

Four Symmetry Classes of Plane Partitions under One Roof

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In a previous paper, the author applied the permanent-determinant method of Kasteleyn and its non-bipartite generalization, the Hafnian–Pfaffian method, to



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cyclically symmetric plane partitions, transpose-complement plane partitions, and the intersection of the last two types) in the representation theory of $sl(2, \mathbb{C})$. The result is a unified proof of the four enumerations. © 1996 Academic Press, Inc.

Stanley [10] and Robbins recognized that there are ten symmetry classes of plane partitions in a box, and, with the aid of computer experiments and other work, that the number of plane partitions in each symmetry class is given by a product formula. The program of proving each of these formulas has recently been completed [1, 6, 11], but although the formulas are very similar, there is no known unified treatment. Moreover, some of the enumerations presently require difficult, ad-hoc calculations with generating functions or matrices. In fact, the enumeration of one of the classes (totally symmetric plane partitions or TSPP's) admits a natural q -analogue which is still open.

In a previous paper, the author [6] discussed the permanent-determinant method of Kasteleyn and its generalization, the Hafnian–Pfaffian method, as the first step towards a possible unified treatment. (The connection between this method and plane partitions was discovered jointly with James Propp.) In this paper, we carry out this strategy for the four symmetry class that are given by determinants and not merely Pfaffians: unrestricted plane partitions (PP's), cyclically symmetric plane partitions (CSPP's), transpose-complement plane partitions (TCPP's), and cyclically symmetric, transpose-complement plane partitions (CSTCPP's). The new

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idea is to find the matrices given by the permanent-determinant method in the representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. The main result of the present paper is the following theorem:

THEOREM 0.1. *A Kasteleyn-flat, weighted adjacency matrix for the graph $Z(a, b, c)$, whose matchings are bijective with plane partitions in an $a \times b \times c$ box, arises as $\alpha(X)|_{-1}$ in a representation α of $\mathfrak{sl}(2, \mathbb{C})$ which is a tensor product of three irreducible representations, with a basis formed from weight bases of irreducible representations. Consequently, $(\det \alpha(X)|_{-1})/m$ is the number of such plane partitions, where m is the value of any term in the determinant.*

The representation theory of the quantum group $U_q(\mathfrak{sl}(2, \mathbb{C}))$ similarly yields the q -enumeration of PP's, but the author could not obtain the known q -enumeration of CSPP's by this method.

Proctor's minuscule method is a competing method that, in various forms, yields the enumeration of five symmetry classes of plane partitions [5, 8, 12]. Proctor's method partly explains Stembridge's $q = -1$ phenomenon [13], and it is applicable to any symmetry class not involving cyclic symmetry. Likewise, the method given here is applicable to any symmetry class which does not involve transposition or complementation, but which may involve their product.

The author believes that one of these two methods, or perhaps some combination of the two, should lead to a complete unified enumeration. Both methods involve Cartan–Weyl representation theory. Moreover, many of the enumerations can be analyzed with the Gessel–Viennot non-intersecting lattice-path method [3], which in turn can be recognized as a condensed version of the permanent-determinant method [4]. For example, the Gessel–Viennot method applied to PP's yield Carlitz matrices. The determinants of these matrices are well-known. In entirely independent work, Proctor [9] rederived, the Carlitz determinants by an $\mathfrak{sl}(2, \mathbb{C})$ factoring method which is equivalent to the one presented here.

The author would like to thank James Propp for suggesting numerous corrections and for his continued interest in this work.

1. PRELIMINARIES

1.1. Plane Partitions and Their Symmetries

A *plane partition* in an $a \times b \times c$ box is a collection of unit cubes in the rectangular solid $[0, a] \times [0, b] \times [0, c]$ which is stable under gravitational attraction towards the origin. Equivalently, it is an $a \times b$ matrix of integers

with entries between 0 and c ; the stack of cubes is then a *Ferrer's diagram* (or bar graph) of the matrix. Figure 1 shows a picture of a plane partition together with three back walls of the box. We let $N(a, b, c)$ be the total number of plane partitions in an $a \times b \times c$ box.

There are three natural symmetry operations on plane partitions:

1. *Transposition*, τ , is a reflection through a diagonal plane of the box of a plane partition. A transposition-symmetric plane partition is called, briefly, a *symmetric plane partition*.

2. *Rotation*, ρ , is a rotation by 120° about the long diagonal of the box of a plane partition. Symmetry under rotation is also called *cyclic symmetry*.

3. *Complementation*, κ , takes the point (x, y, z) to the point $(a - x, b - y, c - z)$ for an $a \times b \times c$ box. The operation κ is the same for individual cubes, but for a plane partition P , $\kappa(P)$ is the set of all cubes C such that $\kappa(C) \notin P$.

By convention, the operations τ and ρ are only defined when the hexagon or box possesses them as symmetries, while the operation κ always exists. The group of all symmetries has ten conjugacy classes of subgroups, and for a subgroup G , let $N_G(a, b, c)$ be the number of plane partitions with that symmetry. We describe G by generators as $G = \langle g_1, g_2, \dots \rangle$ and we often omit the angle brackets, so that $N_{\rho, \tau}(a, a, a)$ is the number of totally symmetric plane partitions. (By convention, the phrase "total symmetry" does not include the symmetry operation of complementation.)

As mentioned in the introduction, we will consider the four symmetry classes whose symmetry groups are subgroups of $\langle \rho, \kappa \tau \rangle$. In other words, CSPP's are invariant under ρ , TCPP's are invariant under $\kappa \tau$, and CSTCPP's are invariant under both.

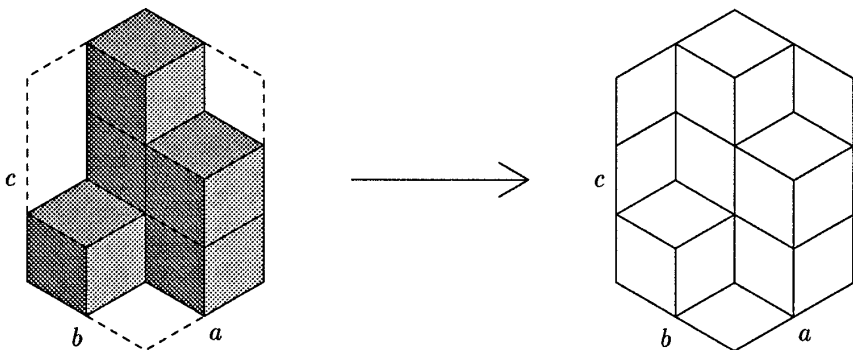


FIG. 1. A plane partition in a box and its lozenge tiling.

In addition to ordinary enumeration, we will also consider the weighted enumeration of plane partitions, where the weight of a plane partition with n cubes is q^n . Specifically, let $N_G(a, b, c)_q$ be the total weight of all plane partitions in the given symmetry class. The polynomials $N(a, b, c)_q$, $N_\tau(a, a, b)_q$, and $N_\rho(a, a, a)_q$ have nice product formulas, but we will only give a proof of the formula for $N(a, b, c)_q$. (There is an alternate weighting whereby the weight of a plane partition is q^n if it has n orbits; the corresponding polynomial $N'_\tau(a, a, b)_q$ is known to have a product formula and $N'_{\rho, \tau}(a, a, a)$ is conjectured to do so, but we will not consider these interesting cases.)

1.2. Product Formulas

We define some products related to Stanley’s formulas [10].

If a_1, a_2, \dots, a_k are positive integers and n is an integer, define the *box product* $C(a_1, a_2, \dots, a_k; n)$ by

$$C(a_1, a_2, \dots, a_k; n) = \prod_{x_1=0}^{a_1-1} \prod_{x_2=0}^{a_2-1} \cdots \prod_{x_k=0}^{a_k-1} \max\left(n - \sum_{i=1}^k x_i, 1\right).$$

A typical box product is the product of the numbers shown in Fig. 2. Define the *simplex product* $T(k, n)$ by

$$T(k, n) = C(\overbrace{\infty, \infty, \dots, \infty}^k; n).$$

Thus, $T(1, n) = n!$, while

$$T(2, n) = n!(n - 1)! (n - 2)! \cdots 2!1!$$

Simplex products can also be defined inductively by the following rules:

$$\begin{aligned} T(0, n) &= n \\ T(k, 0) &= 1 \\ T(k, n) &= T(k - 1, n) T(k, n - 1) \end{aligned} \tag{1}$$

for non-negative integers k and n , with $T(k, n) = 1$ for all other cases.

4	3	2	1	1	1
5	4	3	2	1	1
6	5	4	3	2	1
7	6	5	4	3	2

FIG. 2. Factors of the box product $C(6, 4; 7)$.

It follows immediately from the definition that

$$C(a_1, a_2, \dots, a_k; n) = \frac{C(\infty, a_2, a_3, \dots, a_k; n)}{C(\infty, a_2, a_3, \dots, a_k; n - a_1)}.$$

Applying this formula to each variable in turn yields a multiplicative inclusion-exclusion formula:

$$\begin{aligned} C(a_1, a_2 \dots, a_k; n) &= T(k, n) T(k, n - a_1)^{-1} T(k, n - a_2)^{-1} \dots T(k, n - a_k)^{-1} \\ &\quad T(k, n - a_1 - a_2) T(k, n - a_1 - a_3) \dots T(k, n - a_{k-1} - a_k) \\ &\quad \vdots \\ &\quad T\left(k, n - \sum_{i=1}^k a_i\right)^{(-1)^k}. \end{aligned}$$

In particular,

$$C(a, b, c; n) = \frac{T(3, n) T(3, n - a - b) T(3, n - a - c) T(3, n - b - c)}{T(3, n - a) T(3, n - b) T(3, n - c) T(3, n - a - b - c)}. \tag{2}$$

Similarly, define a box q -product by

$$C(a_1, a_2, \dots, a_k; n)_q = \prod_{x_1=0}^{a_1-1} \prod_{x_2=0}^{a_2-1} \dots \prod_{x_k=0}^{a_k-1} \left(\max \left(n - \sum_{i=1}^k x_i, 1 \right) \right)_q,$$

where $(n)_q = 1 + q + \dots + q^{n-1}$, and a simplex q -product by

$$T(k, n)_q = C(\infty, \dots, \infty; n)_q.$$

The formula for $N(a, b, c)$ given by Stanley [10] and originally proved by MacMahon, is

$$N(a, b, c) = \frac{C(a, b, c; a + b + c - 1)}{C(a, b, c; a + b + c - 2)}.$$

Combining this equation with equation (2) and equation (1), we obtain an expression suggested by Propp [6] for the same numbers:

$$N(a, b, c) = \frac{T(2, a + b + c - 1) T(2, a - 1) T(2, b - 1) T(2, c - 1)}{T(2, a + b - 1) T(2, a + c - 1) T(2, b + c - 1)}.$$

For either formulation of MacMahon’s enumeration, we can q -ify throughout.

2. THE ENUMERATION

2.1. *The Permanent-Determinant Method*

Following reference [6], a plane partition in an $a \times b \times c$ box is equivalent to a tiling of a certain hexagon $H(a, b, c)$ by unit lozenges, where a *lozenge* is a rhombus with a 60 degree angle, as shown in Figure 1. The hexagon $H(a, b, c)$ has angles of 120 degrees and edge lengths of $a, b, c, a, b,$ and $c,$ going clockwise around the perimeter. The hexagon has a unique tiling by unit equilateral triangles, and a unit lozenge tiling is therefore equivalent to a perfect matching of a bipartite graph $Z(a, b, c)$, shown in Fig. 3, whose vertices are the triangles and whose edges are given by adjacency.

The number of perfect matchings of an arbitrary bipartite graph is given by the permanent of its bipartite adjacency matrix. Kasteleyn's *permanent-determinant method* dictates that, if the graph is planar, then there is a way to change the signs of the permanent to convert it to a determinant. (More accurately, Kasteleyn considered the more general non-bipartite case and produced a Pfaffian; the bipartite case was clarified by Percus [7].) Indeed, the permanent of the adjacency matrix $A(a, b, c)$ of $Z(a, b, c)$ equals the determinant up to a global sign, and the global sign is in any case ambiguous because the rows and columns are unordered. One way to demonstrate the equality is to note that two matchings differ by a 3-cycle (carried by a hexagon of $Z(a, b, c)$) if their plane partitions differ by a cube. Since a 3-cycle is an even permutation, it incurs no sign change between the corresponding terms of $\text{Det } A(a, b, c)$. Since any two plane partitions are connected by the operation of adding or taking away an individual cube, it follows that all terms have the same sign in the determinant.

More generally, suppose that $Z(a, b, c)$ is weighted with non-zero weights, and that M is its weighted adjacency matrix. Then the non-vanishing

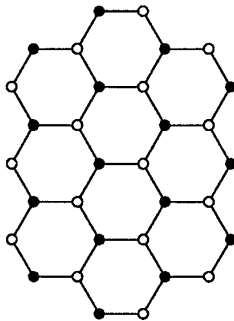


FIG. 3. The graph $Z(2, 2, 3)$.

terms in the determinant of M also correspond to matchings of $Z(a, b, c)$, but they are no longer necessarily equal. If, for every hexagon of $Z(a, b, c)$, the weights $x_1, y_1, z_1, x_2, y_2,$ and z_2 shown in Fig. 4 satisfy

$$x_1 y_1 z_1 = x_2 y_2 z_2,$$

then the non-vanishing terms of $\text{Det } M$ are equal. In this case, we say that M and the weighting are *Kasteleyn-flat*. The number of matchings is therefore

$$(\text{Det } M)/m, \tag{3}$$

where m is the value of any single term. This principle can be extended to the q -enumeration problem: Define the *Kasteleyn curvature* of a hexagon as

$$\frac{x_1 y_1 z_1}{x_2 y_2 z_2}.$$

(This statistic is called curvature by analogy with the curvature of a connection on a line bundle or the coboundary operation in homology; see reference [4] for details.) If the Kasteleyn curvature is q everywhere and t is the weight of the matching of the empty plane partition, then

$$N(a, b, c)_q = (\text{Det } M)/m.$$

In this case, the value of a term goes up by q if we add a cube to the corresponding plane partition.

The symmetry operations $\tau, \rho,$ and κ act on hexagons $H(a, b, c)$ respectively as rotation by $120^\circ,$ rotation by $180^\circ,$ and reflection about a diagonal. Therefore $\kappa\tau$ acts as reflection about a bisector (a line which meets two opposite edges of the hexagon in the middle), and in its action on $Z(a, a, 2b)$, it fixes a row of edges which separates the graph into two isomorphic subgraphs. (Note that there are no $\kappa\tau$ -invariant matchings of $Z(a, a, 2b - 1)$, or equivalently $N_{\kappa\tau}(a, a, 2b - 1) = 0.$) The fixed edges must appear in any invariant subgraph, so that TCPP's correspond to matchings of either subgraph, denoted by $Z_{\kappa\tau}(a, a, 2b)$. Similarly, $\langle \rho, \kappa\tau \rangle$ has three

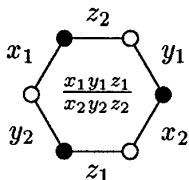


FIG. 4. The weights that determine Kasteleyn curvature.

bisectors as lines of reflection, so that CSTCPP's correspond to matchings of any of the six subgraphs $Z_{\rho, \kappa\tau}(2a, 2a, 2a)$ that remain. Since all the faces of these graphs are also hexagons, the same analysis of Kasteleyn's method applies fully, except that Kasteleyn curvature of q is unnatural for TCPP's and CSTCPP's unless $q = \pm 1$. We will only consider $q = 1$ for these cases in this paper.

A ρ -invariant matching of $Z(a, a, a)$ is simply a matching of the quotient graph $Z_\rho(a, a, a) = Z(a, a, a)/\rho$, and again the same analysis of Kasteleyn's method applies, except that the face in the center is now a 2-gon. This means that $Z_\rho(a, a, a)$ is not a simple graph, the matrix M is not uniquely determined by the weighting of $Z_\rho(a, a, a)$, and the matchings are not bijective with the non-zero terms in the determinant of M . However, if the weight of each edge is a separate variable, then the monomials in the expansion of $\det M$ are bijective with the matchings of $Z_\rho(a, a, a)$. Taking this approach, if the 2-gon has sides with weight x_1 and x_2 , we define its *Kasteleyn curvature* as x_1/x_2 and say that it is flat if $x_1 = x_2$. As before, all terms in the determinant are equal if the weighting is flat at every face. More generally, the assignment of Kasteleyn curvature which corresponds to the q -enumeration $N_\rho(a, a, a)_q$ is q^3 for every hexagon and q for the 2-gon.

2.2. Representation Theory of $\mathfrak{sl}(2, \mathbb{C})$

In this section, we will find a Kasteleyn-flat matrix M for $Z(a, b, c)$ in the representation theory of $\mathfrak{sl}(2, \mathbb{C})$. We begin by reviewing the finite-dimensional representation theory of $\mathfrak{sl}(2, \mathbb{C})$.

The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of traceless 2×2 matrices has an important basis H , X , and Y given by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The entire Lie algebra structure can also be defined by linear extension of the Lie bracket on this basis:

$$[H, X] = 2X$$

$$[H, Y] = -2Y$$

$$[X, Y] = H.$$

Let x and y be the standard basis vector on which the above matrices act. For each positive integer n , $\mathfrak{sl}(2, \mathbb{C})$ has an irreducible representation V_n of dimension $n + 1$ which can be described as the action of $\mathfrak{sl}(2, \mathbb{C})$ on homogeneous polynomials in x and y of degree n . Let α_n be the representation map from $\mathfrak{sl}(2, \mathbb{C})$ to $\text{End}(V_n)$. To derive the action of $\mathfrak{sl}(2, \mathbb{C})$ in the monomial basis $x^n, x^{n-1}y, x^{n-2}y^2, \dots, y^n$, it is convenient to express $X, Y,$ and H formally as

$$X = x \frac{\partial}{\partial y}$$

$$Y = y \frac{\partial}{\partial x}$$

$$H = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$$

The corresponding matrices are then

$$\alpha_n(H) = \begin{pmatrix} n & 0 & 0 & \dots & 0 \\ 0 & n-2 & 0 & \dots & 0 \\ 0 & 0 & n-4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -n \end{pmatrix}$$

$$\alpha_n(X) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\alpha_n(Y) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ n & 0 & 0 & \dots & 0 & 0 \\ 0 & n-1 & 0 & \dots & 0 & 0 \\ 0 & 0 & n-2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

The monomial basis is also known as the (dual) *weight basis*, and we rename the basis vectors $e_n, e_{n-2}, \dots, e_{-n}$, where the eigenvalue of e_i with respect to H is i .

THEOREM 2.1 (Clebsch–Gordan). *Any finite-dimensional, irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ is isomorphic to a direct sum of V_n 's. In particular, their tensor products decompose according to the equation*

$$V_n \otimes V_k \cong V_{n+k} \oplus V_{n+k-2} \oplus V_{n+k-4} \oplus \cdots \oplus V_{|n-k|}.$$

We assume the first half of the theorem without proof. To understand the Clebsch–Gordan decomposition of a tensor product, we recall that if L is an element of an arbitrary Lie algebra, its action on the tensor product of two representations α and β is given by:

$$(\alpha \otimes \beta)(L) = \alpha(L) \otimes I + I \otimes \beta(L), \quad (4)$$

where I is the identity matrix.

Given the first half of the theorem, the following is an argument for the Clebsch–Gordan formula: Let $S(H)$ and $S(\mathfrak{sl}(2, \mathbb{C}))$ be the semi-ring of reducible representations of H with integer eigenvalues and the semi-ring of representations of $\mathfrak{sl}(2, \mathbb{C})$, respectively. The elements of these semi-rings are representations and the algebraic structure is given by \oplus and \otimes . The semi-rings extend to *Grothendieck rings* $R(H)$ and $R(\mathfrak{sl}(2, \mathbb{C}))$ by introducing subtraction. Since each V_n restricts to a reducible representation of H , $S(\mathfrak{sl}(2, \mathbb{C})) \subset S(H)$ and $R(\mathfrak{sl}(2, \mathbb{C})) \subset R(H)$. The ring $R(H)$, as an abelian group, is generated by one-dimensional representations E_n with eigenvalue n for some integer n . By equation (4), eigenvalues add under tensor product, and therefore there exists an isomorphism $\text{ch}: R(H) \rightarrow \mathbb{Z}[t, 1/t]$ given by the formula $\text{ch}(E_n) = t^n$. This isomorphism has a restriction $\text{ch}: R(\mathfrak{sl}(2, \mathbb{C})) \rightarrow \mathbb{Z}[t, 1/t]$ that we will call the *character map*. By the form of $\alpha_n(H)$,

$$\text{ch}(V_n) = t^n + t^{n-2} + \cdots + t^{-n}. \quad (5)$$

Because the $\text{ch}(V_n)$'s are linearly independent, ch is injective. The Clebsch–Gordan formula then follows easily by applying ch to both sides. Indeed, the assertion that the character map is injective on $S(\mathfrak{sl}(2, \mathbb{C}))$ is a convenient restatement of the Clebsch–Gordan theorem.

Since the matrices for $\alpha_n(H)$, $\alpha_n(X)$, and $\alpha_n(Y)$ are nearly permutation matrices, they can be described graphically as in Fig. 5. Each dot in the

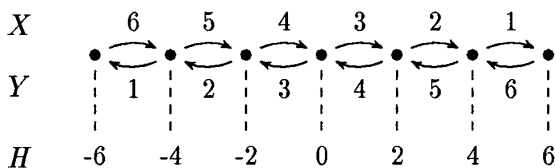


FIG. 5. An irreducible representation of $\mathfrak{sl}(2)$.

figure represents a basis vector which is an eigenvector of $\alpha_n(H)$ with the given eigenvalue. The maps $\alpha_n(X)$ and $\alpha_n(Y)$ send each basis vector to another basis vector times the given scalar factor, except for the vector at the right (resp. left), which is in the kernel of $\alpha_n(X)$ (resp. $\alpha_n(Y)$).

As a warm-up for the main construction, consider the action of X , Y , and H on a tensor product representation $V = V_n \otimes V_k$ in the tensor product basis $\{e_i \otimes e_j\}$. If α is the representation map, the actions of $\alpha(H)$ and $\alpha(X)$ are represented diagrammatically by Fig. 6. Each dot is a basis vector as before, and diagonal strings span eigenspaces of $\alpha(H)$ with the given eigenvectors, because $e_i \otimes e_j$ is an eigenvector of $\alpha(H)$ with eigenvalue $i + j$. By equation (4), most columns of the matrix for $\alpha(X)$ have two terms, which are given by the arrows. (Technically speaking, one might say that the diagram is a directed, weighted graph and the matrix for $\alpha(X)$ is the asymmetric weighted adjacency matrix of the graph.) Let $V|_\lambda$ be the eigenspace of $\alpha(H)$ with eigenvalue λ . Then the matrix for $\alpha(X)$ can be divided into blocks $\alpha(X)|_\lambda: V|_\lambda \rightarrow V|_{\lambda+2}$ which are maps between adjacent eigenspaces. For example, in Fig. 6, $\alpha(X)|_{-1}$, whose domain and target are delineated, has the matrix

$$\begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consider the graph $Z(a, b, c)$ as in Section 2.1, let

$$V = V_{a+b-1} \otimes V_{a+c-1} \otimes V_{b+c-1},$$

and let α be the representation map. The tensor product basis $\{e_i \otimes e_j \otimes e_k\}$ for this representation can be depicted as a 3-dimensional rectangular box

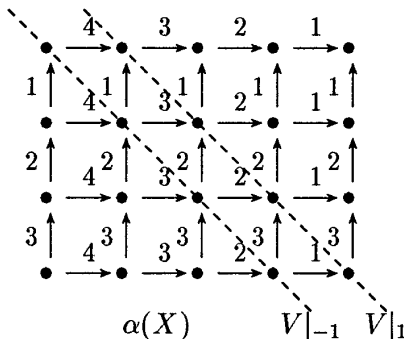
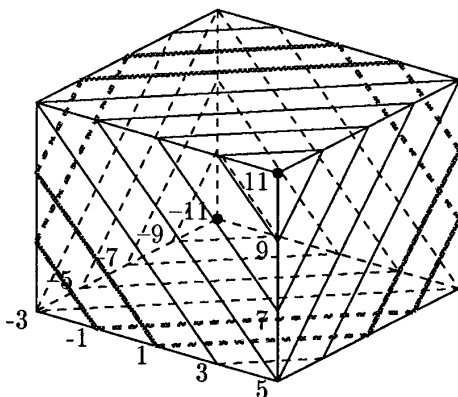
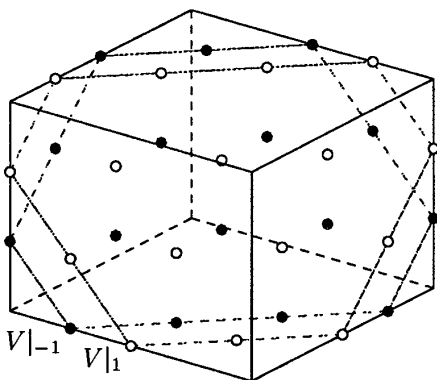


FIG. 6. The representation $V_4 \otimes V_3$.

FIG. 7. The representation $V_4 \otimes V_4 \otimes V_5$.

of points. Unfortunately, it is hard to show all of the points in a 2-dimensional picture. However, $\alpha(H)$ is diagonal as before, and the basis vectors lying in each eigenspace constitute a 2-dimensional slice of the box, because each $e_i \otimes e_j \otimes e_k$ is an eigenvector with eigenvalue $i + j + k$. These slices are given as a sequence for $V_4 \otimes V_4 \otimes V_5$ in Fig. 7, together with the eigenvalue of $\alpha(H)$ for each slice. We will consider the slices $V|_{-1}$ and $V|_1$, which are given separately in Fig. 8, together with points corresponding to their basis vectors.

Most columns of the matrix for $\alpha(X)$ now have three terms, and the matrix divides into submatrices $\alpha(X)|_\lambda: V|_\lambda \rightarrow V|_{\lambda+2}$. Consider the map $\alpha(X)|_{-1}$, which is diagrammed in Fig. 9. If we replace the arrows in this diagram by unoriented edges, we see that $\alpha(X)|_{-1}$ is a weighted, bipartite

FIG. 8. Basis vectors in $V|_{-1}$ and $V|_1$

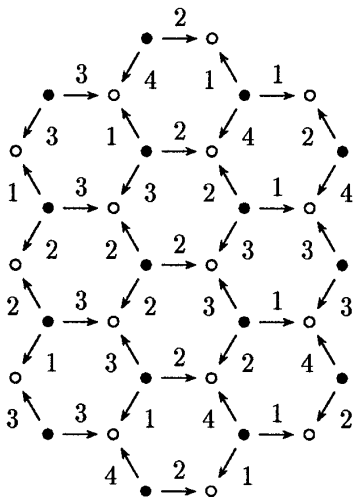


FIG. 9. The weights on $Z(2, 2, 3)$ coming from $\alpha(X)|_{-1}$.

adjacency matrix of the graph $Z(a, b, c)$. Moreover, the weighting is Kasteleyn-flat, because the weights of opposite edges of each face are equal. We have proved the following proposition:

PROPOSITION 2.2. *If the representation $V_{a+b-1} \otimes V_{a+c-1} \otimes V_{b+c-1}$ of $\mathfrak{sl}(2, \mathbb{C})$ has representation map α and is given with the basis which is the tensor product of the weight bases of the three factors, then the matrix for $\alpha(X)|_{-1}$, the restriction of $\alpha(X)$ to the eigenspace of $\alpha(H)$ with eigenvalue -1 , is a Kasteleyn-flat adjacency matrix for $Z(a, b, c)$.*

Applying equation (3),

$$N(a, b, c) = (\det(\alpha(X)|_{-1}))/m, \tag{6}$$

where m is the value of any single term in the expansion of the determinant. Indeed, in this case all terms are not only equal, but are products of the same weights. The product of the weights of the horizontal edges in any matching is

$$\prod_{i=1}^a \prod_{j=1}^b (i+j-1) = C(a, b; a+b-1).$$

Therefore

$$t = \pm C(a, b; a+b-1) C(a, c; a+c-1) C(b, c; b+c-1). \tag{7}$$

The map $\alpha(X)|_{-1}$ is a linear transformation between different vector spaces. For the purpose of computing determinants, it is easier to work with an endomorphism of a single vector space. Observe that the restriction $\alpha(Y)|_1: V|_1 \rightarrow V|_{-1}$ has the same properties as $\alpha(X)|_{-1}$, and that the composition $\alpha(Y)|_1 \alpha(X)|_{-1}$ is an endomorphism of $V|_{-1}$. In particular,

$$N(a, b, c)^2 = \det(\alpha(Y) \alpha(X)|_{-1})/m^2.$$

Since $\alpha(Y) \alpha(X)|_{-1}$ comes from the action of $\mathfrak{sl}(2, \mathbb{C})$ on V , it acts on each summand of a direct-sum decomposition of V separately. Therefore such a decomposition diagonalizes $\alpha(Y) \alpha(X)|_{-1}$ and can be used to find its determinant.

LEMMA 2.3. *The map*

$$\alpha_{2n-1}(Y) \alpha_{2n-1}(X)|_{-1}: V_{2n-1}|_{-1} \rightarrow V_{2n-1}|_{-1}$$

is a map from a 1-dimensional vector space to itself and its effect is multiplication by n^2 .

Compare with Fig. 5 or with the definition of $\alpha_n(X)$ and $\alpha_n(Y)$.

In general if α and β are any two representations of $\mathfrak{sl}(2, \mathbb{C})$,

$$\det((\alpha + \beta)(Y)(\alpha + \beta)(X)|_{-1}) = \det(\alpha(Y) \alpha(X)|_{-1}) \det(\beta(Y) \beta(X)|_{-1}).$$

It follows that the assignment

$$\alpha \mapsto \det(\alpha(Y) \alpha(X)|_{-1})$$

defines an abelian-group homomorphism from the additive subgroup of the Grothendieck ring $R(\mathfrak{sl}(2, \mathbb{C}))$ spanned by V_{2n-1} 's to the multiplicative group \mathbb{Q}^* of non-zero rationals. This homomorphism factors through the character map ch to yield a map D such that

$$D(\text{ch}(V)) = \det(\alpha(Y) \alpha(X)|_{-1})$$

for any odd-weight representation V . By equation (5), we can define D by

$$D(t^{2n-1}) = \frac{n^2}{(n-1)^2}$$

for $n > 1$ and $D(t^{2n-1}) = 1$ otherwise.

Our particular V has character

$$\text{ch}(V) = \sum_{(i, j, k) \in B} t^{2(i+j+k) - 2(a+b+c) - 3},$$

where

$$B = \{1, \dots, a + b\} \times \{1, \dots, a + c\} \times \{1, \dots, b + c\}.$$

The map D therefore yields

$$\begin{aligned} \det(\alpha(Y) \alpha(X)|_{-1}) &= \prod_{(i, j, k) \in B'} \frac{(i + j + k - a - b - c - 1)^2}{(i + j + k - a - b - c - 2)^2} \\ &= \frac{C(a + b, a + c, b + c; a + b + c - 1)^2}{C(a + b, a + c, b + c; a + b + c - 2)^2}, \end{aligned}$$

where $B' \subset B$ is the set of triples (i, j, k) such that $i + j + k \geq a + b + c + 3$. Therefore

$$\det(\alpha(X)|_{-1}) = \pm \frac{C(a + b, a + c, b + c; a + b + c - 1)}{C(a + b, a + c, b + c; a + b + c - 2)}$$

in the given basis. Combining this equation with equations (7) and (6), we obtain

$$N(a, b, c) = \frac{C(a + b, a + c, b + c; a + b + c - 1)}{C(a + b, a + c, b + c; a + b + c - 2) \prod C(a, b; a + b - 1)}.$$

Here, the symbol “ \prod ” denotes repetition of the succeeding factor with cyclic permutation of a, b , and c .

We apply the inclusion-exclusion formula for each cube product and the inductive formula for the resulting simplex products:

$$\begin{aligned} N(a, b, c) &= \frac{T(3, a + b + c - 1) \prod T(3, a - 2) \prod T(2, a - 1)^2}{T(3, a + b + c - 2) \prod T(3, a - 1) \prod T(2, a + b - 1)} \\ &= \frac{T(2, a + b + c - 1) \prod T(2, a - 1)}{\prod T(2, a + b - 1)}. \end{aligned}$$

This is Propp’s expression for $N(a, b, c)$.

Remark. Observe that $\alpha(X)|_{-1}$ is a diagonalizable, integer matrix with integer eigenvalues. The author considered a direct analysis of this matrix without using any representation theory other than the knowledge that the matrix can be decomposed using rational linear algebra. Unfortunately, no obvious pattern for the eigenvectors appeared in small examples. One way to express the coefficients of the eigenvectors with the aid of representation theory is to use Racah $3j$ -symbols, which are complicated and have no obvious derivation using basic linear algebra.

3. QUANTUM REPRESENTATION THEORY OF $\mathfrak{sl}(2, \mathbb{C})$

The ordinary representation theory of $\mathfrak{sl}(2, \mathbb{C})$ admits a q -analogue, the *quantum representation theory*, which can be used to q -enumerate plane partitions. A proper algebraic treatment of the quantum representation theory would present it as the representation theory of a Hopf algebra $U_q(\mathfrak{sl}(2, \mathbb{C}))$ [2]. Briefly, an abstract representation category, or the representation theory of a Hopf algebra, is a category of vector spaces on which some associative algebra or some matrices act, together with suitable definitions of trivial representations, tensor-product representations, and dual representations. The first difference between such a category and the ordinary representation theory of a Lie algebra is that $A \otimes B$ and $B \otimes A$ need not be isomorphic, and if they are isomorphic, they need not be canonically isomorphic. For brevity, we will avoid the definition of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ and give a direct, computational definition of quantum representations of $\mathfrak{sl}(2, \mathbb{C})$.

Let h be a complex number, and define $q = \exp(h)$. More generally, for any number or matrix A , define $q^A = \exp(hA)$. Define the *bracket* of a number or matrix by

$$[A] = \frac{q^{A/2} - q^{-A/2}}{q^{1/2} - q^{-1/2}}.$$

The Laurent polynomial $[n]$ is called a *quantum integer* and is related to the q -integer $(n)_q$ by

$$[n] = q^{(1-n)/2} (n)_q.$$

A *quantum representation* of $\mathfrak{sl}(2, \mathbb{C})$ is defined to be a vector space V and a function $\alpha: \{H, X, Y\} \rightarrow \text{End}(V)$ such that

$$[\alpha(H), \alpha(X)] = 2\alpha(X),$$

$$[\alpha(H), \alpha(Y)] = -2\alpha(Y),$$

$$[\alpha(X), \alpha(Y)] = [\alpha(H)].$$

The tensor product of two representation maps α and β is defined by the equations

$$(\alpha \otimes \beta)(X) = \alpha(X) \otimes q^{\beta(H)/4} + q^{-\alpha(H)/4} \otimes \beta(X),$$

$$(\alpha \otimes \beta)(Y) = \alpha(Y) \otimes q^{\beta(H)/4} + q^{-\alpha(H)/4} \otimes \beta(Y),$$

$$(\alpha \otimes \beta)(H) = \alpha(H) \otimes I + I \otimes \beta(H).$$

These equations are analyzed by Drinfel'd [2], but the diligent reader can check directly that the tensor product matrices again form a valid quantum representation. As a first step, $[(\alpha \otimes \beta)(H)]$ involves $q^{(\alpha \otimes \beta)(H)/2}$, and

$$q^{(\alpha \otimes \beta)(H)/2} = q^{\alpha(H)/2 \otimes I + I \otimes \beta(H)/2} = q^{\alpha(H)/2 \otimes I} q^{I \otimes \beta(H)/2} = q^{\alpha(H)/2} \otimes q^{\beta(H)/2},$$

since the exponential of a sum of commuting matrices is the product of the exponentials.

The classical representation V_n deforms to a quantum representation by the formulas

$$\alpha_n(H) = \begin{pmatrix} n & 0 & 0 & \dots & 0 \\ 0 & n-2 & 0 & \dots & 0 \\ 0 & 0 & n-4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -n \end{pmatrix},$$

$$\alpha_n(X) = \begin{pmatrix} 0 & [1] & 0 & 0 & \dots & 0 \\ 0 & 0 & [2] & 0 & \dots & 0 \\ 0 & 0 & 0 & [3] & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & [n] \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$\alpha_n(Y) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ [n] & 0 & 0 & \dots & 0 & 0 \\ 0 & [n-1] & 0 & \dots & 0 & 0 \\ 0 & 0 & [n-2] & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & [1] & 0 \end{pmatrix}.$$

Finally, although in the quantum representation theory there is no canonical isomorphism between $V_n \otimes V_k$ and $V_k \otimes V_n$, they are isomorphic. Indeed, the Clebsch–Gordan theorem generalizes to the quantum representation theory.

As before, let

$$V = V_{a+b-1} \otimes V_{a+c-1} \otimes V_{b+c-1},$$

and define $V|_\lambda$ to be the eigenspace of $\alpha(H)$ with eigenvalue λ in this representation. In the tensor product basis, the matrix for $\alpha(X)|_{-1}$ is again

a weighted adjacency matrix for $Z(a, b, c)$. In the quantum case, the weighting has Kasteleyn curvature q at every hexagonal face. Therefore

$$N(a, b, c)_q = (\det \alpha(X)|_{-1})/m_q,$$

where m_q is the weight of the matching corresponding to the empty plane partition.

The rest of the derivation is a q -analogue (or quantization) of Section 2. The quantity m_q is given by

$$m_q = \pm q^{-abc/2} C(a, b; a + b - 1)_q C(a, c; a + c - 1)_q C(b, c; b + c - 1)_q,$$

and the Clebsch–Gordan decomposition of V yields

$$\det(\alpha(X)|_{-1}) = \pm q^{-abc/2} \frac{C(a + b, a + c, b + c; a + b + c - 1)_q}{C(a + b, a + c, b + c; a + b + c - 2)_q}.$$

We can q -ify the derivation at the end of Section 2.2 to prove the q -analogue of equation (2):

$$N(a, b, c)_q = \frac{T(2, a + b + c - 1)_q T(2, a - 1)_q T(2, b - 1)_q T(2, c - 1)_q}{T(2, a + b - 1)_q T(2, a + c - 1)_q T(2, b + c - 1)_q}.$$

4. OTHER DETERMINANTAL SYMMETRY CLASSES

The method of Section 2.2 naturally extend to the enumeration of three other symmetry classes of plane partitions.

THEOREM 4.1. *A Kasteleyn-flat, weighted adjacency matrix for each of the graphs $Z_{\kappa\tau}(a, a, 2b)$, $Z_\rho(a, a, a)$, and $Z_{\rho, \kappa\tau}(2a, 2a, 2a)$ arises as $\alpha_{\kappa\tau}(X)|_{-1}$, $\alpha_\rho(X)|_{-1}$, and $\alpha_{\rho, \kappa\tau}(X)|_{-1}$ in representations $\alpha_{\kappa\tau}$, α_ρ , and $\alpha_{\rho, \kappa\tau}$ of $\mathfrak{sl}(2, \mathbb{C})$, with bases formed from weight bases of irreducible representations.*

Recall the weighting of $Z(a, a, 2b)$ induced by $\alpha(X)|_{-1}$ acting on $V = V_{a+2b-1} \otimes V_{a+2b-1} \otimes V_{2a-1}$. The symmetry $\kappa\tau$ preserves this weighting, and is induced by the linear transformation

$$L_{\kappa\tau} \in \text{End}(V_{a+2b-1} \otimes V_{a+2b-1} \otimes V_{2a-1})$$

given by

$$L_{\kappa\tau}(e_i \otimes e_j \otimes e_k) = e_j \otimes e_i \otimes e_k.$$

Define $V_{\kappa\tau}$ to be the eigenspace of $L_{\kappa\tau}$ with eigenvalue -1 , and choose vectors of the form $e_i \otimes e_j \otimes e_k - e_j \otimes e_i \otimes e_k$ with $i < j$ as a basis. Since $L_{\kappa\tau}$ commutes with the action of $\mathfrak{sl}(2, \mathbb{C})$, $V_{\kappa\tau}$ is also a representation.

Recall that $Z_{\kappa\tau}(a, a, 2b)$ is constructed by deleting vertices fixed by $\kappa\tau$ and edges incident to those vertices, and then taking one of the two isomorphic components of the complement. By inspection of the basis for $V_{\kappa\tau}$, the restriction $\alpha_{\kappa\tau}(X)|_{-1}$ of $\alpha(X)|_{-1}$ to $V_{\kappa\tau}$ is a weighted, bipartite adjacency matrix for $Z_{\kappa\tau}(a, a, 2b)$. As in the no-symmetry case, opposite edges of each hexagonal face have the same weight, and therefore it is Kasteleyn-flat. Indeed, the weighting of $Z_{\kappa\tau}(a, a, 2b)$ is induced from that of $Z(a, a, 2b)$. It follows that

$$N_{\kappa\tau}(a, a, 2b) = \det(\alpha_{\kappa\tau}(X)|_{-1})/m_{\kappa\tau}, \tag{8}$$

where $m_{\kappa\tau}$ is the value of a single term in the determinant (i.e., the weight of any matching in $Z_{\kappa\tau}(a, a, 2b)$).

To compute $m_{\kappa\tau}$, observe that a $\kappa\tau$ -invariant matching of $Z(a, a, 2b)$ has two edges of the same weight for each edge in the corresponding matching of $Z_{\kappa\tau}(a, a, 2b)$, plus all of the deleted edges. The weights of the deleted edges are $1, 3, 5, \dots, 2a - 1$. Therefore

$$m_{\kappa\tau} = \sqrt{|m|/(2a - 1)!!}, \tag{9}$$

where $n!! = (n - 1)(n - 3)\dots$ is the odd factorial. The determinant of $\alpha(X)|_{-1}$ is best computed using the D map of Section 2.2. The representation V has two basis vectors $e_i \otimes e_j \otimes e_k$ for each basis vector of $V_{\kappa\tau}$, plus basis vectors of the form $e_i \otimes e_i \otimes e_k$. Therefore

$$\text{ch}(V_{\kappa\tau}) = \frac{1}{2} \left(\text{ch}(V) - \sum_{i=1}^{a+2b} \sum_{k=1}^{2a} t^{4i+2k-4a-4b-3} \right).$$

Let $P(t) = \text{ch}(V) - 2 \text{ch}(V_{\kappa\tau})$ be the double summation; then

$$\begin{aligned} D(P(t)) &= \prod_{i=1}^{a+2b} \prod_{k=1}^{2a} \frac{\max(2i+k-2a-2b-1, 1)^2}{\max(2i+k-1-2a-2b-2, 1)^2} \\ &= \prod_{i=1}^{a+2b} \frac{\max(2i-2b-1, 1)^2}{\max(2i-2a-2b-2, 1)^2} = \frac{(2a+2b-1)!!^2}{4^{b-1}(b-1)!^2}. \end{aligned}$$

Therefore

$$D(\text{ch}(V_{\kappa\tau})) = \sqrt{\frac{4^{b-1}(b-1)!^2 D(\text{ch}(V))}{(2a+2b-1)!!^2}}.$$

Combining with equations (8) and (9),

$$N_{\kappa\tau}(a, a, 2b) = \sqrt{\frac{N(a, a, 2b) 2^{b-1}(b-1)! (2a-1)!!}{(2a+2b-1)!!}}$$

This is equivalent to standard formulas for $N_{\kappa\tau}(a, a, 2b)$ [10].

Similarly, the symmetry ρ on $Z(a, a, a)$ is induced by the linear transformation

$$L_\rho \in \text{End}(V_{2a-1} \otimes V_{2a-1} \otimes V_{2a-1})$$

given by

$$L_\rho(e_i \otimes e_j \otimes e_k) = e_j \otimes e_k \otimes e_i.$$

Define V_ρ to be the eigenspace of L_ρ with eigenvalue 1, and choose as a basis for V_ρ vectors of the form $e_i \otimes e_i \otimes e_i$ and vectors of the form

$$e_i \otimes e_j \otimes e_k + e_j \otimes e_k \otimes e_i + e_k \otimes e_i \otimes e_j$$

when $i, j,$ and k are not all equal. Since $Z_\rho(a, a, a)$ is the quotient graph $Z(a, a, a)/\rho$, the restriction $\alpha_\rho(X)|_{-1}$ of $\alpha(X)|_{-1}$ to V_ρ must be a weighted, bipartite adjacency matrix for $Z_\rho(a, a, a)$. Again, the weighting of $Z_\rho(a, a, a)$ induced from $Z(a, a, a)$ is compatible with $\alpha_\rho(X)|_{-1}$ and is Kasteleyn-flat. If m_ρ is the weight of any matching in $Z_\rho(a, a, a)$, then

$$m_\rho = \sqrt[3]{m}.$$

Meanwhile,

$$D(\text{ch}(V_\rho)) = \sqrt[3]{D(\text{ch}(V)) \prod_{i=1}^{2a} \frac{\max(6i-3a-1, 1)^4}{\max(6i-3a-2, 1)^4}}$$

because

$$\text{ch}(V_\rho)(t) = \frac{1}{3}(\text{ch}(V)(t) + 2 \text{ch}(V_{2a-1})(t^3))$$

by counting basis vectors. The final result is

$$N_\rho(a, a, a) = \sqrt[3]{N(a, a, a) \prod_{i=1}^a \frac{(3i-1)^2}{(3i-2)^2}}$$

This is again easily equivalent to the standard formulas [10].

In $V = V_{4a-1}^{\otimes 3}$, we define

$$V_{\kappa\tau, \rho} = V_{\kappa\tau} \cap V_\rho = \wedge^3 V_{2a-1},$$

together with the preferred basis of vectors $e_i \wedge e_j \wedge e_k$ with $i < j < k$. (Recall that the wedge product is the anti-symmetrized tensor product.) In this basis, $\alpha_{\kappa\tau, \rho}(X)|_{-1}$ is a Kasteleyn-flat matrix for $Z_{\rho, \kappa\tau}(2a, 2a, 2a)$. A product formula for $N_{\rho, \kappa\tau}(2a, 2a, 2a)$ is the inevitable consequence, and in this final case, we omit the details.

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