant derivative of a tensor density is given by

$$\nabla_\mu \sqrt{q} = \partial_\mu \sqrt{q} - \Gamma^\alpha_{\mu\alpha}\sqrt{q}.$$

For the vacuum case ($S_M = 0$), substituting Eq. (3) into (5) gives

$$\nabla_\alpha\left[\sqrt{g}g^{\mu\nu}\right] = 0, \tag{6}$$

which tells us that the connection is given by the Christofell symbols of the metric $g_{\mu\nu}$. That is $q = \lambda g$ and we see that $R_{\mu\nu} = \frac{\lambda-1}{\kappa}g_{\mu\nu}$. If we consider the case $\lambda = 1$ then

$$R_{\mu\nu} = 0.$$

Thus, one sees the equivalence of BI gravity in vacuum with standard GR with cosmological constant.

In the general case, Eq. (5) means that the tensor q plays the role of an auxiliary metric which is compatible with Γ

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}q^{\alpha\beta}(\partial_\mu q_{\nu\beta} + \partial_\nu q_{\mu\beta} - \partial_\beta q_{\mu\nu}). \tag{7}$$

We now propose a modified action containing an arbitrary function $f(R)$, where $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$. Such an action defines the BI-$F(R)$ family of gravity theories and takes the form [5]:

$$S_{\text{EiBI}} = \frac{2}{\kappa}\int d^4x\left[\sqrt{\left|\det\left(g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma)\right)\right|} - \lambda\sqrt{|g|}\right]$$
$$+ \int d^4x\sqrt{|g|}f(R) + S_M[g, \Gamma, \Psi]. \tag{8}$$

Varying the action (8) with respect to the connection we obtain the equation

$$\nabla_\alpha\left[\sqrt{q}(q^{-1})^{\mu\nu} + \sqrt{g}g^{\mu\nu}f_R\right] = 0, \tag{9}$$

where $f_R \equiv df/dR$. The corresponding equation obtained by variation over the metric has the form

$$\sqrt{q}(q^{-1})^{\mu\nu} - \lambda\sqrt{g}g^{\mu\nu} + \frac{\kappa}{2}\sqrt{g}g^{\mu\nu}f(R) - \kappa\sqrt{g}f_R R^{\mu\nu}$$
$$= -\kappa\sqrt{g}T^{\mu\nu}. \tag{10}$$

For simplicity, we now make the (simplifying) assumption that the tensor $q_{\mu\nu}$ for the action (8) is conformally proportional to the metric $g_{\mu\nu}$:

$$q_{\mu\nu} = k(t)g_{\mu\nu}. \tag{11}$$

In this case we have an auxiliary metric $u_{\mu\nu}$ which defines the covariant derivative

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}u^{\alpha\beta}(\partial_\mu u_{\nu\beta} + \partial_\nu u_{\mu\beta} - \partial_\beta u_{\mu\nu}). \tag{12}$$

Here

$$u_{\mu\nu} = \left(k(t) + f_R\right)g_{\mu\nu}. \tag{13}$$

For the condition (11) together with the definition $q_{\mu\nu}$ it is clear that the Ricci tensor must also be proportional to the metric $g_{\mu\nu}$. One can write the relationship between the Ricci tensor and the metric as

$$R_{\mu\nu} = \frac{1}{\kappa}\left(k(t) - 1\right)g_{\mu\nu}. \tag{14}$$

Consider now the spatially-flat FRW universe with metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right). \tag{15}$$

The auxiliary metric then takes the following form

$$u_{\mu\nu} = u(t)\,\text{diag}\left(-1, a(t)^2, a(t)^2, a(t)^2\right), \tag{16}$$

where $u(t) = k(t) + f_R$. Suppose now that $R_{\mu\nu} = r(t)g_{\mu\nu}$ where $r(t)$ is easy to find from Eq. (14). Now, the Christoffel symbols and Ricci tensor of the metric (16) may be constructed leading to

$$r(t) = \frac{3}{2}\left[2\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\frac{\dot{u}}{u} + \frac{\ddot{u}}{u} - \left(\frac{\dot{u}}{u}\right)^2\right], \tag{17}$$

$$r(t) = \left[\frac{\ddot{a}}{a} + \frac{5}{2}\frac{\dot{a}}{a}\frac{\dot{u}}{u} + \frac{\ddot{u}}{2u} + 2\left(\frac{\dot{a}}{a}\right)^2\right], \tag{18}$$

where the upper dot denotes time derivative, i.e., $\dot{} \equiv \frac{d}{dt}$. These two equations for $r(t)$ can be combined to get

$$r(t) = 3\left(H + \frac{\dot{u}}{2u}\right)^2, \tag{19}$$

$$2\dot{H} = H\frac{\dot{u}}{u} + \frac{3}{2}\left(\frac{\dot{u}}{u}\right)^2 - \frac{\ddot{u}}{u}, \tag{20}$$

where we have defined H as the Hubble rate ($H = \frac{\dot{a}}{a}$). Using these two equations, one can verify that $\frac{\dot{u}}{u} = \frac{\dot{r}}{r}$, which leads to

$$u(t) = cr(t) \tag{21}$$

where c is a constant. The remaining equations lead us to

$$H = \pm\sqrt{\frac{u}{3c}} - \frac{\dot{u}}{2u}. \tag{22}$$

From this result, as shown in [5], the form of the function $f(R)$ may be found explicitly

$$f(R) = \frac{2}{\kappa}(\lambda - 1) - R + \frac{c-\kappa}{8}R^2. \tag{23}$$

It should be noted that when we selected the relation between the metric $g_{\mu\nu}$ and the tensor $q_{\mu\nu}$ (11), the form of the function $f(R)$ and Eq. (22) is determined without using Eq. (10). However, Eq. (10) without matter does not contradict the conditions obtained previously and allows us to fix some parameters.

The above consideration shows that this conformal approach allows to obtain the solutions of equations of motion and the form of the function $f(R)$ only in the absence of matter or matter with a constant energy-density $\rho = const$ and $p = -\rho$. For more general forms of matter, the complete non-perturbative formulation given in Ref. [5] should be applied. Hence, we obtain the following form of $f(R)$

$$f(R) = \frac{2}{\kappa}(\lambda - 1) + 2\rho - R + \frac{c-\kappa}{8}R^2. \tag{24}$$

Let us stress once more that the solutions in the presence of arbitrary matter may be constructed, as shown in detail in Ref. [5]. All this leads to the solution of the system of Eqs. (9) and (10) with an arbitrary scale factor, which is determined from the Eq. (22). From the other side, choosing the appropriate metric one finds the relation between $g_{\mu\nu}$ and $q_{\mu\nu}$. Hence, the reconstruction program appears to be completely different from the usual $f(R)$ gravity. Let us consider several cosmologically-viable examples when the relation between the metric $g_{\mu\nu}$ and tensor $q_{\mu\nu}$ is explicitly chosen.
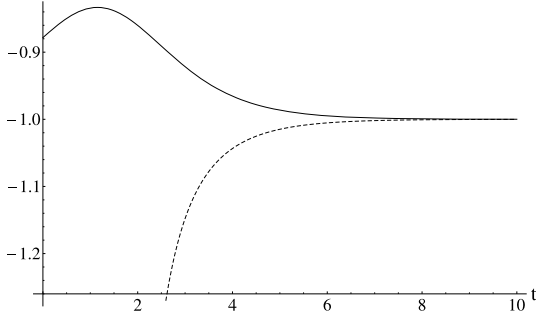
Fig. 1. w_{eff} as a function of t (for $u = e^{ht}$) with the parameters $c = 0.1$, $h = -2$, $u_0 = 3$ (solid line corresponds the sign "+" in (25) and dashed – "−").
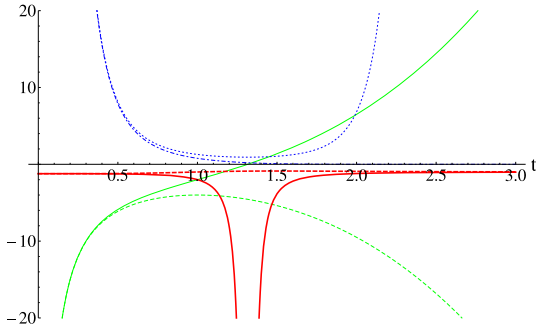

Fig. 2. a (blue (dotted) line), H (green (thin) line) and w_{eff} (red (thick) line) as a function of t ($u = t^h$) with the parameters $c = 1$, $h = 6$, $u_0 = 3$ (continuous line corresponds the sign "+" in (28) and dashed – "−").

1. Little Rip universe

We can define the function u as $u = u_0 e^{ht}$. Then from the Eq. (22) it is easy to find the scale factor

$$a = e^{\pm \frac{2\sqrt{u_0}}{\sqrt{3c}h} e^{\frac{ht}{2}} - \frac{ht}{2}}, \tag{25}$$

and

$$H = -\frac{h}{2} \pm \frac{\sqrt{u_0}}{\sqrt{3}\sqrt{c}} e^{\frac{ht}{2}}. \tag{26}$$

The effective equation of state (EoS) parameter becomes:

$$w_{eff} = -1 - \frac{2\dot{H}}{3H^2} = -1 \mp \frac{4h\sqrt{3}\sqrt{u_0 c}}{(3\sqrt{c}h - 2\sqrt{3}e^{\frac{ht}{2}}\sqrt{u_0})^2} e^{\frac{ht}{2}},$$

where the time-dependence of the effective EoS is presented in Fig. 1 for $h < 0$. The behavior of the effective EoS for $h > 0$ is exactly the same as in Fig. 1, only the red line will meet the "+" sign in Eq. (26) and blue – "−".

It is easy to see that for a positive h (and sign "+" in the expression (25)) we obtain the so-called Little Rip universe [8] ($a \to \infty$ and $H \to \infty$ at future infinity). For negative values of h, the model behaves as de Sitter universe at infinity. If in the expression (25) we choose the sign "+" then we observe the exponentially expanding universe. If we select the sign "−" then the early universe is compressed and then it expands, that is, we have the so-called bounce cosmology [9]. The case with a positive value of the parameter h and sign "−" in Eq. (25) is not interesting, since the scale factor tends to zero. From Fig. 1 it is clear that phantom-like ($w < -1$) as well as quintessence-like ($-1/3 > w > -1$) dark

energy cosmologies maybe obtained as solutions from our BI-$f(R)$ theory.

For the Little Rip scenario, the effective EoS is always less than -1, so that the dark energy density increases with time, but w approaches -1 asymptotically and sufficiently rapidly that a singularity is avoided. But it leads to a dissolution of bound structures at some point in the future (similar to the effect of a Big Rip singularity). As the universe expands, the relative acceleration between two points separated by a distance l is given by $l\ddot{a}/a$. If there is a particle with mass m at each of these points, an observer at one of the masses will measure an inertial force on the other mass, as [8]

$$F_{iner} = ml\ddot{a}/a = ml(\dot{H} + H^2). \tag{27}$$

Let us assume the two particles are bound by a constant force F. If F_{iner} is greater than F, the two particles become unbound. This is the Rip produced by the accelerating expansion. We see that this situation will be realized if H or/and $\dot{H}$ tends to infinity even in the BI-$f(R)$ gravity under consideration. Indeed, we see that for h positive we receive the force F_{iner} exponentially growing with time which tends to infinity. For example, if $c = 1$, $h = 2$, $u_0 = 1$ the disintegration of the Solar System occurs at $F_{iner} \sim 10^{23}$, which corresponds to $t \approx 24.4$ Gyr.

Note that in the model under consideration the curvature is determined by the metric $u_{\mu\nu} = u_0 e^{ht} g_{\mu\nu}$, and R is equal to $\frac{4e^{ht}u_0}{c}$ rather than $3h^2 - \frac{3\sqrt{3}e^{\frac{h}{t}}2h\sqrt{u_0}}{\sqrt{c}} + \frac{4e^{ht}u_0}{c}$ as it should be for the metric (25).

2. Power-law evolution

For $u = u_0 t^h$ one finds

$$a = a_0 e^{\frac{\pm 2t^{1+\frac{h}{2}}\sqrt{u_0}}{\sqrt{3c}(2+h)}} t^{-h/2}, \tag{28}$$

or

$$a = a_0 e^{\frac{\pm 2(t_0-t)^{1+\frac{h}{2}}\sqrt{u_0}}{\sqrt{3c}(h-2)}} (t_0 - t)^{-h/2}, \tag{29}$$

for $u = u_0(t_0 - t)^{-h}$. The Hubble parameter takes the form

$$H = -\frac{h}{2t} \pm \frac{t^{h/2}\sqrt{u_0}}{\sqrt{3}\sqrt{c}},$$

while for $u = u_0(t_0 - t)^h$ one should replace $t \to (t_0 - t)$ and $h \to -h$. In the first case, the effective EoS parameter takes the form

$$w_{eff} = \frac{-3ch(4 + 3h) \pm 8\sqrt{3u_0c}ht^{1+\frac{h}{2}} - 12t^{2+h}u_0}{(3\sqrt{c}h - 2\sqrt{3}t^{1+\frac{h}{2}}\sqrt{u_0})^2},$$

and we can build time-dependence of the EoS parameter (Fig. 2). We see that the effective EoS parameter approaches to minus one. The behavior of the scale factor and the Hubble parameter can be illustrated by Fig. 2.

We see that the expanding universe corresponds the scale factor a appropriate to the plus sign in Eq. (28). This is again Little Rip where the singularity is moved to infinity. In addition, we again observe a bouncing cosmology. For $u = u_0(t_0 - t)^{-h}$ one gets

$$w_{eff} = \left(-3ch(-4 + 3h)(t_0 - t)^h \right.$$
$$\pm 8\sqrt{3}\sqrt{c}h(t_0 - t)^{1+\frac{h}{2}}\sqrt{u_0} - 12(t_0 - t)^2 u_0 \right)$$
$$\left. / \left((3\sqrt{c}h(t_0 - t)^{h/2} - 2\sqrt{3}(t_0 - t)\sqrt{u_0})^2 \right). \right.$$
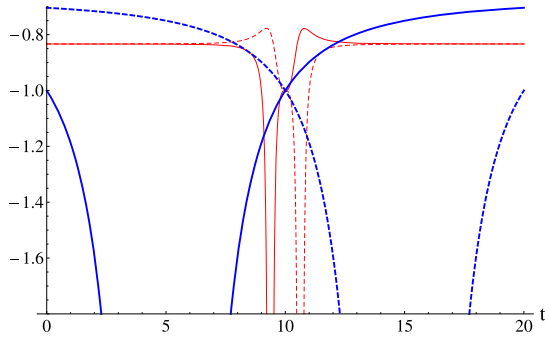
Fig. 3. w_{eff} as a function of t ($u = u_0(t_0 - t)^{-h}$) with the parameters $c = 1$, $h = 8$, $u_0 = 3$, $t_0 = 10$ (red (thin) line) and $c = 1$, $h = 4$, $u_0 = 300$, $t_0 = 10$ (blue (thick) line) (continuous line corresponds the plus sign in (29) and dashed – "−").
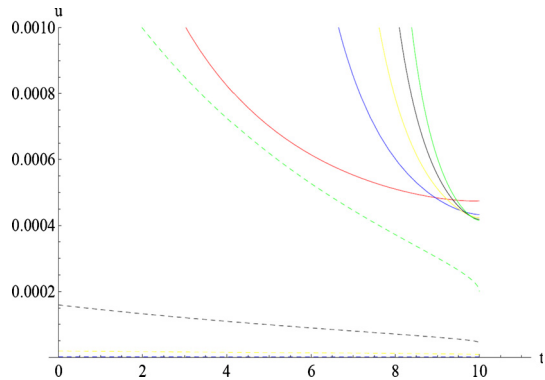

Fig. 4. u as a function of t for metric (30) (continuous line corresponds $y = 3/2$, dashed – $y = 1/2$) with the parameters $c = 1$, $a_0 = 50$, $t_0 = 10$, $a_1 = 1$ (red line), 3 (blue line), 5 (yellow line), 7 (black line) and 9 (green line). (From the bottom to the top.)

This model behaves almost like the model built with the metric (28), but at the moment $t = t_0$ we have a Big Rip singularity [10]. It should be noted that in this case the effective EoS parameter at this time is equal to minus one. The time-dependence of the EoS parameter is drawn in Fig. 3.

In the above examples two types of singularities occur. However, other types of finite-time future singularities are possible. For quintessence dark energy, one can get a singularity for which the pressure goes to infinity at a finite time, but the scale factor and density remain finite (a sudden singularity, or a Type II singularity) [10]. Alternatively, the density and pressure can both become infinite with a finite scale factor at a finite time (a Type III singularity) [10], or higher derivatives of the Hubble parameter H can diverge (a Type IV singularity) [10]. It is known that the occurrence of a singularity at a finite time in the future may lead to some inconsistencies.

Singularities of these types can be obtained by considering the metric

$$a = a_0 - a_1(t_0 - t)^y, \tag{30}$$

where a_0, a_1 and y are positive constants. If $y = 1/2$ then we have Type III singularity. For $y = 3/2$ we have Type II singularity. For $y = 5/2$ a Type IV singularity occurs.

Using Eq. (22) to specify the form of the scale factor one can find the relation between the metric $g_{\mu\nu}$ and the tensor $q_{\mu\nu}$. For example, for $y = 3/2$ we obtain the following form of $u(t)$
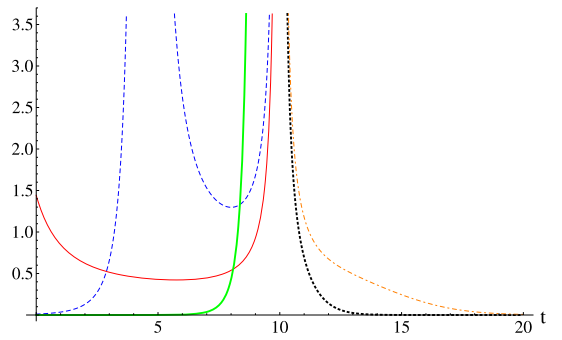

Fig. 5. u as a function of t with the parameters $c = 1$, $c_1 = 1$, $t_0 = 10$ (red (thin) line for $l = -3$, blue (dashed) for $l = -2$, green (thick) for $l = -1$, black (dotted) for $l = 1$ and yellow (dot-dashed) for $l = 2$.

$$u(t) = \left(36 a_0^{2/3} a_1^{4/3} \right)$$
$$/ \left(\left(a_0 - a_1(t_0 - t)^{3/2} \right)^2 \left(-2\sqrt{3} \text{ArcTan} \left[\frac{1 + \frac{2a_1^{1/3}\sqrt{t_0-t}}{a_0^{1/3}}}{\sqrt{3}} \right] \right. \right.$$
$$+ 6 a_0^{1/3} a_1^{2/3} c_1 - -2 \text{Log} \left[a_0^{1/3} - a_1^{1/3}\sqrt{t_0-t} \right]$$
$$\left. \left. + \text{Ln} \left[a_0^{2/3} + a_0^{1/3} a_1^{1/3}\sqrt{t_0-t} + a_1^{2/3}(t_0-t) \right] \right)^2 \right), \tag{31}$$

where c_1 is integration constant, and c has been set to $c = 3/4$ for simplicity.

In Fig. 4 we illustrate the behavior of the relation between the metric $g_{\mu\nu}$ and the tensor $q_{\mu\nu}$ for $y = 1/2$ (Type III singularity) and $y = 3/2$ (Type II singularity).

To the light of the above results, one may ask if a ΛCDM cosmology can also be reconstructed. The answer is positive. To see it, let the scale factor have the following form

$$a = a_0 e^{g(t)},$$
$$g(t) = \frac{2}{3(1 + w)} \log \left(\alpha \sinh \left(\frac{3(1 + w)}{2l}(t - t_0) \right) \right), \tag{32}$$

where w, α, l and a_0 are constant. Such solution corresponds to the ΛCDM-model, which was also reconstructed in $f(R)$ gravity [6]. For this metric, the FRW equation becomes

$$\frac{3}{\kappa^2} H^2 = \rho_0 a^{-3(1+w)} + \frac{3}{\kappa^2 l^2},$$

where $\alpha = \frac{1}{3}\kappa^2 l^2 \rho_0 a_0^{-3(1+w)}$. Note, however, that when this metric is substituted into Eq. (22) (restricting the choice of the positive sign in this expression) then a real solution is found only if $w \leq -1/3$. For example, if we choose $w = -1/3$ we obtain the following expression for u

$$u = \frac{3cl^4 \text{Csch}[\frac{t}{l} - \frac{t_0}{l}]^2}{(c_1 - l^3 \text{Log}[\text{Tanh}[\frac{t-t_0}{2l}]])^2}, \tag{33}$$

where c_1 is an integration constant. The behavior of this function is illustrated in Fig. 5.

We thus see that, similarly as in $f(R)$ gravity, one can get ΛCDM cosmology by means of modifications in the gravitational action. However, there is a qualitative difference between $f(R)$ and BI-$f(R)$ gravity in this respect. The reconstruction in $f(R)$ gravity is achieved by the change of the function $f(R)$. In BI-$f(R)$ gravity, on the contrary, the form of $f(R)$ is uniquely fixed by internal consistency relations produced by the conformal approach.

The reconstruction is possible thanks to the change of the relation between the two metrics. Indeed, we see that the choice of the relation between the metric $g_{\mu\nu}$ and the tensor $q_{\mu\nu}$ in the form of the conformal connection ($g_{\mu\nu} = u(t)q_{\mu\nu}$) uniquely determines the form of the function $f(R)$ (23) in the absence of matter or (24) for matter with a constant energy density $\rho = \text{const}$ and $p = -\rho$. For other types of matter one should use the (more complicated and non-conformal) Palatini formulation developed in Ref. [5]. Hence, we obtain Eq. (22) from which one gets either $a(t)$ or $u(t)$. It is remarkable that one can get an arbitrary FRW cosmology as solution of BI-$f(R)$ gravity using the above scheme. For instance, the unification of early-time inflation with late-time acceleration within BI-$f(R)$ gravity may be done. Of course, the corresponding expressions are a little bit complicated and will not be presented here.

Acknowledgements

References

[1] S. Nojiri, S.D. Odintsov, eConf C 0602061 (2006) 06, Int. J. Geom. Methods Mod. Phys. 4 (2007) 115, arXiv:hep-th/0601213;
S. Nojiri, S.D. Odintsov, Phys. Rep. 505 (2011) 59, arXiv:1011.0544 [gr-qc];
S. Nojiri, S.D. Odintsov, Int. J. Geom. Methods Mod. Phys. 11 (2014) 1460006, arXiv:1306.4426 [gr-qc];
V. Faraoni, S. Capozziello, Beyond Einstein gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics, Fund. Theor. Phys., vol. 170, Springer, 2010;
S. Capozziello, M. De Laurentis, Phys. Rep. 509 (2011) 167, arXiv:1108.6266 [gr-qc];
G.J. Olmo, Int. J. Mod. Phys. D 20 (2011) 413, arXiv:1101.3864 [gr-qc].

[2] S. Nojiri, S.D. Odintsov, Phys. Rev. D 68 (2003) 123512, arXiv:hep-th/0307288.

[3] S. Deser, G.W. Gibbons, Class. Quantum Gravity 15 (1998) L35;
M. Bañados, P.G. Ferreira, Phys. Rev. Lett. 105 (2010) 011101.

[4] X.-L. Du, K. Yang, X.-H. Meng, Y.-X. Liu, arXiv:1403.0083 [gr-qc];
H.-C. Kim, arXiv:1312.0703 [gr-qc];
P.P. Avelino, R.Z. Ferreira, Phys. Rev. D 86 (2012) 041501, arXiv:1205.6676 [astro-ph.CO];
C. Escamilla-Rivera, M. Banados, P.G. Ferreira, Phys. Rev. D 85 (2012) 087302, arXiv:1204.1691 [gr-qc];
I. Cho, H.-C. Kim, T. Moon, Phys. Rev. D 86 (2012) 084018;
J.H.C. Scargill, M. Banados, P.G. Ferreira, Phys. Rev. D 86 (2012) 103533;
F. Fiorini, Phys. Rev. Lett. 111 (2013) 041104;
S.I. Kruglov, Phys. Rev. D 89 (2014) 064004, arXiv:1310.6915 [gr-qc];
K. Yang, X.-L. Du, Y.-X. Liu, Phys. Rev. D 88 (2013) 124037;
C. Escamilla-Rivera, M. Banados, P.G. Ferreira, arXiv:1301.5264 [gr-qc];
T. Harko, F.S.N. Lobo, M.K. Mak, S.V. Sushkov, Mod. Phys. Lett. A 29 (2014) 1450049;
P.P. Avelino, Phys. Rev. D 85 (2012) 104053;
M. Bouhmadi-Lopez, C.Y. Chen, P. Chen, arXiv:1302.5013 [gr-qc];
R. Ferraro, F. Fiorini, J. Phys. Conf. Ser. 314 (2011) 012114;
Dan N. Vollick, Phys. Rev. D 69 (2004) 064030, arXiv:gr-qc/0309101.

[5] A.N. Makarenko, S. Odintsov, G.J. Olmo, arXiv:1403.7409 [hep-th].

[6] S. Nojiri, S.D. Odintsov, Phys. Rev. D 74 (2006) 086005, arXiv:hep-th/0608008;
A. de la Cruz-Dombriz, A. Dobado, Phys. Rev. D 74 (2006) 087501, arXiv:gr-qc/0607118.

[7] G.J. Olmo, D. Rubiera-Garcia, Phys. Rev. D 88 (2013) 084030;
G.J. Olmo, D. Rubiera-Garcia, H. Sanchis-Alepuz, Eur. Phys. J. C 74 (2014) 2804, arXiv:1311.0815 [hep-th].

[8] P.H. Frampton, K.J. Ludwick, R.J. Scherrer, Phys. Rev. D 84 (2011) 063003, arXiv:1106.4996 [astro-ph.CO];
P.H. Frampton, K.J. Ludwick, S. Nojiri, S.D. Odintsov, R.J. Scherrer, Phys. Lett. B 708 (2012) 204–211, arXiv:1108.0067 [hep-th];
I. Brevik, E. Elizalde, S. Nojiri, S.D. Odintsov, Phys. Rev. D 84 (2011) 103508, arXiv:1107.4642 [hep-th];
A.N. Makarenko, V.V. Obukhov, I.V. Kirnos, Astrophys. Space Sci. 343 (2013) 481, arXiv:1201.4742 [gr-qc].

[9] R.H. Brandenberger, Int. J. Mod. Phys. Conf. Ser. 01 (2011) 67, arXiv:0902.4731 [hep-th];
R.H. Brandenberger, AIP Conf. Proc. 1268 (2010) 3, arXiv:1003.1745 [hep-th].

[10] S. Nojiri, S.D. Odintsov, S. Tsujikawa, Phys. Rev. D 71 (2005) 063004, arXiv:hep-th/0501025.