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NOTE

ON A LOGICAL PROBLEM

Pavel M. BLECHER
Institute of Applied Mathematics, Moscow A-47, USSR

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The full solution of a logical problem is given.

In this note I shall consider the following logical problem.

Problem. *There is a group of N persons, some of which are reliable and the rest are unreliable being known that the reliable persons are a majority. A reliable person answers only the truth to all questions while an unreliable one answers sometimes the truth and sometimes a lie. A mathematician (not belonging to the group) wants to find out "who is who" in the group. For that he may ask any person about any other one in the group if the latter is a reliable person or not. What is the least number of questions by which he can find out for sure who is who in the group?*

Let $Q(N)$ be the least number of questions. The first upper bound, $Q(N) \leqslant 2N-3$, was obtained (I believe so) by Konyagin, the author of the problem. A little later I could prove the estimate $Q(N) \leqslant [\frac{3}{2}(N-1)]$. After that another proof of this estimate was found by Shlosman. As concerns the lower bound, Ruzsa proved that $Q(N) \geqslant [\frac{1}{4}(5N-3)]$ and Galvin improved his result to the following: If $N \geqslant 5$, then

$$Q(N) \geqslant \begin{cases} \frac{4}{3}N - 1 & \text{if } N \equiv 0 \pmod 6, \\ [\frac{4}{3}N] & \text{otherwise} \end{cases}$$

(private communications). The purpose of this note is to prove the following result.

Theorem. $Q(N) = [\frac{3}{2}(N-1)]$, $N \geqslant 3$.

Let at first N be odd, $N = 2k+1$. We must prove that $Q(N) = 3k$. For that we shall prove at first that $Q(N) \leqslant 3k$ and next that $Q(N) \geqslant 3k$. As a matter of fact at the first stage we shall give an algorithm which solves the problem for $3k$

questions and at the second one we shall prove that the problem cannot be solved for lesser number of questions.

The upper bound, $Q(N) \leq 3k$

Algorithm. Let us enumerate the persons and ask the 2nd, 3rd, etc. if the 1st is reliable. We shall stop as soon as one of the following two events will occur:

Event A. k persons have said that the first is reliable. Then the first is reliable indeed and all those who said "no" are unreliable. We ask now the first about all the other persons. The direct computation shows that we use $3k$ questions in this procedure.

Event B. The number of those who said "no" exceeds the number of those who said "yes". In this case if m persons said "no", then $(m-1)$ person said "yes", $m = 1, 2, \ldots$. Moreover it is easy to verify that in the group of the first $2m$ persons the number of unreliable persons is not less then the number of reliable ones, so in the rest group of $2k+1-2m = 2(k-m)+1$ persons the reliable persons form a majority. So using $3(k-m)$ questions we can sort out them. Next we choose a reliable person among them and ask about the 1st person and about those among the first $2m$ persons whose answers about the 1st were truthful (others are evidently unreliable). In such a way we sort out all the persons by $2m-1+3(k-m)+1+m = 3k$ questions what was to be shown. The upper bound is proved.

The above mentioned Shlosman's proof of the upper bound is based on another algorithm. It seems to be more complicated but in some sense it is more economic: If the group contains not more than $M\ (<\frac{1}{2}N)$ unreliable persons it ends not later than after $N+M$ questions.

The lower bound, $Q(N) \geq 3k$

Assume $Q(N) \leq 3k-1$. Now we shall give a strategy of answers and show that there exists always at least two dissections of the group of N persons into reliable and unreliable persons which agree with all the $Q(N)$ answers. We divide 'the game' into two stages.

Stage I. The first $(k-1)$ questions.

All the answers $a_1, \ldots, a_{k-1}$ are "no". Let $(s_1, s_1'), \ldots, (s_{k-1}, s_{k-1}')$ be the sequence of the pairs of persons in the game (we ask the s_i about the s_i' in the ith question). Let G be a (not oriented) graph whose vertices are $V = \{s \mid \exists i = 1, \ldots, k-1\colon\ s = s_i \text{ or } s = s_i'\}$, i.e. $V = \bigcup_{i=1}^{k-1} \{s_i, s_i'\}$, and edges are $(s_1, s_1'), \ldots, (s_{k-1}, s_{k-1}')$. Let $G_1, \ldots, G_r$ be the connected components of the graph G and $V_1, \ldots, V_r$ be the sets of vertices of the subgraphs $G_1, \ldots, G_r$. Let W be the complement of V to the whole set of N persons. At last, let $l_1, \ldots, l_r$ be the number of edges of the subgraphs $G_1, \ldots, G_r$, resp. Then

$$(*) \qquad l_1 + \cdots + l_r = k-1.$$

Stage II. The last $2k$ questions.

The answers $a_k, \ldots, a_{3k-1}$ are defined by the rules (as before we ask s_i about s_i', $i=k, \ldots, 3k-1$):

$$a_i = \text{"yes"} \quad \text{if } s_i' \in W,$$
$$a_i = \text{"no"} \quad \text{if } s_i' \in V \setminus V^0,$$
$$a_i = \text{"yes"} \quad \text{if } s_i' \in V^0.$$

Here the set V^0, $V^0 \subset V$, depends on i and is defined in the following way. First at all, $V^0 = V^0[i]$ contains at most one vertex from any V_m, $m=1, \ldots, r$. Let

$$V[i] = \{s \mid \exists j,\ k \leqslant j \leqslant i:\ s = s_j'\} \equiv \bigcup_{j=k}^{i} \{s_j'\}$$

and

$$V_m[i] = V[i] \cap V_m, \quad m = 1, \ldots, r.$$

If $V_m[i] \neq V_m$, then by definition V^0 contains no point of V_m.

If $V_m[i] = V_m$, then we take the minimal number n such that $V_m[n] = V_m$ (surely n depends on m) and put $v_m^0 = s_n'$. In such a case v_m^0 belongs to $V^0[i]$. Thus the set $V^0 = V^0[i]$ is defined.

Now we present a configuration S of persons which agrees with the answers $a_1, \ldots, a_{3k-1}$. To do this we consider the (final) set $V^0 = V^0[3k-1]$ and an auxiliary set $V^{0\prime}$, $V^{0\prime} \subset V$, which is constructed in such a way that each connected component V_m contains exactly one point either from V^0 or from $V^{0\prime}$. More precisely, if V_m has a point $v_m^0 \in V^0$ it contains no point of $V^{0\prime}$; if V_m has not any point of V^0, then the set $V_m \setminus V[3k-1]$ is non-empty and we choose an arbitrary point $v_m^{0\prime} \in V_m \setminus V[3k-1]$ as a representative of V_m in $V^{0\prime}$. Thus the sets V^0, $V^{0\prime}$ are defined and we put in the configuration S:

$$s \text{ is 'reliable'} \quad \text{if } s \in W \cup V^0 \cup V^{0\prime},$$
$$s \text{ is 'unreliable'} \quad \text{if } s \in V \setminus (V^0 \cup V^{0\prime}).$$

We state that

(i) S agrees with the answers $a_1, \ldots, a_{3k-1}$.

(ii) The number of unreliable persons in S does not exceed $k-1$.

(iii) Let $s \notin V[3k-1]$ (the last set contains at most $2k$ points so such an s does exist), moreover if $V^{0\prime}$ is non-empty let $s \in V^{0\prime}$; then the change of the type of the s gives a configuration S' which agrees as well as S with the answers $a_1, \ldots, a_{3k-1}$.

Thus the sequence of the answers $a_1, \ldots, a_{3k-1}$ does not distinguish S and S' what contradicts our assumption. Now we verify the statements (i)–(iii).

(i) The following takes place:

(a) the answers $a_k, \ldots, a_{3k-1}$ are truthful;

(b) $s \in W$ does not participate in the answers $a_1, \ldots, a_{k-1}$;

(c) $s = v_m^0 = V^0 \cap V_m$ (or $s = v_m^{0\prime} = V^{0\prime} \cap V_m$) answers in the $a_1, \ldots, a_{k-1}$ only about $s' \in V_m$, $s' \neq s$, and so his answers "no" are the truth.

Thus all the reliable persons $s \in W \cup V^0 \cup V^{0\prime}$ answer only the truth what was stated.

(ii) Let v_m be the number of points of the set V_m, $m = 1, \ldots, r$. Then $v_m \leq l_m + 1$ (the graph G_m is connected). By construction the number of unreliable persons in V_m is $v_m - 1$, so their total number is

$$\sum_{m=1}^{r} (v_m - 1) \leq \sum_{m=1}^{r} l_m = k - 1$$

(see (*) above) what was stated.

(iii) Let $s \notin V[3k-1]$. By construction $V^0 \subset V[3k-1]$, so $s \notin V^0$. Assume $s \in W$. Then s is reliable in S and unreliable in S'. Moreover there is no question about him. Therefore S' agrees with the answers $a_1, \ldots, a_{3k-1}$ as well as S. Consider now $s \in V \backslash V^0$. Let $s \in V_m$. Then $s \notin V[3k-1]$ implies that V_m does not contain any element of V^0, so V_m consists only of unreliable persons except maybe s itself (if $s = v_m^{0\prime} \in V^{0\prime}$). So the answers of s in the $a_1, \ldots, a_{k-1}$ are truthful, so all his answers are truthful and so we can change his type while not destroying the agreement with the answers $a_1, \ldots, a_{3k-1}$ (we use here that the answers about s are only from unreliable persons from V_m).

This completes the proof. The case $N = 2k + 2$ is considered in the same manner.

In conclusion we mention a generalization of the problem. Assume that it is known in addition that the number of unreliable persons does not exceed M (which is $\leq [\frac{1}{2}(N-1)]$ and >0). What is the least number of questions $Q(N, M)$ in this case? Repeating the proof of the lower bound (with $M-1$ questions at the first stage) one can prove that $Q(N, M) \geq N + M - 1$. Moreover repeating the proof of the upper bound (with M positive answers in the Event A) one can prove that $Q(N, M) \leq N + M - 1$. Thus $Q(N, M) = N + M - 1$.

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