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Commutative Nonassociative Algebras and Cubic Forms

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The purpose of this note is to establish a close relationship between the two algebraic concepts of the title. F will be a field which for simplicity we take to be algebraically closed and of characteristic zero (e.g., the complex numbers). All algebras considered will be commutative, not necessarily associative, with an identity, and finite dimensional over F; we will use the unrestricted term "algebra" in this sense.

An algebra is simple if it has exactly two ideals, and semi-simple if it is a direct sum of simple algebras. The Chinese Remainder Theorem is easily checked to hold, so we let the radical of an algebra be the intersection of its maximal ideals, and have that an algebra has radical zero if and only if it is semi-simple. Let A be an algebra. For $a, b, c \in A$ we write $[a, b, c]$ for the associator $(a \cdot b) \cdot c - a \cdot (b \cdot c)$, and $[A, A, A]$ for the subspace of A spanned by all the associators. Then A is associative if and only if $[A, A, A] = \{0\}$. If the other extreme holds—namely $[A, A, A] = A$—we call A *anti-associative*. If A is simple and associative, no proper square roots of zero exist in the sense that $a \in A$, $a^2 = 0$ imply $a = 0$. In general, we call A *regular simple* if it is simple, it is not anticommutative, and it has no proper square roots of zero. We call an algebra *regular semi-simple* if it is a direct sum of regular simple algebras. It is these algebras we wish to characterize, or at least show that their theory is equivalent to the theory of nonsingular cubic forms over F. Nonsingular cubic forms in three or less variables are of course classically known (X_1^3, $X_1^3 + X_2^3$, $X_2^2X_3 - X_1(X_1 - X_3)(X_1 - \lambda X_3)$, where $\lambda \in F$, $\lambda \neq 0, 1$), but in larger numbers of variables their theory is of course very extensive and very far from complete.

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1. Cubic Forms

By a cubic form we mean a homogeneous polynomial of degree three in some given number of variables over F. Two cubic forms $f(X_1, \dots, X_n)$ and $g(Y_1, \dots, Y_m)$ are called *equivalent*, and we write $f \sim g$, if g can be gotten from f by a linear reversible change of variables; i.e., if $n = m$ and there is an invertible n by n matrix $[\alpha_{i,j}]$ such that if $\sum_i a_{i,j} Y_i = X_j$ for $j = 1, \dots, n$, then $f(X_1, \dots, X_n) = g(Y_1, \dots, Y_n)$. The cubic form $f(X_1, \dots, X_n)$ is called nonsingular if the projective variety defined by f has no singular points; this is equivalent to the fact that the partials $\partial f/\partial X_1, \dots, \partial f/\partial X_n$ do not have a common nontrivial zero in F. Any form equivalent to a nonsingular form is nonsingular. Two forms are viewed as being essentially the same (i.e., isomorphic) if they are equivalent; for this reason it is easiest in anything but final computations to deal with cubic spaces. By a cubic space we mean a pair (V, θ), where V is a finite dimensional vector space over F and θ is a trilinear symmetric map from $V \times V \times V$ to F. Two cubic spaces (V, θ) and (V', θ') are called isomorphic if there is a bijective linear map t from V to V' with

$$\theta'(t(v_1), t(v_2), t(v_3)) = \theta(v_1, v_2, v_3),$$

for all $v_1, v_2, v_3 \in V$. If (V, θ) is a cubic space and $a_1, a_2, \dots, a_n$ is a basis of V, then

$$f = \sum_i \sum_j \sum_k \theta(a_i, a_j, a_k) X_i X_j X_k$$

is a cubic form in n variables. If this formula holds for some basis of V we say f is *associated* to (V, θ). If (V', θ') is a cubic space and g is a cubic form which is associated to (V', θ'), then f is equivalent to g if and only if (V, θ) is isomorphic to (V', θ'). Also every cubic form is associated to some cubic space. Hence rather than working with cubic forms, up to equivalence, we can work with cubic spaces, up to isomorphism. One checks that a form associated to (V, θ) is nonsingular if and only if for each nonzero $u \in V$ there exists a $v \in V$ with $\theta(u, u, v) \neq 0$. If this is true we call (V, θ) nonsingular.

Proposition 1.1. *Let (V, θ) be a nonsingular cubic space over F. Then there exists a u in V such that for each nonzero v in V there is a w in V with $\theta(u, v, w) \neq 0$.*

Proof. Let n be the dimension of V over F. Let $y_1, y_2, \dots, y_n$ be indeterminants over F and let K be an algebraic closure of $F(y_1, y_2, \dots, y_n)$. Let $f = f(X_1, X_2, \dots, X_n)$ be a cubic form over F associated to (V, θ). Then f is nonsingular, so $\partial f/\partial X_1, \dots, \partial f/\partial X_n$ do not have a common nontrivial zero in F. Hence by Hilbert's Nullstellensatz they do not have a common nontrivial zero in K. Thus f is still nonsingular when considered as a polynomial with coefficients in K. This polynomial is associated to some cubic space over K; one checks that $(K \otimes_F V, \theta_K)$ is such a space, where θ_K is the unique map from $(K \otimes_F V) \times (K \otimes_F V) \times (K \otimes_F V)$ to K, where

$$\theta_K(a_1 \otimes v_1, a_2 \otimes v_2, a_3 \otimes v_3) = a_1 a_2 a_3 \theta(v_1, v_2, v_3)$$

for all $a_1, a_2, a_3 \in K$ and $v_1, v_2, v_3 \in V$. Hence, we have that $(K \otimes_F V, \theta_K)$ is a nonsingular cubic space over K.

Let i be an integer from 1 to n. There is a unique F-linear derivation D_i of $F(y_1, \ldots, y_n)$ with $D_i(y_j) = \delta_{i,j}$ for $j = 1, \ldots, n$, where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise. Since K is of characteristic zero, it is a separable algebraic field extension of $F(y_1, \ldots, y_n)$ so D_i can be extended to a unique derivation, which we also call D_i, of K with $D_i(y_j) = \delta_{i,j}$ for all j. Now choose a basis $v_1, v_2, \ldots, v_n$ of V over F and define a map B_i from $K \otimes_F V$ to itself by

$$B_i(a_1 \otimes v_1 + \cdots + a_n \otimes v_n) = D_i(a_1) \otimes v_1 + \cdots + D_i(a_n) \otimes v_n,$$

for all $a_1, a_2, \ldots, a_n \in K$. One can check that this is well defined and is independent of the choice of basis.

LEMMA 1.2. *For all* $w_1, w_2, w_3 \in K \otimes_F V$,

$$D_i(\theta_K(w_1, w_2, w_3)) = \theta_K(B_i(w_1), w_2, w_3) + \theta_K(w_1, B_i(w_2), w_3) + \theta_K(w_1, w_2, B_i(w_3))$$

Proof. This is easily checked directly.

Now let $z = y_1 \otimes a_1 + y_2 \otimes a_2 + \cdots + y_n \otimes a_n$. Note that $B_i(z) = 1 \otimes a_i$.

LEMMA 1.3. *For each* $v \in K \otimes_F V$ *there exists a* $w \in K \otimes_F V$ *with* $\theta_K(z, v, w) \neq 0$.

Proof. Just suppose $v \in K \otimes_F V$ with $\theta_K(z, v, w) = 0$ for all $w \in K \otimes_F V$. Then

$$0 = D_i(0) = D_i(\theta_K(z, v, v))$$

and by the last lemma this is

$$\theta_K(B_i(z), v, v) + \theta_K(z, B_i(v), v) + \theta_K(z, v, B_i(v)),$$

which by the choice of v is $\theta_K(B_i(z), v, v)$. But $B_i(z) = 1 \otimes a_i$ so $0 = \theta_K(1 \otimes a_i, v, v)$ for $i = 1, \ldots, n$. Since $1 \otimes a_1, \ldots, 1 \otimes a_n$ is a basis of $K \otimes_F V$ this gives that $\theta_K(c, v, v) = 0$ for all $c \in K \otimes_F V$. But we proved $(K \otimes_F V, \theta_K)$ is nonsingular, and thus $v = 0$. This proves the lemma.

As a corollary to the lemma we have

$$0 \neq \det([\theta_K(z, 1 \otimes a_i , 1 \otimes a_j]).$$

Since $\theta_K(z, 1 \otimes a_i , 1 \otimes a_j) = \sum_s y_s \theta(a_s , a_i , a_j)$, this nonzero determinant is a homogenious polynomial of $g(y_1 ,..., y_n)$ over F of degree n. Thus there exist $\alpha_1 , \alpha_2 ,..., \alpha_n \in F$ with $g(\alpha_1 , \alpha_2 ,..., \alpha_n) \neq 0$; i.e., with

$$\det\left(\left[\sum_s \alpha_s \theta(a_s , a_i , a_j)\right]\right) \neq 0.$$

Hence letting $u = \sum_s \alpha_s a_s$, $\det([\theta(u, a_i , a_j)]) \neq 0$ which means for each nonzero v in V there is a w in V with $\theta(u, v, w) \neq 0$. Proposition 1.1 is proven.

This proposition 1.1 with its restriction to nonsingular forms is very weak; actually one has to work hard to find any nontrivial cubic form which does not satisfy the conclusion of proposition 1.1. We say a cubic space (V, θ) *has nontrivial Hessian* if the conclusion of Proposition 1.1 holds (i.e., there exists a u in V such that $v \in V$ and $\theta(u, v, w) = 0$ for all $w \in V$ imply $v = 0$). Of course a degenerate cubic space (V, θ) (one where there exists a $v_1 \neq 0$ in V with $\theta(v_1 , v_2 , v_3) = 0$ for all $v_2 , v_3 \in V$) does not have a nontrivial Hessian, but these are trivial because every cubic space is uniquely a direct sum of a zero cubic space and a nondegenerate one. William Adkins has shown that

$$X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_1X_2X_5 + X_1X_3X_6 + X_1X_4X_7 + X_2X_3X_8 + X_2X_4X_9$$

does not have a nontrivial Hessian and is nondegenerate.

PROPOSITION 1.4. *Let (V, θ) be a cubic space which has a nontrivial Hessian (e.g., is nonsingular). Let $u \in V$ be such that $v \in V$ and $\theta(u, v, w) = 0$ for all $w \in V$ imply $v = 0$. Then there exists a unique multiplication on V which makes V into an algebra with u as identity and which satisfies*

$$\theta(a, b, c) = \theta(u, u, (a \cdot b) \cdot c)$$

for all a, b, c, in V. We denote this algebra by $V_{(u)}$.

Proof. For $v \in V$ define $h_v \in \mathrm{Hom}_F(V, F)$ by $h_v(w) = \theta(u, v, w)$ for all $w \in V$. One checks that $v \mapsto h_v$ is an injective linear map from V to $\mathrm{Hom}_F(V, F)$. Since these two spaces have the same dimension this map is actually bijective. For a, $b \in V$ we define a functional f on V by $f(w) = \theta(a, b, w)$ for all $w \in V$. Hence there exists a unique v in V with $h_v = f$. We denote v by $a \cdot b$. Then $a \cdot b$ is that unique element in V with

$$\theta(a, b, w) = \theta(u, a \cdot b, w)$$

for all $w \in V$. With this one checks that multiplication is bilinear, commutative, has u for an identity, and satisfies $\theta(u, u, (a \cdot b) \cdot c) = \theta(a, b, c)$ for all $a, b, c \in V$. Conversely, suppose a new multiplication is given with these properties. Then

$$\theta(u, a \cdot b, c) = \theta(u, u, (u \cdot (a \cdot b)) \cdot c) = \theta(u, u, (a \cdot b) \cdot c) = \theta(a, b, c)$$

for all $a, b, c \in V$, and since this is the relation used to define the first multiplication, the two multiplications are equal.

Note 1.5. Let the notation be as in Proposition 1.4. For $a, b \in V$ let $\varphi(a, b)$ be $\theta(u, a, b)$. Then φ is a nondegenerate inner product on V which satisfies $\varphi(a \cdot b, c) = \varphi(a, b \cdot c)$ and $\varphi(a \cdot b, c) = \theta(a, b, c)$ for all a, b, c in V.

Proof.

$$\begin{aligned}\varphi(a \cdot b, c) &= \theta(u, a \cdot b, c) = \theta(u, u, (u \cdot (a \cdot b)) \cdot c)\\ &= \theta(u, u, (a \cdot b) \cdot c) = \theta(a, b, c) = \theta(b, c, a) = \theta(u, u, (b \cdot c) \cdot a)\\ &= \theta(u, u, (u \cdot (b \cdot c)) \cdot a) = \theta(u, b \cdot c, a) = \theta(u, a, b \cdot c) = \varphi(a, b \cdot c).\end{aligned}$$

That φ is a nondegenerate is immediate from the properties of u.

PROPOSITION 1.6. *Let (V, θ) be a nonsingular cubic space. Let $u \in V$ be such that for each nonzero $v \in V$ there is a $w \in V$ with $\theta(u, v, w) \neq 0$. Make V into an algebra by giving it the unique multiplication which has u for an identity and satisfies $\theta(a, b, c) = \theta(u, u, (a \cdot b) \cdot c)$ for all $a, b, c \in V$. Then V is a regular semi-simple algebra.*

Proof. Let $a \in V$ with $a \cdot a = 0$. Then for all $c \in V$, $\theta(a, a, c) = \theta(u, u, (a \cdot a) \cdot c) = 0$ so $a = 0$ since (V, θ) is nonsingular. This proves no proper square roots of zero exist. Now let I be a mininal nonzero ideal of V. Using the inner product φ of note 1.5, let $I^{\perp}$ be the set of all $c \in V$ such that $\varphi(b, c) = 0$ for all $b \in I$. Then $I \cap I^{\perp} = \{0\}$ for if a were an element in this intersection, for all $v \in V$ we would have $\varphi(a \cdot a, v) = \varphi(a, a \cdot v) = 0$ (and thus $a \cdot a = 0$) since $a \cdot v \in I$. Thus V is the direct sum of I and $I^{\perp}$. One checks directly that $I^{\perp}$ is an ideal, and thus V is the direct sum of I and $I^{\perp}$ as algebras. One can check that I is simple, and by reapplying this procedure to $I^{\perp}$ one can get that V is the direct sum of simple algebras. We now show any one of these simple algebras, say I, is not anti-commutative. Just suppose I were anti-commutative. Then for $a \in I$, $b \in V$, $a \cdot b \in I$ so $a \cdot b$ can be written as a linear combination of associators of elements in I. But for $x, y, z \in I$.

$$\begin{aligned}\theta(u, u, x \cdot (y \cdot z)) &= \theta(u, x, y \cdot z) = \theta(u, y \cdot z, x) = \theta(y, z, x)\\ &= \theta(x, y, z) = \theta(u, u, (x \cdot y) \cdot z)\end{aligned}$$

so $\theta(u, u, [x, y, z]) = 0$. Since $a \cdot b$ is a linear combination of such associators, $0 = \theta(u, u, a \cdot b) = \theta(u, a, b)$. This is true for all $b \in V$. Thus $a = 0$. But a was any element in I. This contradiction proves the proposition.

2. Algebras

Let A be an algebra over F. We call $u \in A$ a *unit* if for each $b \in A$, $u \cdot x = b$ has a solution x in A. Since A is finite dimensional one can check that this is equivalent to u being a nonzero-divisor (i.e., $u \cdot c \neq 0$ for all nonzero c in A). Let u be a unit. One can check that for each $b \in A$ there exists exactly one $x \in A$ with $u \cdot x = b$. Now let $a, b \in A$. Then $a \cdot b \in A$ so there is a unique $c \in A$ with $u \cdot c = a \cdot b$. We denote c by $a * b$. This defines a new multiplication $*$ on A. One checks that A with $*$ is an algebra; we denote this algebra by $A_{(u)}$ and call it a *variant* of A. This new multiplication is characterized in terms of the old one by $u \cdot (a * b) = a \cdot b$ for all $a, b \in A$. One checks that u is the identity of $A_{(u)}$. Hence for each unit there is a unique variant of A with that unit as an identity. We take the point of view that if we really know an algebra then we can compute all its variants; if we want to make a list of all algebras then it would be least cumbersome to list only one algebra from each variant class, rather than to include with each algebra all its variants. If A and B are algebras, we call B *variant-isomorphic* to A and write $B \sim A$ if B is isomorphic to a variant of A. One can check that a variant of a variant is a variant, each algebra is a variant of each of its variants, and the variant with the identity as identity is the algebra itself. Hence variant isomorphism is an equivalence relation. For associative algebras being variant-isomorphic and being isomorphic are equivalent.

One quickly checks that a unit of an algebra is a unit of every variant, an ideal of an algebra is an ideal of every variant, the radical of an algebra is the radical of every variant, a proper square root of zero of an algebra is a proper square root of zero in every variant, a variant of a direct sum of algebras is a direct sum of corresponding variants, a tensor product of variant is a variant of the tensor product of the corresponding algebras, a variant of a factor algebra is a factor algebra of a variant, a variant of a simple algebra is simple, a variant of an associative (respectively antiassociative) algebra is associative (respectively antiassociative), etc. In particular, each variant of a regular semi-simple algebra is regular semi-simple. In the list we will give of the regular semi-simple algebras, we will give only one from each variant-isomorphism class (there are still way too many to make the listing anything more than a shifting of the problem to more familiar ground).

Proposition 2.1. *Let (V, θ) and (V', θ') be cubic spaces which have nontrivial Hessians (e.g., are nonsingular). Choose a $u \in V$ such that for each nonzero*

v in V there is a $w \in V$ with $\theta(u, v, w) \neq 0$. Choose a $u' \in V'$ such that for each nonzero v' in V' there is a $w' \in V'$ with $\theta'(u', v', w') \neq 0$. Let A be the algebra $V_{(u)}$ and B be the algebra $V'_{(u')}$ (see Proposition 1.4). *Then A is variant-isomorphic to B if and only if (V, θ) is isomorphic to (V', θ')* (i.e., *if and only if a cubic form associated to (V, θ) is equivalent to a cubic form* (*or equivalently all cubic forms*) *which is* (*or are*) *associated to (V', θ')*). *In particular, the variant-isomorphism class of A is independent of the choice of u.*

Proof. First suppose (V', θ') is isomorphic to (V, θ) by a bijective linear map t from V to V'. Let $a, b \in V$. Then for any $x \in V$

$$\begin{aligned}
&\theta(u, a \cdot b, x) \\
&\quad = \theta(u, u, (u \cdot (a \cdot b)) \cdot x) = \theta(u, u, (a \cdot b) \cdot x) = \theta(a, b, x) \\
&\quad = \theta'(t(a), t(b), t(x)) = \theta'(u', u', (t(a) \cdot t(b)) \cdot t(x)) \\
&\quad = \theta'(u', u', (u' \cdot (t(a) \cdot t(b))) \cdot t(x)) = \theta'(u', t(a) \cdot t(b), t(x)) \\
&\quad = \theta'(t(t^{-1}(u')), t(t^{-1}(t(a) \cdot t(b))), t(x)) = \theta(t^{-1}(u'), t^{-1}(t(a) \cdot t(b)), x) \\
&\quad = \theta(u, u, (t^{-1}(u') \cdot t^{-1}(t(a) \cdot t(b))) \cdot x) = \theta(u, t^{-1}(u') \cdot t^{-1}(t(a) \cdot t(b)), x).
\end{aligned}$$

Hence $a \cdot b = t^{-1}(u') \cdot t^{-1}(t(a) \cdot t(b))$. If $t^{-1}(u')$ is a unit, then writing $*$ for the multiplication of the variant of A with $t^{-1}(u')$ as identity, the above gives $a * b = t^{-1}(t(a) \cdot t(b))$ or $t(a * b) = t(a) \cdot t(b)$ and we are done. We prove $t^{-1}(u')$ is a unit by supposing $t^{-1}(u') \cdot c = 0$ with $c \in A$. For each $y' \in V'$

$$0 = \theta(u, u, (t^{-1}(u') \cdot c) \cdot t^{-1}(y')) = \theta(t^{-1}(u'), c, t^{-1}(y')) = \theta'(u', t(c), y').$$

Hence $t(c) = 0$, so $c = 0$.

Now conversely suppose A is isomorphic to a variant of B. We will need some lemmas. An algebra D is *indecomposable* if it is not a direct sum of two nonzero algebras. The *center* $Z(D)$ of an algebra D is the set of all $z \in D$ such that $z \cdot (x \cdot y) = (z \cdot x) \cdot y$ for all $x, y \in D$; this center is an associative subalgebra of D.

LEMMA 2.2. *Let x be a unit of an associative algebra D. Then there is a unit y in D with $x = y^3$.*

Proof. D is a direct sum of indecomposable algebras $D_1, \ldots, D_m$. If $x = x_1 + \cdots + x_m$ with $x_i \in D_i$ and $x_i = y_i^3$ for $y_i \in D_i$, $i = 1, \ldots, m$, then $x = (y_1 + \cdots + y_m)^3$. Hence without loss of generality we may assume D is indecomposable. But D is associative, so idempotents can be lifted, so D/I is indecomposable where I is the radical of D. Since D/I is semi-simple, it must be a field, and since F is algebraically closed $D = F \cdot 1 + I$. I is nilpotent so there is a smallest positive integer s with $I^s = 0$. If $s = 1$, $D = F1$ and the lemma is true since F is algebraically closed. Now let $J = I^{s-1}$ and suppose the lemma is true with D replaced by D/J. Then there is a unit $y + J$ in D/J with $x + J = (y + J)^3$ in D/J. Using $J \cdot J = 0$, it is easily checked that y is a unit in D. There is a $b \in J$ with $x = y^3 + b$. $(y^{-2} \cdot b)^2 = 0$ since $J \cdot J = 0$. Hence $(y + 3^{-1}y^{-2} \cdot b)^3 = y^3 + b = x$. The lemma follows by induction on s.

Now let e be the unit of B such that A is isomorphic to the variant of B which has e as identity. Let $*$ be the multiplication of this variant so that $e \cdot (b * b') = b \cdot b'$ for all $b, b' \in B$. Let t be a bijective linear map from A to B with $t(a \cdot a') = t(a) * t(a')$ for all $a, a' \in A$. t must preserve the identities so $t(1) = e$. Multiplying by e we get $t(1) \cdot t(a \cdot t(a \cdot a') = t(a) \cdot t(a')$. Let Γ be the linear map from A to F where $\Gamma(a) = \theta'(t(1), t(1), t(a))$ for all $a \in A$. For $y \in A$ let f_y be the linear map where $f_y(a) = \theta(1, y, a)$ for $a \in A$. Then $y \mapsto f_y$ is an injective linear map from A to $\mathrm{Hom}_F(A, F)$. This map must be surjective since $\mathrm{Hom}_F(A, F)$ has the same dimension as A. Thus there is a unique $d \in A$ with $f_d = \Gamma$, i.e., with $\theta(1, d, a) = \theta'(t(1), t(1), t(a))$ for all $a \in A$. Let x, y be any elements in A. We use the notation of Note 1.5 and write φ and φ' for the inner products corresponding to θ and θ', respectively. For any z in A.

$$\begin{aligned}
&\theta(1, (d \cdot x) \cdot y, z)\\
&\quad = \varphi((d \cdot x) \cdot y, z) = \varphi(d \cdot x, y \cdot z) = \varphi(d, x \cdot (y \cdot z))\\
&\quad = \theta(1, d, x \cdot (y \cdot z)) = \theta'(t(1), t(1), t(x \cdot (y \cdot z)))\\
&\quad = \theta(t(1), t(x \cdot (y \cdot z)), t(1))\\
&\quad = \theta'(1, 1, (t(1) \cdot t(x \cdot (y \cdot z))) \cdot t(1)) = \theta'(1, 1, (t(x) \cdot t(y \cdot z)) \cdot t(1))\\
&\quad = \theta'(t(x), t(y \cdot z), t(1)) = \theta'(t(1), t(y \cdot z), t(x))\\
&\quad = \theta'(1, 1, (t(1) \cdot t(y \cdot z)) \cdot t(x)) = \theta'(1, 1, t(y) \cdot t(z)) \cdot t(x))\\
&\quad = \theta'(t(y), t(z), t(x)) = \theta'(t(x), t(y), t(z)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\theta(1, d \cdot (x \cdot y), z)\\
&\quad = \varphi(d \cdot (x \cdot y), z) = \varphi(d, (x \cdot y) \cdot z) = \theta(1, d, (x \cdot y) \cdot z)\\
&\quad = \theta'(t(1), t(1), t((x \cdot y) \cdot z)) = \theta'(t(1), t((x \cdot y) \cdot z), t(1))\\
&\quad = \theta'(1, 1, (t(1) \cdot t(x \cdot y) \cdot z)) \cdot t(1)) = \theta'(1, 1, t(x \cdot y) \cdot t(z)) \cdot t(1))\\
&\quad = \theta'(t(1), t(x \cdot y), t(z)) = \theta'(1, 1, (t(1) \cdot t(x \cdot y)) \cdot t(z))\\
&\quad = \theta'(1, 1, (t(x) \cdot t(y)) \cdot t(z)) = \theta'(t(x), t(y), t(z)).
\end{aligned}$$

Thus $\theta(1, (d \cdot x) \cdot y - d \cdot (x \cdot y), z) = 0$, and since this is true for all $z \in A$,

$(d \cdot x) \cdot y - d \cdot (x \cdot y) = 0$. Hence d is in the associative algebra $Z(A)$. d is a nonzero-divisor in $Z(A)$, for if $b \in Z(A)$ and $d \cdot b = 0$, then for any $w \in B$

$$\begin{aligned} 0 &= \varphi(d \cdot b, t^{-1}(w)) = \varphi(d, b \cdot t^{-1}(w)) = \theta'(t(1), t(1), t(b \cdot t^{-1}(w))) \\ &= \theta'(t(1), t(b \cdot t^{-1}(w)), t(1)) = \theta'(1, 1, (t(1) \cdot t(b \cdot t^{-1}(w))) \cdot t(1)) \\ &= \theta'(1, 1, (t(b) \cdot w) \cdot t(1)) = \theta'(1, t(b) \cdot w, t(1)) = \varphi'(t(b) \cdot w, t(1)) \\ &= \varphi'(w \cdot t(b), t(1)) = \varphi'(w, t(b) \cdot t(1)), \end{aligned}$$

so $t(b) \cdot t(1) = 0$, and since $t(1) = e$ is a unit, $t(b) = 0$ and thus $b = 0$. Since d is a nonzero-divisor, it is a unit so by Lemma 2.2 there is a unit c in $Z(A)$ with $c^3 = d$. Define a map f from B to A by $f(x') = c \cdot t^{-1}(x')$ for all $x' \in B$. f is linear, and since t^{-1} is injective and c is a unit, f is injective and thus bijective. For any x, y, z in A, we proved above $\theta'(t(x), t(y), t(z)) = \theta(1, d \cdot (x \cdot y), z)$. But

$$\begin{aligned} \theta(1, d \cdot (x \cdot y), z) &= \theta(1, 1, (c^3 \cdot (x \cdot y) \cdot z) = \theta(1, 1, ((c \cdot x) \cdot (c \cdot y)) \cdot (c \cdot z)) \\ &= \theta(c \cdot x, c \cdot y, c \cdot z), \end{aligned}$$

since c is in $Z(A)$. Thus for any x', y', $z' \in B$, we let $x = t^{-1}(x')$, $y = t^{-1}(y')$, $z = t^{-1}(z')$ and have

$$\theta'(x', y', z') = \theta(c \cdot x, c \cdot y. c \cdot z) = \theta(f(x'), f(y'), f(z')).$$

This completes the proof of Proposition 2.1.

We call an algebra A *admissible* if there exists a nonsingular inner product φ on A which is multiplicative in the sense that $\varphi(a \cdot b, c) = \varphi(a, b \cdot c)$ for all a, b, c in A. One checks that a variant of an admissible algebra is admissible.

COROLLARY 2.3. *Let A be an algebra. Then there exists a cubic space (V, θ) with nontrivial Hessian and a $u \in V$ which satisfies the hypothesis of Proposition* 1.4, *such that $V_{(u)}$ is isomorphic to A if, and only if, A is admissible. Hence there is a natural one-one correspondence between equivalence classes of cubic forms with nontrivial Hessian and variant-isomorphism classes of admissible algebras.*

What little of this corollary is not covered by Note 1.5 and Proposition 2.1 is easily checked.

An associative algebra which is a direct sum of algebras, each of which has exactly one maximal ideal and one minimal ideal is called a Frobenius algebra. One can check that these are exactly the associative admissible algebras. By the above result, isomorphism classes of Frobenius algebras correspond bijectively with a class of equivalence classes of cubic forms (which will have as singularities all elements whose squares are zero in the corresponding algebras). I think most ring theorists would say that any attempt to classify (commutative) Frobenius algebras over F is unlikely because they exist in such abundance; this shows the complexity of highly singular cubic forms which exist.

PROPOSITION 2.4. *Let A be a regular semi-simple algebra. Then there exists a nonsingular cubic space (V, θ) (unique up to isomorphism) and a $u \in V$ such that for each nonzero $v \in V$ there is a $w \in V$ with $\theta(u, v, w) \neq 0$, with $V_{(u)}$ isomorphic to A. Hence there is a natural one-one correspondence between equivalence classes of nonsingular cubic forms and variant isomorphism classes of regular semi-simple algebras,*

Proof. Let A be a direct sum of the regular simple algebras $A_1, \ldots, A_m$. For $i = 1, \ldots, m$, $A_i \neq [A_i, A_i, A_i]$ so there is a nonzero linear map f_i from A_i to F which has $[A_i, A_i, A_i]$ in its kernel. We define $\varphi_i(a_i, b_i)$ to be $f_i(a_i \cdot b_i)$ for all $a_i, b_i \in A_i$, and quickly check that φ_i is a multiplicative nonzero inner product on A_i. Letting $I_i = \{a_i \in A_i \mid \varphi_i(a_i, b_i) = 0$ for all $b_i \in A_i\}$, we check that I_i is a proper ideal of A_i and thus is zero. Hence φ_i is a nonsingular inner product on A_i. Now for $a_i, b_i \in A_i$ we define

$$\varphi(a_1 + \cdots + a_m, b_1 + \cdots + b_m) = \varphi_1(a_1, b_1) + \cdots + \varphi_m(a_m, b_m)$$

and check that φ is a nonsingular inner product on A. For $a, b, c \in A$ we define $\theta(a, b, c)$ to be $\varphi(a \cdot b, c)$, and check that (A, θ) is a cubic space. $1 \in A$ and for each nonzero $b \in A$ there is a c in A with $\theta(1, b, c) = \varphi(b, c) \neq 0$. For $x, y, z \in A$, $\theta(x, y, z) = \varphi(x \cdot y, z) = \varphi(1 \cdot (x \cdot y), z) = \varphi(1, (x \cdot y) \cdot z) = \theta(1, 1, (x \cdot y) \cdot z)$ so $A_{(1)} = A$. If a is a nonzero-element in A, then $a^2 \neq 0$ so there is a b with $0 \neq \varphi(a^2, b) = \theta(a, a, b)$. Thus (A, θ) is nonsingular.

Note 2.5. The algebra can be computed explicitly from the form $f(X_1, \ldots, X_n)$ as follows. First it is necessary to find a point

$$a = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in F^n$$

such that $(1/3)(\sum \alpha_i \, \partial f/\partial X_i) = g(X_1, \ldots, X_n)$ is nonsingular (this can be done by computing the determinant of the coefficient matrix of g). Then for $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$, multiplication is given by linearity and that for $i, j = 1, \ldots, n$, $e_i \cdot e_j = \beta_1 e_1 + \cdots + \beta_n e_n$, where $(1/3!)(\partial^2 f/\partial X_i \partial X_j) = (1/2)(\sum \beta_k \, \partial g/\partial X_k)$. For instance if $f = X_2{}^2 X_3 - X_1(X_1 - X_3)(X_1 - \lambda X_3)$, $\lambda \neq 0, 1$, then we can let $a = e_3$ and get

$$e_1 \cdot e_1 = -(1 + 1/\lambda)\, e_1 + (-(1 + 1/\lambda)^2 + 3/\lambda)\, e_3,$$

$e_1 \cdot e_2 = 0$, $e_1 \cdot e_3 = e_1$, $e_2 \cdot e_1 = 0$, $e_2 \cdot e_2 = (-1/\lambda)\, e_1 - (1/\lambda)(1 + 1/\lambda)\, e_3$, $e_2 \cdot e_3 = e_2$, $e_3 \cdot e_1 = e_1$, $e_3 \cdot e_2 = e_2$, $e_3 \cdot e_3 = e_3$.

References

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2. Y. I. Manin, "Cubic Forms," North-Holland, Amsterdam, 1973.