Computers Math. Applic. Vol. 22, No. 1, pp. 1-11, 1991 0097-4943/91 $3.00 + 0.00

NUMERICAL SOLUTION OF SOME INTEGRAL EQUATIONS IN DISTRIBUTIONS

A. G. RAMM AND PEIQING LI
Department of Mathematics, Kansas State University, Manhattan, KS 66506

(Received August 1990)

Abstract—A class $\mathcal{R}$ of integral equations, whose kernels are kernels of positive rational functions of elliptic operators, do not have in general locally integrable solutions. Their solutions belong to a distribution space $\dot{H}^{-\alpha}(D)$, where $H^{\alpha}(D)$ is the Sobolev space. A numerical method is given for computing these solutions. Results of numerical experiments are reported. A typical example of the equation of the class $\mathcal{R}$ is the equation

$$\int_{-1}^{1} \exp(-|x-y|)h(y)\,dy = f(x), \quad -1 \le x \le 1, \quad f \in H^1(-1,1).$$

I. INTRODUCTION

The analytic theory of signal estimation by the criterion of minimum of variance is based on the equation of the type

$$Rh := \int_D R(x,y)h(y)\,dy = f(x), \quad x \in \overline{D} \subset R^r. \tag{1}$$

Here D is a bounded domain with a smooth boundary Γ, $\overline{D} = D \cup \Gamma$, $R(x,y) \in \mathcal{R}$, where $\mathcal{R}$ is the class of kernels of positive rational functions of self-adjoint elliptic operators defined in $L^2(R^r)$. In other words

$$R(x,y) = \int_\Lambda P(\lambda)Q^{-1}(\lambda)\Phi(x,y,\lambda)\,d\rho(\lambda) \tag{1'}$$

where Λ, $\Phi(x,y,\lambda)$, $d\rho(\lambda)$ are, respectively, the spectrum, the spectral kernel and the spectral measure of an elliptic self-adjoint operator $\mathcal{L}$ of order s, $P(\lambda) > 0$ and $Q(\lambda) > 0$, $-\infty < \lambda < \infty$, are polynomials

$$\deg P(\lambda) = p, \quad \deg Q(\lambda) = q, \quad \mathrm{ord}\mathcal{L} = s. \tag{2}$$

Define

$$\alpha := \frac{1}{2}s(q-p). \tag{3}$$

Let $H^{\alpha}(D)$ be the Sobolev space and $\dot{H}^{-\alpha}(D)$ its dual space with respect to the $L^2(D)$ inner product. The space of distributions $\dot{H}^{-\alpha}(D)$ can be described as the space of the elements of $H^{-\alpha}(R^r)$ whose support is $\overline{D}$. Recall that $H^{-\alpha}(R^r)$ is the space of distributions $h(x)$ such that

$$\left(\int_{R^r} |\tilde{h}|^2(1+\lambda^2)^{-\alpha}d\lambda\right)^{1/2} :=\| h \|_{-\alpha} < \infty, \tag{4}$$

A derivation of equation (1) in the framework of signal estimation theory is given in [1] and [2]. In [1] and [2] the class $\mathcal{R}$ has been introduced and studied. The basic result from [1], [2] which we need here is

AGR thanks ONR and NSF for support.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

PROPOSITION 1. *The operator R defined in (1) is an isomorphism between $\dot{H}^{-\alpha}(D)$ and $H^{\alpha}(D)$ where α is defined by (3) and the kernel $R(x,y) \in \mathcal{R}$.*

In particular, Proposition 1 says that for any $f \in H^{\alpha}(D)$ equation (1) with the kernel $R \in \mathcal{R}$ has a unique solution in $\dot{H}^{-\alpha}(D)$ and this solution depends continuously in the norm of $\dot{H}^{-\alpha}(D)$ on $f \in H^{\alpha}(D)$.

For example, consider

$$Rh = \int_{-1}^{1} \exp(-|x-y|)\, h(y)\, dy = f(x), \quad -1 \le x \le 1. \tag{5}$$

In this example $D = (-1,1)$, $r = 1$, $\mathcal{L} = -i\partial$, $\partial := \frac{d}{dx}$, $\Lambda = (-\infty,\infty)$, $d\rho(\lambda) = d\lambda$, $\Phi(x,y,\lambda) = (2\pi)^{-1}\exp\{i\lambda(x-y)\}$, $P(\lambda) = 1$, $Q(\lambda) = \frac{\lambda^2+1}{2}$, $s = 1$, $p = 0$, $q = 2$, $\alpha = 1$.

The solution to (5) in $\dot{H}^{-1}(D)$ exists and is unique for any $f \in H^1(D)$. This solution is given by the formula [3, p. 389]

$$h(x) = \frac{-f'' + f}{2} + \frac{\delta(x+1)[-f'(-1) + f(-1)] + \delta(x-1)[f'(1) + f(1)]}{2} \tag{6}$$

where $\delta(x)$ is the delta-function. It can be characterized as the solution of minimal order of singularity.

It follows from the results in [1], page 19, that if a function $f \in H^{\alpha}(D)$ has an isolated point $x_0 \in D$ of non-smoothness then the solution h will have at x_0 a point of singularity but will be smooth for $x \neq x_0$ in a neighborhood of x_0. For example, if $f(x) := |x|^{\gamma}$ in a neighborhood of $x_0 = 0$, and f is the right-hand side of (5), then $f(x) \in H^1 := H^1(-1,1)$ for $\gamma > 0.5$, and formula (6) shows that the solution $h(x)$ behaves like $\gamma(\gamma-1)|x|^{\gamma-2}$ as $x \to 0$, where the formulas for the derivatives of the functions $|x|^{\gamma}$ and $|x|^{\gamma-1}\, sgn\ x$ (in the sense of distributions) were used: $(|x|^{\gamma})' = \gamma|x|^{\gamma-1}\, sgn\ x$, $(|x|^{\gamma-1}\, sgn\ x)' = (\gamma-1)|x|^{\gamma-2}$ (see [5, §I.3, formulas (9) and (10)]). In the case of equation (5) $\alpha = 1$, $f \in H^1$ and $f \notin H^2$ for $0.5 < \gamma < 1.5$.

In [3] and [2] a theory of equations of class $\mathcal{R}$ is developed and a numerical method for solving these equations is described. Convergence of the method is proved.

In this paper, we present a variant of the numerical method for solving equations of class $\mathcal{R}$ and report the results of numerical experiments which show that the method is stable and practically efficient.

Note that although equations (1) and (5) look like Fredholm equations of the first kind, so that one can think that these equations lead to ill-posed problems, but in fact the situation is totally different. The problem of solving (1) and (5) in $\dot{H}^{-\alpha}(D)$ is well posed as follows from Proposition 1.

Why can one have a wrong idea about ill-posedness of the problem of solving (1)? Because there is a vast literature on the following problem (see e.g. [4]). Let $Au = f$ be a linear equation with an injective compact operator A from an infinite dimensional Banach space U into a Banach space F. Suppose that $f \in \text{Ran}A$, where $\text{Ran}A$ is the range of A. Assume that f_δ is given such that $\| f_\delta - f \| < \delta$, where $\delta > 0$ is a small given number and f_δ does not necessarily belong to $\text{Ran}A$. Given $\{A, \delta, f_\delta\}$ one wants to find u_δ such that $\| u - u_\delta \| \to 0$ as $\delta \to 0$. The ill-posedness of this problem comes from the fact that A^{-1} is an unbounded operator (since A is compact and the space U is infinite dimensional). Therefore even if $f_\delta \in \text{Ran}A = \text{Dom}(A^{-1})$ one cannot say that $u_\delta = A^{-1}f_\delta$ since $\| A^{-1}f_\delta - u \| = \| A^{-1}(f_\delta - f) \|$ does not go to zero as $\delta \to 0$, in general.

The problem of solving equation (1) in $\dot{H}^{-\alpha}(D)$ is well-posed since the operator $R^{-1} : H^{\alpha}(D) \to \dot{H}^{-\alpha}(D)$ is continuous and is defined on all of $H^{\alpha}(D)$. Note that (1) does not have a solution in $L^2(D)$ in general.

Analytic estimates of the norm of R and R^{-1} can be found in [1] and [2]:

$$\| R \|_{\dot{H}^{-\alpha} \to H^{\alpha}} \le \sup_{\lambda \in \Lambda} \left\{ P(\lambda)\, Q^{-1}(\lambda)(1+\lambda^2)^{\alpha/s} \right\} := \gamma_1 \tag{7}$$

$$\| R^{-1} \|_{H^{\alpha} \to \dot{H}^{-\alpha}} \le \left(\inf_{\lambda \in \Lambda} \left\{ P(\lambda)\, Q^{-1}(\lambda)(1+\lambda^2)^{\alpha/s} \right\} \right)^{-1} := \gamma_2. \tag{8}$$

We will denote $\| \cdot \|_{H^{\alpha}(D)} := \| \cdot \|_{\alpha}$ and $\| \cdot \|_{\dot{H}^{-\alpha}(D)} := \| \cdot \|_{-\alpha}$.

In Section II the numerical method is described. In Section III the numerical results are presented.

II. DESCRIPTION OF THE NUMERICAL METHOD

It is proved in [1], [2] that if f is smooth enough, for example, $f \in H^{\beta}(D)$, $\beta \geq (q-p)s$, then the singular support of the solution to (1) is $\partial D = \Gamma$. See, for example, formula (6) which shows that if $f \in H^2$ then the singular (distributional) part of the solution is the second term in the right-hand side of (6).

Therefore, assuming that f is smooth enough, let us look for an approximate solution of the form

$$h = h_{sm} + h_{sing} \tag{9}$$

where the singular part is [3]:

$$h_{sing} = \sum_{|j|=0}^{\alpha-1} b_j(s)\,\delta_\Gamma^{(j)}. \tag{10}$$

The distribution $b_j(s)\,\delta_\Gamma^{(j)}$ is defined by the formula

$$\int_{R^r} b_j(s)\,\delta_\Gamma^{(j)}\,\phi(x)dx = (-1)^{|j|}\int_\Gamma \{b_j(s)\,\phi(s)\}^{(j)}\,ds, \tag{11}$$

The coefficients $b_j(s)$ are sufficiently smooth functions, $b_j(s) \in C^{(j)}(\Gamma)$, j is a multi-index, $|j| = j_1 + \cdots + j_r$. The regular (smooth) part of h in (9) is h_{sm}. Let us choose a complete in $\dot{H}^{-1}(D)$ system $\{\psi_j\}$ of linearly independent elements, which is a union of regular functions $\{\phi_j\}$, where the set $\{\phi_j\}$ forms a basis of $L^2(D)$, and singular functions $\delta_\Gamma^{(j)}$, $|j| \leq \alpha - 1$. We wish to use a projection method, namely a least squares method. If $h_n \in \operatorname{span}(\psi_1, \ldots, \psi_n) := M_n$ is defined to be the solution to the equation

$$\epsilon_n := \| Rh_n - f \|_\alpha = \min \tag{12}$$

then equation (12) has a unique solution, and

$$\| h_n - h \|_{-\alpha} \to 0 \quad \text{as} \quad n \to \infty \tag{13}$$

where $h = R^{-1}f \in \dot{H}^{-\alpha}(D)$.

The first of these two statements is obvious since the distance from an element f to the finite-dimensional subspace M_n in a Hilbert space H^α is attained at a unique element of M_n.

The second statement follows from the inequality

$$\| h_n - h \|_{-\alpha} \leq \gamma_2 \| Rh_n - f \|_\alpha \to 0, \quad n \to \infty, \tag{14}$$

where γ_2 is given by (8). The right-hand side of (14) goes to zero, since the set $\{\psi_j\}$ is complete in $\dot{H}^{-\alpha}$ by the assumption, and therefore, the set $\{R\psi_j\}$ is complete in H^α, since R is an isomorphism between $\dot{H}^{-\alpha}$ and H^α. A practical implication of (14) is the following: one increases n for computing the approximate solution h_n until the discrepancy ϵ_n defined by (12) is sufficiently small.

Let us illustrate this numerical method using equation (5) as an example. In this example $D = (-1, 1)$, $\alpha = 1$. Thus, let us look for an approximate solution of the form

$$h_n = c_{-2}^{(n)}\,\delta(x+1) + c_{-1}^{(n)}\,\delta(x-1) + \sum_{j=0}^{n} c_j^{(n)}\,\phi_j(x) := \sum_{j=-2}^{n} c_j^{(n)}\,\phi_j(x), \tag{15}$$

where $\delta(x)$ is the delta-function, $\phi_{-2} := \delta(x+1)$, $\phi_{-1}(x) := \delta(x-1)$ and $\{\phi_j(x)\}$ is a complete linearly independent system in $L^2(D)$, for example

$$\phi_j = \cos\left\{\frac{j\pi}{2}(x+1)\right\}, \quad j = 0, 1, 2, \ldots \tag{16}$$

The system (16) is orthogonal and complete in $L^2(D)$. For any n the system

$$\{\phi_j\}, \quad -2 \le j \le n \tag{17}$$

is linearly independent in $\dot{H}^{-1}$. If M_n is the linear span of functions (17) then

$$\rho(h, M_n) \to 0 \quad \text{as} \quad n \to \infty, \quad \forall h \in \dot{H}^{-1}, \tag{18}$$

where $\rho(h, M_n) = \inf_{g \in M_n} \| h - g \|_{-1}$ is the distance from h to M_n in $\dot{H}^{-1}$.

The coefficients $c_j^{(n)}$ in (15) should be determined from (12), that is, in the case of equation (5) from

$$\left\| \sum_{j=-2}^{n} c_j^{(n)} g_j(x) - f \right\|_1 = \min \tag{19}$$

where

$$g_j(x) := R\phi_j(x) = \int_{-1}^{1} \exp(-|x-y|)\phi_j(y)\,dy. \tag{20}$$

Problem (19) has a unique solution which can be computed by solving the linear system for $c_j^{(n)}$:

$$\sum_{p=-2}^{n} a_{jp} c_p^{(n)} = f_j, \quad -1 \le j \le n. \tag{21}$$

Here

$$a_{jp} := (g_p, g_j)_1 = \int_{-1}^{1} (g_p \overline{g}_j + g_p' \overline{g}_j')\,dx, \tag{22}$$

where the primes denote derivatives and the bar stands for complex conjugate,

$$f_j = (f, g_j)_1, \quad -2 \le j \le n. \tag{23}$$

The matrix (22) is a Gramian of a linearly independent system $\{g_j\}$ in H^1 and therefore this matrix is positive definite. Therefore the system (21) is uniquely solvable. The function h_n defined by (15) with the coefficients $c_j^{(n)}$ which solve (21) satisfies (13). Let us summarize the result.

THEOREM 1. *The system (21) is uniquely solvable for all $n = 0, 1, 2, \ldots$. If $c_j^{(n)}$ solves (21), then h_n, defined by (15), converges in $\dot{H}^{-1}$ to the unique solution of (5) in the space $\dot{H}^{-1}$.*

If $f(x)$ has an isolated point of non-smoothness such that $f(x) = A|x - x_0|^{\gamma} + g(x)$, $0.5 < \gamma < 1.5$, where $A = \text{const}$ and $g(x)$ is smooth, then the solution to equation (5) will have the form $h = h_{sm}(x) + h_{sing}$. Here h_{sm} is smooth and $h_{sing} = c_{-2}\,\delta(x+1) + c_{-1}\,\delta(x-1) + A\gamma(\gamma-1)|x|^{\gamma-2}$. The constant c_{-2} and c_{-1} are the same as in formula (6) and the distribution $|x|^{\gamma-2}$ is defined as usual ([5], §1.3).

In Section III numerical results are given.

III. NUMERICAL RESULTS

We give six typical examples of our numerical experiments.

(1) f is such a function that the corresponding h_{sing} is zero and h_{sm} can be expanded as a linear combination of $\{\phi_j(x)\}$ with a few terms.

(2) f is such a function that the corresponding h has no singular part, but h_{sm} can not be expanded as a linear combination of $\{\phi_j(x)\}$ with a few terms.

(3) f is such a function that the corresponding h has singular part, but h_{sm} can be expanded as a linear combination of $\{\phi_j(x)\}$ with a few terms.

(4) f is such a function that the corresponding h has singular part, and h_{sm} has a slowly convergent representation by a linear combination of $\{\phi_j(x)\}$.

(5) f belongs to $H^2(-1,1)$, but does not belong to $H^3(-1,1)$, and the corresponding h has the same feature as that in example 4.

(6) f belongs to $H^1(-1,1)$, but does not belong to $H^2(-1,1)$, and the corresponding h has the same feature as that in example 4.

The numerical experiments show that the method given in Section II provides a reasonably accurate approximation of h. It tells us how to choose n, the number of the basis functions $\{\phi_j\}$. Let

$$\epsilon_n := \left\| \sum_{j=-2}^{n} c_j^{(n)} g_j(x) - f \right\|_1 . \tag{24}$$

Suppose $Rh_n = f + \eta_n$, then $h_n = R^{-1}f + R^{-1}\eta_n$. So

$$\| h_n - h \|_{-1} = \| R^{-1}\eta_n \|_{-1} \leq \| R^{-1} \| \cdot \| \eta_n \|_1 = \| R^{-1} \| \cdot \epsilon_n . \tag{25}$$

Hence, when ϵ_n does not change considerably as n grows, for $n > n_0$, say, there is no need to increase n in practical calculations. Also numerical experiments demonstrate that the method works for noisy data. How it will work can be seen from the condition number of $A_n = (a_{jp})_{-2\leq j,p\leq n}$, where a_{jp} is given in (22). Note that a_{jp}, $1 \leq j$, $p \leq n$, can be calculated explicitly in our examples. We will give the condition number for each example below.

Before we give the results, the term pointwise relative error should be defined: the pointwise relative error of a function h at a point x is $\left|\frac{\tilde{h}(x)-h(x)}{h(x)}\right|$, where $\tilde{h}$ is an approximation of h, and this notation is used in the following graphs.

(1) In the first example, the right-hand side is $f(x) = -2 + 2\cos \pi(x+1)$, and the exact solution is $h(x) = -1 + (1+\pi^2)\cos \pi(x+1)$, see graph 1. If n is 2, then the condition number of $A = (a_{jp})_{-2\leq j,p\leq 2}$ is $O(10)$, and $\epsilon_2 = O(10^{-13})$. So it is not necessary to increase n. For $n = 2$, the pointwise relative error of h_2 is $O(10^{-15})$. As we see in Graph 1, $h(x)$ and $h_2(x)$ coincide practically.

(2) In the second example, the right-hand side is $f(x) = -2\exp(x-1) + \frac{2}{\pi}\sin \pi(x+1) + 2\cos \pi(x+1)$ and the exact solution is $h(x) = (\frac{1}{\pi} + \pi)\sin \pi(x+1) + (1+\pi^2)\cos \pi(x+1)$, see Graph 2 which shows the Graphs of h and h_{340}. If $n = 340$, then the condition number of $A_{340} = (a_{jp})_{-2\leq j,p\leq 340}$ is $O(10^6)$, and $\epsilon_{340} = O(10^{-11})$. The pointwise relative error of h_{340} is less than $O(10^{-6})$ for $-0.98 \leq x \leq 0.98$ and is not more than $O(10^{-3})$ for $-1 \leq x < -0.98$ and $0.98 < x \leq 1$. In Graph 2.1 we can see a small deviation between $h(x)$ and $h_{340}(x)$ near two end points.

(3) In the third example, one has

$$f(x) = \cos\frac{\pi(x+1)}{2} + 4\cos 2\pi(x+1) - 1.5\cos\frac{7\pi(x+1)}{2}$$

and the exact solution

$$h(x) = \frac{1}{2}\left(1+\frac{\pi^2}{4}\right)\cos\frac{\pi(x+1)}{2} + (2+8\pi^2)\cos 2\pi(x+1) - 0.75(1+12.25\pi^2) \cdot \cos\frac{7\pi(x+1)}{2} + 1.75\,\delta(x+1) + 2.25\delta(x-1),$$

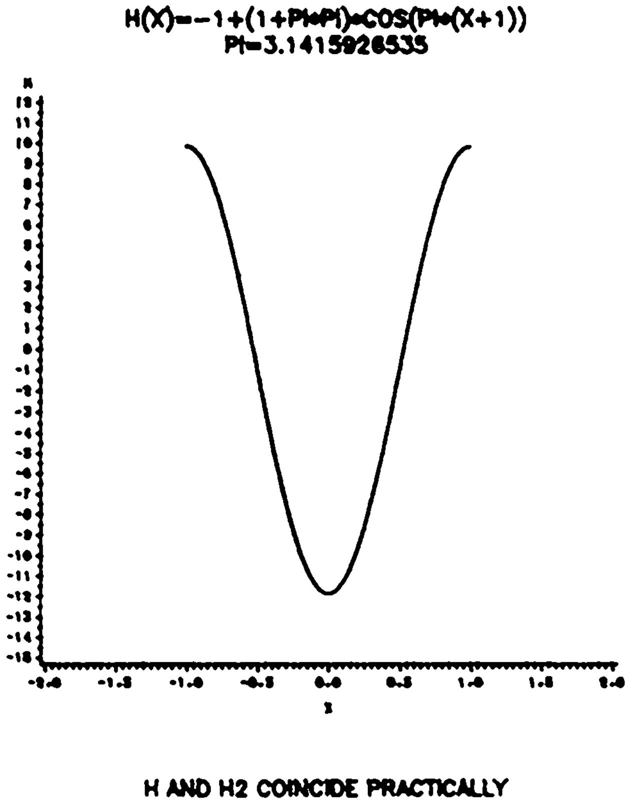
see Graph 3. If $n = 7$, then the condition number of $A_7 = (a_{jp})_{-2\leq j,p\leq 7}$ is $O(10^2)$, and $\epsilon_7 = O(10^{-11})$. The pointwise relative error of h_7 is $O(10^{-11})$ for $-0.94 \leq x \leq 0.94$, and is not more that $O(10^{-9})$ if $-1 \leq x < -0.94$ and $0.94 < x \leq 1.0$. The relative error for the coefficients of the singular part is $O(10^{-9})$. Again we can not tell the difference between $h(x)$ and $h_7(x)$ in Graph 3.

(4) In the fourth example, one has

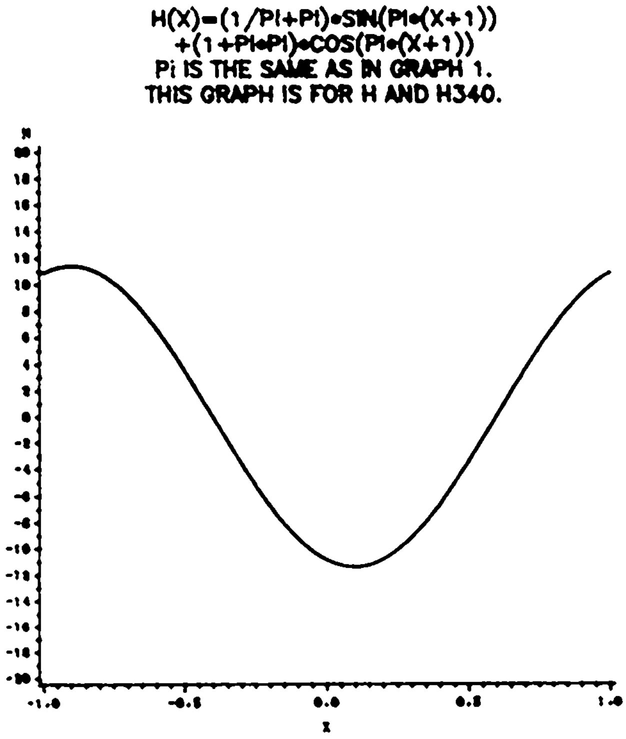
$$f(x) = \exp(-x) + 2\sin 2\pi(x+1)$$

and the exact solution is

$$h(x) = (1+4\pi^2)\sin 2\pi(x+1) + (e-2\pi)\delta(x+1) + 2\pi\,\delta(x-1),$$



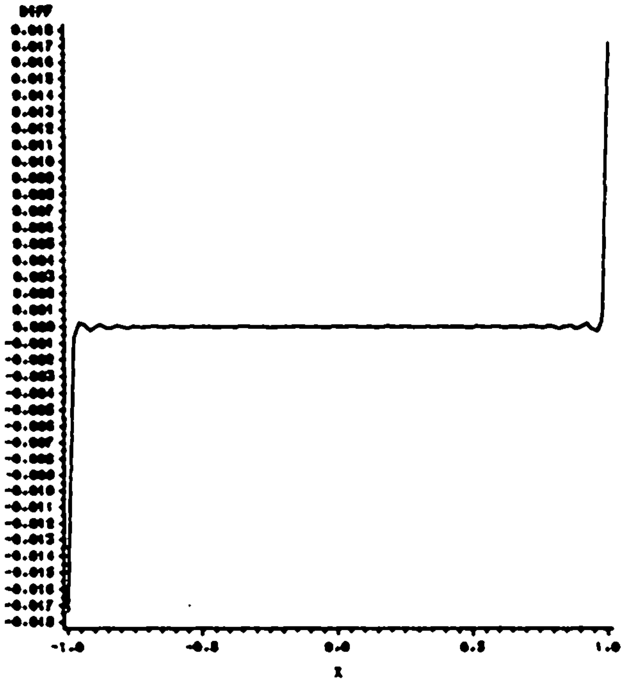
Graph 1



Graph 2

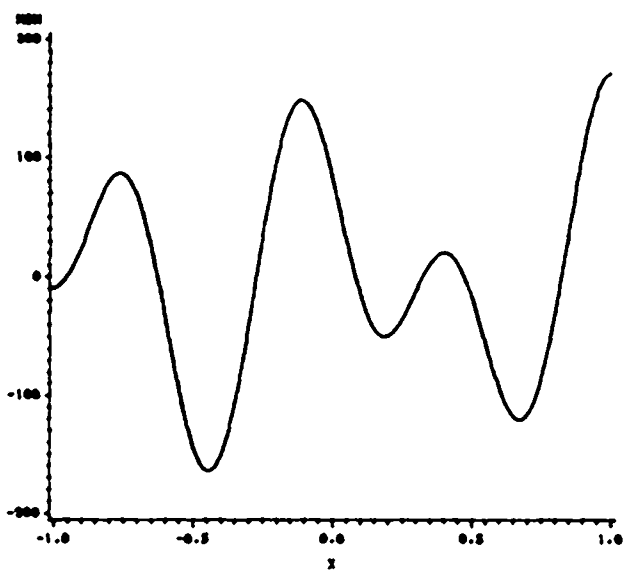
H(X)=(1/PI+PI)*SIN(PI*(X+1))
+(1+PI*PI)*COS(PI*(X+1))

DIFF IS THE DIFFERENCE BETWEEN H AND H340.



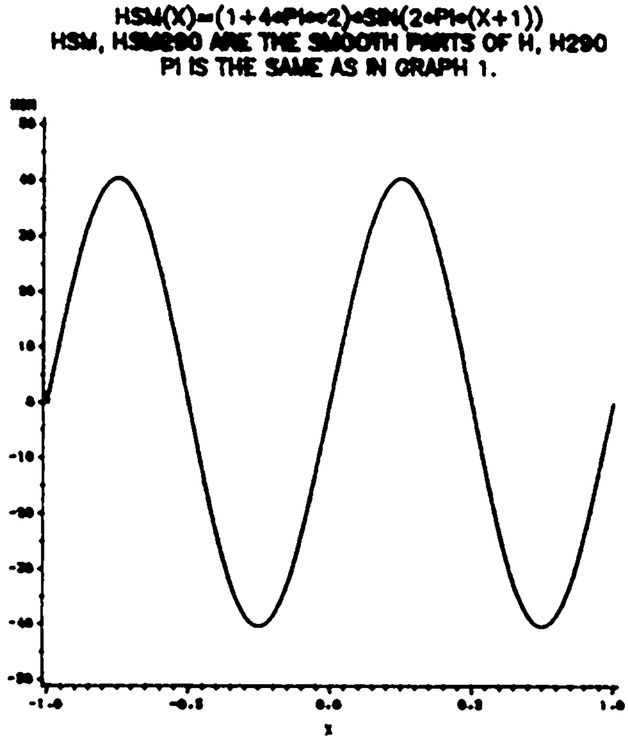
Graph 2.1

HSM(X)=(0.5+PI*PI/8)*COS(PI*(X+1))
+(2+8*PI*PI)*COS(2*PI*(X+1))
-0.75*(1+(3.5*PI)**2)*COS(7*PI*(X+1)/2
HSM, HSM7 ARE THE SMOOTH PARTS OF H, H7
PI IS THE SAME AS IN GRAPH 1.

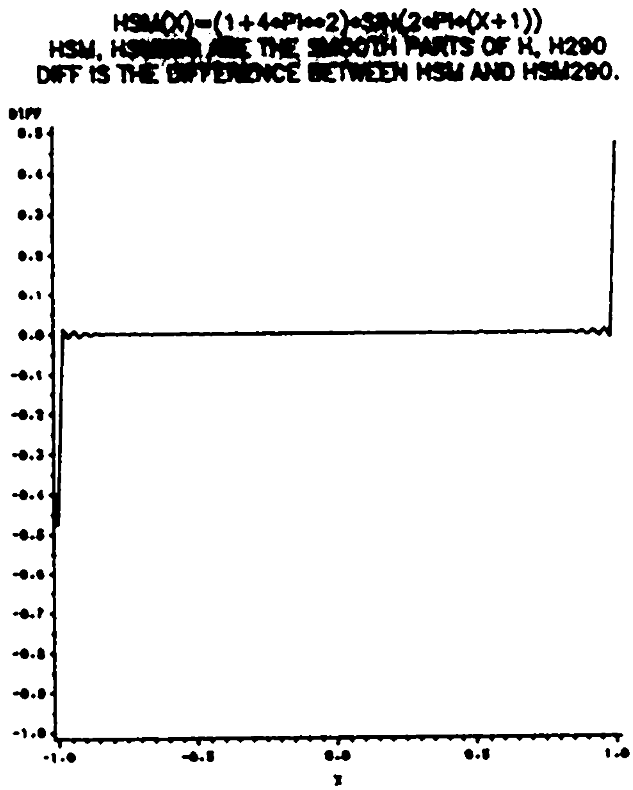


HSM AND HSM7 COINCIDE PRACTICALLY

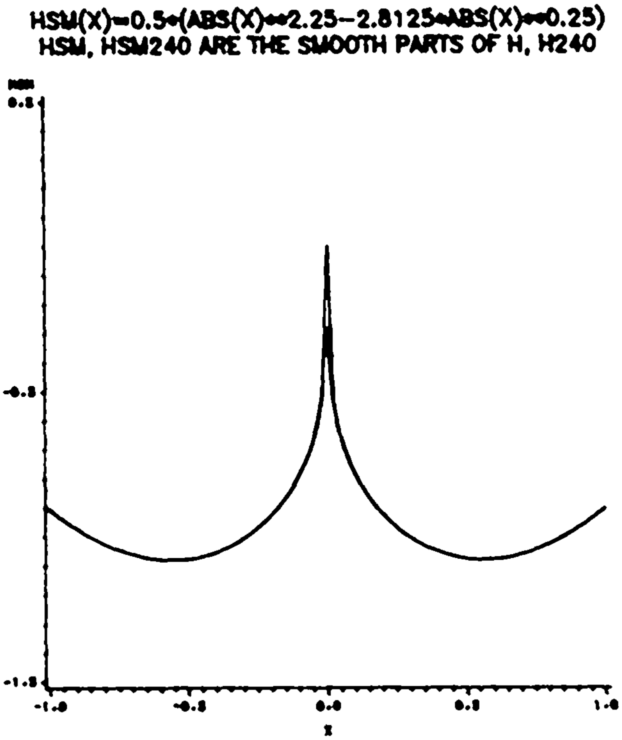
Graph 3



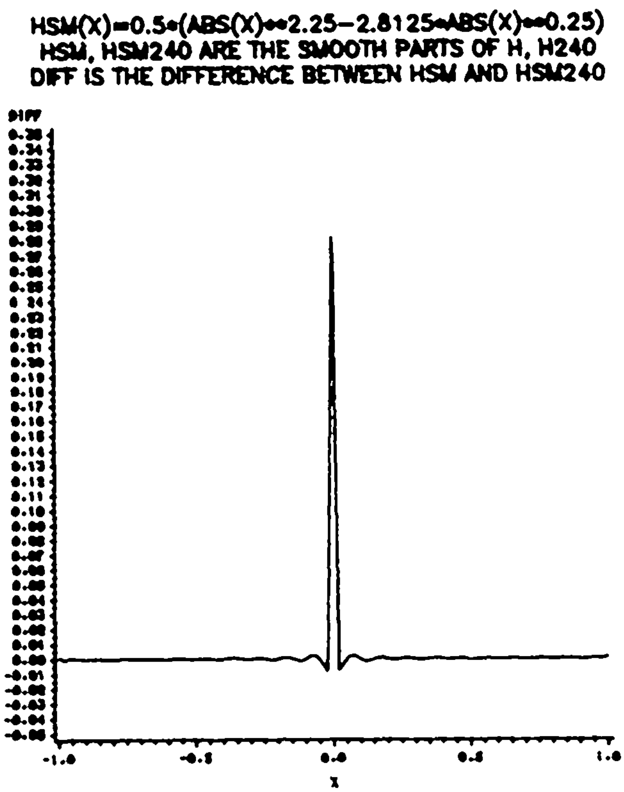
Graph 4



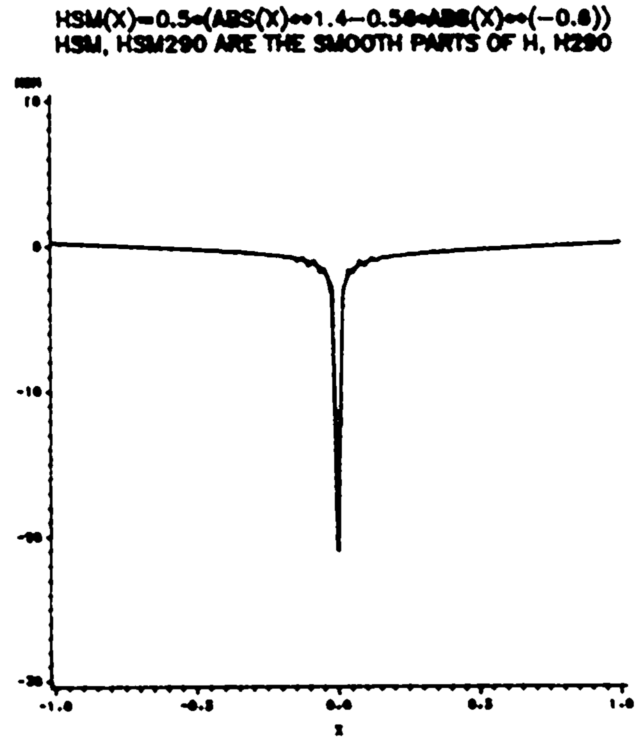
Graph 4.1



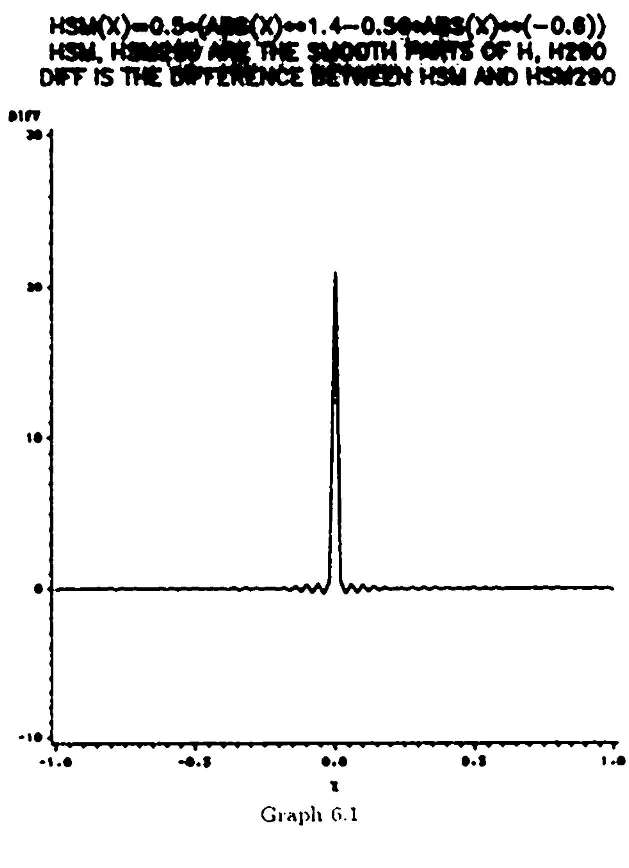
Graph 5



Graph 5.1



Graph 6



Graph 6.1

see Graph 4. If $n = 290$, then the condition number of $A_{290} = (a_{jp})_{-2\le j,p\le 290}$ is $O(10^6)$ and $\epsilon_{290} = O(10^{-9})$. The pointwise relative error of h_{290} is less than $O(10^{-4})$ for $-0.98 \le x \le 0.98$, and the pointwise error is between 0.47 and 0.014 in the interval $[-1.0, -0.98) \cup (0.98, 1.0]$ which is shown in the Graph 4.1 clearly. For the coefficients of the singular part, the relative error is $O(10^{-4})$.

(5) In the fifth example, one has

$$f(x) = |x|^{2.25}$$

and the exact solution is

$$h(x) = \frac{|x|^{2.25} - 2.8125|x|^{0.25}}{2} + 1.625\,(\delta(x+1) + \delta(x-1)).$$

If $n = 240$, then the condition number of A_{240} is $O(10^6)$ and $\epsilon_{240} = O(10^{-10})$. The relative error for the singular part is $O(10^{-7})$. In the interval $[-1, -0.12) \cup (0.12, 1]$, the pointwise relative error is $O(10^{-4})$ and in the interval $[-0.12, 0.12]$ it is $O(10^{-3})$ except at the origin. In particular, at the points near the origin, ± 0.02, the accuracy is $O(10^{-3})$, see Graph 5 for both smooth parts of h and h_{240}.

(6) In the sixth example, one has

$$f(x) = |x|^{1.4}$$

and the exact solution is

$$h(x) = \frac{|x|^{1.4} - 0.56|x|^{-0.6}}{2} + 1.2\,(\delta(x+1) + \delta(x-1))$$

where $x \in [-1, 0) \cup (0, 1]$. The result we got is for $n = 290$. In this case, the condition number of A_{290} is $O(10^6)$ and $\epsilon_{290} = O(10^{-5})$. The relative error for the singular part is $O(10^{-4})$. Qualitatively, approximation in the interval $[-1, 0) \cup (0, 1]$ here is similar to that in Example 5. The relative error is $O(10^{-2})$ in $[-1, -0.16) \cup (0.16, 1]$ and $O(10^{-1})$ in $[-0.16, 0.16] \setminus \{0\}$. It is no surprise to see that the error is much bigger than before. However we think that the result is reasonable, since $f(x)$ is less smooth than in the examples above.

From these examples, it is seen that the big error occurs either at the origin or at the end points. However it is observed in the computation that this can be overcome by increasing the number n of basis functions and the accuracy of the computation of f_j, $j = -2, -1, \ldots$ in (23). The reduction of the computational error requires additional computer time.

We conclude that this method is accurate and easy to implement numerically.

References

1. A.G. Ramm, *Theory and Applications of Some New Classes of Integral Equations*, Springer-Verlag, New York, (1980).
2. A.G. Ramm, *Random Fields Estimation Theory*, Pitman, Boston, (1989).
3. A.G. Ramm, Numerical solution of integral equations in a space of distributions, *J. Math. Anal. Appl.* **110**, 384–390 (1985).
4. V. Ivanov *et al.*, *Theory of Linear Ill-posed Problems and Applications*, in Russian, Nauka, Moscow, (1978).
5. I. Gelfand and G. Shilov, *Generalized Functions*, Vol. 1, Acad. Press, New York, (1964).