Weighted norm inequalities of Bochner–Riesz means

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Abstract

Let $w$ be a Muckenhoupt weight and $H^p_w(\mathbb{R}^n)$ be the weighted Hardy spaces. We use the atomic decomposition of $H^p_w(\mathbb{R}^n)$ and their molecular characters to show that the Bochner–Riesz means $T^\delta_R$ are bounded on $H^p_w(\mathbb{R}^n)$ for $0 < p \leq 1$ and $\delta > \max\{n/p - (n + 1)/2, [n/p]r_w(r_w - 1)^{-1} - (n + 1)/2\}$, where $r_w$ is the critical index of $w$ for the reverse Hölder condition. We also prove the $H^p_w - L^p_w$ boundedness of the maximal Bochner–Riesz means $T^\delta_*$ for $0 < p \leq 1$ and $\delta > n/p - (n + 1)/2$.

Keywords: $A_p$ weights; Atomic decomposition; Bochner–Riesz means; Molecular characterization; Weighted Hardy spaces

1. Introduction

The Bochner–Riesz means of order $\delta > 0$ are defined for testing functions $f$ on $\mathbb{R}^n$ by

$$T^\delta_R f(x) = \int_{|\xi| < R} \hat{f}(\xi) \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta e^{2\pi i x \cdot \xi} d\xi, \quad 0 < R < \infty.$$ 

They were first studied by Bochner [1] in connection with summation of multiple Fourier series. Questions concerning the convergence of multiple Fourier series have led to the study of their $L^p$ boundedness. As for their $H^p$ boundedness, Sjölin [11] and Stein, Taibleson and Weiss [12] showed:

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Theorem A. Suppose that $0 < p \leq 1$ and $\delta > n/p - (n + 1)/2$. Then the operator $f \mapsto T_R^\delta f$ is bounded on $H^p$, and satisfies
\[ \|T_R^\delta f\|_{H^p} \leq C \|f\|_{H^p}, \]
where $C$ is independent of $f$ and $R$.

A weighted weak type estimate for the maximal operator $T_*^\delta$, defined by
\[ T_*^\delta f(x) = \sup_{R > 0} |T_R^\delta f(x)|, \]
was given by Sato [9]:

Theorem B. Let $0 < p < 1$, $w \in A_1$, and $\delta = n/p - (n + 1)/2$. Then
\[ \sup_{\lambda > 0} \lambda^p w \left( \{ x \in \mathbb{R}^n : T_*^\delta f(x) > \lambda \} \right) \leq C_{w,n,p} \|f\|_{H^p_w}^{p} \]
holds for all $f \in H^p_w$.

For $w$ equal to a constant function, the above inequality is sharp when $\delta$ is the critical index (cf. [12, p. 90]). Thus, we shall consider values of $\delta$ greater than $n/p - (n + 1)/2$, and prove that the maximal operator $T_*^\delta$ have the following strong type boundedness.

Theorem 1. Let $0 < p \leq 1$ and $\delta > n/p - (n + 1)/2$. If $w \in A_1$, then the operator $f \mapsto T_*^\delta f$ is of type $(H^p_w, L^p_w)$.

The scheme of the proof is to express $T_R^\delta$ as a convolution operator and get an estimate for the kernel of this convolution operator. Then we transfer the estimate of the kernel to the estimate of $T_*^\delta$, from which we get the $H^p_w - L^p_w$ boundedness of the operator $f \mapsto T_*^\delta f$. We also use the estimate of $T_*^\delta$ to show the $H^p_w$ boundedness of Bochner–Riesz means.

Theorem 2. Let $w \in A_1$ with critical index $r_w$ for the reverse Hölder condition. Suppose that $0 < p \leq 1$ and $\delta > \max\{n/p - (n + 1)/2, [n/p]r_w - 1, 1 - (n + 1)/2\}$. Then the operator $f \mapsto T_R^\delta f$ is bounded on $H^p_w$, and satisfies
\[ \|T_R^\delta f\|_{H^p_w} \leq C \|f\|_{H^p_w}, \]
where $C$ is independent of $f$ and $R$.

Remark. It is known (cf. [10]) that, for $n \geq 2$, there exists $f \in H^1_w \cap L^1$, $w \in A_1$, such that
\[ \limsup_{R \to \infty} |T_R^{(n-1)/2} f| = \infty \quad \text{almost everywhere}. \]
Hence, when $\delta = n/p - (n + 1)/2$ is the critical index, Theorem 2 is not true for $p = 1$ and $n \geq 2$.

Throughout this paper $C$ denotes a positive constant not necessarily the same at each occurrence, and a subscript is added when we wish to make clear its dependence on the parameter in the subscript.
2. $A_p$ weights

The definition of class $A_p$ was first used by Muckenhoupt [8], Hunt, Muckenhoupt and Wheeden [6], and Coifman and Fefferman [2] in the investigation of weighted $L^p$ boundedness of Hardy–Littlewood maximal function and Hilbert transform. Let $w$ be a nonnegative function defined on $\mathbb{R}^n$. We say that $w \in A_p$, $1 < p < \infty$, if
\[
\left( \int_I w(x) \, dx \right) \left( \int_I w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C |I|^p
\]
for every cube $I \subseteq \mathbb{R}^n$, where $C$ is independent of $I$ and $0 \cdot \infty$ is taken to be 0. For $p = 1$, $w \in A_1$ if
\[
\frac{1}{|I|} \int_I w(x) \, dx \leq C \cdot \inf_{x \in I} w(x)
\]
for every cube $I \subseteq \mathbb{R}^n$.

A function $w \in A_\infty$ if it satisfies the condition $A_p$ for some $p > 1$. It is well known that $w \in A_p$, $1 \leq p < \infty$, implies $w \in A_r$ for all $r > p$. Also, if $w \in A_p$, $1 < p < \infty$, implies $w \in A_q$ for some $1 < q < p$. We thus write $q_w \equiv \inf(q > 1: w \in A_q)$ for the critical index of $w$ and set weighted measure $w(E) = \int_E w(x) \, dx$.

A close relation to $A_p$ is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant $C > 0$ such that
\[
\left( \frac{1}{|I|} \int_I w^r(x) \, dx \right)^{1/r} \leq C \left( \frac{1}{|I|} \int_I w(x) \, dx \right)
\]
for every cube $I \subseteq \mathbb{R}^n$.

we say that $w$ satisfies reverse Hölder condition of order $r$ and write $w \in RH_r$. It is known that if $w \in RH_r$, $r > 1$, then $w \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$. We thus write $r_w \equiv \sup\{r > 1: w \in RH_r\}$ to denote the critical index of $w$ for the reverse Hölder condition. For a function $f$ and $v \in \mathbb{R}^n$, we define $\tau_{v,f}$ the translation of $f$ given by $\tau_{v,f}(x) = f(x - v)$. Then $\tau_{v,w} \in A_p$ for $w \in A_p$, $1 \leq p \leq \infty$, and all $v \in \mathbb{R}^n$, and $q_{\tau_{v,w}} = q_w$, $r_{\tau_{v,w}} = r_w$.

The following result provides us the comparison between the Lebesgue measure of a set $E$ and its weighted measure $w(E)$.

Theorem C. [5] Let $w \in A_q$, $q \geq 1$. Then there exists constant $C > 0$ such that
\[
\left( \frac{|E|}{|I|} \right)^q \leq C \frac{w(E)}{w(I)}
\]
for any measurable subset $E$ of a cube $I$.

3. Weighted Hardy spaces and their atomic decompositions

Given a weight function $w$ on $\mathbb{R}^n$, as usual we use $L_q^w(\mathbb{R}^n)$ to express the space of all functions satisfying $||f||_{L_q^w}^q \equiv \int_{\mathbb{R}^n} |f(x)|^q w(x) \, dx < \infty$. When $q = \infty$, $L_\infty^w$ will be taken to mean $L_\infty$. Analogous to the classical Hardy spaces, the weighted Hardy spaces $H_p^q(\mathbb{R}^n)$, $p > 0$, can be defined in terms of maximal functions. Namely, let $\varphi$ be a function in $\mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing smooth functions, satisfying $\int_{\mathbb{R}^n} \varphi(x) \, dx = 1$. Define
\[
\varphi_r(x) = r^{-n} \varphi(x/r), \quad r > 0, \quad x \in \mathbb{R}^n,
\]
and the maximal function \( f^* \) by
\[
f^*(x) = \sup_{r>0} |f \ast \varphi_r(x)|.
\]
Then \( H^p_w(\mathbb{R}^n) \) consists of those tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) for which \( f^* \in L^p_w(\mathbb{R}^n) \) with
\[
\|f\|_{H^p_w} = \|f^*\|_{L^p_w}.
\]
We also can characterize these weighted Hardy spaces \( H^p_w \) in terms of atoms in the following way.

**Definition.** On \( \mathbb{R}^n \), let \( 0 < p \leq 1 \leq q \leq \infty \) and \( p \neq q \) such that \( w \in A_q \) with critical index \( q_w \). Set \( [\cdot] \) the integer function. For \( s \in \mathbb{Z} \) satisfying \( s \geq [n(q_w/p - 1)] \), a real-valued function \( a \) is called \((p, q, s)\)-atom centered at \( x_0 \) with respect to \( w \) (or \( w-(p, q, s)\)-atom centered at \( x_0 \)) if

(i) \( a \in L^q_w \) and is supported in a cube \( I \) centered at \( x_0 \);
(ii) \( \|a\|_{L^q_w} \leq w(I)^{1/q - 1/p} \);
(iii) \( \int_{\mathbb{R}^n} a(x) x^\alpha \, dx = 0 \) for every multi-index \( \alpha \) with \( |\alpha| \leq s \).

In the sequel we always use \( N \) to denote the integer \([n(q_w/p - 1)]\).

Let \( H^p_{w,q,s} \) denote the space consisting of tempered distributions admitting a decomposition
\[
f = \sum \lambda_i a_i,
\]
where \( a_i \)'s are \( w-(p, q, s)\)-atoms and \( \sum |\lambda_i|^p < \infty \). For fixed weight function \( w \) and \( f \in H^p_w(\mathbb{R}^n) \), we also set
\[
N_{p,q,s}(f) = \inf \left\{ \left( \sum_i |\lambda_i|^p \right)^{1/p} : \sum_i \lambda_i a_i \text{ is a decomposition of } f \right. 
\]
into \((p, q, s)\)-atoms with respect to \( w \) \}

We have the following atomic decomposition for \( H^p_w \) (cf. [3,7]).

**Theorem D.** If the triple \((p, q, s)\) satisfies the conditions of \( w-(p, q, s)\)-atom, then \( H^p_w = H^p_{w,q,s} \). Moreover, both \( \|f\|_{H^p_w} \) and \( N_{p,q,s}(f) \) are equivalent.

If we allow an atom to have support outside of a cube and also replace its size condition, then we get a generalized atom. Such generalized atoms are called molecules and are useful in certain applications. Denote by \( I^0_{r,x_0} \) the cube centered at \( x_0 \) with side length \( 2r \), and denote simply by \( I^0_r \) to be \( I^0_{r,x_0} \). We now define the molecules corresponding to the atoms mentioned above.

**Definition.** For \( 0 < p \leq 1 \leq q \leq \infty \) and \( p \neq q \), let \( w \in A_q \) with critical index \( q_w \) and critical index \( r_w \) for the reverse Hölder condition. Set \( s \geq N, \epsilon > \max\{sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, 1/p - 1\}, a = 1 - 1/p + \epsilon \), and \( b = 1 - 1/q + \epsilon \). A \((p, q, s, \epsilon)\)-molecule centered at \( x_0 \) with respect to \( w \) (or \( w-(p, q, s, \epsilon)\)-molecule centered at \( x_0 \)) is a function \( M \in L^q_w(\mathbb{R}^n) \) satisfying

(i) \( M(x) \cdot w(I^0_{|x-x_0|})^b \in L^q_w(\mathbb{R}^n) \);
(ii) \( \|M\|_{L^q_w}^{a/b} \cdot \|M(x) \cdot w(I^0_{|x-x_0|})^b\|_{L^q_w}^{1-a/b} = M_w(M) < \infty \);
(iii) \( \int_{\mathbb{R}^n} M(x)x^\alpha \, dx = 0 \) for every multi-index \( \alpha \) with \(|\alpha| \leq s\).

The above \( \mathcal{N}_w(M) \) is called the molecular norm of \( M \) with respect to \( w \) (or \( w \)-molecular norm of \( M \)). If there is no ambiguity, we still use \( \mathcal{N}(M) \) to denote the \( w \)-molecular norm of \( M \).

We have the following molecular characterization of weighted Hardy spaces.

**Theorem E.** [7] Let \((p, q, s, \varepsilon)\) be the quadruple in the definition of molecule, and let \( w \in A_q \). Every \((p, q, s, \varepsilon)\)-molecule \( M \) centered at any point with respect to \( w \) is in \( H^p_w(\mathbb{R}^n) \), and \( \| M \|_{H^p_w} \leq C \mathcal{N}(M) \) where the constant \( C \) is independent of the molecule.

4. Weighted norm inequalities of \( T_\delta^R \) and \( T_\delta^* \)

The Bochner–Riesz means can be expressed as convolution operators

\[
T_\delta^R f(x) = (f * \phi_1/R)(x),
\]

where \( \phi_\varepsilon(x) = \varepsilon^{-n} \phi(x/\varepsilon) \) and \( \phi(x) = \{(1 - |\cdot|^2)^\delta \} \hat{\phi}(x) \). The kernel of these convolution operators has the following property.

**Lemma 1.** [9] If \( \delta = n/p - (n + 1)/2 \) and \( 0 < p < 1 \), then \( \phi \) satisfies the inequality

\[
\sup_{x \in \mathbb{R}^n} (1 + |x|)^{n/p} |D_\alpha^\phi(x)| \leq C_{\alpha, n, p} \text{ for all multi-indices } \alpha.
\]

Recall the maximal Bochner–Riesz means

\[
T_\delta^* f(x) = \sup_{R > 0} |T_\delta^R f(x)|.
\]

Using the estimate in Lemma 1, we get the following inequality.

**Lemma 2.** Let \( 0 < p < 1 \) and \( \delta = n/p - (n + 1)/2 \). For any \( w-(p, \infty, N) \)-atom \( a \) centered at \( 0 \) with \( \text{supp}(a) \subseteq I_r \), we have the inequality

\[
T_\delta^* a(x) \leq C \left( \frac{|I_r|}{w(I_r)} \right)^{1/p} (r + |x|)^{-n/p},
\]

where \( C \) depends only on \( n \) and \( p \).

**Proof.** It suffices to show (1) holds for \( r = 1 \) only. Given \( w-(p, \infty, N) \)-atom \( a \) with \( \text{supp}(a) \subseteq I_r \), if set \( a_0(x) \equiv w(I_r)^{-1/p} w(I_1)^{-1/p} a(rx) \), then \( a_0 \) is a \( w-(p, \infty, N) \)-atom with \( \text{supp}(a_0) \subseteq I_1 \). We have

\[
(a * \phi_\varepsilon)(x) = \varepsilon^{-n} \int a(x - y) \phi \left( \frac{y}{\varepsilon} \right) \, dy
\]

\[
= w(I_r)^{-1/p} w(I_1)^{1/p} \varepsilon^{-n} \int a_0 \left( \frac{x - y}{r} \right) \phi \left( \frac{y}{\varepsilon} \right) \, dy
\]

\[
= w(I_r)^{-1/p} w(I_1)^{1/p} (a_0 * \phi_\varepsilon/r) \left( \frac{x}{r} \right).
\]
If we assume (1) true for \( r = 1 \), then
\[
T_\delta^* a(x) \leq w(I_r)^{-1/p} w(I_1)^{1/p} T_\delta^* a_0(x/r) \leq C(w(I_r)^{-1} |I_r|)^{1/p} (r + |x|)^{-n/p}.
\]

Let \( a \) be any \( w-(p, \infty, N) \)-atom with \( \text{supp}(a) \subseteq I_1 \). From the inequality
\[
|a*x(\phi_\varepsilon)(x)| \leq \|a\|_{\infty} \|\phi_\varepsilon\|_1 \leq w(I_1)^{-1/p} \|\phi_\varepsilon\|_1 \leq C w(I_1)^{-1/p},
\]
Lemma 2 holds for \( |x| \leq 2n \). As for \( |x| > 2n \), we are going to prove
\[
|a*x(\phi_\varepsilon)(x)| \leq C_{n,p} w(I_1)^{-1/p} |x|^{-n/p} \quad \text{for} \quad |x| > 2n,
\]
so that we take the supremum over all \( \varepsilon > 0 \) to complete the proof. By Lemma 1, we obtain
\[
|a*x(\phi_\varepsilon)(x)| = \varepsilon^{-n} \left| \int_{|y| \leq \sqrt{n}} a(y) \phi \left( \frac{x-y}{\varepsilon} \right) dy \right|
\]
\[
\leq C_{n,p} w(I_1)^{-1/p} \varepsilon^{-n} \int_{|y| \leq \sqrt{n}} \varepsilon^{n/p} dy
\]
\[
\leq C_{n,p} w(I_1)^{-1/p} \varepsilon^{n(1/p-1)} |x|^{-n/p}
\]
which implies (2) for \( 0 < \varepsilon \leq 1 \). For \( \varepsilon > 1 \), we consider \( n/(n+k+1) \leq p < n/(n+k) \), \( k = 0, 1, 2, \ldots \). Let \( P(y) \) be the \( k \)th order Taylor polynomial of \( \phi(y) \) at \( x/\varepsilon \). Note that \( k < n(1/p - 1) \leq n(qw/p - 1) \) and \( n(1/p - 1) \leq k + 1 \). From Lemma 1 and the moment conditions of atom, we obtain
\[
|a*x(\phi_\varepsilon)(x)| = \varepsilon^{-n} \left| \int_{|y| \leq \sqrt{n}} a(y) \left\{ \phi \left( \frac{x-y}{\varepsilon} \right) - P \left( \frac{x-y}{\varepsilon} \right) \right\} dy \right|
\]
\[
\leq C_{n,p} w(I_1)^{-1/p} \varepsilon^{-n} \int_{|y| \leq \sqrt{n}} \left( 1 + \frac{|x-\theta y|}{\varepsilon} \right)^{-n/p} \left| \frac{y}{\varepsilon} \right|^{k+1} dy
\]
\[
\leq C_{n,p} w(I_1)^{-1/p} \varepsilon^{n/p-n-k-1} |x|^{-n/p}
\]
\[
\leq C_{n,p} w(I_1)^{-1/p} |x|^{-n/p},
\]
where \( 0 < \theta < 1 \). \( \Box \)

In the following, we shall prove that when \( \delta \) is greater than the critical index, the conclusion of Theorem B can be strengthened. Before proving, we need the following estimate.

**Lemma 3.** [4, p. 412] Let \( w \in A_q \), \( q > 1 \). Then, for all \( r > 0 \), there exists a constant \( C \) independent of \( r \) such that
\[
\int_{|x| \geq r} \frac{w(x)}{|x|^{nq}} dx \leq C \frac{w(I_r)}{r^{nq}}.
\]

**Theorem 1.** Let \( 0 < p \leq 1 \) and \( \delta > n/p - (n+1)/2 \). If \( w \in A_1 \), then the operator \( f \mapsto T_\delta^* f \) is of type \((H^p_w, L^p_w)\).
Proof. Let \( p_1 = 2n/(n+1+2\delta) < p \). Then it follows from Lemma 1 that
\[
\sup_{x \in \mathbb{R}^n} (1 + |x|)^{n/p_1} |D^\alpha \phi(x)| \leq C_{\alpha,n,p_1} \quad \text{for all multi-indices } \alpha.
\]
By Theorem D, we need to show that, for \( w-(p, \infty, [n(1/p_1 - 1)])-\text{atom } a \) centered at any point,
\[
\|T^\delta_a\|_{L^p_w} \leq C \quad (C \text{ independent of } a).
\]
We first consider the weighted \( p \)-atom centered at 0. Let \( a = \text{any } w-(p, \infty, [n(1/p_1 - 1)])-\text{atom with supp}(a) \subseteq I_r \). It is easy to see that
\[
\bar{a}(x) := w(I_r)^{1/p - 1/p_1} a(x)
\]
is a \( w-(p_1, \infty, [n(1/p_1 - 1)]) \)-atom with supp(\( \bar{a} \)) \( \subseteq I_r \). Thus, by Lemma 2,
\[
T^\delta_{\bar{a}}(x) \leq C_{n,p_1} \left( \frac{|I_r|}{w(I_r)} \right)^{1/p_1} (r + |x|)^{-n/p_1},
\]
that is,
\[
T^\delta_{\bar{a}}(x) \leq C_{n,p} w(I_r)^{-1/p_1} r^{n/p_1} (r + |x|)^{-n/p_1}. \tag{3}
\]
We then use Lemma 3 to get
\[
\|T^\delta_{\bar{a}}\|_{L^p_w}^p = \left( \int_{|x|<r} + \int_{|x|\geq r} \right) |T^\delta_{\bar{a}}(x)|^pw(x) \, dx
\leq C_{n,p} \left( \int_{|x|<r} w(I_r)^{-1} w(x) \, dx + \int_{|x|\geq r} w(I_r)^{-1} r^{np/p_1} |x|^{-np/p_1} w(x) \, dx \right)
\leq C_{n,p}.
\]
Let \( a = \text{a } w-(p, \infty, [n(1/p_1 - 1)]) \)-atom centered at \( x_0 \in \mathbb{R}^n \). Then \( \tau^{-\delta}_{-x_0} a \) is a \( w_1-(p, \infty, [n(1/p_1 - 1)]) \)-atom centered at 0, where \( w_1 = \tau^{-\delta}_{-x_0} w \in A_1 \). Since \( \tau^{-\delta}_{-x_0}(T^\delta_R a) = T^\delta_R(\tau^{-\delta}_{-x_0} a) \), we have \( \|T^\delta_{\bar{a}}\|_{L^p_{w_1}}^p = \|T^\delta_{\tau^{-\delta}_{-x_0} a}\|_{L^p_{w_1}}^p \leq C_{n,p} \) and the proof is complete. \( \square \)

Using the estimate of \( T^\delta_{\bar{a}} \), we have the following \( H^P_w \) boundedness of Bochner–Riesz means.

**Theorem 2.** Let \( w \in A_1 \) with critical index \( r_w \) for the reverse Hölder condition. Suppose that \( 0 < p \leq 1 \) and \( \delta > \max\{n/p - (n+1)/2, n/p\}r_w^{-1} - (n+1)/2 \). Then the operator \( f \mapsto T^\delta_R f \) is bounded on \( H^P_w \), and satisfies
\[
\|T^\delta_R f\|_{H^P_w} \leq C\|f\|_{H^P_w},
\]
where \( C \) is independent of \( f \) and \( R \).

**Proof.** As a consequence of Theorems D and E, it suffices to show that if \( f \) is a \( w-(p, \infty, s) \)-atom for some \( s \geq N \), then \( T^\delta_R f \) is a \( w-(p, \infty, N, \epsilon) \)-molecule with \( \mathcal{M}(T^\delta_R f) \leq C \), where the constant \( C \) is independent of \( f \) nor \( R \).

Let \( p_1 = 2n/(n+1+2\delta) < p \) and \( f = \text{a } w-(p, \infty, [n(1/p_1 - 1)]) \)-atom centered at \( 0 \) with \( \text{supp}(f) \subseteq I_r \). Set \( a = 1 - 1/p + \epsilon, b = 1 + \epsilon, \) where \( \max\{Nr_w(r_w-1)^{-1}n^{-1} + (r_w-1)^{-1}, 1/p - 1\} < \epsilon < 1/p_1 - 1 \). From (3) we have
\[
|T^\delta_R f(x)| \leq C_{n,p} w(I_r)^{-1/p}. \tag{4}
\]
On the other hand, Theorem C and (3) imply
\[ |T^\delta_R f(x)| \cdot w(I_{|x|})^b \leq C_w |T^\delta_R f(x)| w(I_r)^b |x|^{nb} r^{-nb} \]
\[ \leq C_{w,n,p} w(I_r)^{b-1/p} r^{n(1/p_1 - b)} (r + |x|)^{nb/n_1 p_1} \]
\[ \leq C_{w,n,p} w(I_r)^{b-1/p} \quad \text{if } |x| > r, \]
and inequality (4) yields
\[ |T^\delta_R f(x)| \cdot w(I_{|x|})^b \leq C_{n,p} w(I_r)^{b-1/p} \quad \text{if } |x| \leq r. \]
We thus have
\[ |T^\delta_R f(x)| \cdot w(I_{|x|})^b \leq C_{w,n,p} w(I_r)^{b-1/p}. \]
Hence,
\[ \mathcal{M}(T^\delta_R f) \leq C_{w,n,p} (w(I_r)^{-1/p})^{a/b} (w(I_r)^{b-1/p})^{1-a/b} = C_{w,n,p}. \]
To verify the vanishing moments, we note that, for $|v| \leq N$,
\[ \int T^\delta_R f(x) x^v \, dx = \left[ T^\delta_R f(x)x^v \right] ^{\wedge} (0) = C_n D^v (T^\delta_R \hat{f})(0) = C_n D^v (\hat{f} \phi_1^{1/R})(0) \]
\[ = C_n \sum_{|\alpha|+|\beta|=|v|} C_{\alpha,\beta} \left( D^\beta \hat{f} \phi_1^{1/R} \right)(0) \]
\[ = 0 \]
since $f$ has the higher order vanishing moments.

Let $f$ be a $w$-$(p, \infty, s)$-atom centered at $x_0 \in \mathbb{R}^n$. Then $\tau_{-x_0} f$ is a $(p, \infty, s)$-atom centered at 0 with respect to $\tau_{-x_0} w$. Since $\tau_{-x_0} w \in A_1$ and $r_{\tau_{-x_0} w} = r_w$, the above arguments show that $\tau_{-x_0} (T^\delta_R f) = T^\delta_R (\tau_{-x_0} f)$ is a $(p, \infty, N, \varepsilon)$-molecule centered at 0 with respect to $\tau_{-x_0} w$. Therefore, $T^\delta_R f$ is a $(p, \infty, N, \varepsilon)$-molecule centered at $x_0$ with respect to $w$ and $\mathcal{N}_w(T^\delta_R f) = \mathcal{N}_{\tau_{-x_0} w}(T^\delta_R (\tau_{-x_0} f)) \leq C$, and the proof is finished. \( \square \)

References