A polynomial-time algorithm for the paired-domination problem on permutation graphs

T.C.E. Cheng\textsuperscript{a,}*, Liying Kang\textsuperscript{b}, Erfang Shan\textsuperscript{a,b}

\textsuperscript{a}Department of Logistics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
\textsuperscript{b}Department of Mathematics, Shanghai University, Shanghai 200444, China

Received 19 April 2007; received in revised form 6 September 2007; accepted 16 February 2008
Available online 9 April 2008

Abstract

A set \( S \) of vertices in a graph \( H = (V, E) \) with no isolated vertices is a \textit{paired-dominating set} of \( H \) if every vertex of \( H \) is adjacent to at least one vertex in \( S \) and if the subgraph induced by \( S \) contains a perfect matching. Let \( G \) be a permutation graph and \( \pi \) be its corresponding permutation. In this paper we present an \( O(mn) \) time algorithm for finding a minimum cardinality paired-dominating set for a permutation graph \( G \) with \( n \) vertices and \( m \) edges.

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Keywords: Algorithm; Permutation graph; Paired-domination

1. Introduction

In this paper we in general follow [14] for notation and graph theory terminologies. Specifically, let \( G = (V, E) \) be a graph with \textit{vertex set} \( V \) and \textit{edge set} \( E \), and let \( v \) be a vertex in \( V \). The order of \( G \) is given by \( n = |V| \) and its size by \( m = |E| \). The \textit{open neighborhood} of \( v \) is defined by \( N(v) = \{u \in V \mid uv \in E\} \) and the \textit{closed neighborhood} of \( v \) is defined by \( N[v] = N(v) \cup \{v\} \). In general, let \( N(S) \) and \( N[S] \) denote, respectively, \( \cup_{v \in S} N(v) \) and \( \cup_{v \in S} N[v] \).

For subsets \( S, T \subseteq V \), the set \( S \) dominates the set \( T \) in \( G \) if \( N[T] \subseteq N[S] \). Each vertex \( v \) of \( G \) dominates itself and every vertex adjacent to \( v \), i.e., all vertices in its closed neighborhood. For \( S \subseteq V \), let \( \langle S \rangle \) denote the subgraph of \( G \) induced by \( S \).

A set \( S \subseteq V \) is a \textit{dominating set} of \( G \) if every vertex not in \( S \) is adjacent to at least one vertex in \( S \). The \textit{domination number} of \( G \) is the minimum cardinality of a dominating set of \( G \). A \textit{matching} in a graph \( G \) is a set of independent edges in \( G \). A \textit{perfect matching} \( M \) in \( G \) is a matching in \( G \) such that every vertex of \( G \) is incident to a vertex of \( M \).

A \textit{paired-dominating set} of a graph \( G \) is a set \( S \) of vertices of \( G \) such that every vertex is adjacent to some vertex in \( S \) and the subgraph induced by \( S \) contains a perfect matching \( M \) (not necessarily induced). Two vertices joined by an edge of \( M \) are said to be \textit{paired} and are also called \textit{partners} in \( S \). Every graph without isolated vertices has a paired-dominating set since the end-vertices of any maximal matching form such a set. The \textit{paired-domination number} of
\[ \gamma_{pr}(G), \] is the minimum cardinality of a paired-dominating set. The minimum paired-dominating set problem, abbreviated as MPDS, is to find a paired-dominating set \( S \) of \( G \) such that \( |S| \) is minimized. Paired-domination was introduced by Haynes and Slater [14] as a model for assigning backups to guards for security purposes, and has been studied from the theoretic point of view, for example, in \([2–4,7,8,10,11,15–19,21,25–27,29]\), among others.

The aim of this paper is to investigate the problem of determining \( \gamma_{pr}(G) \) for a permutation graph \( G \) from the algorithmic point of view. The decision problem to determine a minimum cardinality paired-dominating set of an arbitrary graph has been known to be NP-complete (see [13]). For the special case of trees, Qiao et al. [26] presented a linear time algorithm. Cheng et al. [8] proposed an \( O(m + n) \) and \( O(m(m + n)) \) time algorithms to solve the MPDS problem for interval graphs and circular-arc graphs, respectively. The literature on algorithmic aspects of domination in graphs has been surveyed and detailed by Chang [5].

Let \( \pi = [\pi_1, \pi_2, \ldots, \pi_n] \) be a permutation on the set \( V_n = \{1, 2, \ldots, n\} \). Then the permutation graph \( G[\pi] = (V, E) \) is the undirected graph such that \( V = V_n \) and \( (i, j) \in E \) if and only if
\[ (i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0, \]
where \( \pi^{-1}(i) \) is the position of \( i \) in \( \pi = [\pi_1, \pi_2, \ldots, \pi_n] \).

A permutation graph is an intersection graph based upon the permutation diagram Fig. 1, which is defined as follows: Write the number 1, 2, \ldots, \( n \) horizontally from left to right. Under every \( i \), write the number \( \pi(i) \). Draw line segments connecting \( i \) in the top row and \( i \) in the bottom row, for each \( i \). It is easy to see that two vertices \( i \) and \( j \) of \( G[\pi] \) are adjacent if and only if the corresponding line segments of \( i \) and \( j \) intersect. Fig. 1 shows the permutation graph \( G[\pi] \) where its corresponding permutation diagram of a permutation \( \pi[3, 1, 5, 7, 4, 2, 6] \).

Fig. 1. A permutation graph and its permutation diagram.

In this paper, we propose an efficient \( O(mn) \) algorithm for solving the MPDS problem on permutation graphs. Our algorithm is based on a recursive formula using the dynamic programming method. In Section 2, we describe our recursive formula of the dynamic programming. Our algorithm is described in Section 3. Section 4 contains some conclusions.

2. A dynamic programming approach

In this section, we shall describe our basic approach based upon the dynamic programming approach. Essentially, we want to find an MPDS of \([ \pi_1, \pi_2, \ldots, \pi_n ]\) dominating \([1, 2, \ldots, n]\). In the following, we may assume that the permutation graph \( G[\pi] \) discussed below is connected; otherwise, we look at each (connected) component separately.

For convenience, we introduce more notation as follows:

1. For any \( 1 \leq i, j \leq n \), and \( V_i = \{ \pi_1, \pi_2, \ldots, \pi_i \} \), denote \( V_{i,j} \) as the subset of \( V_i \) containing all elements smaller than or equal to \( j \), i.e., \( V_{i,j} = \{ \pi_k \in V_i \mid \pi_k \leq j \} \). Clearly, \( V_{i,j} \subseteq V_i \).

2. For each \( 1 \leq i \leq n \), denote \( \pi_i^* \) as the minimum number over the suffix \( \pi_i, \pi_{i+1}, \ldots, \pi_n \), i.e., \( \pi_i^* = \min(\pi_i, \pi_{i+1}, \ldots, \pi_n) \), and set \( V_i^* = V_i \cup \{ \pi_i^* \} \).

3. For any vertex set \( S \), define max(\( S \)) as the maximum number in \( S \).

4. For a family \( \mathcal{F} \) of sets of vertices, Min(\( \mathcal{F} \)) denotes a minimum cardinality set \( S \) in \( \mathcal{F} \) and max(\( S \)) is as large as possible if \( \mathcal{F} \) is not the empty set; Min(\( \mathcal{F} \)) denotes a set of infinite cardinality otherwise. Min(\( \mathcal{F} \)) may not be unique. If there is more than one candidate for Min(\( \mathcal{F} \)), we select arbitrarily one of the candidates.
Lemma 1. For a permutation graph $G[\pi]$ with no isolated vertices, $\langle V_i^* \rangle$ has no isolated vertices for each $i$, $1 \leq i \leq n$.

Proof. Suppose to the contrary that there exists an $i_0$ ($1 \leq i_0 \leq n$) such that $\langle V_{i_0}^* \rangle$ has an isolated vertex $\pi_l$ ($l \leq i_0$). Then $\pi_l \leq \pi_{i_0}^*$, for otherwise $(\pi_i, \pi_{i_0}^*) \in E(G)$. If $\pi_l = \pi_{i_0}^*$ ($= \min(\pi_{i_0}, \pi_{i_0+1}, \ldots, \pi_n)$), then $\pi_l = \pi_{i_0}$. Hence, $\pi_{i_0}$ is an isolated vertex in $G$, contradicting the assumption of the lemma. If $\pi_l < \pi_{i_0}^*$, then $\pi_l = l$. Thus, for $1 \leq i < l$, $\pi_i < l$, and for $l < i \leq n$, $\pi_i > l$. This implies that $\pi_i$ is an isolated vertex in $G$, contradicting our assumption again. □

By Lemma 1, we see that $\langle V_i^* \rangle$ has no isolated vertices, so it is clear that for each $i$ and $j$, $1 \leq i$, $j \leq n$, there exists a subset $D$ of $V_i^*$ such that $D$ dominates all the vertices of $V_i$ and $\langle D \rangle$ has a perfect matching in $\langle V_i^* \rangle$.

Based on Lemma 1, for each $i$ and $j$, $1 \leq i$, $j \leq n$, we define $PD_{i,j}$ as follows:

(i) $PD_{i,j}$ is a minimum cardinality subset $S$ of $V_i^*$ such that $S$ is a dominating set of $\langle V_i, j \rangle$ and $\langle S \rangle$ has a perfect matching in $\langle V_i^* \rangle$;

(ii) $\max(PD_{i,j})$ is as large as possible.

In particular, we define $PD_{0,j} = \emptyset$ for $1 \leq j \leq n$. Clearly, $PD_{n,n}$ is a desired minimum cardinality paired-dominating set for $G[\pi]$.

We define $X = \{S : S \subseteq V_i^* \text{ such that } S \text{ is a dominating set of } \langle V_{i,j} \rangle \}$. Following the above definitions, we have

$PD_{i,j} = \begin{cases} \emptyset & \text{if } V_{i,j} = \emptyset, \\ \min(X) & \text{otherwise.} \end{cases}$

Consider the case $i = 1$. If $j < \pi_1$, then $V_{1,j} = \{\pi_1\} \cap \{1, 2, \ldots, j\} = \emptyset$, and so $PD_{1,j} = \emptyset$. Otherwise, $V_{1,j} = \{\pi_j\}$. According to our assumption that $G$ contains no isolated vertices, we have $\pi_1 \neq 1$. Then $\pi_1^* = 1$ and $V_{1,j}^* = \{1, \pi_1\}$. Hence $PD_{1,j} = \{1, \pi_1\}$. So we obtain

$PD_{1,j} = \begin{cases} \emptyset & \text{if } j < \pi_1, \\ \{1, \pi_1\} & \text{otherwise.} \end{cases}$

We first give several basic lemmas that will be useful for the proof of our recursive formula $PD_{i,j}$.

Lemma 2 (Chao et al. [6]). For positive integers $i_1, i_2$ and $j$, if $1 \leq i_1 < i_2 \leq n$ and $1 \leq j \leq n$, then $V_{i_1,j} \subseteq V_{i_2,j}$ and $V_{i_1}^\ast \subseteq V_{i_2}^\ast$.

Lemma 3. For $1 \leq i < j < k \leq n$ and $\pi_k < \pi_j < \pi_i$, if $w$ is adjacent to $\pi_j$, then $w$ is adjacent to at least one of $\pi_k$ and $\pi_i$.

Proof. The proof is straightforward and omitted. □

Lemma 4. For $1 < l \leq i$, there exists a $PD_{l-1, \pi_i^*}$ such that $\pi_i^* \notin PD_{l-1, \pi_i^*}$.

Proof. Let $S$ be a $PD_{l-1, \pi_i^*}$. Thus $S \subseteq V_{l-1}^\ast$ is a dominating set of $\langle V_{l-1, \pi_i^*} \rangle$ and $\langle S \rangle$ has a perfect matching in $\langle V_{l-1}^\ast \rangle$. If $\pi_i^* \notin S$, then the desired result follows. If $\pi_i^* \in S$, then $\pi_i^* = \pi_{l-1}^*$ as $S \subseteq V_{l-1}^\ast$. Hence, there exists a vertex $\pi_{i'} \in S$ ($i' \leq l - 1$) such that $\pi_{i'}^* = \pi_i^*$ and $\pi_{i'}$ are paired in $S$. So, we have $\pi_i^{-1}(\pi_{i'}^*) > i'$ and $(\pi_i^{-1}(\pi_{i'}^*) - i')(\pi_i^* - \pi_{i'}^*) < 0$. Thus $\pi_{i'}^* > \pi_i^*$. We claim that $N(\pi_{i'}) \cap V_{l-1}^\ast - S \neq \emptyset$. If this is not so, then $\pi_{i'}$ dominates no vertices of $V_{l-1, \pi_i^*}$, and so does $\pi_i^*$ as $\pi_{i'} > \pi_i^*$. This means that $S - \{\pi_{i'}, \pi_i^*\} \subseteq V_{l-1}^\ast$ is a dominating set of $\langle V_{l-1, \pi_i^*} \rangle$ and $\langle S - \{\pi_{i'}, \pi_i^*\} \rangle$ has a perfect matching in $\langle V_{l-1}^\ast \rangle$. Thus $S - \{\pi_{i'}, \pi_i^*\}$ is a $PD_{l-1, \pi_i^*}$, which contradicts the minimality of $S$. Let $\pi_i^* \in N(\pi_{i'}) \cap V_{l-1}^\ast - S$ and $S' = S \cup \{\pi_i^*\} - \{\pi_i^*\}$. Then $S' \subseteq V_{l-1}^\ast$ is a dominating set of $\langle V_{l-1, \pi_i^*} \rangle$ and $\langle S' \rangle$ has a perfect matching in $\langle V_{l-1}^\ast \rangle$ with $|S'| = |S| + |\pi_i^*|$ and $\max(S') \geq \max(S)$. So $S'$ is a $PD_{l-1, \pi_i^*}$, satisfying $\pi_i^* \notin S'$, as required. □
For $1 < i \leq n$, we define

$$PD_{\pi_i^*} = \text{Min}(\{PD_{l-1, \pi_i^*} \cup \{\pi_i^*, \pi_l\} : \pi_l \in N(\pi_i^*), \pi_i^* \not\in PD_{l-1, \pi_i^*}, l \leq i\})$$

and

$$PD_{\max} = \begin{cases} PD_{l-1,j} \cup \{\pi_i, \text{max}(V_i)\} & \text{if } \pi_i \neq \text{max}(V_i), \\ V_i & \text{otherwise.} \end{cases}$$

By Lemma 4, $PD_{\pi_i^*} \neq \emptyset$. The following Lemmas 5 and 6 assert that $PD_{\pi_i^*}$ and $PD_{\max}$ (if $\text{max}(V_i) \neq \pi_i$ and $\text{max}(PD_{l-1,j}) < \pi_i$) are candidates for computing $PD_{i,j}$.

**Lemma 5.** For any integers $i$ and $j$, $1 < i \leq n$ and $1 \leq j \leq n$, $PD_{\pi_i^*} \in X_1(\subseteq X)$.

**Proof.** By the definition of $PD_{\pi_i^*}$, $\pi_i^* \not\in PD_{l-1,\pi_i^*}$, while $PD_{l-1,\pi_i^*}$ is a minimum dominating set of $(V_{l-1,\pi_i^*})$. We claim $\pi_l \not\in PD_{l-1,\pi_i^*}$. If this is not the case, then it is easy to see that $\pi_l = \pi_{l-1}^* \leq \pi_i^*$. On the other hand, since $\pi_l \in N(\pi_i^*)$ $(l \leq i)$, $\pi_l > \pi_i^*$, which is impossible. From Lemma 2, $V_{l-1}^* \subseteq V_i^*$ as $l \leq i$. Hence, $PD_{l-1,\pi_i^*} \cup \{\pi_i^*, \pi_l\} \subseteq V_i^*$. We next show that each vertex of $V_{i,j} - V_{l-1,\pi_i^*}$ is dominated by $\pi_i^*$ or $\pi_l$. Let $\pi_k \in V_{i,j} - V_{l-1,\pi_i^*}$. If $\pi_k > \pi_i^*$, then $(\pi_k - \pi_i^*)(k - \pi_i^{-1}(\pi_i^*)) < 0$, and so $\langle \pi_k, \pi_i^* \rangle \in E$. If $\pi_k < \pi_i^*$, then $k \geq l$. Since $\pi_l \in N(\pi_i^*)$ and $l \leq i$, $\pi_l > \pi_i^*$, then $\pi_l > \pi_i^* > \pi_k$. This implies that $(\pi_l - \pi_i^*)(k - l) \leq 0$, i.e., $\pi_k = \pi_l$ or $\langle \pi_k, \pi_l \rangle \in E$. Hence, all the vertices in $V_{i,j}$ are dominated by $PD_{l-1,\pi_i^*} \cup \{\pi_i^*, \pi_l\}$. Therefore, $PD_{l-1,\pi_i^*} \cup \{\pi_i^*, \pi_l\} \in X_1$. Note that $PD_{\pi_i^*} = \text{Min}(\{PD_{l-1,\pi_i^*} \cup \{\pi_i^*, \pi_l\} : \pi_l \in N(\pi_i^*), l \leq i\})$, so $PD_{\pi_i^*} \in X_1$, as desired. □

**Lemma 6.** For any integers $i$ and $j$, $1 < i \leq n$ and $1 \leq j \leq n$, if $\text{max}(V_i) \neq \pi_i$ and $\text{max}(PD_{l-1,j}) < \pi_i$, then $PD_{\max} \in X$.

**Proof.** Clearly, $PD_{\max} \subseteq V_i^*$. Since $\text{max}(V_i) \neq \pi_i$ and $\text{max}(PD_{l-1,j}) < \pi_i$, $\pi_i \not\in PD_{l-1,j}$ and $\pi_i \not\in \text{max}(V_i)$, and thus $\text{max}(V_i) \not\in PD_{l-1,j}$ and $(\text{max}(V_i), \pi_j) \in E$. Note that $V_{i,j} - V_{l-1,j} \subseteq \{\pi_l\}$, and we have $PD_{\max} = PD_{l-1,j} \cup \{\pi_l, \text{max}(V_i)\}$ as a dominating set of $(V_{i,j})$ and $(PD_{\max})$ has a perfect matching in $(V_i^*)$, the desired result follows. □

In order to present the recursive formula of $PD_{i,j}$ for the case of $1 < i \leq n$, we further prove the following several lemmas.

**Lemma 7.** For each $S \in \text{Min}(X_1)$, let $\pi_l = \text{max}(S)$. Then $\pi_i^* < \pi_l$ and $\pi_l \in N(\pi_i^*)$.

**Proof.** By the definition of $X_1$, we have $\pi_i^* \in S$. Suppose $\pi_i^* \geq \pi_l$, then $\text{max}(S) = \pi_i^*$. This implies that $\pi_i^*$ is an isolated vertex of $S$, which contradicts the assumption that $S$ has a perfect matching in $(V_i^*)$. So $\pi_i^* < \pi_l$. Furthermore, since $(\pi_l - \pi_i^*)(l - \pi_i^{-1}(\pi_i^*)) < 0$, $\langle \pi_i^*, \pi_l \rangle \in E$, and thus $\pi_l \in N(\pi_i^*)$. □

By the definition of $\text{Min}(X_1)$, all the candidates $S$ for $\text{Min}(X_1)$ have the same $\text{max}(S)$. Let $S \in \text{Min}(X_1)$, $\pi_l = \text{max}(S)$, and let $M$ be a perfect matching in $(S)$. Let $S \in \text{Min}(X_1)$, $\pi_l = \text{max}(S)$, and let $M$ be a perfect matching in $(S)$.

**Lemma 8.** For any integers $i$ and $j$, $1 < i \leq n$ and $1 \leq j \leq n$, if there exist $\pi_{i_1}$ ($i_1 < l$) and $\pi_{l'}$ such that $(\pi_i^*, \pi_{i_1}) \in M$ and $(\pi_l, \pi_{l'}) \in M$, then $\text{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.

**Proof.** By Lemma 5, it suffices to show that there exists an $S^* \in PD_{\pi_i^*} \cap X_1$ such that $\text{max}(S^*) \geq \text{max}(S) = \pi_l$. Note that $\text{max}(S) = \pi_l > \pi_{l'} \in S$ and $(\pi_l, \pi_{l'}) \in M$, so $l' > l$. We distinguish the following two cases depending on whether or not $\pi_{l-1}^* \in S$.

**Case 1.** Suppose first $\pi_{l-1}^* = \pi_i^*$. In this case, we claim that $N(\pi_{i_1}) \cap V_l - S \neq \emptyset$. Otherwise, since $\pi_i^* < \pi_{l'} < \pi_l$ and $l' < l < \pi_i^{-1}(\pi_i^*)$, by Lemma 3, each vertex dominated by $\pi_{l'}$ in $G$ is adjacent to $\pi_l$ or $\pi_i^*$. Furthermore, for each $t > l$, $\pi_l \in V_{i,j}$, it is dominated by $\pi_{l'}$ as $\pi_l > \pi_{l'}^* (= \pi_{l-1}^*)$. This implies that $S - \{\pi_{i_1}, \pi_{l'}\}$ is a dominating set of $(V_{i,j})$ and $(S - \{\pi_{i_1}, \pi_{l'}\})$ has a perfect matching $M \cup \{(\pi_i^*, \pi_{i_1})\} - \{(\pi_i^*, \pi_{l'}), (\pi_l, \pi_{l'})\}$ in $(V_i^*)$ by making a pair of $\pi_l$ and $\pi_{l'}$, contradicting the minimality of $S$. Let $\pi_{l'} \in N(\pi_{i_1}) \cap V_l - S$ and let $S_1 = S \cup \{\pi_{i_1}, \pi_{l'}\} - \{\pi_{l'}\}$. Then $S_1 \subseteq V_i^*$ is a dominating set of $(V_{i,j})$ and $M_1 = (M \cup \{(\pi_i^*, \pi_{i_1}), (\pi_l, \pi_{l'})\}) - \{(\pi_i^*, \pi_{l'}), (\pi_l, \pi_{l'})\}$ is a perfect matching in $(S_1)$. So $S_1 \subseteq X_1$ with $|S_1| = |S|$ and $\text{max}(S_1) \geq \text{max}(S)$ such that $\pi_{l'} \not\in S_1$ and $\pi_{l-1}^* \in S_1$. 
For any \( \pi_k \in S_1 \), where \( l < k \leq i \), there exists \( \pi_{k'} \) such that \((\pi_k, \pi_{k'}) \in M_1\). We claim that \( k' > l \) and 
\[ N(\pi_{k'}) \cap V_l - S_1 \neq \emptyset. \] Indeed, if \( k' > l \), then for each vertex \( \pi_t \in N((\pi_k, \pi_{k'}) \cap V_l - S_1 \), we have \( \pi_t > \pi_k > \pi_{k-1} = \pi_t^* \) or \( \pi_t > \pi_{k'} > \pi_{k-1} = \pi_t^* \), so \( \pi_t \) is dominated by \( \pi_t^* \). Moreover, note that for each vertex \( \pi_t \in V_{ij} \), \( l < t \leq i \), it is also dominated by \( \pi_t^* \) as \( \pi_t \geq \pi_{k'}^* \). This implies that \( S - (\pi_k, \pi_{k'}) \) is a dominating set of \((V_{ij}) \) and \((S_1 - (\pi_k, \pi_{k'})) \) still has a perfect matching in \((V_i^*)\), which contradicts the minimality of \( S_1 \). So \( k' < l \). We further show that \( N(\pi_{k'}) \cap V_l - S_1 \neq \emptyset \). Otherwise, since \( k' < l < k \) and \((\pi_k, \pi_{k'}) \in E, \pi_{k'} > \pi_k > \pi_{k-1} = \pi_t^* \), then \( \pi_{k'} \) is dominated by \( \pi_t^* \). As above, we deduce that \( S_1 - (\pi_k, \pi_{k'}) \) is a dominating set of \((V_{ij}) \) and \((S_1 - (\pi_k, \pi_{k'})) \) has a perfect matching in \((V_i^*)\), a contradiction. Let \( \pi_{k''} \in N(\pi_{k'}) \cap V_l - S_1 \) and let \( S_2 = S_1 \cup (\pi_{k''} - \{\pi_k\}) \). Then \( S_2 \subset V_i^* \) is a dominating set of \((V_{ij}) \) with \( |S| = |S_1| \) and \( |S_2| \) has a perfect matching in \((V_i^*)\) and \( |S_2| \geq |S_1| \). For any \( \pi_1 \in S_2 \), where \( l < s < i \), continuing the process as above, we can obtain after a finite number of steps a set \( S^* \subset V_i^* \) satisfying the following conditions:

(i) \( S^* \cap \{(\pi_{l+1}, \pi_{l+2}, \ldots, \pi_i) - \{\pi_t^*\}\} = \emptyset; \)

(ii) \( S^* \subset V_i^* \) is a dominating set of \((V_{ij}) \) with \( |S^*| = |S| \) and \( |S^*| \) in \((V_i^*)\) has a perfect matching in which \( \pi_t^* \) and \( \pi_t \) are paired;

(iii) \( \max(S^*) \geq \max(S) \).

Then \( S^* \in X_1 \). Since \( \pi_t^* < \pi_t \), it follows that \( V_{ij} - \pi_t^* \) is dominated by \( \pi_t^* \) or \( \pi_t \), so \( S^* - \{\pi_t^*, \pi_t\} \) is a dominating set of \((V_{ij}) \) and \((S^* - \{\pi_t^*, \pi_t\}) \in (V_i^*)\) has a perfect matching. By the minimality of \( S^* \), we deduce that \( S^* - \{\pi_t^*, \pi_t\} \subset V_{ij} \) is a minimum cardinality dominating set of \((V_{ij}) \) and contains a perfect matching. Then \( \pi_t^* - \{\pi_t^*, \pi_t\} \) is a \( PD\pi_t^* \). Hence, \(|S| = |S^*| = |PD\pi_t^*| + 2 \). Note that \(|PD\pi_t^*| = |PD\pi_t^*| + 2 \). Then \( |PD\pi_t^*| = |PD\pi_t^*| + 2 \), then \( \max(PD\pi_t^*) = \max(S^*) \geq \max(S) \). So \( \text{Min}(X_1 \cup \{PD\pi_t^*\}) = PD\pi_t^* \).

Case 2. Suppose \( \pi_{l-1}^* \neq \pi_t^* \). As in Case 1, we first find a set \( S_1 \in X_1 \) with \(|S_1| = |S| \) and \( \max(S_1) \geq \max(S) \) such that \( \pi_t \notin S_1 \) and \( \pi_{l-1}^* \notin S_1 \).

Suppose \( \pi_{l-1}^* \notin S_1 \). Since \( \pi_{l-1}^* < \pi_t \), it follows that \( \pi_{l-1}^* \notin S_1 \). Clearly, \( S_1 \subset V_i^* \). We further show that \( S_1 \) is a dominating set of \((V_{ij}) \). It suffices to show that all the vertices dominated by \( \pi_t^* \) can be dominated by \( S_1 \). Indeed, let \( \pi_t \in N(\pi_t^*) \). If \( l > l' \), it follows from \( \pi_t > \pi_t^* \) that \( \pi_t < \pi_t^* \) or \( \pi_t > \pi_t^* \). Observe that \( \pi_t < \pi_t^* \) and \( l < l' \leq i \leq \pi_t - \pi_t^* \), then \( \pi_t \) is dominated by \( \pi_t \) or \( \pi_t^* \). If \( l < l' \leq i \leq \pi_t - \pi_t^* \), then \( \pi_t \) is dominated by \( \pi_t^* \). Therefore, \( S_1 \) is a dominating set of \((V_{ij}) \) and \( M_1 = M \cup \{\pi_{l+1}, \pi_{l+2}, \pi_i, \pi_{l+1} \} \) is a perfect matching in \((S_1) \). So \( S_1 \subset X_1 \) and \( \max(S_1) = \max(S_1) \) such that \( \pi_t \notin S_1 \) and \( \pi_{l-1}^* \notin S_1 \).

Suppose \( \pi_{l-1}^* \in S_1 \). Let \( \pi_{l-1}^* \in X_1 \). We claim that \( N(\pi_{l-1}^*) \cap V_l - S_1 \neq \emptyset \). If this is not so, then, for each vertex \( \pi_t \in N(\pi_{l-1}^*) \cap V_l - S_1 \), \( l < t \leq i \). This implies that \( \pi_{l-1}^* \notin S_1 \) such that \( \pi_t \notin S_1 \) and \( \pi_{l-1}^* \notin S_1 \).

For any \( \pi_k \neq \pi_{l-1}^*, \pi_k \in S_1 \), where \( l < k \leq i \), there exists \( \pi_{k'} \) such that \((\pi_k, \pi_{k'}) \in M_1\). We claim that \( k' > l \) and 
\[ N(\pi_{k'}) \cap V_l - S_1 \neq \emptyset. \] Indeed, if \( k' > l \), then for each vertex \( \pi_t \in N((\pi_k, \pi_{k'}) \cap V_l - S_1 \), we have \( \pi_t > \pi_k > \pi_{k-1} = \pi_t^* \) or \( \pi_t > \pi_{k'} > \pi_{k-1} = \pi_t^* \), so \( \pi_t \) is dominated by \( \pi_t^* \). Moreover, for each vertex \( \pi_t \in V_{ij} \), \( l < t \leq i \), we have \( \pi_t < \pi_t^* \) or \( \pi_t > \pi_t^* \), so \( \pi_t \) is dominated by \( \pi_t^* \). This implies that \( S_1 - (\pi_k, \pi_{k'}) \) is a dominating set of \((V_{ij}) \) and \((S_1 - (\pi_k, \pi_{k'})) \) still has a perfect matching in \((V_i^*)\), which contradicts the minimality of \( S_1 \). So \( k' < l \).

Similar to the discussion in Case 1, we can deduce that \( N(\pi_{k'}) \cap V_l - S_1 \neq \emptyset \).

Let \( \pi_{k''} \in N(\pi_{k'}) \cap V_l - S_1 \) and let \( S_2 = S_1 \cup \{\pi_{k''} - \{\pi_k\} \). Then \( S_2 \subset V_i^* \) is a dominating set of \((V_{ij}) \) with \( |S_2| = |S_1| \) and \( |S_2| \) has a perfect matching in \((V_i^*)\) and \( |S_2| \geq |S_1| \). Proceeding as above, we get a set \( S^* \subset V_i^* \) satisfying the following conditions:

(i) \( S^* \cap \{(\pi_{l+1}, \pi_{l+2}, \ldots, \pi_i) - \{\pi_t^*\}\} = \emptyset; \)

(ii) \( S^* \subset V_i^* \) is a dominating set of \((V_{ij}) \) with \( |S^*| = |S| \) and \( |S^*| \) in \((V_i^*)\) has a perfect matching in which \( \pi_t^* \) and \( \pi_t \) are paired;

(iii) \( \max(S^*) \geq \max(S) \).
Then $S^* \in X_1$. As in Case 1, it can be verified that no vertex in $V_{l-1, \pi^*_l}$ is dominated by $\pi^*_l$ or $\pi_l$ since $\pi^*_l < \pi_l$, so $S^* - \{\pi^*_l, \pi_l\}$ is a dominating set of $(V_{l-1, \pi^*_l})$ and $(S^* - \{\pi^*_l, \pi_l\})$ in $(V_{l-1, \pi^*_l})$ has a perfect matching. By the minimality of $S^*$, it follows that $S^* - \{\pi^*_l, \pi_l\} \subseteq V_{l-1, \pi^*_l}$ is a minimum cardinality dominating set of $(V_{l-1, \pi^*_l})$. Then $S^* - \{\pi^*_l, \pi_l\}$ is a $PD_{l-1, \pi^*_l}$, and thus $S^*$ is a $PD_{\pi^*_l}$. Hence, $|S| = |S^*| = |PD_{l-1, \pi^*_l}| + 2$. Note that $|PD_{\pi^*_l}| \leq |PD_{l-1, \pi^*_l}| + 2 = |S|$ and if $|PD_{\pi^*_l}| = |PD_{l-1, \pi^*_l}| + 2$, then max$(PD_{\pi^*_l}) = max(S^*) \geq max(S)$. Therefore, $Min(X_1 \cup \{PD_{\pi^*_l}\}) = PD_{\pi^*_l}$. 

**Lemma 9.** For any integers $i$ and $j$, $1 \leq i \leq n$ and $1 \leq j \leq n$, if there exist $\pi_{i_1}$, $i > l$ and $\pi_{i_2}$ such that $(\pi^*_l, \pi_{i_1}) \in M$ and $(\pi_l, \pi_{i_2}) \in M$, then $Min(X_1 \cup \{PD_{\pi^*_l}\}) = PD_{\pi^*_l}$. 

**Proof.** Similar to **Lemma 8**, we need to show that there exists an $S^* \subseteq PD_{\pi^*_l} \cap X_1$ such that $max(S^*) \geq max(S)$. We claim that $\pi^*_l \neq \pi^*_l, \pi^*_l - 1 \not\subseteq S$, and $N(\pi^*_l - 1) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} \not\subseteq \emptyset$. We first show that $\pi^*_l - 1 \neq \pi^*_l$. Suppose to the contrary that $\pi^*_l - 1 = \pi^*_l$, then it is easy to see that $\pi^*_l < \pi^*_l < \pi_l$ and $\pi^*_l < \pi_l < \pi^*_l$. Hence, by **Lemma 3**, $S - \{\pi^*_l, \pi_{l-1}\}$ is a dominating set of $(V_{i,j})$ and $(S - \{\pi^*_l, \pi_{l-1}\})$ has a perfect matching in $(V_{i,j})$ by pairing $\pi^*_l$ with $\pi_l$ which contradicts the minimality of $S$. So $\pi^*_l - 1 \neq \pi^*_l$. Second, we show that $\pi^*_l - 1 \not\subseteq S$. Suppose this is not the case, $\pi^*_l - 1 \subseteq S$. For any vertex $\pi_l \in N(\pi^*_l), r < i$, then $\pi_l > \pi^*_l$. By our assumption that $(\pi^*_l, \pi_{l-1}) \in M$, we have $\pi^*_l > \pi^*_l$ as $i_1 < \pi^*_l$ (i.e. $\pi^*_l$). Hence, $(\pi^*_l, \pi_{l-1}) \in E$. If $t > i_1 (l > 1)$, then $\pi^*_l < \pi_l < \pi^*_l$ and $(\pi^*_l, \pi_l) \in E$. So $N[\pi^*_l] \subseteq N[\pi^*_l] \cup N[\pi^*_l]$. For any vertex $\pi_l \in N[\pi^*_l], t \leq l - 1$, then $\pi_l > \pi^*_l \geq \pi^*_l$ and $t < l - 1 \leq \pi^*_l (\pi^*_l), so \{(\pi^*_l, \pi^*_l), \pi^*_l\} \not\subseteq \emptyset$. Then $S^* - \{\pi^*_l, \pi_{l-1}\}$ is a dominating set of $(V_{i,j})$ and $M' = M \cup \{(\pi^*_l, \pi^*_l), \pi^*_l\}$ is a perfect matching in $(S')$. This contradicts the minimality of $S$. So $\pi^*_l - 1 \not\subseteq S$. Second, we show that $\pi^*_l - 1 \not\subseteq S$. If $N(\pi^*_l - 1) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} = \emptyset$, then $N(\pi^*_l) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} = \emptyset$, so we have $N[\pi^*_l] \subseteq N[\pi^*_l] \cup N[\pi^*_l]$. Hence, $S - \{\pi^*_l, \pi_{l-1}\}$ is a dominating set of $(V_{i,j})$ and $(S - \{\pi^*_l, \pi_{l-1}\})$ has a perfect matching in $(V_{i,j})$, contradicting the minimality of $S$.

Let $\pi_l \in N(\pi^*_l - 1) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\}$ and $S_1 = S \cup \{\pi^*_l - 1, \pi^*_l\} - \{\pi^*_l, \pi_l\}$. Since $N[\pi^*_l] \subseteq N[\pi^*_l] \cup N[\pi^*_l]$, and $N[\pi^*_l] \subseteq N[\pi^*_l] \cup N[\pi^*_l] \cup N[\pi^*_l]$, $S_1$ is a dominating set of $(V_{i,j})$ and $(S_1)$ has a perfect matching in $(V_{i,j})$ by pairing $(\pi^*_l, \pi^*_l - 1)$ and $(\pi^*_l, \pi^*_l)$. So $S_1 \subseteq X_1$ with $|S_1| = |S|$ and $max(S_1) \geq max(S)$ such that $\pi^*_l \not\subseteq S_1$ and $\pi^*_l \not\subseteq S_1$. Using analogous arguments as in **Lemma 8**, we can get a set $X_1 \subseteq S_1$ such that $S^* - \{\pi^*_l, \pi^*_l\} = PD_{l-1, \pi^*_l}$ and $S^*$ is a $PD_{\pi^*_l}$. Hence, $|S| = |S^*| = |PD_{l-1, \pi^*_l}| + 2$. Note that $|PD_{\pi^*_l}| \leq |PD_{l-1, \pi^*_l}| + 2 = |S|$ and if $|PD_{\pi^*_l}| = |PD_{l-1, \pi^*_l}| + 2$, then max$(PD_{\pi^*_l}) = max(S^*) \geq max(S)$. Therefore, $Min(X_1 \cup \{PD_{\pi^*_l}\}) = PD_{\pi^*_l}$. 

**Lemma 10.** For any integers $i$ and $j$, $1 < i \leq n$ and $1 < j \leq n$, if $(\pi^*_l, \pi_{l-1}) \in M$, then $Min(X_1 \cup \{PD_{\pi^*_l}\}) = PD_{\pi^*_l}$. 

**Proof.** Similar to **Lemma 8**, we again need to show that there exists an $S^* \subseteq PD_{\pi^*_l} \cap X_1$ such that $max(S^*) \geq max(S)$. We consider the following two cases depending on whether or not $\pi_{l-1} \subseteq S$ is equal to $\pi^*_l$.

**Case 1.** Suppose $\pi^*_l - 1 = \pi^*_l$. Then, for any $\pi_k \in S$ for $l < k < i$, there exists $\pi_k$ such that $(\pi_k, \pi_k) \in M$. Similar to the discussion for $S_1$ in **Case 1 of Lemma 8**, we can obtain a set $S^* \subseteq X_1$ satisfying the conditions (i)–(iii) in **Case 1 of Lemma 8** and $S^*$ is a $PD_{\pi^*_l}$ with $(max(PD_{\pi^*_l}) \geq max(S))$. Therefore, $Min(X_1 \cup \{PD_{\pi^*_l}\}) = PD_{\pi^*_l}$. 

**Case 2.** Suppose $\pi^*_l - 1 \not\subseteq S$. If $\pi^*_l - 1 \not\subseteq S$, then we deal with $S$ as in **Case 2 of Lemma 8** for $S_1$. Finally, we can obtain a set $S^* \subseteq X_1$ satisfying the conditions (i)–(iii) in **Case 2 of Lemma 8** and $S^*$ is a $PD_{\pi^*_l}$ with $(max(PD_{\pi^*_l}) \geq max(S))$. Hence, $Min(X_1 \cup \{PD_{\pi^*_l}\}) = PD_{\pi^*_l}$.

Suppose $S \subseteq \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} - \{\pi^*_l\}$, then it follows that no vertex in $V_{i,j}$ is dominated by $\pi^*_l$ or $\pi_l$, so $S = \{\pi^*_l, \pi_l\}$ is a dominating set of $(V_{l-1, \pi^*_l})$ and $(S - \{\pi^*_l, \pi_l\})$ has a perfect matching in $(V_{l-1, \pi^*_l})$ by pairing $\pi^*_l$ with $\pi_l$ which contradicts the minimality of $S$. Then $S - \{\pi^*_l, \pi_l\}$ is a $PD_{l-1, \pi^*_l}$, and thus $S$ is a $PD_{\pi^*_l}$. Hence, $|S| = |PD_{l-1, \pi^*_l}| + 2$. Note that $|PD_{\pi^*_l}| \leq |PD_{l-1, \pi^*_l}| + 2 = |S|$ and if $|PD_{\pi^*_l}| = |PD_{l-1, \pi^*_l}| + 2$, then max$(PD_{\pi^*_l}) = max(S^*) \geq max(S)$. Therefore, $Min(X_1 \cup \{PD_{\pi^*_l}\}) = PD_{\pi^*_l}$.
Suppose to the contrary that Lemma 2 holds. Let \( \pi \) and \( \pi_i \) be any two vertices in \( S \). Then \( \min(\pi, \pi_i) > \max(\pi, \pi_i) \) and \( \max(\pi, \pi_i) \leq \min(\pi, \pi_i) \). Therefore, \( \pi \) and \( \pi_i \) are not in the same connected component. From Lemma 1, we can conclude that \( \pi \) and \( \pi_i \) are not in the same connected component.

Proof. Note that \( \pi \) and \( \pi_i \) are not in the same connected component. Therefore, \( \pi \) and \( \pi_i \) are not in the same connected component. This means that \( \pi \) and \( \pi_i \) are not in the same connected component. Hence, \( \pi \) and \( \pi_i \) are not in the same connected component. By Lemma 1, we can conclude that \( \pi \) and \( \pi_i \) are not in the same connected component.

Lemma 11. For any integers \( i, j \), if \( 1 < i \leq n \) and \( 1 \leq j \leq n \), \( \min(X_1 \cup \{PD_{\pi_i}^*\}) = PD_{\pi_i}^* \).

Lemma 12. For any integers \( i, j \), \( 1 < i \leq n \) and \( \pi_i \leq j \leq n \), if \( \max(V_i) = \pi_i \), then \( X_3 = \emptyset \).

Proof. Suppose to the contrary that \( X_3 \neq \emptyset \). Let \( S \subseteq X_3 \). Then \( \pi_i, \pi_i^* \notin S \) and \( S \subseteq \langle V_i \rangle \) is a dominating set of \( \langle V_i, j \rangle \) and \( \langle S \rangle \) has a perfect matching in \( \langle V_i \rangle \). Since \( \pi_i \leq j \leq n \), \( \pi_i \in V_i \), so \( \pi_i \) is dominated by a vertex \( \pi_i (l < i) \) in \( S \). Then \( (\pi_i, \pi_i) \in E \), i.e., \( (\pi_i, \pi_i) (i-l) \leq 0 \). This implies that \( \pi_i > \pi_i \), contradicting the assumption of \( \max(V_i) = \pi_i \).

Lemma 13. For any integers \( i, j \), \( 1 < i \leq n \) and \( \pi_i \leq j \leq n \), if \( \max(PD_{\pi_i-1,j}) < \pi_i \), then \( \min(X_3 \cup \{PD_{\pi_i}^\max\}) = PD_{\pi_i}^\max \).

Proof. If \( \max(V_i) = \pi_i \), by Lemma 12, \( X_3 = \emptyset \). The result follows. So we may assume that \( \max(V_i) \neq \pi_i \). Let \( Z \) denote the set \( \{S \subseteq V_{i-1}, S \subseteq V_i \} \) and \( \langle S \rangle \) has a perfect matching in \( \langle V_{i-1} \rangle \). Let \( A \) be any set of \( X_3 \). Since \( \pi_i \notin S \) and \( \pi_i^* \notin S, A \subseteq V_{i-1} \). By Lemma 2, we have \( V_{i-1} \subseteq V_i \), so \( A \) is a set of \( X_3 \). Since \( \pi_i \leq j \), \( \pi_i \in V_{i,j} \), \( \max(A) > \pi_i \). Thus \( \max(A) > \pi_i > \max(PD_{\pi_i-1,j}) \). Note that \( PD_{\pi_i-1,j} = \min(Z) \) and, by our definition, \( \max(PD_{\pi_i-1,j}) \) is as large as possible. Then it must be the case that \( |A| = |PD_{\pi_i-1,j}| \). Hence, \( |A| > |PD_{\pi_i-1,j}| + 2 = |PD_{\pi_i-1,j} \cup \{\pi_i\}| \). Furthermore, \( \max(A) = \max(V_i) = \max(PD_{\pi_i-1,j} \cup \{\pi_i\}) \). Therefore, \( \min(X_3 \cup PD_{\pi_i}^\max) = PD_{\pi_i}^\max \).

Lemma 14. For any integers \( i, j \), \( 1 < i \leq n \) and \( 1 \leq j \leq n \), then \( \min(X_3 \cup \{PD_{\pi_i-1,j}\}) = PD_{\pi_i-1,j} \).

Proof. Define \( Z \) as in Lemma 13. Let \( A \) be any set of \( X_3 \). As in the proof of Lemma 13, we can verify that \( A \subseteq Z \). Note that \( PD_{\pi_i-1,j} = \min(Z) \). So \( \min(X_3 \cup \{PD_{\pi_i-1,j}\}) = PD_{\pi_i-1,j} \).

Lemma 15. For any integers \( i, j \), \( 1 < i \leq n \) and \( 1 \leq j \leq n \), then \( \min(X_1 \cup X_2) = \min(X_1) \).

Proof. Let \( S_1 = \min(X_2) \). According to the definition of \( X_2 \), \( \pi_i^* \notin X_2, \pi_i \notin X_2 \). Therefore, \( S_1 \) is a perfect matching \( M \). So there exists a vertex \( \pi_i \in X_2 \) such that \( (\pi_i, \pi_i) \in M \). Then \( (\pi_i - \pi_i)(l - i) < 0 \), and thus \( \pi_i > \pi_i \). Hence \( \pi_i^* \leq \pi_i < \pi_i \) and \( 1 < i < \pi_i^\max \).

This means that \( (\pi_i^* - \pi_i)(\pi_i^\max - \pi_i) < 0 \), i.e., \( (\pi_i, \pi_i^*) \in E \). Let \( S_2 = (S_1 - \{\pi_i\}) \cup \{\pi_i^*\} \). From (1) and Lemma 13, we can conclude that \( S_2 \subseteq V_i^* \) is a dominating set of \( \langle V_i \rangle \) and \( \langle S_2 \rangle \) has a perfect matching by pairing \( \pi_i \) and \( \pi_i^* \). So \( S_2 \subseteq X_1 \). \( |S_2| = |S_1| \) and \( \max(S_2) \geq \max(S_1) \). Consequently, \( \min(X_1 \cup X_2) = \min(\min(X_1), \min(X_2)) = \min(\min(X_1), \min(X_1)) = \min(X_1, S_1) = \min(X_1) \).

In the following, we present the recursive formula of our dynamic programming.

Theorem 16. For any integers \( i, j \), \( 1 < i \leq n \) and \( 1 \leq j \leq n \), the following recursive formula correctly computes \( PD_{i,j} \):

\[
PD_{i,j} = \begin{cases} 
\min(\{PD_{\pi_i^*,PD_{\pi_i}^\max}\}) & \text{if } j \geq \pi_i \text{ and } \max(PD_{\pi_i-1,j}) < \pi_i, \\
\min(\{PD_{\pi_i^*,PD_{\pi_i-1,j}}\}) & \text{otherwise.}
\end{cases}
\]
Proof. According to our definitions, \( X = X_1 \cup X_2 \cup X_3 \). By Lemmas 5 and 6, we have \( PD_{\pi^*} \in X_1 \subseteq X \), \( PD_{\max} \in X \). To complete our proof, we distinguish the following two cases.

Case 1. Suppose that \( j \geq \pi_i \) and \( \max(PD_{i,j}) < \pi_i \). If \( \max(V_i) = \pi_i \), then, by Lemmas 11, 12 and 15, we have

\[
\begin{align*}
\min(X) &= \min(X_1 \cup X_2 \cup \{PD_{\pi^*}, PD_{\max}\}) \\
&= \min(X_1 \cup \{PD_{\pi^*}, PD_{\max}\}) \\
&= \min(\{\min(X_1 \cup \{PD_{\pi^*}\}), PD_{\max}\}) \\
&= \min(\{PD_{\pi^*}, PD_{\max}\}).
\end{align*}
\]

If \( \max(V_i) \neq \pi_i \), then, by Lemmas 11, 13 and 15, we have

\[
\begin{align*}
\min(X) &= \min(X \cup \{PD_{\pi^*}, PD_{\max}\}) \\
&= \min(X_1 \cup X_2 \cup X_3 \cup \{PD_{\pi^*}, PD_{\max}\}) \\
&= \min(X_1 \cup X_3 \cup \{PD_{\pi^*}, PD_{\max}\}) \\
&= \min(\{\min(X_1 \cup \{PD_{\pi^*}\}), \min(X_3 \cup \{PD_{\max}\})\}) \\
&= \min(\{PD_{\pi^*}, PD_{\max}\}). \quad \Box
\end{align*}
\]

Case 2. Suppose that \( j < \pi_i \) or \( \max(PD_{i-1,j}) \geq \pi_i \). We first show that \( PD_{i-1,j} \in X \). If \( j < \pi_i \), then \( V_{i,j} = V_{i-1,j} \), so \( PD_{i-1,j} \in X \). If \( \max(PD_{i,j}) \geq \pi_i \), then \( \pi_i \) is dominated by \( PD_{i-1,j} \), so \( PD_{i-1,j} \in X \). Note that \( PD_{i-1,j} \subseteq PD_{\max} \). From Lemmas 11, 14 and 15, it follows that

\[
\begin{align*}
\min(X) &= \min(X \cup \{PD_{\pi^*}, PD_{i-1,j}\}) \\
&= \min(X_1 \cup X_2 \cup X_3 \cup \{PD_{\pi^*}, PD_{i-1,j}\}) \\
&= \min(X_1 \cup X_3 \cup \{PD_{\pi^*}, PD_{i-1,j}\}) \\
&= \min(\{\min(X_1 \cup \{PD_{\pi^*}\}), \min(X_3 \cup \{PD_{i-1,j}\})\}) \\
&= \min(\{PD_{\pi^*}, PD_{i-1,j}\}). \quad \Box
\end{align*}
\]

3. An algorithm for MPDS on permutation graphs

Based on the recursive formula in Section 2, we next present the algorithmic steps to solve MPDS on permutation graphs. The overall structure of our algorithm is outlined as follows:

**Algorithm:** Finding an MPDS on a Permutation Graph.

Input: A permutation \( \pi = [\pi_1, \pi_2, \ldots, \pi_n] \).

Output: A minimum cardinality paired-dominating set of \( G[\pi] \).

Step 1. Initialize \( PD_{0,j} = \emptyset \).

\[
PD_{i,j} = \begin{cases} 
\emptyset & \text{if } j < \pi_i, \\
\{1, \pi_1\} & \text{otherwise}.
\end{cases}
\]

for \( j = 1, 2, \ldots, n \).

Step 2. for \( i \leftarrow 2 \) to \( n \) do

Step 3. \( PD_{\pi^*} = \min(PD_{i-1,\pi^*_i} \cup \{\pi^*_i, \pi_i\} : \pi_i \in N(\pi^*_i), \pi^*_i \notin PD_{i-1,\pi^*_i}, l \leq i\} \)

Step 4. for \( j \leftarrow 1 \) to \( n \) do

Step 5. \( PD_{\max} = \begin{cases} 
PD_{i-1,j} \cup \{\pi_i, \max(V_i)\} & \text{if } \pi_i \neq \max(V_i), \\
V_i & \text{otherwise}.
\end{cases}
\]

Step 6. \( PD_{i,j} = \begin{cases} 
\min(PD_{\pi^*}, PD_{\max}) & \text{if } j \geq \pi_i \text{ and } \max(PD_{i-1,j}) < \pi_i, \\
\min(PD_{\pi^*}, PD_{i-1,j}) & \text{otherwise}.
\end{cases}
\]
Step 7. END
Step 8. END
Step 9. Output $PD_{n,n}$.

The time complexity of the above algorithm can be analyzed as follows. The time required in Step 3 is at most $d(\pi^*_7)$. The operations of Steps 5 and 6 can be performed in constant time. The time required in the loop from Step 4 to Step 7 is at most $O(n)$. Consequently, the overall running time of the algorithm is $O(mn)$ in an amortized sense.

**Theorem 17.** Given any permutation $\pi$, the algorithm finds a minimum cardinality paired-dominating set of the permutation graph $G[\pi]$.

**Example.** To illustrate our algorithm, we compute the example shown in Fig. 1. as follows:

1. $PD_{0,j} = \emptyset$;
2. $PD_{\max} = V_1, PD_{1,1} = PD_{1,2} = \emptyset, PD_{1,3} = \cdots = PD_{1,7} = \{1, 3\};$
3. $\pi^*_7 = 2, PD_{\pi^*_7} = \{3, 2\}, PD_{\max} = \{1, 3\}, PD_{2,1} = \cdots = PD_{2,7} = \{3, 2\}$ or $\{1, 3\};$
4. $\pi^*_7 = 2, PD_{\pi^*_7} = \{3, 2\}, PD_{\max} = V_3, PD_{3,1} = \cdots = PD_{3,4} = \{3, 2\}$ or $\{1, 3\}, PD_{3,5} = \cdots = PD_{3,7} = \{3, 2\};$
5. $\pi^*_7 = 2, PD_{\pi^*_7} = \{3, 2\}, PD_{\max} = V_4, PD_{4,1} = \cdots = PD_{4,4} = \{3, 2\}$ or $\{1, 3\}, PD_{4,5} = \cdots = PD_{4,7} = \{3, 2\};$
6. $\pi^*_7 = 2, PD_{\pi^*_7} = \{3, 2\}, PD_{\max} = \{2, 3, 7, 4\}$ or $\{1, 3, 7, 4\}, PD_{5,1} = \cdots = PD_{5,3} = \{3, 2\}$ or $\{1, 3\}, PD_{5,4} = \cdots = PD_{5,7} = \{3, 2\};$
7. $\pi^*_7 = 2, PD_{\pi^*_7} = \{3, 2\}, PD_{\max} = \{1, 3, 2, 7\}, PD_{6,1} = \cdots = PD_{6,3} = \{3, 2\}$ or $\{1, 3\}, PD_{6,4} = \cdots = PD_{6,7} = \{3, 2\};$
8. $\pi^*_7 = 6, PD_{\pi^*_7} = \{3, 2, 7, 6\}, PD_{\max} = \{3, 2, 7, 6\}$ or $\{1, 3, 7, 6\}, PD_{7,1} = \cdots = PD_{7,3} = \{3, 2, 7, 6\}$ or $\{1, 3, 7, 6\}, PD_{7,4} = \cdots = PD_{7,7} = \{3, 2, 7, 6\}.$

In light of our algorithm, $PD_{7,7} = \{3, 2, 7, 6\}$ is a minimum cardinality paired-dominating set of the graph.

4. Conclusions

In this paper we presented an $O(mn)$ algorithm for finding a minimum cardinality paired-dominating set for a permutation graph with order $n$ and size $m$. Our algorithm is based on a recursive formula in conjunction with applying the dynamic programming method. The idea was previously used by Chao et al. [6] for finding the minimum cardinality dominating set on permutation graphs. We speculate that the time complexity of the MPDS problem on permutation graphs can be reduced to $O(n \log n)$ and we suggest that researchers investigate such a possibility. It is also interesting to determine whether there exist some other classes of graphs in which the minimum paired-domination problem is polynomially solvable.

Acknowledgments

We are grateful to the referees for their valuable comments, which have led to improvements in the presentation of the paper. This research was supported in part by The Hong Kong Polytechnic University under grant number G-YX69, the National Natural Sciences Foundation of China under grant numbers 10571117, 60773078 and Shu Guang Plan of the Shanghai Education Development Foundation under grant number 06SG42.

References


