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Total Blow-Up versus Single Point Blow-Up

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1. INTRODUCTION

Traditional thermal explosion theory is used to describe reaction initiation in condensed explosives and is limited formally to nondeformable materials. Kassoy and Poland [4] significantly extended this theory to develop an ignition model for a reactive gas in a bounded container in order to describe the induction period. During this induction period there is a spatially homogeneous pressure rise in the system which causes a compressive heating effect in the constant volume container. Mathematically this compressibility of the gas is expressed by means of an integral term in the induction model for the temperature perturbation $\theta(x, t)$. This model is given by

$$\begin{aligned} \theta_t - \Delta\theta &= \delta e^{\theta} + \frac{\gamma-1}{\gamma}\cdot\frac{1}{\operatorname{vol}\Omega}\int_{\Omega}\theta_t(x,t)\,dx \\ \theta(x,0) &= \phi(x) \geqslant 0, \qquad x \in \Omega \\ \theta(x,t) &= 0,\ x\in\partial\Omega, \qquad t \geqslant 0, \end{aligned} \tag{D}$$

and (D) motivates this paper.

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In an earlier paper, Bebernes and Bressan [2] analyzed this ignition model (D) for a compressible reactive gas and proved the following using semigroup techniques. For any positive value of the Frank–Kamenetski parameter δ and any value of the gas constant $\gamma \geqslant 1$, (D) has a unique classical solution $\theta(x, t)$ on $\Omega \times [0, T)$ where Ω is an arbitrary bounded container and $T = +\infty$ or $T < +\infty$. In the latter case, $\theta(x, t)$ blows up as t approaches T. If $\delta > \delta_{\text{FK}}$, the Frank–Kamenetski critical value, then $T < \infty$ and blow-up or thermal runaway occurs in finite time.

The purpose of this paper is to describe where blow-up occurs in the container Ω for a more general problem

$$\begin{aligned} &u_t - \Delta u = f(u) + g(t) \\ &u(x, 0) = \phi(x), \qquad x \in \Omega \\ &u(x, t) = 0, \qquad x \in \partial\Omega,\ t > 0. \end{aligned} \tag{G}$$

2. Statement of Problem

Consider

$$u_t - \Delta u = f(u) + g(t) \tag{2.1}$$

$$u(x, 0) = \phi(x), \qquad x \in \Omega, \tag{2.2}$$

$$u(x, t) = 0, \qquad x \in \partial\Omega,\ t > 0,$$

where we assume throughout the paper that

$$\begin{aligned} &\Omega = B_R = \{x \in \mathbb{R}^n : |x| < R\}, \\ &\phi \in C^2(\bar{\Omega}),\ \phi = 0 \text{ on } \partial\Omega,\ \phi \geqslant 0,\ \Delta\phi + f(\phi) \geqslant 0, \\ &\phi \text{ is radially symmetric and radially decreasing on } \Omega, \\ &f \in C',\ f(u) \geqslant 0 \text{ for } u \geqslant 0, \\ &g \in C,\ g(t) \geqslant 0 \text{ on its domain of existence, or} \\ &g(t) = \frac{K}{\operatorname{vol}\Omega} \int_\Omega u_t(x, t)\, dx \text{ with } 0 < K < 1. \end{aligned} \tag{2.3}$$

Then the following facts are known [2]:

(i) a unique solution $u(x, t)$ of (2.1)–(2.2) exists on $\Omega \times [0, \sigma)$, $\sigma > 0$ sufficiently small;

(ii) $u(x, t) \geqslant 0$;

(ii′) if $g'(t) \geqslant 0$ or if $g(t) = (K/\operatorname{vol}\Omega) \int_\Omega u_t(x, t)\, dx$ with $0 < K < 1$, then $u_t \geqslant 0$ and $u_0(t) = \max_\Omega u(x, t)$ is increasing;

(iii) if $u(x, t)$ exists on $0 \leqslant t < \sigma < \infty$ and $u_0(\sigma^-) < \infty$, then the solution u can be uniquely extended to $0 < t < \sigma + \varepsilon$, $\varepsilon > 0$.

Let $T \equiv \sup\{\sigma : u(x, t) \text{ exists on } 0 \leqslant t < \sigma\}$.

(iv) $u(x, t)$ is radially symmetric and $u(\cdot, t)$ is radially decreasing for $t \in [0, T)$.

(v) If $T < \infty$, then $u_0(T^-) = \max_{\bar{\Omega}} u(x, T^-) = +\infty$.

We always will assume that $R > 0$ and $\phi(x) \geqslant 0$ are such that $T < \infty$; then $u(x, t)$ blows up in finite time. When $g(t) = (K/\text{vol}\,\Omega) \int_\Omega u_t(x, t)\, dx$, we always must assume $0 \leqslant K < 1$ to assure local existence for (2.1)–(2.2).

DEFINITION. A point $x \in \bar{\Omega}$ is a *blow-up point* if there exists $\{(x_n, t_n)\}$ such that $t_n \to T^-$, $x_n \to x$, and $u(x_n, t_n) \to +\infty$ as $n \to \infty$.

The purpose of this paper is to determine the set of blow-up points for (2.1)–(2.2) primarily when $\Omega = B_R$.

In the next section we review and extend one of the known results for (2.1)–(2.2) when $g(t) \equiv 0$.

In Section 4, we prove our key results which can be summarized as follows. Assume $\Omega = B_R$ and blow-up occurs at T.

(I) If $\int_0^T g(t)\, dt = +\infty$, then blow-up occurs everywhere (Theorem 4.1).

(II) If $\int_0^T g(t) < +\infty$ and $f(u) = e^u$ or $f(u) = (u + \lambda)^p$, $\lambda \geqslant 0$, $p > 1$, then blow-up occurs only at $x = 0$ (consequence of Theorems 4.5 and 4.6).

(III) If $g(t) = (K/\text{vol}\,\Omega) \int_\Omega u_t(x, t)\, dx$, $K < 1$ and $f(u) = (u + \lambda)^p$, $\lambda \geqslant 0$, $1 < p < 1 + 2/n$, then blow-up occurs everywhere (Theorem 4.4).

(IV) If $g(t) = (K/\text{vol}\,\Omega) \int_\Omega u_t(x, t)\, dx$, $K < 1$, and $f(u) = e^u$, then blow-up occurs at a single point (Theorem 4.5).

(V) If $g(t) = (K/\text{vol}\,\Omega) \int_\Omega u_t(x, t)\, dx$ and $f(u) = (u + \lambda)^p$, $\lambda \geqslant 0$, $p > 1 + 2/n$, then blow-up occurs only at $x = 0$ provided $K < 1$ is sufficiently small (Theorem 4.7).

3. $g(t) \equiv 0$—REVIEW OF KNOWN RESULTS

Baras and Cohen [1], Friedman and McLeod [3], Weissler [6], and Weissler and Mueller [7] have considered (2.1)–(2.2) when $g(t) \equiv 0$ and for particular $f(u)$.

When $f(u) = e^u$ and $g(t) \equiv 0$, Friedman and McLeod [3] proved

THEOREM 3.1. (a) *$x=0$ is the only blow-up point in B_R.*

(b) *Given $\alpha \in (0,1)$, there exists $K_\alpha > 0$ such that*

$$u(x,t) \leqslant -\frac{2}{\alpha} \ln x + K_\alpha \tag{3.1}$$

on $\bar{B}_R \times [0, T]$.

They proved this result by using a maximum principle argument applied to $J(r,t) = w + \varepsilon r^n F(u)$ where $r = |x|$, $w = r^{n-1} u_r$, and F is a positive function with F_u, $F_{uu} \geqslant 0$. If, for $\varepsilon > 0$ sufficiently small,

$$f'F - F'f \geqslant 2\varepsilon n F F', \tag{3.2}$$

then J satisfies

$$J_t + \frac{n-1}{r} J_r - J_{rr} - bJ \leqslant 0, \qquad 0 < r < R,\ 0 < t < T, \tag{3.3}$$

where b is a bounded function for $0 < r < R$, and moreover J cannot take a positive maximum on $r = R$. Also, for ε small anough $J \leqslant 0$ at $t = 0$. By the maximum principle $J \leqslant 0$ on $\Omega \times [0, T)$. If $F(u) = e^{\alpha u}$ with $\alpha \in (0,1)$, then (3.2) holds and

$$r^{n-1} u_r \leqslant -\varepsilon r^n F(u) \tag{3.4}$$

follows. Setting $H(u) = (1/\alpha)\, e^{-\alpha u}$, then $H_r = -e^{-\alpha u} u_r \geqslant \varepsilon r$. Integrating, they obtain

$$u(r,t) \leqslant \frac{2}{\alpha} \ln r^{-1} + K_\alpha \tag{3.1}$$

and hence blow-up occurs at $x = 0$ only.

This same argument can be used to improve the upper bound estimate on u.

THEOREM 3.2. *If $f(u) = e^u$, then*

$$u(r,t) \leqslant 2 \ln r^{-1} + \ln(\ln r^{-1}) + C. \tag{3.5}$$

Proof. The function $F(u) = e^u \cdot (1+u)^{-1}$ satisfies (3.2) for $0 < \varepsilon \leqslant 1/2n$. Then, since $\int_u^\infty ds/F(s) = (u+2)\, e^{-u}$, we have from

$$u_r \leqslant -\varepsilon r \frac{e^u}{u+1}$$

that

$$(u+2)\,e^{-u} \geqslant \frac{\varepsilon r^2}{2}.$$

Hence, $-u+\ln(u+2) \geqslant \ln(\varepsilon r^2)/2$ implying

$$u-\ln(u+2) \leqslant 2\ln\frac{1}{x}+C'$$

and

$$u-\ln u \leqslant 2\ln\frac{1}{r}+C''.$$

Thus, $u \leqslant 2\ln 1/r + C'' + \ln(2/\alpha \ln \frac{1}{r} + K_\alpha)$ by (3.1) and we conclude

$$u(r,t) \leqslant 2\ln\frac{1}{r}+\ln\ln\frac{1}{r}+C.$$

Using the same type of argument as for Theorem 3.1, Friedman and McLeod also proved for $f(u)=u^p$, $p>1$:

THEOREM 3.3. *Let $f(u)=u^p$, $p>1$. Consider IBVP* (2.1)–(2.2).

(a) *$x=0$ is the only blow-up point in B_R.*

(b) *Given $\alpha \in (1, p)$, there is a $C>0$ such that*

$$u(x,t) \leqslant \frac{C}{x^{2/(\alpha-1)}}.$$

Baras and Cohen [1] have recently proven that the solution $u(x,t)$ of (2.1)–(2.2), with $g(t)\equiv 0$ and under some special assumptions on f, blows up completely after T and hence, in a sense to be made precise later, everywhere in Ω after T.

Consider the following sequence of approximating IBVP's to (2.1)–(2.2):

$$w_t - \Delta w = f_n(w) \tag{3.6}$$

$$\begin{aligned} w(x,t)&=0, && x\in\partial\Omega,\ t>0\\ w(x,0)&=\phi(x), && x\in\Omega, \end{aligned} \tag{3.7}$$

where f_n is uniformly Lipschitz continuous, nondecreasing, $f_n(0)=0$, and $f_n \uparrow f$. For each n, (3.6)–(3.7) has a solution $u_n(x,t)$ on $\Omega\times[0,\infty)$.

THEOREM 3.4. *Let $f(0)=0$. If one of the following three hypotheses holds,* H$_1$. *Ω is convex and if $n \geqslant 2$, there exists $p \in (1, n/(n-2))$ such that* $0 \leqslant f'(u) \leqslant c(|u|^{p-1}+1)$, $u \geqslant 0$;

H$_2$. *f is convex and there exists $q > 1$, $a \geqslant 0$ such that $f(u)/u^q$ is non-decreasing on $(a, +\infty)$;*

H$_3$. *f is convex and there exists $p \in (1, (n+2)/(n-2))$ such that* $0 \leqslant \lim_{u \to \infty} f(u)/u^p < \infty$, *then*

(a) $\lim_{n \to \infty} u_n(x, t) = u(x, t)$, $(x, t) \in \Omega \times [0, T)$;

(b) $\lim_{n \to \infty} u_n(x, t) = +\infty$, $(x, t) \in \Omega \times (T, \infty)$;

(c) $\lim_{n \to \infty} u_n(x, T) = \lim_{t \to T^-} u(x, t)$, $x \in \Omega$.

For IBVP (2.1)–(2.2) with $g(t) \geqslant 0$, we will prove that $u(x, t)$ blows up at a single point $x = 0$ or everywhere as $t \to T$.

4. $g(t) \geqslant 0$—New Results

For IBVP (2.1)–(2.2) assuming the hypotheses (2.3) we now show that $u(x, t)$ blows up everywhere at T or at a single point.

THEOREM 4.1. *If $\int_0^T g(t)\, dt = +\infty$, then*

$$\lim_{t \to T^-} u(x, t) = +\infty \qquad \text{for all} \quad x \in B_R$$

and $u(x, t)$ blows up everywhere.

Proof. Fix any $\bar{x} \in B_R$ and let $\rho = R - |\bar{x}|$. On the ball $B(\bar{x}, \rho) \subseteq B_R$, the solution $u(x, t)$ of (2.1)–(2.2) is an upper solution for

$$\begin{aligned} &v_t = \Delta v + g(t) \\ &v(x, 0) = 0, \qquad |x - \bar{x}| < \rho \\ &v(x, t) = 0, \qquad |x - \bar{x}| = \rho, \qquad t \in [0, T). \end{aligned} \tag{4.1}$$

Using the Green's function for (4.1), the solution $v(\bar{x}, t)$ of (4.1) can be expressed as

$$\begin{aligned} v(\bar{x}, t) &= \int_0^t \int_{B(\bar{x}, \rho)} G(\bar{x}, y, t-s)\, g(s)\, dy\, ds \\ &\geqslant \int_0^t g(s) \int_{B(\bar{x}, \rho)} G(\bar{x}, y, T-0)\, dy\, ds \\ &\geqslant K(\rho) \int_0^t g(s)\, ds, \end{aligned}$$

where $K(\rho) = \int_{B(\bar{x}, \rho)} G(\bar{x}, y, T)\, dy$.

Clearly, as $t \to T^-$, $v(\bar{x}, t) \to \infty$ since $\int_0^T g(s)\, ds = +\infty$. Since $v(x, t) \leqslant u(x, t)$ on $|x - \bar{x}| < \rho$, $t \in [0, T)$, $u(\bar{x}, t) \to +\infty$ as $t \to T^-$. But $\bar{x} \in B_R$ was arbitrary and thus blow-up occurs everywhere.

Remark. Note that Theorem 4.1 holds for arbitrary domains and not just for radially symmetric problems.

THEOREM 4.2. *Consider IBVP* (2.1)–(2.2) *with* $g(t) = (K/\text{vol})\Omega \int_\Omega u_t(x, t)\, dx$, $0 < K < 1$. *If the solution* $u(x, t)$ *of* (2.1)–(2.2) *blows up at some* $\bar{x} \neq 0$, *then it blows up everywhere in* B_R.

Proof. First observe

$$\int_0^t g(s)\, ds = \frac{K}{\text{vol}\,\Omega} \int_0^t \int_\Omega u_t(x, s)\, dx\, ds = \frac{K}{\text{vol}\,\Omega} \int_\Omega (u(x, t) - \phi(x))\, dx.$$

If $\lim_{t \to T^-} u(\bar{x}, t) = +\infty$, by the radial monotonicity of u,

$$\int_\Omega u(x, t)\, dx \geqslant \int_{|x| \leqslant |\bar{x}|} u(x, t)\, dx \geqslant \text{vol}\, B_{|\bar{x}|}\, u(\bar{x}, t)$$

$\to \infty$ as $t \to T^-$. Hence $\int_0^T g(s)\, ds = +\infty$ and by Theorem 4.1 blow-up occurs everywhere.

We now prove a theorem, similar to Theorem 3.1 of Friedman and McLeod [3, p. 432], which allows us to get lower bounds on $u(x, t)$ and the integral of $u(x, t)$ over Ω.

THEOREM 4.3. *Assume* $\int_0^\infty f(u)\, du = +\infty$. *Let* $u(x, t)$ *be a solution of IBVP* (2.1)–(2.2) *which blows up only at* $x = 0$. *Then there exists a* $t^* \in [0, T)$ *such that*

$$|\nabla u(x, \bar{t})|^2 \leqslant 2[-F(u(x, \bar{t})) + F(u_0(\bar{t})) + Lf(u_0(\bar{t}))] \tag{4.2}$$

for all $\bar{t} \in (t^*, T)$ *where* $F(w) = \int_0^w f(u)\, du$, $L = \int_0^T g(t)\, dt < \infty$, *and* $u_0(t) = u(0, t) = \max_\Omega u(x, t)$.

Proof. By the assumptions and Theorem 4.1, it follows that $L < \infty$. Since u blows up only at the origin, both u and ∇u are uniformly bounded on the parabolic boundary $\partial \tilde{Q}$ of the cylinder $\tilde{Q} = \{(x, t): |x| \leqslant R/2, 0 \leqslant t < T\}$. This follows from classical interior estimates. For let $v = u - G(t)$ where $G(t) = \int_0^t g(s)\, ds$, then $\nabla v = \nabla u$ and $v_t = \Delta v + f(v + G(t))$ where $G(t)$ is bounded on $[0, T]$. Hence,

$$\max_{(x, t) \in \partial \tilde{Q}} \left\{ \frac{|\nabla u(x, t)|^2}{2} + F(u(x, t)) \right\} = M < \infty.$$

Choose $t^* < T$ so that $F(u_0(t)) > M$ for all $t \in [t^*, T)$ (such a t^* exists because $u_0(t)$ is increasing to $+\infty$ (recall (ii′) and $F(w) \to +\infty$ as $w \to \infty$).

For any $\bar{t} \in [t^*, T)$, define

$$J(x, t) = \frac{|\nabla u(x, t)|^2}{2} + F(u(x, t)) - F(u_0(\bar{t})) - f(u_0(\bar{t})) \int_0^t g(s)\, ds. \quad (4.3)$$

We will show by a maximum principle argument that $J(x, t) \leqslant 0$ on $\{(x, t): |x| < R/2,\ 0 \leqslant t \leqslant \bar{t}\}$. From this, (4.2) follows immediately.

Notice that, on $\partial\tilde{Q}$, we have

$$J(x, t) \leqslant M - F(u_0(\bar{t})) - f(u_0(\bar{t})) \int_0^t g(s)\, ds < 0.$$

Moreover, for $x = 0 \in \mathbb{R}^n$, $t \in [0, \bar{t})$,

$$J(0, t) \leqslant F(u(x, t)) - F(u_0(\bar{t})) \leqslant 0.$$

A direct computation yields

$$\begin{aligned} J_t(x, t) &= \nabla u \cdot \nabla(\Delta u) + f'(u)|\nabla u|^2 + f(u)\, \Delta u \\ &\quad + f^2(u) + f(u)\, g(t) - f(u_0(\bar{t}))\, g(t), \\ \nabla J(x, t) &= \Delta u(\nabla u) + f(u)\, \nabla u = (\Delta u + f(u))\, \nabla u, \end{aligned}$$

and

$$\Delta J(x, t) = (\Delta u)^2 + \nabla u \cdot \nabla(\Delta u) + f'(u)|\nabla u|^2 + f(u)\, \Delta u.$$

Using the fact that

$$\begin{aligned} |\nabla J - (\Delta u)\, \nabla u|^2 &= |\nabla u|^2\, (\Delta u)^2 + \nabla J[\nabla J - 2(\Delta u)\nabla u] \\ &= f^2(u)\, |\nabla u|^2, \end{aligned}$$

we obtain

$$\begin{aligned} J_t(x, t) - \Delta J - \frac{[\nabla J - 2(\Delta u)\, \nabla u] \cdot \nabla J}{|\nabla u|^2} \\ = [f(u(x, t)) - f(u_0(\bar{t}))]\, g(t) \leqslant 0. \end{aligned}$$

Noting that $\nabla u = 0$ only at $x = 0 \in \mathbb{R}^n$, the Maximum Principle implies $J(x, t) \leqslant 0$ for $|x| < R/2$, $0 \leqslant t \leqslant \bar{t}$. In particular, $J(x, \bar{t}) \leqslant 0$ and (4.2) follows.

COROLLARY. *In addition to the hypotheses of Theorem* 4.3, *if* $f'(u) \geqslant 0$ *for* $u > 0$, *then*

$$|\nabla u(x, t)|^2 \leqslant 2f(u_0(t))(u_0(t) - u(x, t) + L) \tag{4.4}$$

for t sufficiently close to T.

Proof. Take $t = \bar{t}$ in Theorem 4.3 and use $F(u_0(t)) - F(u) = \int_u^{u_0} f(s)\, ds \leqslant (u_0 - u) f(u_0)$.

We now are in a position to prove one of the key results of this paper.

THEOREM 4.4. *Assume* $f'(u) \geqslant 0$ *for* $u > 0$, $f(u) = o(u^{1+2/n})$ as $u \to \infty$, *and let* $g(t) = (K/\text{vol}\,\Omega) \int u_t(x, t)\, dx$ *with* $0 < K < 1$. *Then the solution* $u(x, t)$ *of IBVP* (2.1)–(2.2) *satisfies*

$$\lim_{t \to T^-} u(x, t) = +\infty \qquad \textit{for all} \quad x \in B_R$$

and blow-up occurs everywhere.

Proof. If the conclusion were false, then single point blow-up must occur by Theorems 4.1 and 4.2.

Using the facts that u is radially symmetric and (4.4), we can derive a lower bound on $u(x, t)$:

$$\int_0^r \frac{|u_r(r, t)|}{(u_0(t) - u(r, t) + L)^{1/2}}\, dr \leqslant \int_0^r 2^{1/2} f(u_0(t))^{1/2}\, dr$$

$$(u_0(t) - u(r, t) + L)^{1/2} \leqslant 2^{-1/2} f(u_0(t))^{1/2} r + L^{1/2}$$

$$u_0(t) + L - u(r, t) \leqslant f(u_0(t))\, r^2 + 2L.$$

Thus,

$$u(r, t) \geqslant u_0 - L - f(u_0)\, r^2. \tag{4.5}$$

We now use (4.5) to get a lower bound on the integral of $u(x, t)$ over Ω. Let w_n denote the area of the surface of the n-dimensional ball. From (4.5), $u = u_0/2$ on $r > r_1 \equiv [(u_0/2 - L)/f(u_0)]^{1/2}$.

Thus, $u \geqslant u_0/2$ if $r \leqslant r_1$. Integrating over Ω, we get

$$\int_\Omega u(x, t)\, dx = w_n \int_0^R u(r, t)\, r^{n-1}\, dr$$

$$\geqslant w_n \int_0^{r_1} \frac{u_0}{2} r^{n-1}\, dr$$

$$= \frac{w_n u_0}{2n} \left[\left(\frac{u_0}{2} - L \right) \Big/ f(u_0) \right]^{n/2}.$$

Since $f(s)=o(s^{1+2/n})$ as $s \to \infty$, this last term tends to ∞ as $u_0 \to \infty$, that is, as $t \to T^-$. This implies that

$$\lim_{t \to T^-} \int_0^t g(s)\, ds = \lim_{t \to T^-} \left[\frac{K}{\text{vol}\, \Omega} \int_\Omega [u(x,t) - \phi(x)]\, dx \right] = +\infty,$$

and is a contradiction with Theorem 4.1. We conclude that $u(x, t)$ blows up everywhere at $t = T$.

Remark. The conditions on f in Theorem 4.4 are satisfied for all f of the form $(u+\lambda)^p$ with $\lambda \geqslant 0$ and $1 < p < 1 + 2/n$.

An obvious question is: what happens if $p > 1 + 2/n$? We first consider $f(u) = e^u$.

THEOREM 4.5. *If $f(u)=e^u$, $g(t)=(K/\text{vol}\,\Omega)\int_\Omega u_t(x,t)\,dx$ with $0<K<1$, then the solution $u(x, t)$ of IBVP* (2.1)–(2.2) blows up only at $x = 0$.

Proof. The proof is similar to that given by Friedman and McLeod [3, pp. 427–429]. Set $w = r^{n-1}u_r$, $c(r) = \varepsilon r^n$ where $\varepsilon > 0$ is to be determined, $J(r, t) = w + cF(u, t)$ where $F(u, t) = e^{\alpha(u - G(t))}$ with $\alpha \in (0, 1)$ and $G(t) = \int_0^t g(s)\, ds$.

We claim $J \leqslant 0$ on $B_R \times [0, T)$. This will be accomplished by again using a maximum principle argument applied to J.

It is observed immediately that

$$w_t + \frac{n-1}{r} w_r - w_{rr} - f'(u) w = 0. \tag{4.6}$$

Using (4.6), $u_r = w/r^{n-1}$, and $w = -cF + J$, a direct computation gives

$$\begin{aligned} &J_t + \frac{n-1}{r} J_r - J_{rr} - bJ \\ &\qquad \leqslant -c(f'F - F_u f - 2\varepsilon n F F_u) + c(F_u g + F_t), \end{aligned} \tag{4.7}$$

where $b = f'(u) - 2\varepsilon F_u$. If (i) $f'F - F_u f \geqslant 2\varepsilon n F F_u$ and (ii) $F_u g + F_t \leqslant 0$, then

$$J_t + \frac{n-1}{r} J_r - J_{rr} - bJ \leqslant 0 \tag{4.8}$$

on $[0, R] \times [0, T)$. (ii) is immediate and (i) holds if $\varepsilon \leqslant (1-\alpha)/2n\alpha$. To apply the maximum principle to J knowing (4.8), we need only check behavior of J on the parabolic boundary of $(0, R) \times (0, T)$. At $r = 0$, $J(0, t) = 0$. Next we observe that J cannot take a positive maximum on $r = R$ since $J_r \leqslant w_r + c'F$ and thus $J_r(R, r) \leqslant -R^{n-1}[f(0) + g(t)] + c'(R)\, F(0, t) = -R^{n-1}[1 + g(t)] + \varepsilon n R^{n-1} e^{\alpha(-G(t))} \leqslant R^{n-1}[\varepsilon n - 1 - g(t)] < 0$ if $\varepsilon < 1/n$. Finally, note that $J(r, 0) = r^{n-1}\phi'(r) + cF(\phi, 0) < 0$ on

$0 \leqslant r < R$ provided $\varepsilon > 0$ is sufficiently small and $\phi'(r) < 0$. (As noted in [3], this can be relaxed to $\phi'(r) \leqslant 0$.) We conclude $J(r, t) \leqslant 0$ on $[0, R] \times (0, T]$.

Thus

$$r^{n-1} u_r \leqslant -\varepsilon r^n e^{\alpha(u - G(t))}$$

and

$$-e^{-\alpha(u - G(t))} u_r \geqslant \varepsilon r.$$

Set $H(u, t) = (1/\alpha)\, e^{-\alpha[u - G(t)]}$, then

$$H_r(u, t) \geqslant \varepsilon r \tag{4.9}$$

and integrating, we have

$$e^{-\alpha[u(r, t) - G(t)]} \geqslant \frac{\alpha \varepsilon r^2}{2}.$$

From this, we have

$$u(r, t) \leqslant \frac{2}{\alpha} \ln r^{-1} - \frac{1}{\alpha} \ln \frac{\alpha\varepsilon}{2} + G(t). \tag{4.10}$$

Integrating over $\Omega = B_R$, recalling that

$$G(t) \leqslant \frac{K}{\operatorname{vol} \Omega} \int_\Omega u(x, t)\, dx, \tag{4.11}$$

we get from (4.10)

$$\int_\Omega u(x, t)\, dx \leqslant \int_\Omega \left(\frac{2}{\alpha} \ln \frac{1}{|x|} - \frac{1}{\alpha} \ln \frac{\alpha\varepsilon}{2} \right) dx + K \int_\Omega u(x, t)\, dx \tag{4.12}$$

or

$$(1 - K) \int_\Omega u(x, t)\, dx \leqslant \int_\Omega \left(\frac{2}{\alpha} \ln \frac{1}{|x|} - \frac{1}{\alpha} \ln \frac{\alpha\varepsilon}{2} \right) dx < \infty$$

and blow-up occurs at a single point provided $K < 1$.

Remark. From the proof of Theorem 4.5 if $f(u) = e^u$ and $\int_0^T g(s)\, ds < \infty$, then $u(x, t)$ blows up at a single point $x = 0$.

In fact, we can prove more if $\int_0^t g(s)\, dx < \infty$.

THEOREM 4.6. *If* $G(t) = \int_0^t g(s)\, ds < \infty$ *for* $t \in [0, T]$, *if* $f'(u) \geqslant 0$ *for* $u > 0$, *if* $F > 0$, F', $F'' \geqslant 0$, *and if* *for all* $K>0$, *there exists* $\bar{u}$ *such that*

$$f'(u)\,F(u-\xi)-f(u)\,F'(u-\xi)\geqslant 2n\varepsilon F(u-\xi)\,F'(u-\xi) \tag{4.13}$$

for all $u\geqslant\bar{u}$, $0\leqslant\xi\leqslant K$, *and* $\varepsilon>0$ *sufficiently small,*

then the solution $u(x,t)$ *of IBVP* (2.1)–(2.2) *blows up only at* $x=0$.

Remark. If $f(u)=(u+\lambda)^p$, $p>1$, and $F(u)=(u+\lambda)^q$, $1<q<p$, $\lambda\geqslant 0$, then (4.13) is satisfied and single point blow-up occurs.

Proof. The proof proceeds as for Theorem 4.5. Set $w=r^{n-1}u_r$, $c(r)=\varepsilon r^n$, $\varepsilon>0$, and let

$$J(r,t)=w+cF(u-G(t)).$$

Then

$$J_t+\frac{n-1}{r}J_r-J_{rr}-bJ=-c(f'(u)\,F(u-G)-F'(u-G)\,f(u)$$
$$-2\varepsilon nF(u-G)\,F'(u-G)). \tag{4.14}$$

Let $L=G(T)$. By assumption (4.13), there exists $\bar{u}>0$ such that (4.13) holds for $u\geqslant\bar{u}$, $0\leqslant\xi\leqslant L$, and $\varepsilon>0$ sufficiently small. Thus, for all $(r,t)\in[0,R]\times[0,T]$ such that $u(r,t)\geqslant\bar{u}$,

$$J_t+\frac{n-1}{r}J_r-J_{rr}-bJ\leqslant 0. \tag{4.15}$$

If blow-up occurs at some $r^*>0$, then $\lim_{t\to T^-}u(r,t)=+\infty$ for all $r\in[0,r^*)$ since u is radially decreasing. In particular, there exists a $t_1\in[0,T)$ such that $u(r^*/2,t)\geqslant\bar{u}$ for all $t\in[t_1,T)$.

By the implicit function theorem, there exists a continuously differentiable function $r_1(t)$ on $[t_1,T)$ with range contained in $[r^*/2,R)$ such that $u(r_1(t),t)=\bar{u}$ for $t\in[t_1,T)$. We claim that

$$u_r(r_1(t),t)\leqslant -M<0$$

for some constant $M>0$ and all $t\in[t_1,T)$. Indeed, since $f,g\geqslant 0$, the solution v of the IBVP,

$$v_t=v_{rr}+\frac{n-1}{r}v_r$$

$$v(r,t_1)=u(r,t_1),\qquad r\in\left(\frac{r^*}{2},R\right)$$

$$v\left(\frac{r^*}{2},t\right)=\bar{u},\ v(R,t)=0,\qquad t\in[t_1,T),$$

provides a lower bound for u. Hence

$$-M = \max_{s \in [t_1, T]} v_r(R, s) \geqslant u_r(R, t)$$

for all $t \in [t_1, T)$.

The function u_r satisfies the IBVP:

$$(u_r)_t = u_{rrr} + \frac{n-1}{r} u_r + f'(u) u_r$$

$$u_r(R, t) \leqslant -M < 0, \qquad u_r\left(\frac{r^*}{4}, t\right) \leqslant 0 \qquad \text{for } t \in [t_1, T),$$

$$u_r(r, t_1) \leqslant -M', \qquad r \in \left[\frac{r^*}{4}, R\right], \qquad \text{for some } M' > 0.$$

Since $f'(u) \geqslant 0$ and $u_r \leqslant 0$, $(u_r)_t \leqslant u_{rrr} + ((n-1)/r) u_r$ and we conclude that there exists a constant M'' such that $u_r(r, t) \leqslant -M'' < 0$ uniformly on the set $A = [r^*/2, R] \times [t_1, T)$. Using this bound, we see that by choosing $\varepsilon > 0$ sufficiently small,

$$J(r_1(t), t) \leqslant 0 \qquad \text{and} \qquad J(r, t_1) \leqslant 0$$

for $t \in [t_1, T)$, $r \in [0, R)$. At $r = 0$, $J(0, t) \equiv 0$. An application of the maximum principle to J knowing (4.15) yields $J \leqslant 0$ on the set $B = \{(r, t): 0 \leqslant r \leqslant r_1(t), t_1 \leqslant t < T\}$. Hence $r^{n-1} u_r \leqslant -\varepsilon r^n F(u - G(t))$ for all $(r, t) \in B$. We then have

$$-\frac{u_r}{F(u-L)} \geqslant \varepsilon r$$

and integrating

$$-\int_0^r \frac{u_r \, dr}{F(u-L)} \geqslant \frac{\varepsilon r^2}{2}$$

or

$$-\int_{u_0 - L}^{u(r, t) - L} \frac{dz}{F(z)} \geqslant \frac{\varepsilon r^2}{2}.$$

Set $H(s) = +\int_{u_0 - L}^{s} dz/F(z)$, then

$$-H(u(r, t) - L) \geqslant \frac{\varepsilon r^2}{2}$$

and

$$u(r, t) \leqslant H^{-1}\left(-\frac{\varepsilon r^2}{2}\right) + L \qquad \text{for} \quad (r, t) \in B.$$

By this contradicts our assumption that blow-up occurs at some $r^* > 0$. We conclude that blow-up occurs at a single point.

THEOREM 4.7. *If* $f(u) = (u+\lambda)^p$ *where* $\lambda \geqslant 0$ *and* $p > 1 + 2/n$, *and* $g(t) = (K/\text{vol}\,\Omega) \int_\Omega u_t(x, t)\,dx$, *then the solution of* (2.1)–(2.2) *blows up only at* $x = 0$ *if* $0 \leqslant K \leqslant K_1$ *and* $K < 1$ *where* K_1 *is a constant depending on* n, λ, p, *and* ϕ.

Proof. The idea of the proof is exactly the same as that of Theorem 4.5. Set $w = r^{n-1}u_r$, $c(r) = \varepsilon r^n$ with $\varepsilon > 0$ and $F(u, t) = e^{-\alpha G}(u+\mu)^q$ where $\alpha > 0$, $\mu > 0$, and $\mu \geqslant \lambda$, and $1 + 2/n < q < p$.

Consider $J(u, t) = w + cF(u, t)$. Again, (4.8) holds for J provided that (i) $f'F - F_u f \geqslant 2\varepsilon n F F_u$ and (ii) $F_u g + F_t \leqslant 0$.

For (ii), we note that

$$gF_u + F_t = e^{-\alpha G}(u+\mu)^{q-1}\{qg - \alpha g(u+\mu)\}$$

so condition (ii) holds if

$$\mu\alpha \geqslant q. \tag{4.16}$$

Now

$$\begin{aligned} f'F - fF_u &= (u+\lambda)^{p-1}(u+\mu)^{q-1}e^{-\alpha G}\{(p-q)u + (p\mu - q\lambda)\} \\ &\geqslant 2n\varepsilon q(u+\mu)^{2q-1}e^{-2\alpha G} \\ &= 2n\varepsilon F F_u \end{aligned}$$

provided

$$\varepsilon \leqslant \lambda^{p-1}(p\mu - q\lambda)/2nq\mu^q. \tag{4.17}$$

Certainly $J = 0$ or $r = 0$ while $J_r \leqslant 0$ on $r = R$ if $\varepsilon \leqslant 1/n$. Moreover at $t = 0$, $J = r^{n-1}\phi' + \varepsilon r^n(\phi+\mu)^q \leqslant 0$ if

$$\varepsilon \leqslant \inf\{-\phi'/r(\phi+\mu)^q\}. \tag{4.18}$$

Then from the maximum principle $J \leqslant 0$ in $[0, R] \times [0, T)$. Thus

$$r^{n-1}u_r \leqslant -\varepsilon r^n e^{-\alpha G}(u+\mu)^q.$$

Integrating from 0 to r, we have

$$(u+\mu)^{-(q-1)} \geqslant (q-1)\,\varepsilon e^{-\alpha G}\frac{r^2}{2}$$

which gives

$$u \leqslant [2e^{\alpha G}/\varepsilon(q-1)]^{1/(q-1)} r^{n-1-2/(q-1)}. \tag{4.19}$$

Integrating (4.19) over Ω and using (4.11) we have

$$G \leqslant \frac{nK}{w_n R_n}\int_0^R \left[\frac{2e^{\alpha G}}{\varepsilon(q-1)}\right]^{1/(q-1)} r^{n-1-2/(q-1)}\,dr$$
$$= nK\left[\frac{2}{\varepsilon(q-1)\,R^2}\right]^{1/(q-1)} e^{\alpha G/q-1}/[n-2/(q-1)]. \tag{4.20}$$

For K sufficiently small, say $K \leqslant K_2(n, \varepsilon, \alpha, q)$, there is some G_2 giving equality in (4.20). Then since $G(0)=0$, $G(t)$ remains bounded above by G_2. Hence $\int_\Omega u\,dx$ is also bounded and u only blows up at $x=0$.

Note that K_1 is given by the maximum value of K_2 for $\mu \geqslant \lambda$, $\mu > 0$, $1+2/n < q < p$, $\alpha \geqslant q/\mu$, and $0 < \varepsilon \leqslant 1/n$ satisfying (4.17) and (4.18).

Remarks. (1) From the proof of Theorem 4.7, if $f(u)=(u+\lambda)^p$, $\lambda \geqslant 0$, $p > 1$, and $\int_0^T g(s)\,ds < \infty$ we again have that u blows up at the single point $x=0$.

(2) We conjecture, but cannot prove, that Theorem 4.7 is true for all $p > 1 + n/2$ and $0 < K < 1$. Our result is considerably more restrictive.

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