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Computational benefit of smoothness: Parameterized bit-complexity of numerical operators on analytic functions and Gevrey's hierarchy[☆]

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ABSTRACT

The synthesis of (discrete) Complexity Theory with Recursive Analysis provides a quantitative algorithmic foundation to calculations over real numbers, sequences, and functions by approximation up to prescribable absolute error $1/2^n$ (roughly corresponding to n binary digits after the radix point). In this sense Friedman and Ko have shown the seemingly simple operators of maximization and integration 'complete' for the standard complexity classes NP and #P — even when restricted to smooth ($=\mathcal{C}^\infty$) arguments. Analytic polynomial-time computable functions on the other hand are known to get mapped to polynomial-time computable functions: non-uniformly, that is, disregarding dependences other than on the output precision n.

The present work investigates the uniform parameterized complexity of natural operators Λ on subclasses of smooth functions: evaluation, pointwise addition and multiplication, (iterated) differentiation, integration, and maximization. We identify natural

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1 Author dedicates (his contribution to) this work to the memory of his late father.

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integer parameters $k = k(f)$ which, when given as enrichment to approximations to the function argument f, permit to computably produce approximations to $\Lambda(f)$; and we explore the asymptotic worst-case running time sufficient and necessary for such computations in terms of the output precision n and said k.

It turns out that Maurice Gevrey's 1918 classical hierarchy climbing from analytic to (just below) smooth functions provides for a quantitative gauge of the uniform computational complexity of maximization and integration that, non-uniformly, exhibits the phase transition from tractable (i.e. polynomial-time) to intractable (in the sense of NP-'hardness'). Our proof methods involve Hard Analysis, Approximation Theory, and an adaptation of Information-Based Complexity to the bit model.

1. Motivation

Smoothness helps — but in what sense precisely? Optimization problems (like integer programming) over discrete, but also over Lipschitz-continuous, functions regularly turn out as NP-complete; whereas for $\mathcal{C}^1$ or $\mathcal{C}^2$ functions algorithms such as bisection or Newton–Raphson promise efficiently computing maxima with superlinear convergence, that is, to yield approximations up to absolute error $1/2^n$ (roughly corresponding to n valid binary digits) within time polynomial in n. On the other hand iterative methods generally leave open the problem of 'guessing' appropriate starting points. And in fact Harvey Friedman and Ker-I Ko have proven [33,29,19,30,31].

Fact 1. *Real (smooth) function maximization and integration are NP-hard in the following sense:*

(a) *For every polynomial-time computable $f : [0; 1] \to \mathbb{R}$, the function* $\mathrm{MAX}(f): [0; 1] \to \mathbb{R}$, $x \mapsto \max_{0\le t\le x} f(t)$ *is again polynomial-time computable in case $P = NP$;*
(b) *and there exists a polynomial-time computable smooth (meaning $\mathcal{C}^\infty$) $f : [0; 1] \to \mathbb{R}$ such that polynomial-time computability of* $\mathrm{MAX}(f)$ *implies $P = NP$.*
(c) *For every polynomial-time computable $f : [0; 1] \to \mathbb{R}$, the function $\int f : [0; 1] \to \mathbb{R}$, $x \mapsto \int_0^x f(t)$ is again polynomial-time computable in case $FP = \#P$;*
(d) *and there exists a polynomial-time computable smooth (meaning $\mathcal{C}^\infty$) $f : [0; 1] \to \mathbb{R}$ such that polynomial-time computability of $\int f$ implies $FP = \#P$.*
(e) *On the other hand, on the class $\mathcal{C}^\omega[0; 1]$ of real analytic functions (= local power series) $f : [0; 1] \to \mathbb{R}$, both* MAX *and $\int$ do map polynomial-time computable instances to polynomial-time computable ones* [43]*;*
(f) *and* [38, §5.2] *extends this to Gevrey's hierarchy of function classes $\mathcal{G}^\gamma$ starting off at $\mathcal{C}^\omega = \mathcal{G}^1$, including the quasi-analytic functions, and extending with $\gamma \to \infty$ towards (but not 'reaching') $\mathcal{C}^\infty$; see the formal* Definition 17.

Observe the generality of the non-uniform lower complexity bounds in (b) and (d): They do not refer to any specific algorithm computing said maximum or integral, nor do they suppose the function f be 'represented' in any way — since it is fixed and every possible finite information (such as the polynomial-time algorithm computing f) thus available as a discrete constant. Similarly, solving ordinary first-order differential equations with $\mathcal{C}^1$ right-hand side is PSPACE-complete [32,25,27] but maps polynomial-time computable analytic functions to polynomial-time computable ones [45,6].

Note that this and the upper complexity bounds in Fact 1(e) and Fact 1(f) are non-uniform,[2] too: They fix an arbitrary polynomial-time computable analytic or Gevrey input function f and consider the worst-case complexity of the output function $g = \Lambda(f)$ in terms of the precision parameter n only while disregarding the running time's dependence on (parameters of) f and the information about f employed by the algorithm.

1.1. Overview

The present work refines the non-uniform results from Fact 1(e) and Fact 1(f) on the polynomial-time computability of natural operators on increasing subclasses of smooth functions f on the interval $[-1; 1]$: evaluation, pointwise addition and multiplication, differentiation, integration, and maximization.

Starting in Section 3.1 with power series $f(z) = \sum_j a_j z^j$ converging on the compact unit disc, we consider (Definition 10) natural parameters $k, A \in \mathbb{N}$ quantitatively describing the asymptotic decay of $(a_j)_j \subseteq \mathbb{C}$; and we establish that these integers, when provided as input in addition to said coefficients, render the above operations uniformly computable in time polynomial in the binary output precision n, the binary length of A, and the value (= unary length) of k; see Theorem 12. Moreover explicit bounds show each operation to increase the joint parameter $k + \lceil \log_2(A+1) \rceil$ at most polynomially, thus asserting our parameterized polynomial-time computations closed under composition.

Section 3.2 generalizes these investigations to functions f analytic on the real interval $[-1; 1]$: Theorem 16(a) and (b) establish two mathematically equivalent notions – local power series expansions with A, k as above and approximate oracle access to smooth functions with parameters B, ℓ describing a polynomial bounding the growth of iterated derivatives (Definition 14) – as in fact parameterized polynomial-time equivalent. This, together with parameterized polynomial-time computability of addition and multiplication (Theorem 16(c)), maximization (Theorem 16(f)), differentiation (Theorem 16(d)), and integration (Theorem 16(e)), indicates the choice of parameters as natural.

Building on the results from Approximation Theory collected in Section 4.1, Section 4.2 generalizes the above considerations further to functions $f : [-1; 1] \to \mathbb{R}$ on Gevrey's Hierarchy (Definition 17): with parameters B, ℓ, γ describing the growth of iterated derivatives (Definition 22a); or with parameters C, δ describing the growth of degrees of polynomials providing uniform approximations in the sense of Weierstraß(Definition 22b). Theorem 23 establishes the level parameters $\gamma, \delta \in \mathbb{N}$ as linearly related, and to exponentially control the asymptotic running time. For constant γ, δ this amounts to parameterized polynomial time; while growing γ, δ calibrate a uniform increase in complexity — in agreement with the phase transition of the non-uniform Items (a) to (d) in Fact 1.

We first recall the precise notions of computability and complexity on real numbers, sequences, and metric spaces in Section 2. Section 2.1 generalizes and extends these to multivalued (aka non-extensional) functions, to multiparametric complexity theory, and to functionals and operators: receiving (Lipschitz-continuous) real functions as arguments via approximate oracle access, similarly to *Information-Based Complexity* (IBC) but with bit costs.

2. Algorithmic foundations of rigorous numerics

Our investigations build on the theory of real computing by rational approximations, initiated by Alan M. Turing [64]; cmp. also [22,36,30,69,10]. Based on the bit-cost model, it captures variable-precision calculations with prescribable output error $1/2^n$ in dependence on this precision parameter $n \in \mathbb{N}$. To this end, denote by $\mathbb{D}_n := \{a/2^n : a \in \mathbb{Z}\}$ the set of *dyadic rationals* of precision n. Furthermore let $\mathbb{D} := \bigcup_{n \in \mathbb{N}} \mathbb{D}_n$.

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2 Referring to [38, Définition 2.2.9], [38, Corollaire 5.2.14] establishes 'sequentially uniform' polynomial-time computability of the operators on a fixed Gevrey level in the sense of mapping every polynomial-time computable sequence of functions to a polynomial-time computable result sequence. This may be viewed as a higher-type and complexity-theoretic counterpart to Banach–Mazur computability, cf. e. g.[69, §9.1].

Definition 2. (a) Computing a real number x means printing (technically onto a one-way tape) some infinite sequence a_n of integers (in binary without leading zeros) as mantissae/numerators to dyadic rationals $a_n/2^n$ approximating x up to absolute error $1/2^n$.

(b) Computing a real sequence (x_j) similarly means producing an integer double sequence $a_{j,m}$ with $|x_j - a_{j,m}/2^m| \leq 2^{-m}$. Formally, the elements of said sequence occur in order according to the *Cantor pairing function*

$$\mathbb{N} \times \mathbb{N} \ni (j, m) \;\mapsto\; \langle j, m\rangle := j + (j+m) \cdot (j+m+1)/2 \in \mathbb{N} := \{0, 1, 2, \ldots\} \tag{1}$$

that is, the output consists of the single integer sequence (a_n) with $n = \langle j, m\rangle$.

(c) Computing a univariate and possibly partial real function $f \;:\subseteq \mathbb{R} \to \mathbb{R}$ amounts to converting every given sequence $(a_m) \subseteq \mathbb{Z}$ with $|x - a_m/2^m| \leq 1/2^m$ for any $x \in \mathrm{dom}(f)$ into a sequence $(b_n) \subseteq \mathbb{Z}$ with $|f(x) - b_n/2^n| \leq 1/2^n$.

(d) For some mapping $t : \mathbb{N} \to \mathbb{N}_+ := \{1, 2, \ldots\}$, the above computations are said to run in time $t(n)$ if the n-th integer output appears within at most $t(n)$ steps. *Polytime* means running time bounded by some polynomial $t \in \mathbb{N}[X]$.

(e) For metric spaces (X, d) and (Y, e) and $f : X \to Y$, a *modulus of continuity* of f is a mapping $\mu : \mathbb{N} \to \mathbb{N}$ such that $d(x, x') \leq 2^{-\mu(n)}$ implies $e\big(f(x), f(x')\big) \leq 2^{-n}$. Let $\mathcal{L}_L(X, Y) := \{f : X \to Y : e\big(f(x), f(x')\big) \leq L \cdot d(x, x')\}$ denote the class of L-Lipschitz functions; $\mathcal{L}(X, Y) := \bigcup_{L\in\mathbb{N}} \mathcal{L}_L(X, Y)$. We may omit Y in case $Y = \mathbb{R}$.

The above notions are computably – but not polytime – equivalent to standard ones; cmp., e. g., [69, Lemmas 4.2.1 + 6.1.2]. Polynomial-time computation has reasonable properties and according to the *Cobham–Edmonds*Thesis formalizes practical tractability — arguably also in the real setting:

Fact 3. (a) *Every algebraic real is polytime computable; and so are transcendental e and π.*

(b) *Examples of polytime sequences include the following:*

$$(j!)_j, \qquad (1/j!)_j, \qquad \left\{\begin{matrix} 0 & : j \text{ even} \\ (-1)^{(j-1)/2}/j! & : j \text{ odd}\end{matrix}\right\}_j, \qquad \left\{\begin{matrix} (-1)^{j/2}/j! & : j \text{ even} \\ 0 & : j \text{ odd}\end{matrix}\right\}_j$$

(c) *On any bounded real interval, both addition and multiplication are polytime; and so is reciprocal $x \mapsto 1/x$ on any compact interval avoiding 0.*

(d) *Polytime functions are closed under composition. (This relies crucially on both output and input given approximately* [73, *p. 325*].*)*

(e) *Any computable function must necessarily be continuous* [69, *Theorem 4.3.1*]*; and every $f \;:\subseteq \mathbb{R} \to \mathbb{R}$ computable in time $t(n)$ has $\mu(n) := t(n+1)+1$ as modulus of continuity* [30, *Theorem 2.19*].

(f) *The smooth but non-analytic 'pulse' function*

$$h_1(x) := \exp\big(\tfrac{4x^2}{4x^2-1}\big) \quad \textit{for } |x| < \tfrac{1}{2}, \qquad h_1(x) := 0 \quad \textit{for } |x| \geq \tfrac{1}{2}$$

is polytime on $[-1; 1]$ and vanishes outside $\big[-\frac{1}{2}; \frac{1}{2}\big]$; cmp. Fig. 1 a).

(g) *The function $h_{\exp} : [0; 1] \to [0; 1]$ with $h_{\exp}(0) := 0$ and $0 < x \mapsto 1/\ln(e/x)$ indicated in Fig. 1 b) is computable in exponential time but, having no subexponential modulus of continuity, not in polytime.*

The restriction to bounded intervals in Fact 3c) arises from Definition 2d) requiring the running time to be bounded in terms of the output precision only but independent of $x \in \mathrm{dom}(f)$. This requirement will be relaxed in Section 2.1.

We include the easy proof of Fact 3e) as preparation for more advanced arguments later: Suppose $\mathcal{M}$ computes f within time $t(n)$. For any $n \in \mathbb{N}$ let $m := t(n+1)$. To every $x \in \mathrm{dom}(f)$ there exists an integer a_m such that $|x - a_m/2^m| \leq 2^{-m-1}$ (and not just $\leq 2^{-m}$). (Well-)define $a_j := \lceil a_m/2^{m-j}\rceil$ for $0 \leq j < m$ and observe that the finite sequence $(a_0, \ldots, a_m)$ can be extended to a name $(a_j)_j \subseteq \mathbb{Z}$ of any real $x' \in [0; 1]$ with $|x - x'| \leq 2^{-m-1} = 2^{-\mu(n)}$. If $\mathcal{M}$ is given such a name then the integer b_{n+1} output will depend only on the finite part $(a_j)_{j\leq m}$ of said sequence, simply because by definition $\mathcal{M}$ can make at most $t(n+1)$ steps and in particular read at most m input elements before producing

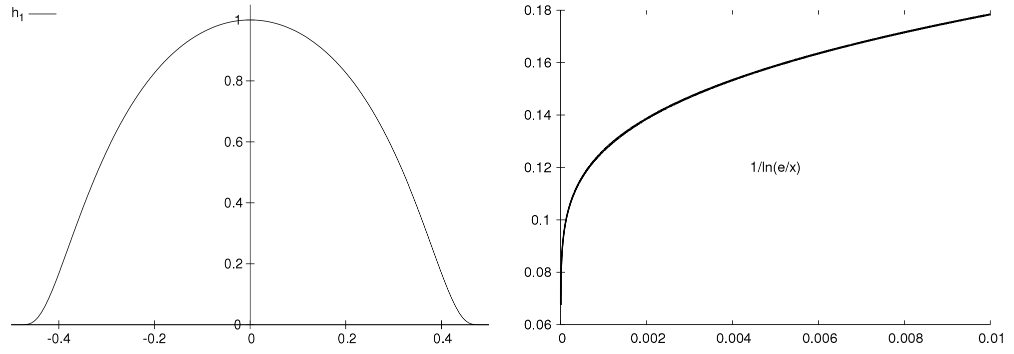

Fig. 1. (a) Smooth but non-analytic function h_1 with compact support $[-1/2; 1/2]$. (b) Function $h_{\exp}(x) = 1/\ln(e/x)$ from Fact 3g).

b_{n+1}. Therefore $y := b_{n+1}/2^{n+1}$ must approximate $f(x')$ up to error 2^{-n-1} for every $x' \in \mathrm{dom}(f)$ with $|x - x'| \leq 2^{-m-1}$, leading to $|f(x) - f(x')| \leq 2^{-n}$.

Concerning Fact 3g) observe that $2^{-n} \geq |h_{\exp}(2^{-\mu(n)}) - h_{\exp}(0)| = 1/(1 + \mu(n) \cdot \ln 2)$ requires $\mu(n) \geq (2^n - 1)/\ln 2$.

2.1. Multifunctions, operators, and parameterized complexity

In many applications one cannot (nor needs to) find, given x, some *specific* value $y = f(x)$, but any y satisfying a certain condition is sufficient. For instance the ceiling function $\mathbb{R} \ni x \mapsto \lceil x \rceil \in \mathbb{Z}$ is, and in fact any function $f : \mathbb{R} \to \mathbb{Z}$ witnessing the Archimedean property of the reals in the sense of satisfying $f(x) \geq x$ must be, discontinuous and thus uncomputable; whereas, given a sequence of approximations $a_m/2^m$ to x up to $1/2^m$, $a_0 + 1$ constitutes an integer upper bound to x. Note that both $a_0 = 0$ and $a_0 = 1$ lead to different upper bounds for the same $x = 1/2$: The assignment $x \mapsto a_0 + 1$ is not extensional/single-valued, but does computably provide *some* $y \in g(x) := \{z \in \mathbb{Z} : z \geq x\}$. Such relations or *multifunctions* in the discrete setting correspond to *Search problems*; their relevance in the theory of real computing is well-known [58,42].

Definition 4. (a) A (possibly partial) multifunction $f \;:\subseteq X \rightrightarrows Y$ is a relation $f \subseteq X \times Y$, identified with the mapping $X \ni x \mapsto f(x) := \{y \in Y : (x, y) \in f\} \in 2^Y$ on $\mathrm{dom}(f) := \{x \in X : f(x) \neq \emptyset\}$. For multifunctions $f \;:\subseteq X \rightrightarrows Y$ and $g \;:\subseteq Y \rightrightarrows Z$, their composition is (subtly) defined [70] as

$$g \circ f := \big\{(x, z) \big| x \in X, z \in Z, f(x) \subseteq \mathrm{dom}(g),\ \exists y \in Y : (x, y) \in f \text{ and } (y, z) \in g\big\} \tag{2}$$

(b) Computing a partial multifunction $f \;:\subseteq \mathbb{R}^\omega \times \mathbb{N} \rightrightarrows \mathbb{R}^\omega \times \mathbb{N}$ means converting any $(\bar{x}, k) \in \mathrm{dom}(f)$ into *some* $(\bar{y}, \ell) \in f(\bar{x}, k)$; where $\bar{x}$ and $\bar{y}$ are input and output as respective integer sequences $(a_m), (b_n)$ with $\left|\bar{x}_j - a_{\langle j,i\rangle}/2^i\right| \leq 2^{-i}$ and $\left|\bar{y}_j - b_{\langle j,i\rangle}/2^i\right| \leq 2^{-i}$.

(c) Such a computation runs in parameterized time $t(n, k)$ if ℓ and b_n appear within at most $t(n, k)$ steps, independently of $\bar{x}$. *Fully polytime* means both a running time bounded by some polynomial in $n + k$ and the value ℓ bounded by some polynomial in k.

The latter condition, together with Eq. (2), ensures that fully polytime computable multifunctions are closed under composition. We shall silently invoke type conversion and consider for example complex power series evaluation $\mathbb{C}^\omega \times \mathbb{C} \ni (\bar{a}, z) \mapsto \sum_j a_j z^j \in \mathbb{C}$ as a partial mapping from $(\mathbb{R}^2)^\omega \times \mathbb{R}^2 \cong \mathbb{R}^\omega$ to $\mathbb{R}^2 \subseteq \mathbb{R}^\omega$ or as partial mapping $\mathbb{C}^\omega \to \mathcal{L}\big([0; 1], \mathbb{C}\big)$, $\bar{a} \mapsto (z \mapsto \sum_j a_j z^j)$. Observe that, due to properties of the pairing function from Eq. (1), accessing the approximation $a_{\langle j,i\rangle}$ up to error $1/2^n$ to x_j in input $(x_j)_j \subseteq [-2^k; 2^k]$ requires, and suffices with, time polynomial in $n + k + j$. Note however that some common identifications may fail for multifunctions; for instance $(f, g) : X \rightrightarrows Y \times Z$ cannot in general be recovered from its components $f : X \rightrightarrows Y$ and $g : X \rightrightarrows Z$!

Remark 5. Classical discrete complexity theory considers algorithmic cost (time, memory) in the worst-case over all inputs $\vec{x}$ of binary length n asymptotically for $n \to \infty$. This worst-case may well be attained by very rare (e.g. one out of 2^n) instances $\vec{x}$ and input lengths n (e.g. Ackermann numbers). A refined approach with more realistic predictions, parameterized complexity theory [18] considers the algorithmic cost in dependence on n *and* on further, secondary parameter(s) k — whose meaning may vary from problem to problem:

(i) A prototypical example, NP-complete Vertex Cover can be decided in time polynomial in $n + 2^k$, where n denotes the number of vertices of the input graph and k the size of the desired cover: for 'small' (i.e. constant or logarithmic) values of k this is polynomial in the input size n, i.e. *fixed-parameter tractable*.

(ii) A large class of NP-complete decision problems on graphs is fixed-parameter tractable with respect to the *tree width* as secondary parameter k [46, §10].

(iii) The Knapsack Problem for inputs consisting of n packets with integer weights bounded by k can be decided in time polynomial in $n + k$, that is in *fully polynomial time*.

The theory of computing over the reals similarly benefits (in the sense of yielding more realistic predictions) from a refined approach taking into account further parameters in addition to the binary output precision n only. For instance in Numerical Analysis the performance of an algorithm considered efficient is permitted to degrade on ill-conditioned inputs: This violates the paradigm of single-parameter worst-case complexity but becomes in accordance with parameterized complexity for the *matrix condition number* [65] as parameter k. For generalized condition numbers of partial function $f \;:\subseteq\; \mathbb{R}^d \to \mathbb{R}$ with singularities/diverging behavior on $\vec{x} \notin \mathrm{dom}(f)$ see [11]. In fact a real problem on a *non*-compact domain [69, Theorem 7.1.5] may be computable but not within time bounded in terms of the output precision only: Consider for instance the exponential function (Example 6d), reciprocals $(0; 1] \ni x \mapsto 1/x$ [69, Exercise 7.2.10 + Theorem 7.3.12], or polynomial root finding [24].

Our approach of applying parameterized complexity theory to real number problems includes and generalizes these particular choices; cmp. Definitions 14 and 22.

Both qualitative and quantitative notions of continuity have been extended to multifunctions [48]. Regarding the proof of Fact 3e) we also point out the similarity to adversary-type arguments regularly employed in IBC [61]. The latter however generally pertains to the algebraic or unit-cost model of real computation [72,47]: IBC considers an algorithm computing some operator or functional to receive the function argument f as blackbox/oracle returning, given $x \in \mathrm{dom}(f)$, $f(x)$ in one step. In the bit model on the other hand a continuous real function f is given by its approximate values on a countable dense subset. More precisely an oracle for f returns, upon query of a tuple $(2^n, 2^m, q) \in \mathbb{Z}^3$ in binary, a dyadic rational approximation $p/2^n$ to $f(q/2^m)$ up to error $1/2^n$ while incurring cost of order $m + n + \log(p)$. Note that padding q or p reduces this in linear-time to the case $m = n$:

Definition 4 (Continued). (d) For $\mathfrak{O} : \{0, 1\}^* \to \{0, 1\}^*$ let $\mathcal{M}^{\mathfrak{O}}$ denote an oracle Turing machine (additional query state, query tape, and answer tape) equipped with $\mathfrak{O}$ as oracle. The machine $\mathcal{M}^?$, i.e. with contingent oracle, computes a (partial, mixed,[3] multi-) functional $\Lambda \;:\subseteq \mathcal{L}[-1; 1] \times \mathbb{R} \times \mathbb{N} \rightrightarrows \mathbb{R}^\omega$ if,

- for every $(f, \bar{x}, k) \in \mathrm{dom}(\Lambda)$ and
- for every sequence $(a_m) \subseteq \mathbb{Z}$ with $|x_j - a_{\langle j,i\rangle}/2^i| \le 1/2^i$ and
- for every oracle $\mathfrak{O}$ representing f in the sense of answering binary queries $\langle 2^m, q\rangle \in \mathbb{N}$ satisfying $-1 \le q/2^m \le 1$ with some $p \in \mathbb{Z}$ such that $|f(q/2^m) - p/2^m| \le 1/2^m$,
- $\mathcal{M}^{\mathfrak{O}}$, on input of (a_m) and of k, outputs a sequence $(b_n) \subseteq \mathbb{Z}$ with $|y_j - b_{\langle j,i\rangle}/2^i| \le 1/2^i$ for some $\bar{y} \in \Lambda(f, \bar{x}, k)$.

3 In the sense of receiving both function/second-order and first-order real number arguments.

(e) For some $t\colon \mathbb{N}\times\mathbb{N}\times\mathbb{N}\to\mathbb{N}_+$, the (parameterized) running time of $\mathcal{M}^?$ is bounded by t if b_n as above appears within $t(n,k,\lceil\|f\|\rceil)$ steps, independently of x, where $\|f\| := \max\{|f(x)| : -1\leq x\leq 1\}$. Λ is *fully polytime* if some oracle Turing machine can compute it within time polynomial in $n+k+\mathrm{lb}\,\|f\|$, where $\mathrm{lb}(t) := \lceil\log_2(t+1)\rceil$ for $t\geq 0$ and each oracle query $\langle 2^m, q\rangle \mapsto p$ counts with $m+\mathrm{lb}(q)+\mathrm{lb}(p)$ steps.

Real computability theory alternatively considers computation of functionals $f \mapsto \Lambda(f)$ without oracles based on encoding f as sequence of, say, (degrees and coefficient of) approximating dyadic Weierstraß polynomials. This notion is uniformly equivalent regarding computability [69, top of p. 161] but differs under the refined perspective of complexity: According to Bernstein's Theorem, functions as simple as the absolute value $[-1;1]\ni x\mapsto |x|$ and even certain $f\in\mathcal{C}^\infty[-1;1]$ require polynomials of degree exponential in N for uniform approximation up to error $1/2^N$ [30, §8.2]. Another natural encoding of $f\in\mathcal{C}[0;1]$, namely via its values on the dense sequence $(0,1,1/2,1/4,3/4,1/8,3/8,5/8,7/8,1/16,\ldots)$ of dyadic rationals, requires a machine to sequentially skip over exponentially many data in order to extract, say, $f(1/2^n)$. In fact even for the compact space $\mathcal{L}_1([-1;1],[-1;1])$, having exponential *metric entropy* implies via a combined counting and adversary argument that *no* encoding over infinite binary strings can render the evaluation operator polytime computable [71, §6]. So granting oracle (as opposed to sequential) access to f seems both theoretically and practically reasonable.

We record closure under composition: Applying a fully polytime functional Λ to a polytime Lipschitz function f (or to a curried polytime family $\bar{a}\mapsto\big(z\mapsto f(\bar{a},z)\big)$ of Lipschitz functions) yields a fully polytime family of functions $(x,k)\mapsto\Lambda(f,x,k)$ (or $(\bar{a},x,k)\mapsto\Lambda\big(f(\bar{a},\cdot),x,k\big)$). This and the above notions extend straightforwardly to functionals involving, say, two function arguments (f,g) and/or two integer parameters (k,ℓ). Observe that the latter according to Definition 4d) and Definition 4e) also provide the algorithm with discrete enrichment [37, p. 238/239] that for reasons of continuity may not be obtainable computationally; cmp. [76] or Item (e) in the following.

Example 6. (a) Multiplication of several bounded real numbers $[-2^k;2^k]^d \ni (x_1,\ldots,x_d)\mapsto x_1\cdots x_d\in[-2^{dk};2^{dk}]$ is computable in time polynomial in $n+k+d$; similarly for addition.

(b) Reciprocals $x\mapsto 1/x$ are computable on the *un*bounded interval $[2^{-k};\infty)$ within time polynomial in $k+n$:
For $x<2^{n+1}$, Newton–Raphson Division yields rational approximations to $1/x$ using time and bitlength polynomial in $n+k$; whereas the case $x>2^n$ can be detected without even reading the entire (integer part of the) first dyadic approximation to x.

(c) Bounded matrix determinants

$$[-2^k;2^k]^{d\times d} \ni A=(a_{j,k}) \mapsto \det(A)=\sum_{\sigma\in\mathcal{S}_d}\mathrm{sign}(\sigma)\cdot\prod_{j=1}^{d}a_{j,\sigma(j)} \in \mathbb{R} \tag{3}$$

are computable within time polynomial in $n+d+k$:
Apply Gaussian Elimination (Bareiss Algorithm) to 2^{-m}-approximations to A within time $\mathrm{poly}(m+k+d)$; and extract from Leibniz' Formula (3) the error bound $2^{\mathcal{O}(k\cdot d-m)}\leq 2^{-n}$ for $m\geq\mathrm{poly}(n+d+k)$.

(d) The exponential function (family) on $[-2^k;2^k]$ is computable in time polynomial in 2^k+n:
On integer arguments $K=|x|\leq 2^k$ this can be done by repeated multiplication, which allows to reduce evaluation on real arguments $x\in[-2^k;2^k]$ to the case $x\in[-1;1]$ up to improved precision $2^{-n}/\exp(K)$ by virtue of $\exp(y+K)=\exp(y)\cdot\exp(K)$. Evaluating the first J terms of the Taylor series $\sum_{j=0}^{J-1}x^j/j!$, each up to error 2^{-m}, takes time polynomial in $J+m$ and incurs total error $\leq\mathcal{O}(J2^{-m})$ plus the tail bound $\sum_{j\geq J}1/j!\leq\sum_{j\geq J}2^{-j+1}=2^{-J+2}$.

(e) The evaluation functional $\mathcal{L}[-1;1]\times[-1;1]\ni(f,x)\mapsto f(x)\in\mathbb{R}$ is incomputable; whereas its parameterized variant $\mathrm{Eval}\ :\subseteq\mathcal{L}[-1;1]\times\mathbb{R}\times\mathbb{N}\ni(f,x,\ell)\mapsto f(x)\in\mathbb{R}$ with $\mathrm{dom}(\mathrm{Eval}) := \big\{(f,x,\ell): f\in\mathcal{L}_{2^\ell}[-1;1], x\in[-1;1]\big\}$ is fully polytime computable:
The negative claim follows from the following observation easily formalized as an adversary argument: An algorithm cannot know or deduce any upper bound on the Lipschitz constant of

f from oracle queries at dyadic arguments only and hence is unable to predict the precision with which x must be known in order to approximate $f(x)$ up to $1/2^n$. For the positive claim obtain a dyadic approximation $q/2^m$ to x where $m := n+\ell+1$ and query the oracle for approximation $p/2^m$ to $f(q/2^m)$ up to error $1/2^m$. By triangle inequality and Lipschitz continuity, this approximates $f(x)$ up to $1/2^n$.

(f) Parametric Lipschitz maximization $\mathrm{Max}\colon \mathcal{L}_{2^\ell}([-1;1],[-2^k;2^k]) \times [-1;1]^2 \to [-2^k;2^k]$,

$$(f,u,v) \mapsto \max\{f(x) : \min(u,v) \le x \le \max(u,v)\}$$

is computable within time polynomial in $2^{\ell+n}+k$

(g) but not within subexponential time, even restricted to bounded real analytic 1-Lipschitz functions $f\colon [-1;1] \to [-1;1]$.

(h) The same holds for parametric integration, that is the functional

$$\int : \mathcal{L}_{2^\ell}([-1;1],[-2^k;2^k]) \times [-1;1] \ni (f,v) \mapsto \int_0^v f(x)\,dx \in \mathbb{R}$$

[17] characterizes polynomial-time computable norms on $\mathcal{C}[-1;1]$. Computability (without complexity considerations) of maximization and integration is well-known [69, Corollary 6.2.5 + Theorem 6.4.1]. We postpone the proofs of Items (f) to (h) to Section 3.2.

Remark 7. Identifying 'natural' parameters is an important part of parameterized complexity theory [46, §5]!

(a) Already in the discrete setting proceeding from binary to unary input representation renders the 'hard' integer factorization problem (used in RSA cryptosystems) polynomial-time computable. Similarly the running time in Items (a) to (e) of Example 6 depends on the binary output precision n and on the *binary length* k (i. e. essentially the logarithm!) of some upper bound to the real input in cases (a)+(c)+(d), to its reciprocal in (b), and to a Lipschitz constant in (e). The running time turns out as polynomial in (n and) said k for (a) to (c) and (e); but only in 2^k (that is, polynomial in the *value* of the upper bound on the real input) for the exponential function — reasonably enough. The dimension d on the other hand enters with its value (i. e. the 'unary' length) for cases (a) and (c).

(b) For analytic f (and k and ℓ) fixed, Example 6f) and Example 6h) fail to yield polynomial-time computability and thus does not – indeed cannot according to Example 6g) – imply Fact 1e). This demonstrates that a Lipschitz constant is insufficient and unsuitable as complexity parameter of real analytic functions with respect to maximization and integration! In the sequel we will explore more appropriate and purportedly natural such parameters.

(c) In the discrete realm, the equivalence of several seemingly unrelated notions of computability (μ-recursion, Turing machine, `WHILE` program, λ calculus) is generally considered as strong evidence for them as 'natural'. Similarly [22] has proven several ad-hoc notions of real function computability from the literature as equivalent, thus establishing them as reasonable. Related investigations are known concerning encodings of real numbers [7], Euclidean closed [8,40] and regular subsets [77,78]. Recent work shows them partly different under the refined view of polynomial-time complexity — but equivalent when restricting to convex sets [53]. Similarly the proposed encodings and parameters of analytic and Gevrey functions may be considered natural for the following reasons:
   (i) They are mutually parameterized polynomial-time equivalent.
   (ii) They induce closure properties such as parameterized polynomial-time computability of addition, multiplication, differentiation, integration — and of course evaluation.
   (ii) The resulting uniform parameterized complexities of maximization and integration, for any fixed choice of the parameters, boil down to the non-uniform Fact 1e) and Fact 1f).

(d) The *Type-2 Theory of Effectivity* (TTE) provides a convenient framework for formalizing and systematically comparing notions of computability over continuous universes via so-called representations, that is, encodings over the Cantor space of infinite binary sequences [69, §3]. It applies also to complexity investigations [71,56], preferably for spaces of polynomial metric entropy. For 'larger' universes second-order representations have been proposed, that are essentially encodings over the Baire space with a certain graduation [26,28].

We aim for a broad community without relying on these particular concepts and therefore prefer to spell out the encodings and parameters *in extenso*: A reader familiar with TTE will easily see the representation they induce.

3. Parameterized complexity of operators on analytic functions

For real numbers $u < v$ let $\mathcal{C}^\infty[u; v]$ denote the class of functions $f : (u; v) \to \mathbb{R}$ which are infinitely often differentiable and all derivatives continuously extend to $[u; v]$. For $z \in \mathbb{C}$ and $r > 0$, abbreviate $\mathrm{ball}(z, r) := \{ w \in \mathbb{C} : |w - z| < r \}$ and $\overline{\mathrm{ball}}(z, r) := \{ w \in \mathbb{C} : |w - z| \le r \}$. Let $U \subseteq \mathbb{C}$ be a non-empty open set of complex numbers and denote by $\mathcal{C}^\omega(U)$ the class of functions $g : U \to \mathbb{C}$ complex differentiable in the sense of Cauchy–Riemann. For sets $D, Y \subseteq \mathbb{C}$ that may or may not be open, define $\mathcal{C}^\omega(D, Y)$ to consist of those functions $f : D \to Y$ that can be extended to $g \in \mathcal{C}^\omega(U)$ for some open set $U \supseteq D$; $\mathcal{C}^\omega(D) := \mathcal{C}^\omega(D, \mathbb{C})$. A function $f : [-1; 1] \to \mathbb{C}$ thus belongs to $\mathcal{C}^\omega[-1; 1]$ if it is the restriction of a complex function differentiable on some open complex neighborhood U of $[-1; 1]$. Let us record for subsequent use the following folklore fact.

Fact 8. (a) *By virtue of* [35, *Proposition 1.2.12*], *a function* $f : [-1; 1] \to \mathbb{C}$ *belongs to* $\mathcal{C}^\omega[-1; 1]$ *iff, for some* $B, \ell \in \mathbb{N}_+$, *it holds*

$$\forall x \in [-1; 1],\ \forall j \in \mathbb{N} : \quad |f^{(j)}(x)| \le B \cdot \ell^j \cdot j!. \tag{4}$$

(b) *By Cauchy's Theorem, each* $f \in \mathcal{C}^\omega(U)$ *can be represented locally around* $z_0 \in U$ *by some power series* $f_{\bar{a}}(z - z_0) := \sum_{j=0}^\infty a_j (z - z_0)^j$. *More precisely, Cauchy's Differentiation Formula yields*

$$a_j = f^{(j)}(z_0)/j! = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{j+1}}\, dz, \qquad \overline{\mathrm{ball}}(z_0, r) \subseteq U$$

(c) *The General Leibniz Rule asserts* $(f \cdot g)^{(J)}(x) = \sum_{j=0}^J \binom{J}{j} f^{(j)}(x) \cdot g^{(J-j)}(x)$.

(d) *The Formula of Faà di Bruno expresses* J*-th derivatives of function composition,* $J \in \mathbb{N}_+$*:*

$$\big(g \circ f\big)^{(J)}(x) = \sum_{\substack{j_1,\dots,j_J \in \mathbb{N} \\ j_1 + 2j_2 + \cdots + Jj_J = J}} \frac{J!}{j_1! \cdot j_2! \cdots j_J!} \cdot g^{(j)}\big(f(x)\big) \cdot \Big(\frac{f^{(1)}(x)}{1!}\Big)^{j_1} \cdot \Big(\frac{f^{(2)}(x)}{2!}\Big)^{j_2} \cdots \Big(\frac{f^{(J)}(x)}{J!}\Big)^{j_J}$$

where $j := j_1 + j_2 + \cdots + j_J$.

(e) *With same notation,* $\sum_{j_1,\dots,j_J} \frac{j!}{j_1! \cdot j_2! \cdots j_J!} \cdot x^j = x \cdot (1 + x)^{J-1}$ *for* $J \in \mathbb{N}_+$ *and* $x > 0$ [35, *Lemma 1.4.1*].

(f) *Stirling's Approximation:* $\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n} \le n! \le n^{n+1/2} \cdot e^{1-n}$.

(g) $\sum_{n=1}^N |x_n|^\gamma \le \big(\sum_{n=1}^N |x_n|\big)^\gamma \le N^{\gamma-1} \cdot \sum_{n=1}^N |x_n|^\gamma$ *for* $\gamma \in \mathbb{N}_+$.

Now for a fixed power series with polytime computable coefficient sequence $\bar{a} = (a_j)_j$, its anti-derivative can be calculated termwise and also its maximum can be approximated efficiently from the Taylor expansion. And since $[-1; 1]$ is compact, finitely many such power series with rational centers z_0 suffice to describe f — and yield the drastic improvements for the analytic over the smooth case. However this proof sketch to the polynomial upper complexity bound is highly nonuniform (recall Section 1.1) and thus far from yielding an actual algorithm, not to mention its running time's dependences. For example the mere evaluation of a power series requires, in addition to the coefficient sequence $\bar{a}$, further information [44, Theorem 4.1]. This section presents, and analyzes the parameterized running times of, uniform algorithms for primitive operations on analytic functions. It begins with single complex power series, w.l.o.g. around 0 with radius of convergence $R > 1$ (Section 3.1), and then proceeds to real functions analytic on $[-1; 1]$ (Section 3.2).

Remark 9 (*Some Related Work*)**.**

(a) Numerical Analysis traditionally considers running times' dependence on the inverse error $1/\epsilon$. Adding closure under composition this leads to *Hereditarily Polynomial Bounded Analysis* [34] and corresponds to real versions of Andrzej Grzegorczyk's subrecursive hierarchy [60]. Note that in both cases a value (i. e. unary, as opposed to binary, length) is regarded as complexity parameter (recall Remark 7a).

(b) The notion of polynomial-time computation prevalent in *Real Complexity Theory* on the other hand roughly corresponds to so-called *exponential* convergence; cmp., e. g., [63]. Rooting in contemporary resource-oriented complexity theory it amounts to approximations up to error 2^{-n}, that is, roughly n correct digits after the radix point; recall Remark 7a).

(c) Uniform algorithms and parameterized upper running time bounds for evaluation have been obtained for instance in [15, Theorem 28] on a subclass of polynomial-time computable power series, namely on the hypergeometric ones whose coefficient sequences obey an explicit recurrence relation and thus can be described by finitely many real parameters. While roots of smooth functions may be abundant yet incomputable [59], it is known that roots of (polynomial-time) computable analytic functions are again (polynomial-time) computable [30, Theorem 4.11]. This non-uniform claim has recently been strengthened uniformly in [74].

(d) Further complexity considerations, and in particular lower bounds, are established in [3,49–51]. There is a vast literature on computability in complex analysis such as [23,1] or [16, §6]. [20] treats computability questions in the complementing, algebraic (aka BSS) model of real number computation [5]; see also [9] concerning their complexity theoretic relation.

(e) For practicality issues refer, e. g., to [66–68].

3.1. Complex power series on the closed unit disc

[69, Theorem 4.3.11] asserts complex power series evaluation $(\bar{a}, z) \mapsto f_{\bar{a}}(z)$ to be uniformly computable when providing, in addition to approximations to z and to $\bar{a} = (a_j)_j \subseteq \mathbb{C}$, some $r \in \mathbb{Q}$ with $|z| < r < R$ and some $A \in \mathbb{N}_+$ such that it holds

$$\forall j: \quad |a_j| \;\le\; A/r^j, \tag{5}$$

where $R := R(\bar{a}) := 1/\limsup_j |a_j|^{1/j}$ denotes the coefficient sequence's radius of convergence. Note that R can be highly incomputable [75, Theorem 6.2]. Moreover, such A does exist for $r < R$ but not necessarily for $r = R$: consider for instance $a_j := j$. Also, any power series has a singular point somewhere on its complex circle of convergence. We thus record that uniform power series evaluation requires discrete enrichment and has a complexity depending on both the output precision n and the distance between $|z|$ and R, parameterized appropriately. By scaling we can w.l.o.g. restrict to the closed unit disc, i. e. presume $|z| \le 1 < R$; and introduce two integer parameters A, k providing both the necessary additional discrete information and the running time bounds:

Definition 10. Abbreviate

$$\widetilde{\mathcal{C}_1^\omega} \quad := \quad \big\{(\bar{a}, A, k) \in \mathbb{C}^\omega \times \mathbb{N}_+ \times \mathbb{N}_+ \;:\; |a_j| \le A/2^{j/k} \text{ for all } j \in \mathbb{N}\big\}.$$

To $\bar{a} \in \mathbb{C}^\omega$ assign $f_{\bar{a}}(z) := \sum_j a_j z^j$ when defined, thus rendering $\widetilde{\mathcal{C}_1^\omega}$ a covering space for $\mathcal{C}^\omega\big(\overline{\text{ball}}(0,1)\big)$.

Indeed $|a_j| \le A/2^{j/k}$ implies $R(\bar{a}) \ge 2^{1/k} \searrow 1$ as $k \to \infty$. Conversely for $R(\bar{a}) > 1$ there exist $k \in \mathbb{N}_+$ with $R(\bar{a}) > r \ge 2^{1/k}$ and $A \in \mathbb{N}_+$ satisfying Eq. (5). Observe that $k = \Theta(\frac{1}{R-1})$ according to Lemma 11c). Now our operators on $\widetilde{\mathcal{C}_1^\omega}$ will in general transform all three, $\bar{a}$, A, and k. For instance, proceeding to the derivative's coefficient sequence $a'_j = (j+1) \cdot a_{j+1}$ does not affect the radius of convergence $R(\bar{a}) = R(\bar{a}')$: classically; but a bound A' on $|a'_j|/r^j$ cannot computably be derived from $\bar{a}$, A, and k! Instead, the below proof of Theorem 12d) will proceed from $r \ge 2^{1/k}$ closer towards 1 via $r' := \sqrt{r} \ge 2^{1/(2k)} =: 2^{1/k'}$ and from A to $A' := A \cdot (1 + 2k)$ of value polynomial in A and in k and computable in binary/unary within time polynomial in their binary/unary lengths, respectively. But let us first record some estimates:

Lemma 11. (a) *Let $r > 1$. Then $t \le C \cdot r^t$ holds for all $t > 0$, where $C := 1/(e \cdot \ln r)$.*
(b) *More generally, $t^s \le (\frac{s}{e \ln r})^s \cdot r^t$ for all $t > 0$ and $s \ge 0$ with the convention of $0^0 = 1$.*

(c) *For all* $N \in \mathbb{N}_+$, $1 + \frac{\ln 2}{N} \le \sqrt[N]{2} \le 1 + \frac{1}{N}$ *holds. In particular,* $1/\ln(r) = \Theta(\frac{1}{r-1})$ *as* $r \searrow 1$.

(d) *With* $(\bar{a}, A, k) \in \widetilde{\mathcal{C}_1^\omega}$ *and* $|z| \le r' := \sqrt{2^{1/k}} = \sqrt{r}$, *it holds* $|f_{\bar{a}}(z)| \le A \cdot \frac{r'}{r'-1}$.
And d-fold derivatives, $d \ge 1$, *satisfy* $|f_{\bar{a}}^{(d)}(z)| \le A \cdot d!/(r'-1)^{d+1}$.

(e) *For all* $\ell \ge 2$ *and* $x \ge \ell^2$ *it holds* $x^\ell \le \exp(x)$.
For all $a, \ell \ge 1$ *and* $b > 0$ *and* $x \ge \ell^2 \cdot a^{1/\ell}/b^2 \ge 4$ *it holds* $a \cdot x^\ell \le \exp(x \cdot b)$.

(f) *For all* $j, d \in \mathbb{N}$ *and* $\ell \in \mathbb{N}_+$ *it holds* $(j+1) \cdot (j+2) \cdots (j+d) \le d^d \cdot \exp(j)$
and $(j+\ell)^d \le (j+1)^d \cdot \ell^d$.

(g) *On the other hand,* $j \cdot (j-1) \cdots (j-d+1) \ge j^d/\exp(d)$ *for* $j, d \in \mathbb{N}_+$.

Proof. Any local extreme point x_0 of $0 < x \mapsto x \cdot r^{-x}$ is a root of $\frac{d}{dx} x \cdot r^{-x} = r^{-x} - \ln(r) \cdot x \cdot r^{-x}$, i.e. $t \cdot r^{-t} \le C := \max\{x \cdot r^{-x} : x > 0\}$ attained at $x_0 := 1/\ln(r)$. Replacing r with $r^{1/s}$ yields (b). For the first part of (c), write $\sqrt[N]{2} = \exp(\ln 2/N)$ and apply $1 + x \le \exp(x) \le 1 + x/\ln 2$ on $[0; \ln 2]$ to $x := \ln(2)/N$. For the second part of (c), Taylor expansion yields $\ln(1-x) \approx -\frac{1}{x} - \frac{1}{2x^2} - \cdots$ for $-1 \ll x := 1 - r < 0$.
Regarding (d), $|f_{\bar{a}}(z)| \le \sum_j |a_j| \cdot |z|^j \le \sum_j A \cdot (r'/r)^j = A \frac{1}{1 - r'/r} = A \cdot \frac{r'}{r'-1}$ and $|f_{\bar{a}}^{(d)}(z)| \le \sum_{j \ge d} j \cdot (j-1) \cdots (j-d+1) \cdot |z|^{j-d} \cdot A/r^j = A \cdot r'^{-2d} \cdot \partial_t^d \sum_j t^j \big|_{t=|z|/r} \le A \cdot r'^{-(d+1)} \cdot \partial_t^d 1/(1-t)\big|_{t=1/r'} = A \cdot d!/(r'-1)^{d+1}$.
Turning to (e), first record that $\ell^2 \le \exp(\ell)$ holds for all $\ell \ge 2$. In particular $x \le \exp(x/\ell)$ is true for $x = \ell^2$; and monotone in x because of $\partial_x x = 1 \le \exp(x/\ell)/\ell = \partial_x \exp(x/\ell)$ for all $x \ge \ell^2$.
Concerning the second claim, substitute $y := x \cdot a^{1/\ell}$ and conclude from the first one that $y^{\ell \cdot a^{1/\ell}/b} \le \exp(y)$ for all $y \ge \ell^2 \cdot a^{2/\ell}/b^2 \ge 4$.
Regarding (f), $(j+1) \cdots (j+d) \le (j+d)^d = d^d \cdot (1 + \frac{j}{d})^d \le d^d \cdot \exp(j/d)^d$.
Finally for (g) take natural logarithms and observe $\ln(j) + \ln(j-1) + \cdots + \ln(j-d+1) \ge \int_{j-d}^{j} \ln(t)\, dt = j \cdot \ln(j) - (j-d) \cdot \ln(j-d) - d \ge d \cdot \ln(j) - d$. □

Theorem 12. (a) *Evaluation* $\widetilde{\mathcal{C}_1^\omega} \times \overline{\mathrm{ball}}(0,1) \ni (\bar{a}, A, k, z) \mapsto f_{\bar{a}}(z) \in \mathbb{C}$ *is computable within time polynomial in* $n + k + \mathrm{lb}(A)$. *Moreover,* $f_{\bar{a}} \in \mathcal{L}_L\big(\overline{\mathrm{ball}}(0,1), \mathbb{C}\big)$ *holds for* $L := 17Ak^2$.

(b) *Pointwise addition* $\widetilde{\mathcal{C}_1^\omega} \times \widetilde{\mathcal{C}_1^\omega} \ni (\bar{a}, A, k, \bar{b}, B, \ell) \mapsto (\bar{a} + \bar{b}, A + B, \max\{k, \ell\}) \in \widetilde{\mathcal{C}_1^\omega}$ *is well-defined and computable within time polynomial in* $n + \mathrm{lb}(k + \ell) + \mathrm{lb}(A + B)$.

(c) *Pointwise multiplication considered as map* $\widetilde{\mathcal{C}_1^\omega} \times \widetilde{\mathcal{C}_1^\omega} \ni (\bar{a}, A, k, \bar{b}, B, \ell) \mapsto (\bar{c}, C, m) \in \widetilde{\mathcal{C}_1^\omega}$ *with* $c_j := \sum_{i=0}^{j} a_i \cdot b_{j-i}$ *and* $C := A \cdot B \cdot (1 + m)$ *where* $m := 2\max(k, \ell)$, *is computable within time polynomial in* $n + \mathrm{lb}(k + \ell) + \mathrm{lb}(A \cdot B)$.

(d) *Differentiation on* $\widetilde{\mathcal{C}_1^\omega}$ *is computable within time polynomial in* $n + \mathrm{lb}(k) + \mathrm{lb}(A)$. *More generally, d-fold differentiation*

$$\widetilde{\mathcal{C}_1^\omega} \times \mathbb{N} \ni \big((a_j)_j, A, k, d\big) \mapsto \Big(\big(a_{j+d} \cdot (j+1) \cdot (j+2) \cdots (j+d)\big)_j, A \cdot d^d \cdot (1+2k)^d, 2k\Big) \in \widetilde{\mathcal{C}_1^\omega}$$

is computable within time polynomial in $n + d + \mathrm{lb}(k) + \mathrm{lb}(A)$.

(e) *Similarly, d-fold anti-differentiation on* $\widetilde{\mathcal{C}_1^\omega}$ *is computable within time polynomial in* $n + d + \mathrm{lb}(k) + \mathrm{lb}(A)$.

(f) *Parametric maximization is polytime computable; more precisely, the mappings* Max ∘ Re *and* |Max| *from* $\widetilde{\mathcal{C}_1^\omega} \times [-1; 1]^2$ *to* $\mathbb{R}$ *are computable within time polynomial in* $n + k + \mathrm{lb}(A)$, *where*

$$\begin{aligned} \mathrm{Max} \circ \mathrm{Re} : (f, u, v) &\mapsto \max\big\{\mathrm{Re} f(x) : \min(u,v) \le x \le \max(u,v)\big\} \quad \textit{and} \\ |\mathrm{Max}| : (f, u, v) &\mapsto \max\big\{|f(x)| : \min(u,v) \le x \le \max(u,v)\big\}. \end{aligned}$$

(g) *As a converse to (a), the parameterized Taylor series expansion around 0,* $(f_{\bar{a}}, k, A) \mapsto (\bar{a}, k, A) \in \widetilde{\mathcal{C}_1^\omega}$ *is well-defined on the domain*

$$\Big\{(f, k, A) : f \in \mathcal{C}^\omega\big(\overline{\mathrm{ball}}(0, 2^{1/k})\big) \subseteq \mathcal{L}\big(\overline{\mathrm{ball}}(0, 2^{1/k}, \mathbb{C})\big), |f(z)| \le A\ \forall |z| \le 2^{1/k}\Big\}$$

and computable (by an oracle Turing machine, recall Definition 4 e) within time polynomial in $n + \mathrm{lb}(k) + \mathrm{lb}(A)$.

Note that, for real-valued f, it holds $\min\{f(z) : z\} = -\max\{-f(z) : z\}$ and $\max\{|f(z)| : z\} = \max\big(\max\{f(z) : z\}, -\min\{f(z) : z\}\big)$. Hence the above Items (a) to (f) indeed constitute a natural choice of basic primitive operations on $\mathcal{C}^\omega\big(\overline{\mathrm{ball}}(0,1)\big)$ parameterized according to $\widetilde{\mathcal{C}_1^\omega}$.

Remark 13. (a) Notice that the integer bound A enters logarithmically, that is, in terms of its binary length; whereas the parameter $k = \Theta(\frac{1}{R-1})$ governing convergence for $|z| \to 1$ may enter both logarithmically (Items b to e) and directly (Items a and f), that is, in terms of its unary length; cmp. Remark 7a).

(b) In fact $\Omega\big(k \cdot (n + \log k + \log A)\big)$ constitutes a lower complexity bound to evaluation, thus establishing Item (a) as asymptotically optimal: Dyadic approximations $b/2^n$ to some $y \neq 0$ up to error $1/2^n$ have binary length $\mathrm{lb}(b) \approx n$ and therefore require at least n steps to output in the bit model. The power series $\sum_j a_j z^j$ with $a_j := A/2^{j/k}$ has parameters A and k; and its evaluation at $z := 1$ yields a value $y = 2A \cdot 2^{1/k}/(2^{1/k} - 1) = \Theta(A \cdot k)$ according to Lemma 11c), amounting to roughly $\log A + \log k$ bits in front of the radix point that is, in addition the n digits aforementioned. Finally $\sum_{j\geq N} a_j z^j = A \cdot 2^{-N/k}/(1 - 2^{-1/k}) = \Theta(A \cdot 2^{-N/k} \cdot k)$ (Lemma 11c) is $\leq 2^{-n}$ iff $N \geq \Omega\big(k \cdot (n + \log k + \log A)\big)$: demonstrating that approximate evaluation requires to 'look' at at least that many coefficients.

(c) All operations are computable in time polynomial in $n + k + \mathrm{lb}\,A$ and return parameter values k' and $\mathrm{lb}\,A'$ polynomial in $k + \mathrm{lb}\,A$. One may thus combine both secondary parameters into a single one $k + \mathrm{lb}\,A$ that still serves as joint enrichment (upper bounding both k and $\mathrm{lb}\,A$) and yields fully polynomial-time computability.

(d) Four parameters R, A, B, ℓ on the other hand are employed in [4, Definition 2.2.1] in order to describe bounds of the form $|a_j| \leq (A + B \cdot j^\ell)/R^j$. And indeed such a refined modeling might allow for a more accurate prediction of the algorithms' behavior: a goal paradigmatic for Algorithm Engineering in (discrete) Computer Science but to the best of our knowledge new to rigorous numerics and real complexity.

Proof of Theorem 12. (a) Given k, evaluate the first roughly $N := k \cdot M$ terms of the power series on the given z for $M := n + \mathrm{lb}(k) + \mathrm{lb}(A)$: according to Eq. (5) the tail $\sum_{j\geq N} |a_j| \cdot |z|^j \leq A \cdot \sum_{j\geq N} (|z|/r)^j = A \cdot (|z|/r)^N \frac{1}{1-|z|/r}$ is then small of order $(|z|/r)^{k\cdot M} = \mathcal{O}(2^{-M})$ for $r = 2^{1/k}$ and $\frac{1}{1-|z|/r} = \Theta(k)$ because of $|z| \leq 1$. For the second claim apply Lemma 11d) and Lemma 11c) to conclude $\sqrt{r} - 1 \geq \ln(2)/(2k)$ and $|f'(z)| \leq A/(\sqrt{r} - 1)^2 \leq A \cdot (2k/\ln 2)^2$. The Mean Value Theorem turns this into a Lipschitz bound up to a factor of two to take into account the diameter of $\overline{\mathrm{ball}}(0,1)$.

(b) Computing $c_j := a_j + b_j$ suffices with time polynomial in $n + \mathrm{lb}(A + B) + j$. The bound $|c_j| \leq (A + B)/2^{j/\max\{k,\ell\}}$ is immediate.

(c) Abbreviate $r_{\bar{a}} := 2^{1/k}$ and $r_{\bar{b}} := 2^{1/\ell}$ such that $|a_j| \leq A/r_{\bar{a}}^j$ and $|b_j| \leq B/r_{\bar{b}}^j$. Then $r := \min(r_{\bar{a}}, r_{\bar{b}}) =: 2^{2/m}$ and $1 < r_{\bar{c}} := 2^{1/m} \leq \sqrt{r}$ implies $j \leq \sqrt{r}^j/(e \cdot \ln \sqrt{r}) \leq m \cdot \sqrt{r}^j/(e \cdot \ln 2) \leq m \cdot \sqrt{r}^j$ according to Lemma 11a) and $e \ln 2 \geq 1$; hence $|c_j| \leq \sum_{i=0}^j A/r_{\bar{a}}^i \cdot B/r_{\bar{b}}^{j-i} = A \cdot B \cdot (j+1)/r^j \leq C/\sqrt{r}^j$.

(d) Differentiate termwise: For $r := 2^{1/k}$ observe $j \leq \frac{2dk}{e\ln 2} \cdot \sqrt{r}^{j/d}$ according to Lemma 11a); which implies $(j+1) \cdot (j+2) \cdots (j+d) \leq (1 + \frac{2kd}{e\ln 2}) \cdot (2 + \frac{2kd}{e\ln 2}) \cdots (d + \frac{2kd}{e\ln 2}) \cdot \sqrt{r}^j \leq d^d \cdot (1+2k)^d \cdot \sqrt{r}^j$; hence $|a_{j+d} \cdot (j+1) \cdot (j+2) \cdots (j+d)| \leq A \cdot d^d \cdot (1+2k)^d/\sqrt{r}^j$.

(e) Integrate termwise: Given $(\bar{a}, k, A)$ with $r := 2^{1/k}$, output $a'_0 := a'_1 := \cdots a'_{d-1} := 0$ and $a'_j := a_{j-d}/j/(j-1)/\cdots/(j-d+1)$ for $j \geq d$ as well as $k' := k$ and some $A' \geq A \cdot r^d$.

(f) First suppose that $f|_{[-1;1]}$ is real, i.e. $a_j \in \mathbb{R}$. Similar to (a), the first $N := k \cdot \mathcal{O}(n + \mathrm{lb}\,k + \mathrm{lb}\,A)$ terms of the series yield a polynomial $p \in \mathbb{D}_N[X]$ of $\deg(p) < N$ with dyadic coefficients which uniformly approximates f up to error 2^{-n-1}. In particular it suffices to approximate the maximum of p on $[u', v']$ up to 2^{-n-1} (for $u', v' \in \mathbb{D}$ sufficiently close to u and v, respectively). This can be achieved by bisection on y w.r.t. the following existentially quantified formula in the first-order equational theory of the reals with dyadic parameters which, involving only a constant number of polynomials and quantifiers, can be decided in time polynomial in the degree and binary coefficient length [2, Exercise 11.7]:

$$\begin{aligned} &\Phi(u', v', p_0, \dots, p_{N-1}) \\ :=\ & \text{“}\exists x, r, s, t \in \mathbb{R} :\ \underbrace{x = u' + r^2}_{x \ge u'} \wedge \underbrace{x = v' - s^2}_{x \le v'} \wedge \underbrace{p(x) = y + t^2}_{p(x) \ge y}\text{”}. \end{aligned}$$

In the general case of a complex-valued $f|_{[-1;1]}$, $|f|^2 = \mathrm{Re}(f)^2 + \mathrm{Im}(f)^2$ is uniformly approximated by the real polynomial $q := \mathrm{Re}(p)^2 + \mathrm{Im}(p)^2$, thus $\max |f|^2$ is polytime computable as above. Since both $\mathbb{R} \ni t \mapsto t^2$ and $\mathbb{R}_+ \ni s \mapsto \sqrt{s}$ are monotonic and polytime computable, the same follows for $\max |f| = \sqrt{\max |f|^2}$.

(g) The bound $|a_j| \le A/2^{j/k}$ follows from Cauchy's differentiation formula (Fact 8b). For real analytic $f : [-1; 1] \to \mathbb{R}$, polytime computability of the sequence $a_j = f^{(j)}(0)/j!$ had been established in [43] using evaluation and interpolation; cf. also the proof of [30, Theorem 6.9] and the more general Theorem 23b). For the complex case, treat $\mathrm{Re} f|_{[-1;1]}$ and $\mathrm{Im} f|_{[-1;1]}$ separately. □

3.2. Analytic functions on the real unit interval

We now consider $f \in \mathcal{C}^\omega[-1; 1]$, that is, complex-valued functions admitting local power series expansions at every $x \in [-1; 1]$. In view of Fact 8a) one arrives at two natural parameterized encodings — that Theorem 16a) and Theorem 16b) establish as parameterized polynomial-time equivalent:

Definition 14. (a) $\widehat{\mathcal{C}^\omega}[-1; 1] := \left\{ \left(f|_{[-1;1]}, A, k\right) \ :\ f \in \mathcal{C}^\omega(\overline{\mathcal{R}}_k),\ |f(z)| \le A \quad \forall z \in \overline{\mathcal{R}}_k \right\} \subseteq \mathcal{L}([-1; 1], \mathbb{C}) \times \mathbb{N}_+^2$, where $\overline{\mathcal{R}}_k := \{x + iy : -\frac{1}{k} \le y \le \frac{1}{k}, -1 - \frac{1}{k} \le x \le 1 + \frac{1}{k}\}$.

(b) Let $\widetilde{\mathcal{C}^\omega}[-1; 1] := \left\{ \left(\overset{\ell}{\tilde{f}}, B, \ell\right) \ :\ f \in \mathcal{C}^\omega[-1; 1],\ \|f^{(j)}\| \le B \cdot \ell^j \cdot j! \right\} \ \subseteq\ \mathbb{C}^\omega \times \mathbb{N}_+^2$, where

$$\begin{aligned} \overset{\ell}{\tilde{f}} := \Big(& f(-1), f(-1 + 1/\ell), f(-1 + 2/\ell), \dots, f(1 - 1/\ell), f(1),\ \ f'(-1), \\ & f'(-1 + 1/\ell), \dots, f'(1), f''(-1), f''(-1 + 1/\ell), \dots, f''(1), \dots\dots, \\ & f^{(j)}(-1), \dots, f^{(j)}(1), \dots\dots \Big). \end{aligned}$$

Mathematically, $f \in \mathcal{C}^\omega[-1; 1]$ is already determined by a germ, that is, one single power series. In fact our proof of Theorem 16d) employs analytic continuation (by one round: The computational complexity of its iteration is deferred to a different work.) So $\widetilde{\mathcal{C}^\omega}[-1; 1]$ encodes f via its power series expansions at $2\ell + 1$ equidistant points in $[-1; 1]$ and the growth condition from Fact 8a). $\widehat{\mathcal{C}^\omega}[-1; 1]$ on the other hand encodes f via its values on dyadic rationals enriched with an integer A bounding the continuation of f to a complex rectangle in $\mathrm{dom}(f)$ extending beyond $[-1; 1]$ by at least $1/k$; put differently, $1/k$ lower bounds the distance from $[-1; 1]$ to any singularity of f. Recall that $\Theta(1/k)$ in Definition 10 would lower bound the distance from $\overline{\mathrm{ball}}(0, 1)$ to any singularity of $f_{\bar{a}}$ according to Lemma 11c).

Example 15. (a) For $x_m \in [-1; 1]$ and $y_m > 0\ (1 \le m \le M)$ the function $z \mapsto \prod_m \left((z - x_m)^2 + y_m^2\right)^{-1}$ is analytic on $[-1; 1]$ with complex singularities at $x_m \pm iy_m$.

(b) The Gaussian function $g_1(x) := \exp(-x^2)$ has

$$|g_1^{(j)}(x)| \ \le\ \tfrac{j!}{2\pi} \cdot \int_{|z-x|=1} |\exp(-z^2)|/1^{j+1}\, dz \ \le\ j! \cdot e$$

according to Cauchy's differentiation formula (Fact 8b), because $|\exp(-z^2)| = \exp\big(-\mathrm{Re}^2(z) + \mathrm{Im}^2(z)\big) \le \exp(1)$ due to $|z - x| = 1$ with $x \in \mathbb{R}$.

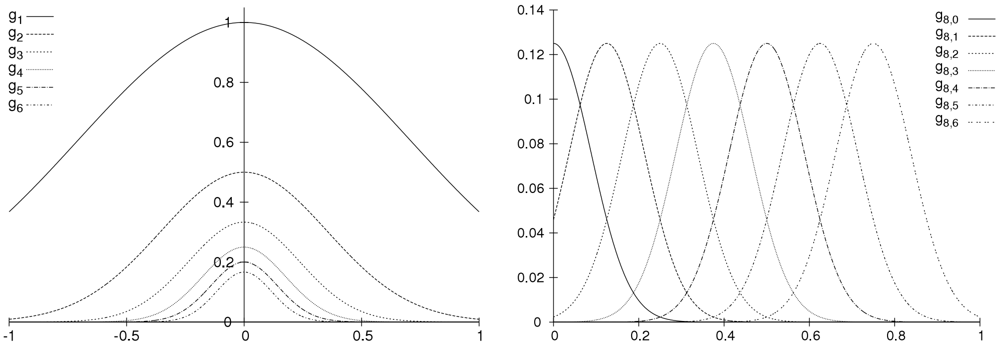

Fig. 2. (a) Gaussian functions g_J and (b) their shifts $g_{J,j}$ as employed in the proof of Example 6g and Example 6h).

Now we can catch up on the postponed

Proof of Example 6. (f) Max: $\mathcal{L}_{2^\ell}([-1;1],[-2^k;2^k]) \times [-1;1]^2 \ni (f,u,v) \mapsto \max\{f(x) : \min(u,v) \le x \le \max(u,v)\} \in [-2^k;2^k]$ is computable within time polynomial in $k+2^{\ell+n}$: Subdivide the interval $[\min(u,v);\max(u,v)]$ into subintervals of width $2^{-n-\ell}$ and evaluate f on each up to error 2^{-n}: Taking the maximum of these $\le 2^{n+\ell+1}$ dyadic rationals of binary length $\le k+n$ yields an approximation to $\mathrm{Max}(f,u,v)$ up to error 2^{-n+1}.

(g) The restriction Max : $\mathcal{L}_1([-1;1],[-1;1]) \cap \mathcal{C}^\omega[-1;1] \to \mathbb{R}$ is not computable in subexponential time: Shift and scale the real analytic Gaussian function g_1 from Example 15b) as depicted in Fig. 2:

$$g_J(x) := g_1(J\cdot x)/J = \exp(-J^2x^2)/J, \quad J \in \mathbb{N}_+; \qquad g_{J,j}(x) := g_J(x - j/J), \quad 0 \le j < J.$$

For $J \gg 1$ these functions are 'high' and 'thin' but not too 'steep' in the sense that

$$g_J(0) = 1/J, \qquad \forall |x| \ge m/J : |g_J(x)| \le 2^{-m}/J, \qquad \forall x : |g_J'(x)| \le 1$$

and in particular 1-Lipschitz. Now any algorithm computing $f \mapsto \mathrm{Max}(f,0,1) = \max\{f(x) : 0 \le x \le 1\}$ on the set $\{0, g_{J,0}, g_{J,1}, \ldots, g_{J,J-1}\} \subseteq \mathcal{C}^\omega[-1;1] \cap \mathcal{L}_1[-1;1]$ up to error $2^{-n} =: 1/(2J)$ must distinguish (every oracle representing) the identically zero function from (all oracles representing) some of the $g_{J,j}$ $(0 \le j < J)$ because the first has $\mathrm{Max}(0,0,1) = 0$ and the others $\mathrm{Max}(g_{J,j},0,1) = 1/J$. Yet, since the $g_{J,j}$ are 'thin', any evaluation up to error 2^{-m} at some $x = q/2^m$ with $|x - j/J| \ge m/J$ (i.e. a query $\langle 2^m, q\rangle$ to the given oracle) may return 0 as approximation to $g_{J,j}(x)$. For a sequence $\langle 2^{m_i}, q_i\rangle$ of queries that unambiguously distinguishes the zero function from the $g_{J,j}$, the intervals $\left[q_i/2^{m_i} - \frac{m_i}{J}; q_i/2^{m_i} + \frac{m_i}{J}\right]$ therefore must necessarily cover $[0;1]$ and in particular satisfy $\sum_i m_i \ge J/2 = 2^{n-2}$. On the other hand each such query incurs cost $\Omega(m_i)$.

(h) The same holds for $\int : \mathcal{L}_{2^\ell}([-1;1],[-2^k;2^k]) \times [-1;1] \ni (f,v) \mapsto \int_0^v (x)\,dx \in \mathbb{R}$: For the positive claim similarly to (f) return the sum of the approximate values of f on the appropriate subintervals of width $2^{-n-\ell}$, divided by the number of subintervals. For the counterpart to (g) observe $\int_0^\infty g_1(x) = \sqrt{\pi}/2$ and $\int_J^\infty g_1(x) = \mathcal{O}\big(\exp(-J^2)\big)$; therefore $\int_0^1 g_{J,j}(x) \ge \int_0^1 g_J(x) \ge 1/(2J^2)$ for all sufficiently large J. Hence, as previously, it is necessary to distinguish $g_{J,j}$ (for unknown j) from the zero function in order to approximate $f \mapsto \int(f,0,1)$ on $\{0, g_{J,0}, \ldots, g_{J,J-1}\}$ up to error $2^{-n} =: 1/(4J^2)$; which requires a collection $\langle 2^{m_i}, q_i\rangle$ of queries of total cost at least $\sum_i m_i \ge J/2 = 2^{n/2-2}$. $\square$

Observe how the bit model, other than IBC, requires taking into account queries of varying precision and cost in order to establish the lower bound in item (g) and the second part of (h). A refined adversary analysis will be employed in the proof of Theorem 23g). We now state the main result of this subsection:

Theorem 16. (a) *The parameterized Taylor series expansion functional*

$$\widehat{\mathcal{C}^\omega}[-1;1] \times [-1;1] \ni (f,A,k,x) \mapsto \big(f^{(j)}(x)\big)_j \in \mathbb{C}^\omega$$

is computable within time polynomial in $n + \mathrm{lb}(k) + \mathrm{lb}(A)$. *Moreover* $(f, A, k) \in \widehat{\mathcal{C}^\omega}[-1;1]$ *implies* $\forall |x| \le 1 : |f^{(j)}(x)| \le A \cdot k^j \cdot j!$. *In particular* $\widehat{\mathcal{C}^\omega}[-1;1] \ni (f, A, k) \mapsto (\overset{k}{\widetilde{f}}, A, k) \in \widetilde{\mathcal{C}^\omega}[-1;1]$ *is well-defined and computable in time polynomial in* $n + k + \mathrm{lb}(A)$.

(b) *Evaluation* $\widetilde{\mathcal{C}^\omega}[-1;1] \times [-1;1] \ni (\overset{\ell}{\widetilde{f}}, B, \ell) \mapsto f(x) \in \mathbb{C}$ *is computable within time polynomial in* $n+\ell+\mathrm{lb}(B)$. *Moreover, for* $(\overset{\ell}{\widetilde{f}}, B, \ell) \in \widetilde{\mathcal{C}^\omega}[-1;1]$ *it holds* $f \in \mathcal{C}^\omega\big(\overline{\mathcal{R}}_{2\ell}\big)$ *and* $\forall z \in \overline{\mathcal{R}}_{2\ell} : |f(z)| \le 4B$.

(c) *Pointwise addition* $+$ *and multiplication* $\cdot$ *on* $\mathcal{C}^\omega[-1;1]$, *both considered as mappings*

$$\star : \widetilde{\mathcal{C}^\omega}[-1;1] \times \widetilde{\mathcal{C}^\omega}[-1;1] \to \widetilde{\mathcal{C}^\omega}[-1;1]$$
$$\Big((\overset{\ell}{\widetilde{f}}, B, \ell), (\overset{m}{\widetilde{g}}, C, m)\Big) \;\mapsto\; \Big(\overset{\ell+m}{\widetilde{f \star g}}, B \star C, \ell + m\Big),$$

are well-defined and computable within time polynomial in $n + \ell + m + \mathrm{lb}(B \star C)$.

(d) *Iterated differentiation on* $\mathcal{C}^\omega[-1;1]$, *considered as the mapping*

$$\widetilde{\mathcal{C}^\omega}[-1;1] \times \mathbb{N}_+ \;\ni\; \Big(\overset{\ell}{\widetilde{f}}, B, \ell, d\Big) \;\mapsto\; \Big(\overset{3\ell}{\widetilde{f^{(d)}}}, B \cdot \ell^d \cdot d^d, 3\ell\Big) \;\in\; \widetilde{\mathcal{C}^\omega}[-1;1],$$

is well-defined and computable within time polynomial in $n + d + \ell + \mathrm{lb}(B)$.

(e) *Anti-differentiation on* $\mathcal{C}^\omega[-1;1]$, *considered as the mapping*

$$\widetilde{\mathcal{C}^\omega}[-1;1] \;\ni\; \Big(\overset{\ell}{\widetilde{f}}, B, \ell\Big) \;\mapsto\; \Big(\overset{\ell}{\widetilde{\textstyle\int f}}, B, \ell\Big) \;\in\; \widetilde{\mathcal{C}^\omega}[-1;1],$$

is well-defined and computable within time polynomial in $n + \ell + \mathrm{lb}(B)$, *where* $f^{(-1)} := \int f : x \mapsto \int_0^x f(t)\,dt$.

(f) *Parametric maximization on* $\mathcal{C}^\omega[-1;1]$, *considered as the mapping* Max $\circ$ Re*:*

$$\widetilde{\mathcal{C}^\omega}[-1;1] \times [-1;1]^2 \;\ni\; \big(\overset{\ell}{\widetilde{f}}, B, \ell, u, v\big) \;\mapsto\; \max\big\{\mathrm{Re} f(x) : \min(u,v) \le x \le \max(u,v)\big\},$$

is computable within time polynomial in $n + \ell + \mathrm{lb}(B)$*; and similarly for* $|\mathrm{Max}|$.

(g) *Concerning composition, for* $(f, A, k), (g, C, m) \in \widehat{\mathcal{C}^\omega}[-1;1]$ *with* $\mathrm{range}(f|_{[-1;1]}) \subseteq [-1;1]$ *it holds* $(g \circ f, C, 2Akm) \in \widehat{\mathcal{C}^\omega}[-1;1]$.

Note that Claims (a) and (b) allow to convert forth and back between $\widetilde{\mathcal{C}^\omega}[-1;1]$ and $\widehat{\mathcal{C}^\omega}[-1;1]$ within parameterized polynomial time, that is, to switch between operating on the power series expansions (Items d and e and f) or on oracle access to function arguments (Items c and g). Similarly to Remark 13c), the secondary parameters employed in $\widehat{\mathcal{C}^\omega}[-1;1]$ and $\widetilde{\mathcal{C}^\omega}[-1;1]$ can be combined into the single $k + \mathrm{lb}(A)$ and $\ell + \mathrm{lb}(B)$, respectively, yielding fully polytime algorithms (Definition 4c and Definition 4e) for most natural primitive operations on analytic functions — with the notable exception of composition whose output value $2Akm$ for the 'unary' parameter is exponential in the 'binary' input $\mathrm{lb}(A)$; recall Remark 13a). For fixed parameter values, however, all operations are polytime in the output precision n: thus recovering the nonuniform Fact 1e).

Proof of Theorem 16. (a) The bound $|f^{(j)}(x)| \le A \cdot k^j \cdot j!$ follows from Cauchy's differentiation formula (Fact 8b) in view of $\overline{\mathrm{ball}}(x, 1/k) \subseteq \overline{\mathcal{R}}_k$. Theorem 12g) thus applies to the translated and scaled function $f_x(z) := f\big(x + \frac{z}{2k}\big)$ analytic on ball$(0,2)$ with Taylor coefficients $a_{x,j} := f_x^{(j)}(0)/j! = f^{(j)}(x)/(2k)^j/j!$ bounded by $A/2^j$, where proceeding from f to f_x increases the running time by a factor at most polynomial in $\mathrm{lb}(k)$. Hence the sequence $((a_{x,j})_j, 1, A)$ is uniformly computable in time polynomial in $n + \mathrm{lb}(A) + \mathrm{lb}(k)$; from which $f^{(j)}(x) = a_{x,j} \cdot (2k)^j \cdot j!$ can be recovered in time polynomial in $j + \mathrm{lb}(k) + \mathrm{lb}(A)$. Applying this $2k + 1$ times yields $\big(f^{(j)}(i/k)\big)_j$ for $i \in \{-k, \ldots, k\}$.

(b) Uniformly bounded growth of derivatives $\|f^{(j)}\| \le B \cdot \ell^j \cdot j!$ shows (i) that $1/2^n$-approximations to entry $f^{(j)}(i/\ell)$ in $\overset{\ell}{\widetilde{f}}$ can be accessed in time polynomial in $\mathrm{lb}(B) + \mathrm{lb}(\ell) + j + n$ and (ii) that f

is real analytic on $[-1; 1]$ (recall Fact 8a) with radius of convergence around every x at least $1/\ell$ (thus $f \in \mathcal{C}^\omega(\overline{\mathcal{R}}_{\sqrt{2}\ell})$). Moreover $f(z) = \sum_j f^{(j)}(x)(z-x)^j/j!$ is bounded on $\overline{\text{ball}}\big(x, 1/\sqrt{2}\ell\big)$ by $\sum_j B \cdot \big(\ell/\sqrt{2}\ell\big)^j \leq 4B$ independent of $x \in [-1; 1]$ and therefore also on $\bigcup_{-1\leq x\leq 1} \overline{\text{ball}}\big(x, 1/\sqrt{2}\ell\big) \supseteq \overline{\mathcal{R}}_{2\ell}$. Given $\big(f^{(j)}(x)\big)_j$, $-1 \leq x \leq 1$, evaluation of f on $\overline{\text{ball}}(x, 1/\sqrt{2}\ell)$ is computable within time polynomial in $n+\text{lb}(\ell)+\text{lb}(B)$: Similarly to (a) consider the translated and scaled function $f_x(z) := f\big(x + \frac{z}{\sqrt{2}\ell}\big)$ analytic on $\text{ball}(0, \sqrt{2})$ with Taylor coefficients $a_{x,j} := f_x^{(j)}(0)/j! = f^{(j)}(x)/(\sqrt{2}\ell)^j/j!$ bounded by $B/\sqrt{2}^j$, thus Theorem 12a) applies with $A := B$ and $k := 2$. In the general case given $\overset{\ell}{\widetilde{f}}$ and $-1 \leq z \leq 1$, it suffices to find some $i \in \{-\ell, \ldots, +\ell\}$ with $|z - i/\ell| \leq 1/(\sqrt{2}\ell)$: the 'overlap' for, say, $z \in [0.3/\ell; 0.7/\ell]$ working with both $i = 0$ and $i = 1$ guarantees this possible in time polynomial in $\text{lb}(\ell)$; and accessing the appropriate subsequence $\big(f^{(j)}(i/\ell)\big)_j$ in $\overset{\ell}{\widetilde{f}}$ takes time polynomial in $\text{lb}(B) + \ell + n$.

(c) Concerning well-definition recall the General Leibniz Rule (Fact 8c):

$$|(f \cdot g)^{(J)}(x)| \leq \sum_{j=0}^{J} \binom{J}{j} \cdot |f^{(j)}(x)| \cdot |g^{(J-j)}(x)|$$

$$\leq \sum_{j=0}^{J} \binom{J}{j} \cdot B \cdot \ell^j \cdot j! \cdot C \cdot m^{J-j} \cdot (J-j)! \leq B \cdot C \cdot J! \cdot (\ell+m)^J.$$

Algorithmically we elegantly argue that polynomial-time computability of real addition and multiplication trivially carry over pointwise to bounded functions (Example 6a) given by oracle access in the sense of $\widehat{\mathcal{C}^\omega}[-1; 1]$; which according to (a) and (b) is polynomial-time equivalent in the claimed parameters to $\widetilde{\mathcal{C}^\omega}[-1; 1]$.

(d) Regarding well-definition observe $|f^{(j+d)}(x)| \leq B \cdot \ell^d \cdot \ell^j \cdot j! \cdot \big((j+1)\cdots(j+d)\big)$ where the latter rising factorial is bounded by $d^d \cdot 3^j$ according to Lemma 11f). These bounds show that the shifts of the given multi-sequences $f^{(j)}(i/\ell)$ constituting $\overset{\ell}{\widetilde{f}}$ by d elements are easy to compute in time polynomial in $\text{lb}(B) + \text{lb}(\ell) + j + n$; but the factor-three increase of ℓ requires the introduction of one new expansion point left and right to each previous one. Put differently, we need to calculate $f^{(j)}\big(\frac{3i\pm 1}{3\ell}\big)$ for all $i \in \{-\ell, \ldots, +\ell\}$ and all $j \in \mathbb{N}$. These can be obtained from the given $a_j := f^{(j)}(i/\ell)$ by evaluating the power series $f^{(j)}(z) = \sum_{j'} a_{j'}^{(j)} z^{j'}$ at $z := \pm 1/(3\ell)$, where we have already seen $a_{j'}^{(j)} := a_{j+j'} \cdot (j'+1)\cdots(j'+j)$ to satisfy $|a_{j'}^{(j)}| \leq (B \cdot \ell^j) \cdot (3\ell)^{j'} \cdot j'!$. So according to (b) this evaluation is feasible within time polynomial in $n + (3\ell) + \text{lb}(B \cdot \ell^j)$, that is, polynomial in $n + \ell + j + \text{lb}(B)$.

(e) By the Fundamental Theorem of Calculus, $\|(\int f)^{(j)}\| = \|f^{(j-1)}\| \leq B \cdot \ell^{j-1} \cdot (j-1)!$ for $j \geq 1$ and $\big|\big(\int f\big)(x)\big| \leq B$, thus establishing well-definition. Computation-wise, $(\int f)^{(j)}(x) = f^{(j-1)}(x)$ for $j \geq 1$ amounts to a shift in the input sequence $\overset{\ell}{\widetilde{f}}$ of entries of binary length $\approx \text{lb}(B)+j\,\text{lb}(\ell)+\ell\,\text{lb}(\ell)+n$; and for $j = 0$, $\big(\int f\big)(\frac{i\pm 1}{\ell}) = \sum_{j'\geq 1} f^{(j'+1)}(i/\ell) \cdot (\pm 1/\ell)^{j'}/j'!$ can be obtained by evaluating the shifted power series within time polynomial in $n + \ell + \text{lb}(B)$ similarly to (b).

(f) We treat the case of $|\text{Max}|$, $\text{Max} \circ \text{Re}$ proceeds similarly. For $i/\ell \leq u \leq v \leq (i+1)/\ell$, $|\text{Max}|(f, u, v)$ can be computed from the power series expansion around i/ℓ provided by $\overset{\ell}{\widetilde{f}}$ within time polynomial in $n + \ell + \text{lb}(B)$ according to Theorem 12f). For general $-1 \leq u \leq v \leq 1$, identify (similarly to b) $-\ell \leq i' \leq i'' \leq \ell$ with $|u - i'/\ell|, |v - i''/\ell| \leq 1/(\sqrt{2}\ell)$. Now consider the (at most 2ℓ) intervals $\big[u, \frac{i'+1/\sqrt{2}}{\ell}\big] \subseteq \overline{\text{ball}}\big(i'/\ell, 1/(\sqrt{2}\ell)\big)$, $\big[\frac{i'+1/\sqrt{2}}{\ell}, \frac{i'+1+1/2}{\ell}\big] \subseteq \overline{\text{ball}}\big((i'+1)/\ell, 1/(\sqrt{2}\ell)\big), \ldots, \big[\frac{i''-1-1/2}{\ell}, \frac{i''-1/\sqrt{2}}{\ell}\big] \subseteq \overline{\text{ball}}\big((i''-1)/\ell, 1/(\sqrt{2}\ell)\big)$, $\big[\frac{i''-1/\sqrt{2}}{\ell}, v\big] \subseteq \overline{\text{ball}}\big(i''/\ell, 1/(\sqrt{2}\ell)\big)$: Together they cover $[u, v]$, and each allows to compute $|\text{Max}|(f, \cdot)$ using the local power series expansion around the appropriate center i/ℓ. The case $i'+1 = i''$ can be handled similarly.

(g) Employ Cauchy's differentiation formula (Fact 8b) to deduce $|f'(z)| \le A \cdot (2k)$ on $\overline{\mathcal{R}}_{2k}$. Now the Mean *Inequality* (rather than Mean Value) Theorem for vector-valued functions implies $|f(x) - f(x+z)| \le 2Ak \cdot |z| \le 1/m$ for $x \in [-1; 1]$ and $z \in \overline{\text{ball}}\big(x, 1/(2Akm)\big)$. Together with the hypothesis of f mapping $[-1; 1]$ back to $[-1; 1]$, this shows f to map $\overline{\mathcal{R}}_{2Akm}$ to $\overline{\mathcal{R}}_m \subseteq \text{dom}(g)$. Therefore, $g \circ f$ is analytic on some open neighborhood of $\overline{\mathcal{R}}_{2Akm}$ and thereon bounded (like g itself) by C. □

We close this section with remarking that no choice of finitely many integer parameters can render the evaluation of power series of *entire* functions computable: There exists a computable coefficient sequence with infinite radius of convergence corresponding to an entire function which is computable on any compact subset but not on the entire complex plane [12].

4. Complexity on Gevrey's scale from analytic towards smooth

This section explores in more detail the complexity-theoretic phase transition of the operators of maximization and integration when climbing from smooth to analytic functions. More precisely, we present a uniform refinement of Fact 1f), investigating the growth of computational complexity of operators in dependence on the level on Gevrey's Hierarchy. Historically this latter notion was introduced when investigating the regularity of solutions to partial differential equations [21].

Definition 17. Write $\mathcal{G}^{\gamma}_{B,\ell}[u; v]$ for the subclass of those real functions $f \in \mathcal{C}^\infty[u; v]$ satisfying

$$\forall x \in [u; v],\ \forall j \in \mathbb{N}: \quad |f^{(j)}(x)| \le B \cdot \ell^j \cdot j^{j\gamma}; \tag{6}$$

for the level-parameter $\gamma \in [1, \infty)$, and set $\mathcal{G}^{\gamma}[u; v] := \bigcup_{B,\ell \ge 1} \mathcal{G}^{\gamma}_{B,\ell}[u; v]$ and $\mathcal{G}[u; v] := \bigcup_{\gamma \ge 1} \mathcal{G}^{\gamma}[u; v]$.

Example 18. (a) For $\gamma = 1$, Definition 17 is equivalent to Eq. (4) by virtue of Stirling (Fact 8f). Therefore it holds $\mathcal{G}^1[-1; 1] = \mathcal{C}^\omega([-1; 1], \mathbb{R})$.

(b) For every $\gamma > 0$ the smooth but non-analytic function $H_\gamma : [-1; 1] \ni x \mapsto \exp\big(-1/\sqrt[\gamma]{|x|}\big)$ belongs to $\mathcal{G}^{1+\gamma}[-1; 1]$ but not to $\mathcal{G}^{1+\delta}[-1; 1]$ for any $\delta < \gamma$. In particular Gevrey's Hierarchy is strict. Moreover it holds $h_1 \in \mathcal{G}^2[-1; 1]$ for the pulse function from Fact 3f).

(c) The polytime computable smooth functions f according to Fact 1b) and Fact 1d) as constructed by Friedman and Ko do *not* belong to Gevrey's Hierarchy. In particular $\mathcal{G} \subsetneq \mathcal{C}^\infty$.

(d) Extending the function family g_J constructed in the proof of Example 6g) and Example 6h), fix $\gamma, J \in \mathbb{N}_+$ and let $g_{\gamma,J}(x) := g_1\big(J^\gamma \cdot x\big)/e^{1+J\cdot\gamma/e}$ with $g_1(x) = \exp(-x^2)$.
Then $g_{\gamma,J} \in \mathcal{G}^1_{\exp(-J\gamma/e),J^\gamma}[-1; 1] \subseteq \mathcal{G}^{\gamma+1}_{1,1}[-1, 1]$.

Proof. (b) For $H_\gamma \in \mathcal{G}^{1+\gamma}[-1; 1]$ see [39, Lemme 1]. According to [39, Lemme 2], $H_\gamma \in \mathcal{G}^{1+\delta}[-1; 1]$ implies that for every $\epsilon > \delta$ the function $v(x) := H_\gamma(x)/H_\epsilon(x) = \exp\big(-\sqrt[\gamma]{|x|} + \sqrt[\epsilon]{|x|}\big)$ is smooth and vanishes at 0: which obviously requires $\epsilon > \gamma$ and thus $\delta \ge \gamma$.
Finally, on $[-1/4; 1/4]$, $h_1(x) = \exp\big(\frac{4x^2}{4x^2-1}\big)$ is the composition of analytic exp with analytic $\frac{4x^2}{4x^2-1}$ and thus belongs to $\mathcal{G}^1[-1/4; 1/4]$ by (a). On $[1/4; 1/2]$, $h_1(x) = H_1(\frac{1-4x^2}{4x^2})$ is the composition of a $\mathcal{G}^2$ with an analytic function and therefore in $\mathcal{G}^2$ again; see Lemma 19a) and Lemma 19d).

(c) In [30, p. 80] this function f is constructed as an infinite join of scaled and shifted pulses $h_1\big(2^{p(n)} \cdot (x - y_{s,t})\big)/2^{q(n)}$ for some non-constant polynomials p, q. (In fact $\sup_n j \cdot p(n) - q(n) < \infty$ seems necessary for $f^{(j)}$ to be bounded, see below.) Let us first record a lower bound on $\|h_1^{(j)}\|$: $h_1 \notin \mathcal{C}^\omega \supseteq \mathcal{G}^1_{1,1}$ (recall Items a and b) yields an $j \in \mathbb{N}$ with $\|h_1^{(j)}\| > j^j$; but we prefer a bound holding for all j. To this end observe that $h_1(0) = 1$ while $h_1(-2) = 0 = h_1(2)$ requires $h_1'(x) \ge 1/2$ for some $x \in [-2; 2]$ by the Mean Value Theorem. More generally $h_1^{(j)}(-2) = h_1'(-2) = 0 = h_1'(2) =$

$h_1^{(j)}(2)$ iteratively implies $h_1^{(j)}(x) \geq 1/2^j$ for some $x = x(j)$; hence $\|h_1^{(j)}\| \geq 1/2^j$. Now by the chain rule $\|f^{(j)}\| = \sup_n \|h_1^{(j)}\| \cdot 2^{j\cdot p(n)-q(n)} \geq \sup_n 2^{j\cdot\tilde{p}(n)-q(n)}$ for $\tilde{p}(n) := p(n) - 1$; and it remains show that $\sup_n j \cdot \tilde{p}(n) - q(n)$ grows asymptotically strictly faster than $\log_2(B \cdot \ell^j \cdot j^{j\gamma}) \leq \mathcal{O}(j \cdot \log j)$ by virtue of Stirling. Indeed in case $\deg(\tilde{p}) = \deg(p) = \deg(q) \geq 1$ it follows for $n := j$ that $j \cdot \tilde{p}(j) - q(j) \geq \Omega(j^2)$; while in case $1 \leq d := \deg(\tilde{p}) = \deg(p) < \deg(q) =: k$ choosing $n := j^{1/(k-d+\epsilon)}$ yields in Bachmann–Landau[4] notation

$$j \cdot \tilde{p}(n) - q(n) \;\geq\; \Omega\big(j^{1+d/(k-d+\epsilon)}\big) - \mathcal{O}\big(j^{k/(k-d+\epsilon)}\big) \;\geq\; \omega(j \cdot \log j)$$

(d) Note $\frac{d^j}{dx^j} g_{\gamma,J}(x) = J^{\gamma j} \cdot g_1^{(j)}\big(J^\gamma \cdot x\big)/\exp(1 + J\gamma/e)$ with $|g_1^{(j)}(y)| \leq j! \cdot e \leq j^j \cdot e$ according to Example 15b). Hence $g_{\gamma,J} \in \mathcal{G}^1_{\exp(-J\gamma/e),J^\gamma}[-1;1] \subseteq \mathcal{G}^{1+\gamma}_{1,1}[-1;1]$ according to Lemma 19a). □

Lemma 19. (a) *For $\gamma \leq \gamma'$, $B \leq B'$, $\ell \leq \ell'$, it holds $\mathcal{G}^\gamma_{B,\ell}[u;v] \subseteq \mathcal{G}^{\gamma'}_{B',\ell'}[u;v]$. Moreover, $f \in \mathcal{G}^\gamma_{B,\ell}[u;w]$ iff both $f|_{[u;v]} \in \mathcal{G}^\gamma_{B,\ell}[u;v]$ and $f|_{[v;w]} \in \mathcal{G}^\gamma_{B,\ell}[v;w]$ hold. Finally, $\mathcal{G}^\delta_{B,\ell\cdot J^\gamma}[u;v] \subseteq \mathcal{G}^{\delta+\gamma}_{B\cdot\exp(J\gamma/e),\ell}[-1;1]$.*

(b) *Each Gevrey level is closed under differentiation; more precisely from $f \in \mathcal{G}^\gamma_{B,\ell}[u;v]$ and $d \in \mathbb{N}_+$ it follows $f^{(d)} \in \mathcal{G}^\gamma_{B_d,\ell_d}[u;v]$, where $B_d := B \cdot \ell^d \cdot (1 + 2d^2 \cdot \gamma/e)^{d\cdot\gamma}$ and $\ell_d := \ell \cdot e \cdot (2d)^\gamma$.*

(c) *Each Gevrey level is closed under pointwise addition and multiplication; more precisely for $f \in \mathcal{G}^\gamma_{A,k}[u;v]$ and $g \in \mathcal{G}^\gamma_{B,\ell}[u;v]$, $f + g \in \mathcal{G}^\gamma_{A+B,\max\{k,\ell\}}[u;v]$ and $f \cdot g \in \mathcal{G}^\gamma_{A\cdot B,k+\ell}[u;v]$.*

(d) *Each Gevrey level is closed under composition [21, §I.2.1]; more precisely for $f \in \mathcal{G}^\gamma_{A,k}[u;v]$ with $f : [u;v] \to [u;v]$ and $g \in \mathcal{G}^\gamma_{B,\ell}[u;v]$ it follows $g \circ f \in \mathcal{G}^\gamma_{B,2k\ell A\exp(\gamma)}[u;v]$.*

Proof. (a) The first claims are immediate from Definition 17. For the last one set $s := j\gamma$, $r := \exp(\gamma/e)$, $t := J$ in Lemma 11b) to conclude $J^{j\gamma} \leq e^{J\gamma/e} \cdot j^{j\gamma}$.

(b) By hypothesis $\|f^{(d+j)}\| \leq B \cdot \ell^{d+j} \cdot (d+j)^{(d+j)\gamma}$; and for $j \geq 1$, $d + j \leq 2dj$ implies $(d+j)^{(d+j)\gamma} \leq j^{d\gamma} \cdot (2d)^{d\gamma} \cdot \big((2d)^\gamma\big)^j \cdot j^{j\gamma}$ where $j^{d\gamma} \leq (d\gamma/e)^{d\cdot\gamma} \cdot e^j$ by virtue of Lemma 11b). The case $j = 0$ is easily verified separately.

(c) Concerning multiplication, apply the General Leibniz Rule (Fact 8c):

$$\|(f \cdot g)^{(J)}\| \;\leq\; A \cdot B \cdot \underbrace{\sum_{j=0}^{J} \binom{J}{j} k^j \cdot \ell^{J-j}}_{=(k+\ell)^J} \cdot \underbrace{j^{j\gamma} \cdot (J-j)^{(J-j)\gamma}}_{\leq J^{j\gamma}\cdot J^{(J-j)\gamma} = J^{J\gamma}}$$

(d) Let us first recall from optimization that a linear function on a convex domain attains its maximum on a vertex. For $J \in \mathbb{N}_+$ the set $D_J := \{(j_1, \ldots, j_J) : j_i \geq 0, j_1 + 2j_2 + \cdots + Jj_J = J\}$ is a simplex with vertices $(J, 0, \ldots, 0), (0, J/2, 0, \ldots, 0), \ldots, (0, \ldots, 0, J/i, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. The linear function

$$\Phi_J : D_J \ni \vec{j} \mapsto (\ln J) \cdot j_1 + (2\ln 2 + \ln J) \cdot j_2 + \ldots + (i \ln i + \ln J) \cdot j_i + \cdots + (J \ln J + \ln J) \cdot j_J$$

is therefore bounded from above by $\max_{1\leq i\leq J}(i \ln i + \ln J) \cdot J/i$ where simple calculus shows $\phi_J : [1;J] \ni i \mapsto (i \ln i + \ln J) \cdot J/i$ to have a minimum at $i = \ln J$ and to attain its maximum at $i = J$: $\max \Phi_J \leq \max \phi_J = (J+1)\ln J$ and by monotonicity of the exponential function for $\vec{j} \in D_j \cap \mathbb{N}^J$ we record

$$J^{j_1+\cdots+j_J} \cdot 1^{j_1} \cdot 2^{2j_2} \cdots J^{Jj_J} \;\leq\; J^{J+1}. \tag{7}$$

[4] Recall that for instance $f \in \omega(g)$ means $g(n)/f(n) \to 0$.

Now bound $\|(g\circ f)^{(J)}\|$ using Faà di Bruno's Formula (and notation from Fact 8d) with

$$\sum_{j_1,\ldots,j_J}\tfrac{J!}{j_1!\cdot j_2!\cdots j_J!}\cdot\big(B\cdot\ell^j\cdot\underbrace{j^{\gamma j}}_{\leq J^{(\gamma-1)j}\cdot j!\cdot e^j}\big)\cdot\big(A\cdot k^1\cdot\tfrac{1^{1\gamma}}{1!}\big)^{j_1}\cdot\big(A\cdot k^2\cdot\underbrace{\tfrac{2^{2\gamma}}{2!}}_{\leq 2^{(\gamma-1)2}\cdot e^2/\sqrt{2\pi}}\big)^{j_2}\cdots$$

$$\times\big(A\cdot k^J\cdot\underbrace{\tfrac{J^{J\gamma}}{J!}}_{\leq J^{(\gamma-1)J}\cdot e^J/\sqrt{2\pi}}\big)^{j_J}$$

$$\leq B\cdot k^J\cdot e^J\cdot\overbrace{\sum_{j_1,\ldots,j_J}\tfrac{j!}{j_1!\cdot j_2!\cdots j_J!}\cdot A^j\cdot\ell^j\cdot e^j/\sqrt{2\pi}^{\,j}}^{\overset{(*)}{=}\frac{A\ell e}{\sqrt{2\pi}}\cdot(1+\frac{A\ell e}{\sqrt{2\pi}})^{J-1}\leq(2A\ell)^J}\cdot\cdot J!\cdot\underbrace{J^{(\gamma-1)j}\cdot 1^{(\gamma-1)j_1}\cdot 2^{2(\gamma-1)j_2}\cdots J^{J(\gamma-1)j_J}}_{\overset{(7)}{\leq}J^{(J+1)\cdot(\gamma-1)}}$$

for $\gamma\geq 1$ where we have employed Fact 8e) in (*) and Stirling (Fact 8f) to estimate $\frac{i^{i\gamma}}{i!}\leq i^{(\gamma-1)i}\cdot e^i/\sqrt{2\pi}$. Finally generously bound $J!\leq J^J$ and $J^{\gamma-1}\leq e^{(\gamma-1)J}$ to conclude $J!\cdot J^{(J+1)\cdot(\gamma-1)}\leq J^{\gamma J}\cdot e^{(\gamma-1)J}$. $\square$

4.1. *Gevrey's classes and approximation theory*

Sections 3.1 and 3.2 have exploited that power series as Taylor expansions of analytic functions converge exponentially fast on any compact subset of their disc of convergence. The smooth function $h(x)=\exp(-1/|x|)$ from Example 18b) on the other hand has a Taylor expansion converging at 0 not at all to h; and any sequence of degree-n polynomials converging to the absolute value function $[-1;1]\ni x\mapsto|x|$ does so at a subexponential rate. More precisely, classical Approximation Theory [13,55,62] provides quantitative refinements of the Stone–Weierstraß Theorem with upper and lower bounds on how well certain functions classes can be approximated by polynomials of prescribed degree. We record

Fact 20. *For a ring R, abbreviate with $R[X]_m$ the R-module of all univariate polynomials over R of degree $<m$. Write $\|a_0+a_1X+\cdots+a_{m-1}X^{m-1}\|_1:=|a_0|+|a_1|+\cdots+|a_{m-1}|$ for the 1-norm of a polynomial (coefficient vector). Let $T_m\in\mathbb{Z}[X]_{m+1}$ denote the m-th Chebyshev polynomial of the first kind, given by the recursion formula $T_0\equiv 1$, $T_1(x)=x$, and $T_{m+1}(x)=2xT_m(x)-T_{m-1}(x)$.*

(a) *It holds $\|T_{m+1}\|_1\leq 3^m$ and $T_m(x)=\cos(m\cdot\arccos x)\in[-1;1]$ for $x\in[-1;1]$. With respect to the scalar product*

$$\langle f,g\rangle\ :=\ \int_{-1}^{1}f(x)\cdot g(x)\cdot(1-x^2)^{-1/2}\,dx=\int_0^{\pi}f(\cos t)\cdot g(\cos t)\,dt \tag{8}$$

on $\mathcal{C}[-1;1]$, the family $\mathcal{T}=(T_m)_m$ forms an orthogonal system, namely satisfying $\langle T_0,T_0\rangle=\pi$ and $\langle T_m,T_m\rangle=\pi/2$ and $\langle T_m,T_n\rangle=0$ for $0\leq n<m$. The orthogonal projection $\mathbb{P}_m(f)$ of $f\in\mathcal{C}[-1;1]$ onto $\mathbb{R}[X]_m$ w.r.t. (8) is given by

$$\mathbb{P}_m(f):=c_0(f)/2+\sum_{k=1}^{m-1}c_k(f)\cdot T_k,\qquad c_k(f):=\tfrac{2}{\pi}\cdot\langle f,T_k\rangle.$$

Moreover, $\|f-\mathbb{P}_m(f)\|\leq\sum_{k\geq m}|c_k(f)|$ and $|c_{m+1}(f)|\leq\frac{4}{\pi}\cdot\|f-g\|$ for every $g\in\mathbb{R}[X]_{m+1}$.

(b) *On the Chebyshev Nodes $x_{m,j}:=\cos\big(\frac{\pi}{2}\frac{2j+1}{m}\big)$, $0\leq j<m$, the Chebyshev polynomials T_m satisfy the discrete orthogonality condition*

$$\sum_{j=0}^{m-1}T_k(x_{m,j})\cdot T_\ell(x_{m,j})=\begin{cases}0 & : k\neq\ell\\ m & : k=\ell=0\\ m/2 & : k=\ell\neq 0\end{cases}.$$

The unique polynomial $\mathbb{I}_m(f) \in \mathbb{R}[X]_m$ *interpolating* $f \in \mathcal{C}[-1;1]$ *at* $x_{m,0},\ldots,x_{m,m-1}$ *is*

$$\mathbb{I}_m(f) := \sum_{j=0}^{m-1} y_{m,j}(f)\cdot T_j, \qquad y_{m,j}(f) := \tfrac{1}{m}\cdot f(x_{m,0})\cdot T_j(x_{m,0}) + \tfrac{2}{m}\cdot\sum_{i=1}^{m-1} f(x_{m,i})\cdot T_j(x_{m,i})$$

with $|y_{m,j}(f)| \le 2\|f\|$ *and* $\|\mathbb{I}_m(f)\|_1 \le \|f\|\cdot 3^m$ *for* $m\in\mathbb{N}$. *Moreover,* $\mathbb{I}_m(f)$ *is a close-to-best polynomial approximation to* f *in the following sense:*

$$\forall g\in\mathbb{R}[X]_m: \quad \|f-\mathbb{I}_m(f)\| \;\le\; \big(2+\tfrac{2}{\pi}\ln m\big)\cdot\|f-g\|.$$

(c) *To* $f\in\mathcal{C}^j[-1;1]$ *and* $m>j$ *there exists a* $g\in\mathbb{R}[X]_m$ *such that*

$$\|f-g\| \;\le\; (\pi/2)^j\cdot\frac{\|f^{(j)}\|}{m\cdot(m-1)\cdots(m-j+1)} \;\le\; \|f^{(j)}\|\cdot\big(\tfrac{e\pi}{2m}\big)^j.$$

(d) *Suppose the sequence of (continuously) differentiable functions* $f_m:[0;1]\to\mathbb{R}$ *converges pointwise to some* f *while* f_m' *converges uniformly to* g. *Then* f *is (continuously) differentiable and satisfies* $f'=g$.

(e) *For* $g\in\mathbb{R}[X]_{m+1}$ *it holds* $\|g^{(j+1)}\| \le \frac{m^2\cdot(m^2-1^2)\cdot(m^2-2^2)\cdots(m^2-j^2)}{1\cdot3\cdot5\cdots(2j+1)}\cdot\|g\| \le \|g\|\cdot m^{2j+2}/(j+1)!$ *and* $\|g\|\le\|g\|_1\le 4^m\cdot\|g\|$.

Proof. (a) From $T_1(x)=x$ and $T_{m+1}(x)=2xT_m(x)-T_{m-1}(x)$, it follows by induction $\|T_{m+1}\|_1 \le 2\|T_m\|_1+\|T_{m-1}\|_1 \le 3^m$. The trigonometric sum-to-product formula $\cos\big(mt\big)+\cos\big((m-2)t\big) = 2\cdot\cos\big((m-1)t\big)\cdot\cos(t)$ implies that $\cos(m\cdot\arccos x)$ satisfies the same recursion as $T_m(x)$, hence both coincide. This implies $T_m(x)\in[-1;1]$ as well as the orthogonality claims using Eq. (8); which in turn follows from integration by substitution. Orthogonality also yields the coefficients of a polynomial in the Chebyshev basis using linear algebra. For $\|f-\mathbb{P}_m(f)\| \le \sum_{k\ge m}|c_k(f)|$ see [13, Theorem 4.4.5iii] or [52, Eq. (3.34)] or [38, Section 5.2.2, Eq. (7)]; and each [13, Theorem 4.4.5i], [52, Eq. (3.35)], and [38, Section 5.2.2, Eq. (4)] asserts $|c_{m+1}(f)| \le \frac{4}{\pi}\cdot\|f-g\|$ for all $g\in\mathbb{R}[X]_{m+1}$.

(b) In view of $T_m(x)=\cos(m\cdot\arccos x)$ from (a), the first claim boils down to the well-known orthogonality of the discrete cosine transform. Linear algebra thus implies the claimed expressions for $\mathbb{I}_m(f)$ and its coefficients $y_{m,j}(f)$. The latter also yields $|y_{m,j}(f)| \le \|f\|\cdot\frac{1}{m}+\frac{2}{m}\cdot(m-1)\le 2\|f\|$ in view of $|T_j(x_{m,i})|\le 1$ according to (a); and $\|\mathbb{I}_m(f)\|_1 \le \max_j|y_{m,j}|\cdot\sum_{j=0}^{m-1}\|T_j\|_1 \le 2\|f\|\cdot\big(1+\sum_{j=1}^{m-1}3^{j-1}\big) = \|f\|\cdot(1+3^{m-1})$ according to (a). The final (and highly non-trivial) claim records a bound on the so-called Lebesgue Constant, cf. [52, Eq. (1.36) and (1.55)].

(c) For the first claim see for instance [13, Section 4.6, Jackson's Theorem Viii] or [38, Section 5.2.2, Eq. (6)]. The second inequality follows from Lemma 11g).

(d) See for instance [54, Theorem 7.17 and 7.12].

(e) The first inequality is due to, and named after, the Markov brothers; cmp. [52, before and after §2.7.4 Remark 3] or [13, p. 228]. For the second observe $m^2\cdot(m^2-1^2)\cdot(m^2-2^2)\cdots(m^2-j^2) \le m^{2j+2}$ and $1\cdot3\cdot5\cdots(2j+1) \ge (j+1)!$. Finally, $g(X)=\sum_{j=0}^m g_j\cdot X^j$ implies $|g(x)| \le \sum_{j=0}^m|g_j| = \|g\|_1$ for $|x|\le 1$; while $\|g\|_1 \le 4^m\cdot\max\big\{\big|g\big(\frac{m+j}{m}\big)\big| : -m\le j\le m\big\}$ according to [57, Lemma 4.1]. □

Proposition 21 refines [38, Théorème 5.2.4]. By means of Fact 20, it quantitatively relates asymptotic growth of derivatives (i.e. membership to a certain Gevrey class with parameters B,ℓ,γ) to the error upon approximation by polynomials of degree m. Weakening the geometric Taylor tail bound for power series (proof of Theorem 12a), these error bounds will be of the form $A\cdot r^{m^q}$ for parameters $A\in\mathbb{N}_+$ and $0<r,q<1$. The proof of Theorem 23 builds on this tool for instance with $q=1/\gamma$ and $r=2^{-(e\pi\ell)^{-q}}$.

Proposition 21. (a) *For* $0<r<1$ *and* $0<q\le 1$ *and* $p\ge 0.6$ *and* $N\in\mathbb{N}_+$ *it holds*

$$\sum_{n>N} r^{n^q}\cdot(n-1)^p \;\le\; \big(\tfrac{2+2p}{q\cdot e\cdot\ln 1/r}\big)^{(1+p)/q}\cdot\sqrt{r}^{N^q} \quad \textit{and} \quad \sum_{n>N} r^{n^q}\cdot n^p \le \big(\tfrac{4+4p}{q\cdot e\cdot\ln 1/r}\big)^{(1+p)/q}\cdot r^{(N+1)^q/4}$$

(b) *Let* $\frac{1}{2} \leq r < 1 < A \in \mathbb{N}$, $0 < q \leq 1$, *and* $N \in \mathbb{N}_+$. *Then* $A \cdot r^{m^q} \leq 2^{-N}$ *holds for all* $m \geq C \cdot N^{1/q}$, *where* $C := \big(2\log_2 A/\log_2(1/r)\big)^{1/q}$. *Conversely, for* $0 < q \leq 1 \leq C$, $\big(\forall m \geq C \cdot N^{1/q} : \epsilon_m \leq 2^{-N}\big)$ *implies* $\epsilon_m \leq 2 \cdot r^{m^q}$ *with* $\frac{1}{2} \leq r := 2^{-C^{-q}} < 1$.

(c) *Suppose* $f \in \mathcal{C}^\infty[-1;1]$ *satisfies Eq. (6) with parameters* $B, \ell, \gamma \geq 1$. *Then to every* $m \in \mathbb{N}_+$ *there exists* $f_m \in \mathbb{R}[X]_m$ *with*

$$\|f - f_m\| \leq A \cdot r^{m^q}, \qquad \|f_m\| \leq A \tag{9}$$

where $q := 1/\gamma$, $1/2 \leq r := 2^{-(e\pi\ell)^{-q}} < 1$, $A := 3B$.

(d) *Conversely suppose* $\frac{1}{2} \leq r < 1 \leq A$, $0 < q \leq 1$, *and* $f : [-1;1] \to \mathbb{R}$ *are such that to every* $m \in \mathbb{N}_+$ *there exists some* $f_m \in \mathbb{R}[X]_m$ *satisfying Eq. (9). Then* f *is smooth and satisfies* $\|f^{(d)} - f_m^{(d)}\| \leq A_d \cdot \sqrt[4]{r}^{m^q}$ *with* $A_d := 2A \cdot \big(\frac{4+8d}{q\cdot e\cdot \ln 1/r}\big)^{(1+2d)/q}/d!$. *Moreover it follows* $f \in \mathcal{G}^\gamma_{B,\ell}[-1;1]$ *with* $\gamma := 2/q - 1$, $B := A \cdot \big(\frac{12}{q^2\cdot e^2 \cdot \ln 1/r}\big)^{1/q}$, *and* $\ell := e^2 \cdot \big(\frac{12}{q\cdot e\cdot \ln 1/r}\big)^{2/q}$.

(e) *To* $f \in \mathcal{G}^\gamma_{B,\ell}[-1;1]$ *and* $m \in \mathbb{N}_+$ *let* $\tilde{f}_m \in \mathbb{D}_m[X]_m$ *denote the polynomial* $\mathbb{I}_m(f)$ *with coefficients 'rounded' to* $\mathbb{D}_m$. *Then it holds* $\|f - \tilde{f}_m\| \leq \tilde{A} \cdot \sqrt{r}^{m^{1/\gamma}}$ *with* $\tilde{A} := (2A+1) \cdot \big(\frac{2\gamma}{e\cdot \ln 1/r}\big)^\gamma$ *for* r *and* A *according to (c).*

Note that, in case $\gamma = 1 = q$, Claims (c) and (d) together characterize $\mathcal{G}^1[-1;1] = \mathcal{C}^\omega[-1;1]$ in terms of function approximability by polynomials [14]; but leave a gap in cases $q < 1 < \gamma$ reflected in Theorem 23a), cmp. [38, Remarques 5.2.5(4)].

Proof of Proposition 21. (a) Since $n - 1 \leq x \leq n$ implies $r^{n^q} \leq r^{x^q}$ and $(n-1)^p \leq x^p$, it follows $r^{n^q} \cdot (n-1)^p \leq \int_{n-1}^n r^{x^q} \cdot x^p\,dx$ and $\sum_{n>N} r^{n^q} \cdot (n-1)^p \leq \int_N^\infty r^{x^q} \cdot x^p\,dx$. In the latter integral substitute $y := x^q \cdot \ln(1/r)$, ranging from $M := N^q \cdot \ln(1/r)$ to ∞; moreover, $x = (y/\ln(1/r))^{1/q}$ and $\frac{dy}{dx} = q \cdot \ln(1/r) \cdot x^{q-1} = q \cdot y/x = y^{1-1/q} \cdot q \cdot \ln(1/r)^{1/q}$. The integral thus transforms into

$$\int_M^\infty e^{-y} \cdot \Big(\frac{y}{\ln 1/r}\Big)^{p/q} \cdot y^{1/q-1} \cdot \tfrac{1}{q} \cdot (\ln \tfrac{1}{r})^{-1/q}\,dy = \tfrac{1}{q} \cdot \big(\ln \tfrac{1}{r}\big)^{-(1+p)/q} \cdot \int_M^\infty e^{-y} \cdot y^s\,dy$$

with $s := (1+p)/q - 1 \geq p$. According to Lemma 11b), $y^s \leq (\frac{s}{e \ln \sqrt{e}})^s \cdot e^{y/2}$; yielding the bound

$$\int_M^\infty e^{-y} \cdot y^s\,dy \leq \big(\tfrac{2s}{e}\big)^s \cdot 2e^{-M/2}, \qquad e^{-M/2} = \sqrt{r}^{N^q}.$$

Finally apply $2 \cdot \big(2 \cdot \frac{1-q+p}{q\cdot e\cdot \ln 1/r}\big)^{(1-q+p)/q}/(q \cdot \ln 1/r) = \big(\frac{2}{q\cdot e\cdot \ln 1/r}\big)^{(1+p)/q} \cdot e \cdot (1-q+p)^{(1-q+p)/q} \leq \big(\frac{2+2p}{q\cdot e\cdot \ln 1/r}\big)^{(1+p)/q}$ since $e^q \cdot (1-q+p)^{1-q+p} \leq (1+p)^{1+p}$ holds for $0 < q \leq 1$ and $p \geq 0.6$. The right inequality follows from index shifting with the observation that $r^{(n-1)^q} = (r^{(1-1/n)^q})^{n^q} \leq \sqrt{r}^{n^q}$ since $(1-1/n)^q \geq (1/2)^q \geq 1/2$ for $n \geq 2$.

(b) Taking binary logarithms on both sides shows the claim equivalent to $m \geq \big(\frac{N+\log_2 A}{\log_2 1/r}\big)^{1/q}$, and the latter is $\leq C \cdot N^{1/q}$ for $N \in \mathbb{N}_+$ and $A \geq 2$. Given m, considering $N' := \lfloor (m/C)^q \rfloor \geq (m/C)^q - 1$, the largest N with $m \geq C \cdot N^{1/q}$, implies $\epsilon_m \leq 2^{-N'} \leq 2^{1-(m/C)^q} = 2 \cdot r^{m^q}$.

(c) Fact 20c) yields $f_m \in \mathbb{R}[X]_m$ with $\|f - f_m\| \leq \|f^{(j)}\| \cdot \big(\frac{e\pi}{2m}\big)^j \leq B \cdot \big(\frac{e\pi\ell}{2m} \cdot j^\gamma\big)^j$ for any $m > j \in \mathbb{N}$. Now choosing $j := (\frac{m}{e\pi\ell})^{1/\gamma}$ implies $\|f - f_m\| \leq B \cdot (1/2)^j = B \cdot r^{m^q}$. For non-integral $j > 1$ take $j' \in \mathbb{N}$ between $j-1$ and j and observe $\big(\frac{e\pi\ell}{2m} \cdot j'^\gamma\big)^{j'} \leq (1/2)^{j-1} = 2r^{m^q}$. Finally $\|f_m\| \leq \|f_m - f\| + \|f\| \leq 2B + B$.

(d) Observe that Fact 20e) implies

$$\|(f_n - f_{n-1})^{(d)}\| \leq (n-1)^{2d} \cdot \|f_n - f_{n-1}\|/d! \leq 2A \cdot (n-1)^{2d} \cdot r^{(n-1)^q}/d!$$

for $d \in \mathbb{N}_+$ and $n \geq 2$ by triangle equality and hypothesis. It thus follows for $M \geq m \in \mathbb{N}_+$

$$\|f_M^{(d)} - f_m^{(d)}\| = \Big\| \sum_{n=m+1}^M f_n^{(d)} - f_{n-1}^{(d)} \Big\| \leq 2A \cdot \sum_{n>m} (n-1)^{2d} \cdot r^{(n-1)^q}/d! \leq A_d \cdot \sqrt[4]{r}^{m^q}$$

by virtue of (a). This shows that $(f_m^{(d)})_m$ is a Cauchy sequence in $\mathcal{C}[-1;1]$ converging uniformly to $f^{(d)}$ according to Fact 20d) with the claimed error bound.
On the other hand, the hypothesis and Fact 20a) assert $|c_n(f)| \leq A \cdot \frac{4}{\pi} \cdot r^{n^q}$ for $n \in \mathbb{N}_+$; from which (a) together with, again, Fact 20a) implies $f = \lim_m \mathbb{P}_m(f) = c_0(f)/2 + \sum_{n\geq 1} c_n(f) \cdot T_n$ with respect to uniform convergence, and thus (Fact 20d) for $j \in \mathbb{N}_+$:

$$\|f^{(j)}\| = \Big\| \sum_{n\geq j} c_n(f) \cdot T_n^{(j)} \Big\| \leq \sum_{n\geq j} |c_n(f)| \cdot \|T_n^{(j)}\| \leq \sum_{n\geq j} \big(A \cdot \tfrac{4}{\pi} \cdot r^{n^q}\big) \cdot (n^{2j}/j!)$$

according to Fact 20a) and Fact 20e); and furthermore, applying (a)

$$\|f^{(j)}\| \leq \overbrace{\Big(\frac{4A}{\pi j!}\Big)}^{\leq A\cdot e^j/j^j} \cdot \Big(\overbrace{\frac{4+8j}{q\cdot e\cdot \ln 1/r}}^{\leq 12j}\Big)^{(1+2j)/q} \cdot \overbrace{\sqrt[4]{r}^{j^q}}^{\leq 1} \leq B \cdot \ell^j \cdot j^{\gamma j}$$

since $j^{1/q} \leq (\frac{1}{q\cdot e})^{1/q} \cdot e^j$ by Lemma 11b). For the case $j = 0$ observe $\|f\| \leq \|f - f_1\| + \|f_1\| \leq A + A \leq B$ since $\big(\frac{12}{q^2\cdot e^2 \cdot \ln 1/r}\big)^{1/q} \geq \frac{12}{e^2\cdot \ln 2} > 2$.

(e) By (c) and Fact 20b), $\|f - \tilde{f}_m\| \leq A \cdot r^{m^{1/\gamma}} \cdot (2 + \frac{2}{\pi} \cdot \ln m) + m \cdot 2^{-m}$ since the rounding changes m coefficients by $\leq 2^{-m}$. Now $2 + \frac{2}{\pi} \ln m \leq 2m$; and $2^{-m} \leq (1/2)^{m^{1/\gamma}} \leq r^{m^{1/\gamma}}$ since $r \geq 1/2$. Finally $m \leq \big(\frac{2\gamma}{e\cdot \ln 1/r}\big)^{\gamma} / \sqrt{r}^{m^{1/\gamma}}$ because $n := m^{1/\gamma}$ has $m = n^\gamma \leq \big(\frac{\gamma}{e\cdot \ln 1/\sqrt{r}}\big)^{\gamma} / \sqrt{r}^n$ according to Lemma 11b). □

4.2. The main result

[38, Corollaire 5.2.14] asserts Max and $\int$ and ∂ to map polytime functions in $\mathcal{G}[-1;1]$ to polytime ones — nonuniformly, that is, for fixed f and in particular presuming γ, B, ℓ according to Definition 17 to be known. This suggests the following encoding:

Definition 22. (a) Let $\widetilde{\mathcal{G}}[-1;1] := \big\{(f,\gamma,B,\ell) : f \in \mathcal{G}^{\gamma}_{B,\ell}[-1;1]\big\} \subseteq \mathcal{G}[-1;1] \times \mathbb{N}_+^3$.

(b) Identifying $a_0 + a_1 X + a_2 X^2 + \cdots + a_{d-1} X^{d-1} \in R[X]_d$ with $(a_0, \ldots, a_{d-1}, 0, \ldots) \in R^\infty$ and abbreviating $\langle (\sum_{j\geq 0} a_{N,j} X^j)_{N\geq 1} \rangle := (a_{N+1,j})_{\langle N,j\rangle \in \mathbb{N}} \in R^\omega$, let $\mathbb{R}^\omega \times \mathbb{N}_+^2 \supseteq \widehat{\mathcal{G}}[-1;1] :=$

$$\Big\{ \Big(\big\langle (f_N)_{N\geq 1} \big\rangle, \delta, C \Big) : C, \delta \in \mathbb{N}_+, f_N \in \mathbb{R}[X]_{C\cdot N^\delta}, \|f_N\| \leq 2^C, \|f_N - f_M\| \leq 2^{-N} + 2^{-M} \Big\}.$$

In order to motivate Item (b) recall that precisely Gevrey functions on $[-1;1]$ can be approximated up to error 2^{-N} in the supremum norm by polynomials of degree polynomial in N, that is, by elements of $\mathbb{R}[X]_{C\cdot N^\delta}$ for some $C, \delta \in \mathbb{N}_+$. A bound like $\|f_N\| \leq 2^C$ on the binary length of the values involved is required for parameterized running time estimates.

Theorem 23. (a) *The mapping $\widehat{\mathcal{G}}[-1;1] \ni \big(\langle (f_N)_N \rangle, \delta, C\big) \mapsto \lim_N f_N \in \mathcal{G}^{\gamma}_{B,\ell}$ is well-defined for $\gamma := 2\delta - 1$, $B := 3^{C+\delta} \cdot \delta^{2\delta}$, $\ell := C^2 \cdot (18\delta)^{2\delta}$. Moreover, evaluation $\widehat{\mathcal{G}}[-1;1] \times [-1;1] \ni \big((f_N)_N, \delta, C, x\big) \mapsto \lim_N f_N(x) \in \mathbb{R}$ is computable within time polynomial in $(n + C + \delta)^\delta$.*

(b) *To $(f,\gamma,B,\ell) \in \widetilde{\mathcal{G}}[-1;1]$ and $N \in \mathbb{N}_+$ there exists $f_N \in \mathbb{R}[X]_{C\cdot N^\gamma}$ with $\|f - f_N\| \leq 2^{-N}$, where $C := 9\ell \cdot (36 + 6\,\mathrm{lb}\,\ell + 6\,\mathrm{lb}\,B + 6\gamma^2)^\gamma$. More precisely there is a total well-defined functional $\widetilde{\mathcal{G}}[-1;1] \ni (f,\gamma,B,\ell) \mapsto \big(\langle (f_N)_N \rangle, \delta, C\big) \in \widehat{\mathcal{G}}[-1;1]$ computable within time polynomial in $(n + \gamma + \ell + \mathrm{lb}\,B)^\gamma$.*

(c) *Pointwise addition and multiplication*

$$\widehat{\mathcal{G}}[-1;1] \times \widehat{\mathcal{G}}[-1;1] \to \widehat{\mathcal{G}}[-1;1],$$
$$\big((f_N)_N, \delta, C, (g_N)_N, \varepsilon, D\big) \mapsto \big((f_{N+1} + g_{N+1})_N, \max\{\delta,\varepsilon\}, (C+D)\cdot 2^{\max\{\delta,\varepsilon\}}\big),$$
$$\big((f_N)_N, \delta, C, (g_N)_N, \varepsilon, D\big) \mapsto \big((f_{N+1+D} \cdot g_{N+1+C})_N, \max\{\delta,\varepsilon\}, E\big)$$

are well-defined for $E := C\cdot(2+D)^{\max\{\delta,\varepsilon\}} + D\cdot(2+C)^{\max\{\delta,\varepsilon\}}$ *and computable within time polynomial in* $(n+C+D+\delta+\varepsilon)^{\max\{\delta,\varepsilon\}}$.

(d) *d-fold iterated anti-differentiation*

$$\widehat{\mathcal{G}}[-1;1]\times\mathbb{N}_+ \ni \big((f_N)_N,\delta,C,d\big) \mapsto \big((f_N^{(-d)})_N,\delta,C+d\big) \in \widehat{\mathcal{G}}[-1;1]$$

is well-defined and computable within time polynomial in $(n+d+C+\delta)^\delta$.

(e) *d-fold iterated differentiation*

$$\widehat{\mathcal{G}}[-1;1]\times\mathbb{N}_+ \ni \big((f_N)_N,\delta,C,d\big) \mapsto \big((f_{4N+D}^{(d)})_N,\delta,C\cdot(4+D)^\delta\big) \in \widehat{\mathcal{G}}[-1;1]$$

is well-defined for $D := 16dC + 48d^2\delta^2$ *and computable within time polynomial in* $(n+d+C+\delta)^\delta$.

(f) *Parametric maximization*

$$\widehat{\mathcal{G}}[-1;1]\times[-1;1]^2 \ni \big((f_N)_N,\delta,C,u,v\big) \mapsto \mathrm{Max}(\lim_N f_N,u,v) \in \mathbb{R}$$

is computable within time polynomial in $(C+n+\delta)^\delta$.

(g) *Fix* $\gamma\in\mathbb{N}$. *Then computing the restriction* $\mathcal{G}_{1,1}^{\gamma+1}[-1;1]\cap\mathcal{L}([-1;1],[-1;1])\ni f\mapsto \mathrm{Max}(f,0,1)$ *requires time at least* $\Omega(n^\gamma)$.

Similarly to Theorem 16, Items (b) to (e) establish natural primitive operations on Gevrey's hierarchy to be fully polytime computable — for fixed δ, which otherwise enters linearly in the degree of the polynomial running time bounds: and this is optimal for (f) according to Item (g). Again, Items (a) and (b) assert mutual 'conversion' between $\widetilde{\mathcal{G}}[-1;1]$ and $\widehat{\mathcal{G}}[-1;1]$ computable within such parameterized time. As remarked in the proof of Theorem 16c), this yields elegant abstract proofs of addition and multiplication within parameterized time, but without explicit expressions for the parameter transformation. Moreover, each round transforming from $\widehat{\mathcal{G}}[-1;1]$ to $\widetilde{\mathcal{G}}[-1;1]$ and back would increase the exponent δ (recall the remark before the proof of Proposition 21), yielding only exponential running time bounds for iterated addition! We therefore prefer direct estimates as more practical, more reasonable, and more insightful.

Proof of Theorem 23. (a) By hypothesis $f := \lim_N f_N$ exists and satisfies Eq. (9) with $m := C\cdot N^\delta$ for $A := 2^C$ and $r := 2^{-C^{-q}}$ and $q := 1/\delta$; cmp. Proposition 21b). Therefore the second part of Proposition 21d) yields well-definition of the mapping, observing $1/\ln(1/r) = C^{1/\delta}/\ln(2)$ and $2^C\cdot\big(\frac{12\delta^2}{e^2\ln 2}C^{1/\delta}\big)^\delta \le B$ and $e^2\cdot\big(\frac{12\delta}{e\ln 2}\cdot C^{1/\delta}\big)^{2\delta}\le\ell$.
Concerning computational evaluation, note that $\|f'\|\le B\cdot\ell$ according to Eq. (6) yields a modulus $\mu(n)$ of continuity of f polynomial in $C+\delta+n$. Hence it suffices to calculate evaluation at $1/2^{\mu(n)}$-approximations $\tilde{x}\in\mathbb{D}_{\mu(n)}$ to x. Moreover Fact 20e) asserts that the coefficients $a_{N,j}$ of f_N have binary length bounded linearly in $C+N$. Therefore, similarly the discussion following Definition 4b), their $1/2^{n'}$-approximations $\tilde{a}_{N,j}$ are located at bit positions polynomial in $n'+N+C$ within the input. These yield $\tilde{f}_N\in\mathbb{D}_{n'}[X]_m$ that can be evaluated at $\tilde{x}$ within time polynomial in $C+n'+\mu(n)+m$; and satisfy $\|f-\tilde{f}_N\|\le\|f-f_N\|+\|f_N-\tilde{f}_N\|\le 2^{-N}+m\cdot 2^{-n'}\le 2^{-n}$ for $N:=n+1$ and $n':=N+\mathrm{lb}(m)$.

(b) For the first claim we could apply Proposition 21c) but, directly aiming for the second claim via Proposition 21e), set $f_N := \mathbb{I}_m(f)$ for $m := C\cdot N^\gamma$: Defined in Fact 20b) via interpolation, these polynomials are easy to compute in time polynomial in $N+n+C$ from evaluating f at the Chebyshev Nodes. Observing $\|f_N\|\le\|f_N-f\|+\|f\|\le 2^{-N}+B\le 2^C$, it remains to verify $\|f-f_N\|\le 1/2^N$. To this end, Proposition 21e) asserts $\|f-\mathbb{I}_m(f)\|\le\tilde{A}\cdot\sqrt{r}^{m^{1/\gamma}}$ for $r := 2^{-(e\pi k)^{-1/\gamma}}$ and $\tilde{A} := (6B+1)\cdot\big(\frac{2\gamma}{e\ln 1/r}\big)^\gamma \le 7B\cdot(2\gamma)^\gamma\cdot e\pi\ell\le 63B\ell(2\gamma)^\gamma$ since $1/\ln(1/r) = (e\pi\ell)^{1/\gamma}/\ln(2)$. Proposition 21b) thus asserts $\|f-f_m\|\le 1/2^N$ for all $m\ge\tilde{C}\cdot N^\gamma$ where $\tilde{C} := \big(2\log_2\tilde{A}/\log_2(1/\sqrt{r})\big)^\gamma \le e\pi\ell\cdot\big((\log_2 63+\log_2\ell+\log_2 B+\gamma+\gamma\log_2\gamma)\cdot 4/\ln 2\big)^\gamma\le C$.

(c) W.l.o.g. presume $\delta=\varepsilon$ for notational simplicity. Concerning addition note that $\deg(f_{N+1}+g_{N+1})\le\max\{C,D\}\cdot(N+1)^\delta\le(C+D)\cdot 2^\delta\cdot N^\delta$ according to Lemma 11f). Moreover, $\|f_{N+1}+g_{N+1}\|\le 2^C+2^D\le 2^{(C+D)\cdot 2^\delta}$ and $\|f_{N+1}+g_{N+1}-f_{M+1}-g_{M+1}\|\le 2\cdot(2^{-N-1}+2^{-M-1})$. Computing

$\langle (f_{N+1}+g_{N+1})_N\rangle$ from $\langle (f_N)_N\rangle$ and $\langle (g_N)_N\rangle$ is easy within time polynomial in $(n+C+D+\delta)^\delta$ in view of the bit length analysis from (a).
Concerning multiplication, $f_{N+1+D}\cdot g_{N+1+C}$ approximates fg up to error 2^{-N} according to

$$\begin{aligned}\|f\cdot g - f_{N+1+D}\cdot g_{N+1+C}\| &= \|f\cdot(g-g_{N+1+C}) + (f-f_{N+1+D})\cdot g_{N+1+C}\|\\ &\le \|f\|\cdot\|g-g_{N+1+C}\| + \|f-f_{N+1+D}\|\cdot\|g_{N+1+C}\|\\ &\le 2^C\cdot 2^{-(N+1+C)} + 2^{-(N+1+D)}\cdot 2^D,\end{aligned}$$

and $\deg(f_{N+1+D}\cdot g_{N+1+C}) \le C\cdot(N+1+D)^\delta + D\cdot(N+1+C)^\delta \le E\cdot N^\delta$ according to Lemma 11f) for $E := C\cdot(2+D)^\delta + D\cdot(2+C)^\delta$. Also, $\|f_{N+1+D}\cdot g_{N+1+C}\| \le 2^C\cdot 2^D \le 2^E$; and $(f_{N+1+D}\cdot g_{N+1+C})_N$ is computable from $(f_N)_N$ and $(g_N)_N$ within time polynomial in $(n+C+D+\delta)^\delta$.

(d) An easy exercise in symbolic manipulation converts input $(f_N)_N$ into output $(f_N^{(-d)})_N$ within time polynomial in $(n+C+\delta)^\delta$. It thus remains to observe inductively that $\|f^{(-d)}\| \le \|\int f_N\| \le \|f_N\| \le 2^C \le 2^{C+d}$ and similarly $\|f^{(-d)} - f_N^{(-d)}\| \le \|f-f_N\| \le 2^{-N}$. Moreover, $\deg f_N^{(-d)} = d + \deg f_N \le d + C\cdot N^\delta \le (C+d)\cdot N^\delta$.

(e) As in (d), computing $(f_{4N+D}^{(d)})_N$ is straight-forward. Concerning the analysis, let $f := \lim_N f_N$ and re-write the hypothesis as $\|f-f_N\| \le 2^{-N} \le A\cdot r^{m^q}$ for $q := 1/\delta$, $A := 2^C \ge 2$, $r := 2^{-C^{-q}}$, and $m := C\cdot N^\delta > \deg(f_N)$; recall Proposition 21b). Now the first part of Proposition 21d) yields $\|f^{(d)} - f_N^{(d)}\| \le A_d\cdot \sqrt[4]{r}^{m^q} = A_d\cdot 2^{-N/4}$ for $d, N \ge 1$ and

$$A_d := 2A\cdot\Big(\tfrac{4d+8d}{q\cdot e\cdot\ln 1/r}\Big)^{(1+2d)/q}/d! = 2^{C+1}\cdot\Big(\tfrac{12d\delta}{e\ln 2}\cdot C^{1/\delta}\Big)^{(1+2d)\delta}/d! \le 2^{D/4}$$

and thus $\|f^{(d)} - f_{4N+D}^{(d)}\| \le 2^{-N}$. Moreover $\deg(f_{4N+D}^{(d)}) < C\cdot(4N+D)^\delta - d \le C\cdot(4+D)^\delta\cdot N^\delta$. Finally $\|f_{4N+D}^{(d)}\| \le \|f_{4N+D}^{(d)} - f^{(d)}\| + \|f^{(d)}\| \le 2^{-N} + B\cdot\ell^d\cdot d^{(2\delta-1)d}$ according to the second part of Proposition 21d), where $B := A\cdot\big(\tfrac{12\delta^2}{e^2\cdot\ln 2}\big)^\delta\cdot C$ and $\ell := e^2\cdot\big(\tfrac{12\delta}{e\ln 2}\big)^{2\delta}\cdot C^2$; hence $\|f_{4N+D}^{(d)}\| \le 2^{C\cdot(4+D)^\delta}$.

(f) Since $\|f-f_N\| \le 1/2^N$ it suffices to approximate $\mathrm{Max}(f_{n+1}, u, v)$ up to error 2^{-n-1}. Moreover $\|f'_{n+1}\| \le \|f_{n+1}\|\cdot\deg(f_{n+1}) < 2^C\cdot\big(C\cdot(n+1)^\delta\big)^2 \le 2^{3C+2\delta\,\mathrm{lb}(n+1)}$ according to Fact 20e); hence we can replace the real arguments u and v by dyadic approximations $u', v' \in \mathbb{D}_{n'}$ up to error $1/2^{n'}$ where $n' \ge n + 3C + 2\delta\,\mathrm{lb}(n+1)$. And for dyadic parameters of binary length thus polynomially bounded in $n+C+\delta$, $\mathrm{Max}(f_{n+1}, u', v')$ can be approximated by eliminating a constant number of quantifiers; recall the proof of Theorem 12f),

(g) Recall from Example 18d) the functions $g_{\gamma,J} \in \mathcal{G}_{1,1}^{\gamma+1}[-1;1]$ with $g_{\gamma,J}(x) = \exp\big(-x^2\cdot J^{2\gamma}\big)/e^{1+J\cdot\gamma/e}$. Now consider, similarly to the proof of Example 6g), their shifts $g_{\gamma,J,j}(x) := \exp\big(-(x\cdot J^\gamma - j)^2\big)/e^{1+J\cdot\gamma/e}$ satisfying $g_{\gamma,J,j}(x) = \exp(-J\cdot\gamma/e - 1) =: y_{\gamma,J}\cdot e$ for $x = j/J^\gamma$, and $|g_{\gamma,J,j}(x)| \le \exp(-J\cdot\gamma/e - 2) = y_{\gamma,J}$ whenever $|x - j/J^\gamma| \ge 1/J^\gamma$. Hence any algorithm computing $f \mapsto \mathrm{Max}(f,0,1) = \max\{f(x) \mid 0 \le x \le 1\}$ on $\{0, g_{\gamma,J,0},\ldots,g_{\gamma,J,J-1}\} \subseteq \mathcal{G}_{1,1}^{\gamma+1}[-1;1]$ up to error 2^{-n}, $n := -\mathrm{lb}(y_{\gamma,J}) = (2+J\cdot\gamma/e)/\ln 2$, must distinguish (every oracle representing) the identically zero function from (all oracles representing) any of the $g_{\gamma,J,j}$ $(0 \le j < J^\gamma)$. Yet, since the $g_{\gamma,J,j}$ are 'thin', any approximate evaluation up to error 2^{-n} at some x with $|x - j/J^\gamma| \ge 1/J^\gamma$ could return 0. □

5. Conclusion and perspectives

While many (and perhaps most 'practical') smooth functions may admit efficient maximization and integration, this does not refute worst-case complexity theory. Instead, it raises the challenge of explicitly specifying the class of instances (here: functions) that some method provably works on, and how efficiently so!

The present work has explored the computational complexity of natural primitive operations on subclasses of smooth functions in the rigorous sense of uniform parameterized worst-case running time. More precisely our research establishes approximate evaluation, pointwise addition and multiplication, maximization, differentiation, and integration to be computable in time polynomial in the binary output precision n and in one further integer parameter. The polynomial running time bounds have degree growing linearly with the level on Gevrey's Hierarchy; in particular they refine, and for fixed function arguments boil down to, previous non-uniform results [38, §5.2]. The underlying algorithms are fully specified with respect to both their continuous (encodings of real numbers, sequences, and functions) and discrete (parameters=enrichment) input and output behavior; and are guaranteed to obey the claimed running time bounds in the worst-case with respect to bit costs. We demonstrate different but mathematically equivalent choices of parameters and encodings as polynomial-time equivalent, thus supporting our claim of them as natural. Moreover the running time's dependence on the parameters is shown asymptotically optimal by an adaptation of the adversary arguments from unit-cost IBC to the bit model with adaptive precision.

Our theoretical results promise realistic predictions on the behavior of practical implementations. This conjectured correspondence will be explored in a separate work. In fact we advertise Real Complexity Theory as a means, and offer the present work as proof of concept, to avoid Peter Linz' valid critique [41].

References

[1] V.V. Andreev, T.H. McNicholl, Computing Conformal Maps onto Canonical Slit Domains in: A. Bauer, R. Dillhage, P. Hertling, K. Ko, and R. Rettinger (Eds.), Proc. 6th Int. Conf. on Computability and Complexity in Analysis (CCA2009), vol. 353 of Informatik Berichte FernUniversität in Hagen, pp.23–34.

[2] S. Basu, R. Pollack, M.-F. Roy, Algorithms in Real Algebraic Geometry, Springer, 2006.

[3] I. Binder, M. Braverman, M. Yampolsky, On computational complexity of Riemann mapping, Ark. Mat. 45 (2) (2007) 221–239.

[4] J. Bitterlich, Data structures and efficient algorithms for power series in exact real arithmetic (Bachelor Thesis in Mathematics), TU Darmstadt, 2012.

[5] L. Blum, F. Cucker, M. Shub, S. Smale, Complexity and Real Computation, Springer, 1998.

[6] O. Bournez, D.S. Graça, A. Pouly, Solving Analytic Differential Equations in Polynomial Time over Unbounded Domains, in: Proc. 36th Int. Symp. on Mathematical Foundations of Computer Science (MFCS'2011), Springer LNCS vol. 6907, pp.170–181.

[7] V. Brattka, P. Hertling, Topological properties of real number representations, Theoret. Comput. Sci. 284 (2) (2002) 241–257.

[8] V. Brattka, K. Weihrauch, Computability on subsets of Euclidean space I: Closed and compact subsets, Theoret. Comput. Sci. 219 (1999) 65–93.

[9] M. Braverman, On the Complexity of Real Functions, in: Proc. 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005), pp.155–164.

[10] M. Braverman, S.A. Cook, Computing over the reals: Foundations for scientific computing, Notices of the AMS 53 (3) (2006) 318–329.

[11] P. Bürgisser, F. Cucker, Condition: The Geometry of Numerical Algorithms, Springer, 2013.

[12] J. Caldwell, M.B. Pour-El, On a simple definition of computable functions of a real variable — with applications to functions of a complex variable, Z. Math. Logik Grundlagen Math. 21 (1975) 1–19.

[13] E.W. Cheney, Introduction to Approximation Theory, McGraw-Hill, 1966.

[14] L. Demanet, L. Ying, On Chebyshev Interpolation of Analytic Functions, preprint, 2010.

[15] Z. Du, C.K. Yap, Uniform Complexity of Approximating Hypergeometric Functions with Absolute Error, in: S.I. Pae, H.-J. Park (Eds.), Proc. 7th Asian Symp. on Computer Math. (ASCM 2005), pp.246–249.

[16] M. Escardó, Algorithmic solution of higher-type equations, J. Logic Comput. 23 (4) (2013) 839–854.

[17] H. Férée, W. Gomaa, M. Hoyrup, Analytical properties of resource-bounded real functionals, J. Complexity 30 (5) (2014) 647–671.

[18] J. Flum, M. Grohe, Parameterized Complexity Theory, Springer, 2006.

[19] H. Friedman, The computational complexity of maximization and integration, Adv. Math. 53 (1984) 80–98.

[20] T. Gärtner, G. Hotz, Representation theorems for analytic machines and computability of analytic functions, Theory Comput. Syst. 51 (1) (2012) 65–84.

[21] M. Gevrey, Sur la nature analytique des solutions des équations aux dérivées partielles. Premier mémoire, Ann. Sci. Ec. Norm. Supèr. (3) 35 (1918) 129–190.

[22] A. Grzegorczyk, On the definitions of computable real continuous functions, Fundamenta Mathematicae 44 (1957) 61–77.

[23] P. Hertling, An effective riemann mapping theorem, Theoret. Comput. Sci. 219 (1999) 225–265.

[24] G. Hotz, On in polynomial time approximable real numbers and analytic functions, in: Informatik als Dialog zwischen Theorie und Anwendung, Vieweg + Teubner, 2009, pp. 155–164.

[25] A. Kawamura, Lipschitz continuous ordinary differential equations are polynomial-space complete, Comput. Complexity 19 (2) (2010) 305–332.

[26] A. Kawamura, S.A. Cook, Complexity Theory for Operators in Analysis, in: Proc. 42nd Ann. ACM Symp. on Theory of Computing (STOC 2010); full version ACM Transactions in Computation Theory vol. 4:2 (2012), article 5, pp. 495–502.

[27] A. Kawamura, H. Ota, C. Rösnick, M. Ziegler, Computational complexity of smooth differential equations, in: Proc. 37th Int. Symp. on Mathematical Foundations of Computer Science (MFCS'2012), in: LNCS, vol. 7464, Springer, 2014, pp. 578–589. full version Logical Methods in Computer Science vol. 10:1.

[28] A. Kawamura, A. Pauly, Function Spaces for Second-Order Polynomial Time Proc. 10th Conf. on Computability in Europe (CiE'14) Springer, LNCS, vol. 8493, pp. 245–254.

[29] K.-I. Ko, The maximum value problem and NP real numbers, J. Comput. System Sci. 24 (1982) 15–35.
[30] K.-I. Ko, Computational Complexity of Real Functions, Birkhäuser, 1991.
[31] K.-I. Ko, in: Yu.L. Ershov, et al. (Eds.), Polynomial-Time Computability in Analysis, in: Handbook of Recursive Mathematics, vol. 2, 1998, pp. 1271–1317.
[32] K.-I. Ko, On the computational complexity of ordinary differential equations, Inf. Control 58 (1983) 157–194.
[33] K.-I. Ko, H. Friedman, Computational complexity of real functions, Theoret. Comput. Sci. 20 (1982) 323–352.
[34] U. Kohlenbach, Proof theory and computational analysis, Proc. 3rd Workshop on Computation and Approximation (Comprox III), Electronic Notes in Theoretical Computer Science 13 (1998) 124–157.
[35] S.G. Krantz, H.R. Parks, A Primer of Real Analytic Functions, second ed., Birkhäuser, 2002.
[36] V. Kreinovich, A. Lakeyev, J. Rohn, P. Kahl, Computational Complexity and Feasibility of Data Processing and Interval Computations, Academic Press, 1998.
[37] G. Kreisel, A. Macintyre, Constructive logic versus algebraization I, in: Troelstra, van Dalen (Eds.), Proc. L.E.J. Brouwer Centenary Symposium, North-Holland, 1982, pp. 217–260.
[38] S. Labhalla, H. Lombardi, E. Moutai, Espaces métriques rationnellement présentés et complexité, le cas de l'espace des fonctions réelles uniformément continues sur un intervalle compact, Theoret. Comput. Sci. 250 (2001) 265–332.
[39] N. Lerner, Resultats d'unicite forte pour des operateurs elliptiques a coefficients gevrey, Comm. Partial Differential Equations 6 (10) (1981) 1163–1177.
[40] S. Le Roux, M. Ziegler, Singular coverings and non-uniform notions of closed set computability, Mathematical Logic Quarterly 54 (2008) 545–560.
[41] P. Linz, A critique of numerical analysis, Bull. Am. Math. Soc. 19 (2) (1988) 407–416.
[42] H. Luckhardt, A fundamental effect in computations on real numbers, Theoret. Comput. Sci. 5 (1977) 321–324.
[43] N.T. Müller, Uniform Computational Complexity of Taylor Series, in: Proc. 14th Int Coll. on Automata, Languages, and Programming (ICALP'87) Springer, LNCS, vol. 267 pp. 435–444.
[44] N.T. Müller, Constructive aspects of analytic functions, in: Proc. Workshop on Computability and Complexity in Analysis, (CCA), in: InformatikBerichte FernUniversität Hagen, vol. 190, 1995, pp. 105–114.
[45] N.T. Müller, B. Moiske, Solving initial value problems in polynomial time, Proc. 22nd JAIIO-PANEL (1993) 283–293.
[46] R. Niedermeier, Invitation to Fixed-Parameter Algorithms, Oxford University Press, 2006.
[47] E. Novak, Some Results on the Complexity of Numerical Integration, http://arXiv.org/abs/1409.6714, 2014.
[48] A. Pauly, M. Ziegler, Relative computability and uniform continuity of relations, Journal of Logic and Analysis 5 (2013).
[49] R. Rettinger, Computability and Complexity Aspects of Univariate Complex Analysis, Habilitation thesis FernUniversität Hagen, 2007.
[50] R. Rettinger, Lower Bounds on the Continuation of Holomorphic Functions, in: Proc. 5th Int. Conf. on Computability and Complexity in Analysis (CCA 2008), Electronic Notes in Theoretical Computer Science vol. 221, pp. 207–217.
[51] R. Rettinger, Towards the Complexity of Riemann Mappings (Extended Abstract), in: Proc. 6th Int. Conf. on Computability and Complexity in Analysis (CCA 2009), Dagstuhl Research Online Publication Server.
[52] T.J. Rivlin, The Chebyshev Polynomials, Wiley&Sons, 1974.
[53] C. Rösnick, Closed Sets and Operators thereon: Representations, Computability and Complexity, submitted for publication, 2013.
[54] W. Rudin, Principles of Mathematical Analysis, third ed., McGraw-Hill, 1976.
[55] A. Schönhage, Approximationstheorie, de Gruyter, 1971.
[56] M. Schröder, Spaces allowing type-2 complexity theory revisited, Math. Logic Quart. 50 (2004) 443–459.
[57] A.A. Sherstov, Making polynomials robust to noise, Theory Comput. 9 (18) (2009) 593–615.
[58] E. Specker, The fundamental theorem of algebra in recursive analysis, in: B. Dejon, P. Henrici (Eds.), Constructive Aspects of the Fundamental Theorem of Algebra, Wiley-Interscience, 1969, pp. 321–329.
[59] E. Specker, Der satz vom maximum in der rekursiven analysis, in: A. Heyting (Ed.), Studies in Logic and The Foundations of Mathematics, North-Holland, 1959, pp. 254–265.
[60] K. Tent, M. Ziegler, Computable functions of reals, Münster J. Math. 3 (2010) 43–66.
[61] J.F. Traub, G.W. Wasilkowski, H. Woźniakowski, Information-Based Complexity, Academic Press, 1988.
[62] L.N. Trefethen, Approximation Theory and Approximation Practice, SIAM, 2013.
[63] L.N. Trefethen, J.A.C. Weidemann, The exponentially convergent trapezoid rule, SIAM Rev. 56 (3) (2014) 385–458.
[64] A.M. Turing, On computable numbers, with an application to the entscheidungsproblem. a correction, Proc. Lond. Math. Soc. 43 (2) (1937) 544–546.
[65] A.M. Turing, Rounding-Off errors in matrix processes, Q. J. Mech. Appl. Math. 1 (1) (1948) 287–308.
[66] J. van der Hoeven, Effective analytic functions, J. Symbolic. Comput. 39 (2005) 433–449.
[67] J. van der Hoeven, On effective analytic continuation, Math. Comput. Sci. 1 (1) (2007) 111–175.
[68] J. van der Hoeven, Fast Composition of Numeric Power Series, Université Paris-Sud Technical Report, 2008–09.
[69] K. Weihrauch, Computable Analysis, Springer, 2000.
[70] K. Weihrauch, The computable multi-functions on multi-represented sets are closed under programming, J. Univ. Comput. Sci. 14 (6) (2008) 801–844.
[71] K. Weihrauch, Computational complexity on computable metric spaces, Mathematical Logic Quarterly 49 (1) (2003) 3–21.
[72] H. Woźniakowski, Why does information-based complexity use the real number model?, Theoret. Comput. Sci. 219 (1999) 451–465.
[73] C.-K. Yap, On guaranteed accuracy computation, in: Falai Chen, Dongming Wang (Eds.), Geometric Computation, World Scientific Publishing, 2004, pp. 322–373.
[74] C.-K. Yap, M. Sagraloff, V. Sharma, Analytic root clustering: A complete algorithm using soft zero tests, in: Proc. 9th Conference on Computability in Europe, (CiE), in: LNCS, vol. 7921, Springer, 2013, pp. 434–444.
[75] X. Zheng, K. Weihrauch, The arithmetical hierarchy of real numbers, Mathematical Logic Quarterly 47 (2001) 51–65.
[76] M. Ziegler, Real computation with least discrete advice: A complexity theory of nonuniform computability, Ann. Pure Appl. Logic 163 (8) (2012) 1108–1139.
[77] M. Ziegler, Computability on regular subsets of euclidean space, Mathematical Logic Quarterly (MLQ) 48 (2002) 157–181. Supplement 1.
[78] M. Ziegler, Computable operators on regular sets, Mathematical Logic Quarterly 50 (2004) 392–404.