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On the Combinatorics of Plethysm

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We construct three (large, reduced) incidence algebras whose semigroups of multiplicative functions, under convolution, are anti-isomorphic, respectively, to the semigroups of what we call partitional, permutational and exponential formal power series without constant term, in infinitely many variables $\mathbf{x} = (x_1, x_2, \ldots)$, under plethysm. We compute the Möbius function in each case. These three incidence algebras are the linear duals of incidence bialgebras arising, respectively, from the classes of transversals of partitions (with an order that we define), partitions compatible with permutations (with the usual refinement order), and linear transversals of linear partitions (with the order induced by that on transversals). We define notions of morphisms between partitions, permutations and linear partitions, respectively, whose kernels are defined to be, in each case, transversals, compatible partitions and linear transversals. We introduce, in each case, a pair of sequences of polynomials in x of binomial type, counting morphisms and monomorphisms, and obtain expressions for their connection constants, by summation and Möbius inversion over the corresponding posets of kernels.

Contents.

INTRODUCTION

In [N-R], a set-theoretic interpretation was given for the plethysm of formal power series, in the variables $\mathbf{x} = (x_1, x_2, \ldots)$, of the form

$$f(\mathbf{x}) = \sum_{\lambda} a_\lambda \mathbf{x}^\lambda / \operatorname{aut}(\lambda),$$

which we call *partitional* formal power series. These are the appropriate generating functions for enumerating structures built (equivariantly) on

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partitions of finite sets. In that theory of *partitionals*, the plethysm is obtained via the notion of *transversals* of a partition, which play a role similar to that of partitions in Joyal's theory of species (cf. [J]). The main disadvantage of the purely set-theoretic approach is the absence of a convenient way of dealing with negative coefficients and inversion of the operations of composition (ordinary or plethystic): there are no inverse species of partitionals. In the case of partitions and composition of exponential formal power series in one variable, the gap is bridged by Doubilet, Rota and Stanley's large, reduced incidence algebra of partitions (cf. [D-R-S]).

The present work stems from one of Rota's characteristic insights: the *prima facie* rather preposterous proposition of finding an order on transversals that gives rise to an incidence algebra for the plethysm of partitional formal power series (without constant term). This the author is still amazed to have found to be possible.

More recently, Bergeron (cf. [B]) developed a set-theoretic interpretation, in the spirit of [N-R], for the plethysm of formal power series in $\mathbf{x}$ with denominators $\operatorname{aut}'(\lambda)$, which we call *permutational* formal power series. These are suitable for the enumeration of structures built (functorially) on permutations of finite sets. In Bergeron's theory, the role of transversals is played by partitions which are *compatible* with a given permutation. The methods we developed for finding the incidence algebra in the partitional case were rather easily adaptable to the permutational case, yielding an incidence algebra for the plethysm of permutational formal power series.

Finally, in the same vein as [N-R] and Joyal's linear species (cf. [J])) the author has developed a theory of linear partitionals (whose inclusion in the present work would be unnecessarily digressive) that gives a set-theoretic interpretation of the plethysm of formal power series in $\mathbf{x}$ with denominators $\lambda!$, which we call *exponential*. The ordering of the corresponding *linear transversals* induced by that of transversals, yields the plethystic incidence algebra in this case.

In these three Joyal-type theories, the morphisms of the underlying categories of partitions, permutations and linear partitions (of finite sets) are always and only isomorphisms. In the present work we extend these sets of morphisms and define their kernels so that these are elements of the corresponding posets of transversals, compatible partitions and linear transversals. This allows us, in each case, to carry out the program of [J-R-S] for the pairs of sequences of polynomials in $\mathbf{x}$ of binomial type which count morphisms and monomorphisms, thus finding expressions for the connection constants, which we propose as the plethystic analogues of the Stirling numbers of first and second kind, by summation and differentiation (Möbius inversion) over the posets of kernels.

In the construction of the three plethystic incidence algebras we have made systematic use of Schmitt's (cf. [S]) recent elegant theory of incidence coalgebras of families of posets, which streamlines the constructions and elucidates the general character of certain properties thereof.

The computation of the Möbius functions of these plethystic incidence algebras, which give plethystic inverses of the generating functions of the zeta functions, can be carried out, surprisingly enough, by using basic theorems of [R] in a rather elegant way, one of the payoffs of the fairly laborious analysis of the structure of intervals in the corresponding posets.

Among the *possible* offspring of the present work is a Lagrange-like inversion formula for the plethysm, which could result from an understanding of how cancellations take place in Schmitt's beautiful formula for the antipode of incidence Hopf algebras. Another related one is the development of a plethystic umbral calculus. Our Proposition 1.5 seems to suggest that the Frobenius operators (the terminology was proposed by Rota, in the spirit of [M-R] should play a rather important role, which at present is unclear to the author. Of the three bialgebras we construct, that of linear transversals seems to be the most amenable to such further developements.

I. Preliminaries

In this section we give the basic terminology, notation and facts about partitions that will be extensively used throughout this work. For the sake of the reader's intuition and perspective we summarize the construction of the (large, reduced) incidence bialgebra of partitions (which we view as the classical case) along the lines that will be followed in the construction of three bialgebras related to plethysm in the subsequent sections. This point of view is sketched in [J-R] and has been more fully developed recently in [S]. For completeness' sake we start with pertinent coalgebraic definitions and end by stating basic definitions and facts about plethysm.

1.1. *Incidence Coalgebras*

Let K be a field. A *K-coalgebra* is a K-vector spade C with linear maps

$$\varepsilon\colon\ C \to K$$

$$\Delta\colon\ C \to C \otimes C,$$

called *augmentation* (or *counit*) and *diagonalization* (or *comultiplication*), respectively, such that if

$$I\colon\ C \to C$$

is the identity map, we have:

1. *Counitary property*: $(\varepsilon \otimes I) \circ \Delta = (I \otimes \varepsilon) \circ \Delta = I$, where we identify C with $K \otimes C$ and $C \otimes K$.

2. *Coassociativity*: $(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta$.

If, furthermore, C is also a K-algebra with multiplication map μ: $C \otimes C \to C$ and unit map η: $K \to C$, then the structure $(C, \Delta, \varepsilon, \mu, \eta)$ is a *bialgebra* if Δ and ε are algebra maps:

$$\Delta(1) = 1 \otimes 1, \qquad \Delta(xy) = \Delta(x)\,\Delta(y),$$
$$\varepsilon(1) = 1, \qquad \varepsilon(xy) = \varepsilon(x)\,\varepsilon(y).$$

Now let (A, μ_A, η_A) be a K-algebra and let (C, Δ, ε) be a coalgebra. Let $R = \mathrm{Mod}_K(C, A)$ be the K-vector space of K-linear maps from C to A. Define the *convolution* of $f, g \in R$ by

$$f * g := \mu_A \circ (f \otimes g) \circ \Delta.$$

Then R becomes a K-algebra with the structure maps

$$\mu_R(f \otimes g) = f * g, \qquad \eta_R(\alpha) = \alpha \eta_A \circ \varepsilon.$$

Let $(H, \Delta, \varepsilon, \mu, \eta)$ be a bialgebra, and view $R = \mathrm{Mod}_K(H, H)$ as a K-algebra with convolution as above. When $I = \mathrm{id}_H$ is a unit of R, its inverse (with respect to convolution) S is called the *antipode* of H, and H is called a *K-Hopf algebra*.

We refer the reader to [A] for further details.

The *incidence coalgebra* $C(P)$ of a poset $(P, \leqslant)$ over a field K (of characteristic zero) is the vector space spanned by the indeterminates $[x, y]$ for all intervals (or segments) $[x, y]$ in P. The augmentation ε and the diagonalization Δ are defined by

$$\varepsilon([x, y]) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise}, \end{cases}$$

$$\Delta[x, y] = \sum_{x \leqslant z \leqslant y} [x, z] \otimes [z, y].$$

The *incidence algebra* $I(P)$ (over K) of P is the linear dual $C(P)^* = \mathrm{Mod}_K(C(P), K)$. This coincides with the usual definition of $I(P)$ (cf., v. gr., [R]) since, for $f, g \in C(P)^*$, we have

$$f * g[x, y] = (f \otimes g) \circ \Delta[x, y] = \sum_{x \leqslant z \leqslant y} f[x, z]\, g[z, y],$$

the usual definition of convolution in $I(P)$.

1.2. *Partitions*

For a more detailed discussion of partitions we refer the reader to [N-R].

A *partition* of a set E (which, in the sequel, will always be finite) is a set of non-empty, mutually disjoint subsets of E whose union is E. We denote by $\Pi[E]$ the lattice of partitions of E under the *refinement* ordering: $\pi \leqslant \sigma$ if every block of π contained in a block of σ $(\sigma, \pi \in \Pi[E])$. Note that $\Pi[\varnothing] = \{\varnothing\}$. We denote by $\hat{0}$ and $\hat{1}$ (or $\hat{0}_E$ and $\hat{1}_E$) the minimum and maximum elements, respectively, of $\Pi[E]$. If $E \neq \varnothing$, we have

$$\hat{0}_E = \{\{x\}\ x \in E\},$$
$$\hat{1}_E = \{E\}.$$

We denote by $\pi \wedge \sigma$ and $\pi \vee \sigma$ the meet and join, respectively, in $(\Pi[E], \leqslant)$. We have

$$\pi \wedge \sigma = \{B \cap C: B \in \pi,\ C \in \sigma,\ B \cap C \neq \varnothing\}.$$

If $\pi \in \Pi[E]$ and $D \subseteq E$, then the *restriction* of the partition π to the set D is the partition $\pi_D \in \Pi[D]$ defined by

$$\pi_D = \{B \cap D: B \in \pi,\ B \cap D \neq \varnothing\}.$$

In the sequel we shall often consider restrictions π_D, where D is a union of blocks of π (typically, D will be a block of a coarser partition); in this case we have

$$\pi_D = \{B \in \pi: B \subseteq D\} \subseteq \pi.$$

If $\pi, \sigma \in \Pi[E]$ and $\pi \leqslant \sigma$, then the *quotient partition* σ/π is the partition of the set π defined by

$$\sigma/\pi := \{\pi_B: B \in \sigma\}.$$

Note that $|\sigma/\pi| = |\sigma|$.

If $\pi \in \Pi[E]$, we denote by $\sim_\pi$ the equivalence relation on E whose quotient set (of equivalence classes) is π; i.e., for $x, y \in E$,

$$x \sim_\pi y \leftrightarrow (\exists B \in \pi)\,(x, y \in B).$$

In terms of equivalent relations, the join of two partitions $\pi, \sigma \in \Pi[E]$ can be characterized as follows: for $x, y \in E$, $x \sim_{\pi \vee \sigma} y$ whenever there is a finite sequence $z_0, ..., z_n$ of elements of E such that

$$x = z_0 \sim_1 z_1 \sim_2 \cdots \sim_n z_n = y,$$

where each relation $\sim_k$ is either $\sim_\sigma$ or $\sim_\pi$.

Two partitions $\pi, \sigma \in \Pi[E]$ *commute* if, for all $x, y \in E$, we have

$$(\exists u \in E)(x \sim_\pi u \sim_\sigma y) \leftrightarrow (\exists v \in E)(x \sim_\sigma v \sim_\pi y).$$

Two partitions π and σ of the same set are said to be *independent* if every block of π meets every block of σ.

The following useful proposition expresses the relationship between the concepts of commutativity and independence of partitions.

PROPOSITION 1.1 (Dubreil and Jacotin; cf. [D-J] or [N-R]). *Let $\pi, \sigma \in \Pi[E]$. Then π and σ commute if, and only if, for every $B \in \pi \vee \sigma$, the restrictions π_B and σ_B are independent partitions of the set B.* ∎

If $\pi \in \Pi[E]$ we refer to the pair (E, π) simply as a *partition.* Two partitions (E, π) and (F, σ) are isomorphic if there is a bijection $f\colon E \to F$ between the underlying sets which sends blocks of π to blocks of σ: for all $B \in \pi$, $f(B) = \{f(x)\colon x \in B\} \in \sigma$. The isomorphism class of a partition (E, π) is uniquely determined by the multiset of positive numbers $\{|B| : B \in \pi\}$, or, equivalently, by its multiplicity function, i.e., the sequence $\lambda = (\lambda_1, \lambda_2, ...)$ defined by

$$\lambda_i := \text{number of blocks of size } i \text{ of } \pi.$$

Thus, we call the sequence λ the *class* of the partition (E, π), and write

$$\mathrm{cl}(E, \pi) := \mathrm{cl}(\pi) := \lambda.$$

The number of automorphisms of a partition of class λ depends only on λ and equals

$$\mathrm{aut}(\lambda) := \prod_{i \geqslant 1} (i!^{\lambda_i} \lambda_i!).$$

If λ is a class (i.e., $\lambda = (\lambda_1, \lambda_2, ...)$, $\lambda_i \in N$, $\lambda_i = 0$ for all but finitely many i's, we set

$$\lambda! = \prod_{i \geqslant 1} \lambda_i!$$

The sum of two classes λ and μ is defined by

$$(\lambda + \mu)_i = \lambda_i + \mu_i.$$

The *direct sum* of two partitions (E, π) and (F, σ) is defined as

$$(E, \pi) + (F, \sigma) := (E + F, \pi + \sigma),$$

where the sums on the right-hand side are direct sums (disjoint unions) of sets. Clearly, we have

$$\mathrm{cl}(\pi + \sigma) = \mathrm{cl}(\pi) + \mathrm{cl}(\sigma).$$

We set

$$\mathbf{0} := (0, 0, \ldots),$$

$$\delta_n := (\delta(n, 1), \delta(n, 2), \ldots), \qquad \text{where} \quad \delta(x, y) = \begin{cases} 1 & \text{if } \; x = y \\ 0 & \text{if } \; x \neq y \end{cases}$$

$$n\lambda := (n\lambda_1, n\lambda_2, \ldots).$$

1.3. *The Classical Case*: *The Large Reduced Incidence Bialgebra of Partitions*

The concept of large reduced incidence algebra was introduced in [D-R-S]. The present coalgebraic approach is detailed in [S], and will be used in the subsequent chapters for its convenience and generality.

Consider the class (in the set-theoretic sense) of all the intervals of all the lattices of partitions of finite sets. Call it P. We shall define an (order-compatible) equivalence relation $\sim$ on P whose equivalence classes (*types*) will be identified with monomials in the variables $\mathbf{x} = (x_1, x_2, \ldots)$. We define a product of types which correspond to the usual algebra structure of $K[\mathbf{x}]$. Using the ordering of the segments, we endow $K[\mathbf{x}]$ with a bialgebra structure. The dual $K[\mathbf{x}]^*$ is isomorphic to the large reduced incidence algebra of [D-R-S]. We sketch a proof of Theorem 5.1 of that paper relating multiplicative functions under convolution with formal power series in one variable under composition. This proof, which avoids incidence coefficients, foreshadows the corresponding proofs relating convolution and plethysm in the next chapters.

The crucial fact in the definition of $\sim$ is the natural factorization theorem for intervals of partitions.

THEOREM 1.2 (Natural factorization of intervals of partitions). *Let* $\pi, \sigma \in \Pi[E]$, $\pi \leqslant \sigma$. *Then there is a natural isomorphism* (*of posets*)

$$[\pi, \sigma] \simeq \prod_{B \in \sigma} \Pi[\pi_B],$$

where the product on the right is the usual product of posets.

Proof. The isomorphism is

$$\rho \to (\rho_B / \pi_B : B \in \sigma). \quad \blacksquare$$

We define the *class* of an interval $[\pi, \sigma] \in P$ to be the multiplicity function of the multiset $\{|\pi_B| : B \in \sigma\}$ of positive numbers, which turns out to be $\mathrm{cl}(\sigma/\pi)$:

$$\mathrm{cl}[\pi, \sigma] := \mathrm{cl}(\sigma/\pi).$$

We define the equivalence relation $\sim$ on P by

$$[\pi, \sigma] \sim [\pi', \sigma'] \qquad \text{iff} \quad \mathrm{cl}[\pi, \sigma] = \mathrm{cl}[\pi', \sigma'].$$

It is easy to see that the relation $\sim$ satisfies Smith's criterion for order-compatibility (cf. [D-R-S, Prop. 4.1]), viz.:

PROPOSITION 1.3. *If* $[\pi, \sigma] \sim [\pi', \sigma']$ *then there is a bijection* ϕ: $[\pi, \sigma] \to [\pi', \sigma']$ *such that, for all* $\rho \in [\pi, \sigma]$, *we have*

$$[\pi, \rho] \sim [\phi(\pi), \phi(\rho)]$$

and

$$[\rho, \sigma] \sim [\phi(\rho), \phi(\sigma)]. \quad \blacksquare$$

The *type* of a segment $[\pi, \sigma] \in P$ is the equivalence class $\mathrm{typ}[\pi, \sigma]$ of $[\pi, \sigma]$ under $\sim$.

Let $\mathbf{x} = (x_1, x_2, ...)$ be a sequence of formal variables, and let K be a field of characteristic zero. If $\lambda = (\lambda_1, \lambda_2, ...)$ is a class, set

$$\mathbf{x}^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots.$$

If $[\pi, \sigma] \in P$, we identify $\mathrm{typ}[\pi, \sigma]$ with the monomial $\mathbf{x}^{\mathrm{cl}[\pi, \sigma]}$. Note that $\mathrm{typ}[\varnothing, \varnothing] = \mathbf{x}^{\mathbf{0}} = 1$.

We define augmentation and diagonalization maps on $K[\mathbf{x}]$ by

$$\varepsilon(\mathbf{x}^\lambda) = \begin{cases} 1 & \text{if } \lambda = \mathbf{0} \text{ or } n\delta_1 \ (n \in N), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Delta \mathbf{x}^\lambda = \sum_{\rho \in [\pi, \sigma]} \mathrm{typ}[\pi, \rho] \otimes \mathrm{typ}[\rho, \sigma],$$

where $[\pi, \sigma]$ is any segment of type $\mathbf{x}^\lambda$. Note that Proposition 1.3 is exactly what is needed to ensure that Δ us well defined. A straightforward computation (cf. [S]) show that the counitary and coassociative properties hold. Thus, $(K[\mathbf{x}], \Delta, \varepsilon)$ is a coalgebra.

We define the *product* of types by

$$\operatorname{typ}[\pi, \sigma] \operatorname{typ}[\pi', \sigma'] = \operatorname{typ}[\pi + \pi', \sigma + \sigma'].$$

It is easy to see that this well defined, and that it coincides with the usual algebra structure of $K[\mathbf{x}]$. Also, if $[\pi, \sigma]$, $[\pi', \sigma']$, $[\pi'', \sigma''] \in P$ and

$$[\pi, \sigma] \sim [\pi' + \pi'', \sigma' + \sigma''],$$

then there is a bijection

$$\phi \colon [\pi', \sigma'] \times [\pi'', \sigma''] \to [\pi, \sigma]$$

such that

$$[\pi, \phi(\rho', \rho'')] \sim [\pi' + \pi'', \rho' + \rho'']$$

and

$$[\phi(\rho', \rho''), \sigma] \sim [\rho' + \rho'', \sigma' + \sigma''].$$

Under these circumstances, a straightforward computation (cf. [S]) shows that Δ and ε are algebra maps. Thus, $([K[\mathbf{x}], \Delta, \varepsilon)$ (with the usual algebra structure) is a bialgebra. Localizing at x_1 makes it into a Hopf algebra (cf. [S]).

The linear dual $K[\mathbf{x}]^*$ is isomorphic to $K[[\mathbf{y}]]$, the vector space of formal power series in the variables $\mathbf{y}$. The isomorphism is

$$f \to \sum_{\lambda} F(\mathbf{x}^{\lambda})\, \mathbf{y}^{\lambda},$$

where the sum is over all classes λ.

f is *multiplicative* if it is an algebra map from $K[\mathbf{x}]$ to K, i.e., $f \in \mathrm{Alg}_K(K[\mathbf{x}], K)$. In this case f is determined by the values $f(x_n)$ $(n \geqslant 1)$, or, equivalently, by the exponential formal power series (in one variable x)

$$F_f(x) = \sum_{n \geqslant 1} f(x_n)\, x^n/n!$$

THEOREM 1.4 (Doubilet, Rota, and Stanley; cf. [D–R–S]). *The convolution of multiplicative functions is multiplicative, and*

$$F_{f*g}(x) = F_g(F_f(x)).$$

Proof. That $\mathrm{Alg}_K(B, K)$ is closed under convolution when B is a bialgebra is a general fact: if $f, g \in \mathrm{Alg}_K(B, K)$ then

$$f * g = (f \otimes g) \circ \Delta,$$

and tensor product and composition of algebra maps are algebra maps.

If $[\pi, \sigma] \in P$ and $\mathrm{cl}[\pi, \sigma] = \lambda$, we set

$$f(\pi, \sigma) := f(\mathrm{typ}[\pi, \sigma]),$$

and if $|E| = n$ we set

$$f(E) := f(x_n).$$

Conversely, if a rule $E \to f(E) \in K$ assigns a scalar to every non-empty finite set so that $f(E)$ depends only on $|E|$, then it defines a unique multiplicative function f by

$$f(x_n) := f(E), \qquad |E| = n > 0.$$

Given two multiplicative functions f and g, define a multiplicative function $f \dot{+} g$ (not to be confused with $f + g$, the usual addition in $K[\mathbf{x}]^*$) by the rule

$$(f \dot{+} g)(E) := f(E) + g(E),$$

and a multiplicative function $f \cdot g$ (same caveat) by the rule

$$f \cdot g(E) := \sum_{E_1 + E_2 = E} f(E_1)\, g(E_2),$$

where the sum on the right ranges over all ordered pairs (E_1, E_2) of non-empty subsets of E such that $E_1 \cap E_2 = \varnothing$ and $E_1 \cup E_2 = E$. Then, it is easy to see that we have

$$F_{f \dot{+} g}(x) = F_f(x) + F_g(x),$$

and

$$F_{f \cdot g}(x) = F_f(x)\, F_g(x).$$

Also, define a multiplicative function $\gamma_n(f)$ $(n \geqslant 1)$ by the rule

$$\gamma_n(f)(E) := \sum_{\pi \in \Pi_n[E]} f(\hat{0}_E, \pi),$$

where $\Pi_n[E] := \{\pi \in \Pi[E]: |\pi| = n\}$. If $f^{\cdot n}$ is the nth power of f with respect to the product just defined, then we have

$$f^{\cdot n}(E) = \sum_{E_1 + \cdots + E_n = E} \prod_{i=1}^{n} f(E_i),$$

where the sum is over all ordered n-tuples $(E_1, \ldots, E_n)$ non-empty, mutually disjoint subsets of E whose union is E. It follows from the definitions and Theorem 1.2 that

$$f(\hat{0}, \pi) = \prod_{B \in \pi} f(B).$$

Therefore we see that $F_{\gamma_n(f)}(x) = (1/n!)\, F_{f^{\cdot n}}(x) = (1/n!)\, F_f(x)^n$.

Let $h = f * g$. Then:

$$\begin{aligned} h(E) = f * g(E) &= \sum_{\pi \in \Pi[E]} f(\hat{0}, \pi)\, g(\pi, \hat{1}) \\ &= \sum_{n \geqslant 1} \sum_{\pi \in \Pi_n[E]} f(\hat{0}, \pi)\, g(\pi, \hat{1}) \\ &= \sum_{n \geqslant 1} g(x_n) \sum_{\pi \in \Pi_n[E]} f(\hat{0}, \pi) \\ &= \sum_{n \geqslant 1} g(x_n)\, \gamma_n(f)(E). \end{aligned}$$

So, if $h_n(E) := g(x_n)\, \gamma_n(f)(E)$, then

$$h = \sum_{n \geqslant 1} h_n$$

(the infinite sum is well defined since, for each E, almost all the $h_n(E)$ are zero), and thus

$$\begin{aligned} F_{f*g}(x) = F_h(x) &= \sum_{n \geqslant 1} F_{h_n}(x) \\ &= \sum_{n \geqslant 1} g(x_n)\, F_f(x)^n / n! \\ &= F_g(F_f(x)). \quad \blacksquare \end{aligned}$$

1.4. *Plethysm*

Let K be a field of characteristic zero, and let $\mathbf{x} = (x_1, x_2, \ldots)$ be an infinite sequence of (countably many) formal variables. Consider the algebra $K[[\mathbf{x}]]$ of formal power series in the variables $\mathbf{x}$. For each $n \geqslant x$ define the *nth Frobenius operator* F_n: $K[[\mathbf{x}]] \to K[[\mathbf{x}]]$ by

$$F_n f(\mathbf{x}) := f(x_n, x_{2n}, x_{3n}, \ldots).$$

Note that $F_n F_m = F_{mn} = F_m F_n$.

The *plethysm* of formal power series $f, g \in K[[\mathbf{x}]]$ with $g(\hat{0})=0$ is the f.p.s. $f[g]$ defined by

$$f[g]=f(F_1 g, F_2 g, F_3 g, \ldots).$$

This operation arises in the context of Polya theory (cf., v.gr., [J]) and is related to Littlewood's plethysm of symmetric functions (cf., e.g., [M] or [J–K]) as follows: if $\mathbf{t}=(t_1, t_2, \ldots)$ is a new set of variables and $p_n(\mathbf{t})=\sum_{i\geqslant 1} t_i^n$ $(n\geqslant 1)$ and $f\in K[[\mathbf{x}]]$ has bounded degree, then $\tilde{f}(\mathbf{t}):=f(p_1(\mathbf{t}), p_2(\mathbf{t}), \ldots)$ is a symmetric function of bounded degree, and

$$(f[g])^{\sim}=\tilde{f}[\tilde{g}],$$

where the plethysm on the right is Littlewood's.

As we shall see in the following sections, plethysm plays a role analogous to that of the composition of f.p.s. in one variable.

PROPOSITION 1.5. *In the usual topology of $K[[\mathbf{x}]]$, the continuous algebra endomorphisms ϕ which commute with the Frobenius operators F_n $(n\geqslant 1)$ are exactly those of the form*

$$\phi f=f[g],$$

where $g=\phi x_1$.

Proof. If ϕ is a continuous algebra endomorphism commuting with the F_n's, we have

$$\phi x_n=\phi F_n x_1=F_n \phi x_1=F_n g,$$

so, if $f=\sum_\lambda a_\lambda \mathbf{x}^\lambda$,

$$\phi f=\sum_\lambda a_\lambda (F_1 g)^{\lambda_1} (F_2 g)^{\lambda_2} \cdots =f[g].$$

Conversely, if $\phi f=f[g]$ then clearly $g=\phi x_1$, ϕ is a continuous algebra endomorphism, and

$$\begin{aligned}\phi F_n f &= (F_n f)[g]=(F_n f)(F_1 g, F_2 g, \ldots)=f(F_n g, F_{2n} g, \ldots)\\ &= f(F_n F_1 g, F_n F_2 g, \ldots)\\ &= f(F_1 g(x_n, x_{2n}, \ldots), F_2 g(x_n, x_{2n}, \ldots), \ldots)\\ &= f[g](x_n, x_{2n}, \ldots)=F_n(f[g])=F_n \phi f. \quad \blacksquare\end{aligned}$$

Finally, observe that plethysm is associative, that $f[x_1]=f$, and that if $g(\mathbf{0})=0$ then $x_1[g]=g$. Thus, the set $\{f\in K[[\mathbf{x}]] : f(\mathbf{0})=0\}$ forms a semigroup under plethysm.

II. The Partitional Case

2.1. *Transversals*

The concept of transversal of a partition was introduced in [N–R]. It plays for partitions and plethysm a role similar to that played by partitions for sets and composition (substitution). In particular, it can be thought of as a notion of "partition of a partition." Here we define an *ordering* of the transversals that further extends this analogy and proves that the plethysm of partitional formal power series in infinitely many variables is related to an incidence algebra, a fact whose consequences are yet to be fully developed.

Let (E, σ) be a partition. An ordered pair (π, τ) of partitions of E is said to be a *transversal* of σ, and the triple (σ, π, τ) is called a *transversal*, if the following conditions hold:

(1) $\pi \leqslant \sigma$,

(2) $\pi \wedge \tau = \hat{0}$,

(3) the partitions π and τ commute, and

(4) $\sigma \vee \tau = \pi \vee \tau$.

The reader can visualize this concept as follows. If (E, σ, π, τ) is a transversal, then the elements of E can be arranged in a collection of matrices whose rows are the blocks of τ and whose columns are the blocks of π (because of (2), (3) and Proposition 1.1), the blocks of σ being unions of columns (by (1)) of the same matrix (by (4)). The matrices are the blocks of $\sigma \vee \tau = \pi \vee \tau$. One such collection of matrices is said to be a *representation* of the transversal (σ, π, τ). Conversely, if a triple (σ, π, τ) of partitions of the same set admits one such representation, then the triple is a transversal. Observe that the representation of a transversal is not unique: rows and columns can be arbitrarily relabeled in each matrix. However, if σ is known, any representation compatible with σ defines a unique transversal of σ.

If (σ, π, τ) is a transversal and $\sigma \vee \tau = \pi \vee \tau = \hat{1}$, then it is called a *small transversal* of *length* $|\tau|$.

Let $T[\sigma]$ be the set of all transversals of σ. Observe that the triples

$$\bar{0} := \bar{0}^{\sigma} := (\sigma, \sigma, \hat{0})$$

and

$$\bar{1} := \bar{1}^{\sigma} := (\sigma, \hat{0}, \hat{1})$$

are always in $T[\sigma]$.

We define the *direct sum* (or *disjoint union*) of transversals by

$$(E, \sigma^1, \pi^1, \tau^1) + (F, \sigma^2, \pi^2, \tau^2) := (E+F, \sigma^1+\sigma^2, \pi^1+\pi^2, \tau^1+\tau^2),$$

which is clearly a transversal, as can be seen by considering representations.

Let (E, σ, π, τ) be a transversal. If $B \subseteq E$ is a union of blocks of σ, we define the *restriction of* (σ, π, τ) to B by

$$(\sigma, \pi, \tau)_B := (\sigma_B, \pi_B, \tau_B),$$

which is again a transversal since it has a representation.

Observe that $(\sigma, \pi, \tau) = \sum_{B \varepsilon \rho} (\sigma, \pi, \tau)_B$ for any $\rho \geqslant \sigma \vee \tau$.

Define an auxiliary ordering in $T[\sigma]$ by

$$(\sigma, \pi, \tau) \leqslant' (\sigma, \pi', \tau') \qquad \text{iff} \quad \pi \geqslant \pi' \text{ and } \tau \leqslant \tau'.$$

The relation $\leqslant'$ is clearly an ordering.

Define a relation $\leqslant$ in $T[\sigma]$ by

$$(\sigma, \pi, r) \leqslant (\sigma, \pi', \tau') \qquad \text{iff} \quad (\sigma, \pi, \tau) \leqslant' (\sigma, \pi', \tau') \text{ and } \pi' \text{ and } \tau \text{ commute.}$$

PROPOSITION 2.1. *$(T[\sigma], \leqslant)$ is a poset with minimum $\bar{0}^\sigma$ and maximum $\bar{1}^\sigma$.*

Proof. The only non-trivial thing to verify is the implication that $(\sigma, \pi^1, \tau^1) \leqslant (\sigma, \pi, \tau) \leqslant (\sigma, \pi^2, \tau^2) \rightarrow \pi^2$ and τ^1 commute.

It follows from the antecedent of this implication that

$$\pi^2 \vee \tau^1 \leqslant (\pi \vee \tau^1) \wedge (\pi^2 \vee \tau) \tag{1}$$

since the left-hand side is finer than both terms of the meet on the right.

Let $B \in \pi^2 \vee \tau^1$, and let $A \in \pi^2_B$, $E \in \tau^1_B$. By Proposition 1.1, all we need to show is $A \cap E \neq \varnothing$.

By (1), $B \subseteq C \cap D$, where $C \in \pi \vee \tau^1$, $D \in \pi^2 \vee \tau$. Since $\pi^2 \leqslant \pi$, there is $\bar{A} \in \pi_C$ with $A \subseteq \bar{A}$. Since $\tau^1 \leqslant \tau$, there is $\bar{E} \in \tau_D$ with $E \subseteq \bar{E}$. Since π and τ^1 commute, we have

$$\varnothing \neq E \cap \bar{A} \subseteq B,$$

and since π^2 and τ commute, we have

$$\varnothing \neq \bar{E} \cap A \subseteq B.$$

If $A \cap E = \varnothing$, then A intersects a block $E_1 \in \tau^1_B$, $E_1 \neq E$, $E \subseteq \bar{E}$. Therefore $\bar{A}$ intersects both E and E_1, so $|\bar{A} \cap \bar{E}| > 1$, contradicting $\pi \wedge \tau = \hat{0}$. ∎

The poset $(T[\sigma], \leqslant)$ is not ranked in general (e.g., $T[\{1, 2/3, 4/5\}]$ is not ranked). Nor is it a lattice (e.g., $T[\{\{1,..., 8\}\}]$ is not a lattice; thanks to M. Haiman for this observation).

The author cannot think about this ordering except in terms of the operations on transversals to be described shortly.

PROPOSITION 2.2 (The "local" character of $\leqslant$). (a) *If $\rho \geqslant \pi \vee \tau'$, then*

$$(\sigma, \pi', \tau') \leqslant (\sigma, \pi, \tau) \qquad \textit{iff} \quad \forall B \in \rho\colon (\sigma, \pi', \tau')_B \leqslant (\sigma, \pi, \tau)_B$$

(b) *If $t = (\sigma, \pi, \tau)$ is a transversal and $\rho \geqslant \sigma$ and for each $B \in \rho$ we are given a transversal $s^B = (\sigma_B, \pi^B, \tau^B)$ such that $s^B \leqslant t_B$, then $\sum_{B \in \rho} s^B \leqslant t$.*

Proof. (a) Since $\rho \geqslant \pi$, clearly $\pi' \geqslant \pi$ iff $\forall B \in \rho\colon \pi'_B \geqslant \pi_B$. Since $\rho \geqslant \tau'$, clearly $\tau' \leqslant \tau$ iff $\forall B \in \rho\colon \tau'_B \leqslant \tau_B$. By Proposition 1.1, since $\rho \geqslant \pi \vee \tau'$ we have: π and τ' commute if $\forall B \in \rho\colon \pi_B$ and τ'_B commute.

Part (b) follows from (a) by observing that if $\sum_{B \in \rho} s^B =: (\sigma, \pi', \tau')$, then by construction $\tau' \leqslant \rho$, so $\rho \geqslant \sigma \vee \tau' \geqslant \pi \vee \tau'$. ∎

The ordering of transversals can be visualized with the aid of the following operations. Let $t = (\sigma, \pi, \tau)$ be a transversal. The *descending operations* are of two kinds:

(1) *Splitting.* Given a partition $\rho \geqslant \sigma$, the result of *splitting t according to ρ* is the transversal $\sum_{B \in \rho} (\sigma_B, \pi_B, \tau_B) = (\sigma, \pi, \tau \wedge \rho)$.

If $\sigma \leqslant \rho \lessdot \sigma \vee \tau$ (i.e., $|\rho| = |\sigma \vee \tau| + 1$ and $\sigma \leqslant \rho \leqslant \sigma \vee \tau$) then splitting t according to ρ is an *elementary splitting*. The visual effect of an elementary splitting is to split into two matrices, by a vertical cut, one of the matrices (with conveniently relabeled columns) of a representation of t.

(2) *Thinning.* This operation affects only t_B, where B is the chosen block of $\sigma \vee \tau$, so we shall assume $\sigma \vee \tau = \hat{1}$ in order to simplify notation. Let n be a common divisor of all the $|\pi_C|$ for $C \in \sigma$. An *n-thinning* consists of choosing a small transversal $(\bar{\pi}, \bar{\tau})$ of length n of the partition σ/π, and then forming the transversal (π', τ') of σ whose components are defined by

$$\pi'/\pi = \bar{\pi},$$

and

$$\tau' = \tau'' \wedge \tau,$$

where τ'' is defined by

$$\tau''/\pi = \bar{\tau}.$$

A representation of (π', τ') can be obtained by replacing the entries of a representation of $(\bar{\pi}, \bar{\tau})$, which are blocks of π, by the corresponding columns of a representation of (π, τ).

If n is prime then an n-thinning is an *elementary thinning.*

Note that the effect of any descending operation can be achieved by a sequence of elementary descending operations.

PROPOSITION 2.3. *Let $t^i = (\sigma, \pi^i, \tau^i) \in T[\sigma]$, $i = 1, 2$. Then $t^1 \leqslant t^2$ iff there is a sequence of descending operations transforming t^2 into t^1.*

Proof. That descending operations produce finer transversals follows easily from the definitions and Proposition 2.2.

Now assume $t^1 \leqslant t^2$. Split t^2 according to $\rho = \sigma \vee \tau^1 \geqslant \sigma$ to get $\sum_{B \in \rho} t_B^2$. Since $t^1 = \sum_{B \in \rho} t_B^1$, if we show that we can thin t_B^2 to get t_B^1, we are done.

Thus, we have reduced to the case $t^1 \leqslant t^2$, $\sigma \vee \tau^1 = \pi^1 \vee \tau^1 = \sigma \vee \tau^2 = \pi^2 \vee \tau^2 = \hat{1}$. Then, in every block of $\pi^2 \vee \tau^1$, no two blocks of π^2 are in the same block of π^1 (otherwise $\pi^1 \wedge \tau^1 \neq \hat{0}$, as in the proof of Proposition 2.1) and no two blocks of τ^1 are in the same block of τ^2 (otherwise $\tau^2 \wedge \pi^2 \neq \hat{0}$). From this it follows that there are representations of t^1 and t^2 consisting of matrices for the blocks of $\pi^2 \vee \tau^1$ aligned vertically in the case of t^1 and horizontally in the case of t^2, so the desired thinning is possible. More precisely, we thin t^2 according to the small transversal $(\sigma/\pi^2, \bar{\pi}, \bar{\tau})$ defined by $\bar{\pi} = \pi^1/\pi^2$, $\bar{\tau} = (\pi^2 \vee \tau^1)/\pi^2$. ∎

COROLLARY 2.4 (The cover relations in $T[\sigma]$). *Let $t^1, t^2 \in T[\sigma]$. Then $t^1 \lessdot t^2$ iff t^1 is the result of performing an elementary descending operation on t^2.* ∎

COROLLARY 2.5. *Let $t^i = (\sigma, \pi^i, \tau^i) \in T[\sigma]$, $i = 1, 2$, $t^1 \leqslant t^2$, and $\sigma \vee \tau^2 = \hat{1}$. Then there is a natural poset isomorphism*

$$\phi \colon [t^1, t^2] \simeq T[(\sigma \vee \tau^1)/(\pi^2 \vee \tau^1)].$$

Proof. The isomorphism can be described as follows. Let $\bar{\sigma} = (\sigma \vee \tau^1)/(\pi^2 \vee \tau^1)$. Choose representations of t^1 and t^2 consisting of matrices for the blocks of $\pi^2 \vee \tau^1$, aligned horizontally in a single row in the case of t^2, and vertically in possibly several separate columns in the case of t^1 (cf. proof of Proposition 2.3). Then, given $\bar{t} \in T[\bar{\sigma}]$, substitute the entries of the matrices of a representation of $\bar{t}$ by the corresponding matrices for the blocks of $\pi^2 \vee \tau^1$, thus obtaining a representation of a transversal $t \in [t^1, t^2]$. The correspondence

$$\phi \colon t \leftrightarrow \bar{t}$$

is the desired isomorphism, which can also be described by

$$\phi(\sigma, \pi, \tau) = (\bar{\sigma}, \bar{\pi}, \bar{\tau}),$$

where

$$\bar{\pi} = (\pi \vee \tau^1)/(\pi^2 \vee \tau^1) \qquad \text{and} \qquad \bar{\tau} = (\tau \vee \pi^2)/(\pi^2 \vee \tau^1). \quad \blacksquare$$

THEOREM 2.6. (The Natural Factorization of Intervals of Transversals). *Let* $t^i = (\sigma, \pi^i, \tau^i) \in T[\sigma]$ $(i = 1, 2)$, $t^1 \leqslant t^2$. *Then there is a natural poset isomorphism*

$$[t^1, t^2] \simeq \prod_{B \in \sigma \vee \tau^2} T[(\sigma \vee \tau^1)_B/(\pi^2 \vee \tau^1)_B].$$

Proof. By Proposition 2.2, the correspondence

$$\psi(t) = (t_B : B \in \sigma \vee \tau^2)$$

defines a natural isomorphism of posets

$$\psi: [t^1, t^2] \simeq \prod_{B \in \sigma \vee \tau^2} [t^1_B, t^2_B].$$

Now use Corollary 2.5. $\blacksquare$

COROLLARY 2.7. (Upper and lower intervals in $T[\sigma]$). *Let* $t = (\sigma, \pi, \tau) \in T[\sigma]$. *Then*

$$[\bar{0}^\sigma, t] \simeq \prod_{B \in \sigma \vee \tau} T[\sigma_B/\pi_B],$$

and

$$[t, \bar{1}^\sigma] \simeq T[(\sigma \vee \tau)/\tau]. \quad \blacksquare$$

2.2. *The Large Reduced Incidence Bialgebra of Transversals*

Let T be the class (in the set-theoretic sense) of all the intervals of all the posets of transversals of partitions of finite sets. Given $[t^1, t^2] \in T$, define $M[t^1, t^2]$ to be the multiset of the classes of the partitions appearing in the natural factorization of $[t^1, t^2]$ (cf. Theorem 2.6); i.e., if $t^i = (\sigma, \pi^i, \tau^i)$ $(i = 1, 2)$, then

$$M[t^1, t^2] := \{\mathrm{cl}((\sigma \vee \tau^1)_B/(\pi^2 \vee \tau^1)_B) : B \in \sigma \vee \tau^2\}.$$

Define an equivalence $\sim$ on T by

$$[s, t] \sim [s', t'] \qquad \text{iff} \quad M[s, t] = M[s', t'].$$

Just as for partitions (Proposition 1.3), we clearly have

PROPOSITION 2.8. *If* $[s, t] \sim [s', t']$ *then there is a bijection* $\phi\colon [s, t] \to [s', t']$ *such that, for all* $u \in [s, t]$, *we have*

$$[s, u] \sim [\phi(s), \phi(u)]$$

and

$$[u, t] \sim [\phi(u), \phi(t)]. \quad \blacksquare$$

Note that $M[s+s', t+t'] = M[s, t] \cup M[s', t']$, a union of multisets (i.e., we add multiplicities). As in the case of partitions, we clearly have

PROPOSITION 2.9. *If* $[s, t]$, $[s', t']$, $[s'', t''] \in T$ *and* $[s, t] \sim [s'+s'', t'+t'']$, *then there is a bijection*

$$\phi\colon [s', t'] \times [s'', t''] \to [s, t]$$

such that

$$[s, \phi(u', u'')] \sim [s'+s'', u'+u'']$$

and

$$[\phi(u', u''), t] \sim [u'+u'', t'+t'']. \quad \blacksquare$$

Let K be a field of characteristic zero. Let X be a set of formal variables x_λ, one for each class $\lambda = (\lambda_1, \lambda_2, ...) \neq \mathbf{0}$. Let $K[X]$ be the usual K-algebra of polynomials in the variables X. If M is a finite multiset of classes, define $X^M \in K[X]$ to be the monomial

$$X^M := \prod_\lambda x_\lambda^{M(\lambda)},$$

where $M(\lambda)$ is the multiplicity of λ in M. Observe that $X^{M \cup M'} = X^M X^{M'}$ and $X^\varnothing = 1$.

Identify the equivalence class of $[s, t] \in T$ under $\sim$ with $X^{M[s,t]}$, and define augmentation and diagonalization maps

$$\varepsilon\colon K[X] \to K$$

$$\Delta\colon K[X] \to K[X] \otimes K[X]$$

by

$$\varepsilon(M) = \begin{cases} 1 & \text{if } M = \varnothing \text{ or } M = \{\delta_1, \delta_1, ..., \delta_1\} (M(\delta_1) \text{arbitrary}) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Delta X^M = \sum_{u \in [s,t]} X^{M[s,u]} \otimes X^{M[u,t]},$$

where $[s, t]$ is any segment with $M[s, t] = M$.

Proposition 2.8 and 2.9 ensure that the general results of Schmitt apply (cf. [S]), so we have

THEOREM 2.10. *Let $B(T)$ be the structure $(K[X], \Delta, \varepsilon)$ with the usual algebra structure. Then $B(T)$ is a bialgebra, and localizing at x_{δ_1} makes it into a Hopf algebra.* ∎

The dual $B(T)^* = \mathrm{Mod}_K\ (B(T), K)$ is the (large, reduced oncidence algebra of transversals. As a vector space, it is isomorphic to the vector space $K[[X]]$ of formal power series in the variables X.

A function $f \in B(T)^*$ is *multiplicative* if it is an algebra map, i.e., $f \in \mathrm{Alg}_K(K[X], K)$. As we saw in the proof of theorem 1.4, $\mathrm{Alg}_K(K[X], K)$ is closed under convolution; it is in fact a semigroup, since the convolution in $B(T)^*$ is associative (by the coassociativity of Δ) and its identity is ε (by the counitary property), which is multiplicative.

A multiplicative function $f \in B(T)^*$ is determined by the values $f(x_\lambda)$ $(x_\lambda \in X)$, or, equivalently, by the *partitional generating function*

$$Gf(\mathbf{x}) = \sum_{\lambda \neq \mathbf{0}} f(x_\lambda)\, \mathbf{x}^\lambda / \mathrm{aut}(\lambda),$$

a formal power series in the new variables $\mathbf{x} = (x_1, x_2, \ldots)$ without constant term.

We can now state the main result.

THEOREM 2.11. *Let $f, g \in \mathrm{Alg}_K(K[X], K)$. Then*

$$G(f * g) = Gg[Gf],$$

i.e., the semigroup of multiplicative functions of $B(T)^*$, under convolution, is anti-isomorphic to the semigroup of partitional formal power series in the variables $(x_1, x_2, \ldots)$ without constant term, under plethysm.

Proof. For $f \in B(T)^*$ and $[s, t] \in T$, write

$$f([s, t]) := f(X^{M[s,t]}).$$

For f multiplicative, define, for σ a partition of class $\lambda \neq \mathbf{0}$,

$$f(\sigma) := f(\lambda) := f([\bar{0}^\sigma, \bar{1}^\sigma]).$$

Conversely, if a rule $\sigma \to f(\sigma) \in K$ assigns a scalar to every non-empty partition so that $f(\sigma)$ depends only on $\mathrm{cl}(\sigma)$, then it defines a unique multiplicative function f by

$$f(x_\lambda) := f(\sigma), \qquad \mathrm{cl}(\sigma) = \lambda \neq \mathbf{0}.$$

Given a family $\{f_i : i \in I\}$ of multiplicative functions such that, for every $\lambda \neq \mathbf{0}$, $f_i(\lambda) = 0$ for almost all $i \in I$, define a multiplicative function $\dot{\sum}_{i \in I} f_i$ by the rule

$$\left(\dot{\sum_{i \in I}} f_i\right)(\sigma) := \sum_{i \in I} f_i(\sigma).$$

This sum is not to be confused with the usual sum in $B(T)^*$. Clearly we have

$$G \dot{\sum_{i \in I}} f_i = \sum_{i \in I} Gf_i.$$

Define a product (different from convolution) in $\mathrm{Alg}_K(K[X], K)$ by the rule

$$f \cdot g(\sigma) := \sum_{\sigma_1 + \sigma_2 = \sigma} f(\sigma_1)\, g(\sigma_2),$$

where the sum ranges over all *ordered* pairs (σ_1, σ_2) of non-empty partitions such that $\sigma_1 + \sigma_2 = \sigma$. Since the number of such pairs with $\mathrm{cl}(\sigma_1) = \mu$ and $\mathrm{cl}(\sigma_2) = \nu$ is

$$\binom{\lambda}{\mu, \nu} := \begin{cases} \dfrac{\lambda!}{\mu!\,\nu!} = \dfrac{\mathrm{aut}(\lambda)}{\mathrm{aut}(\mu)\,\mathrm{aut}(\nu)} & \text{if } \mu + \nu = \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

it is easy to see that

$$G(f \cdot g) = (Gf)(Gg).$$

It follows that $f^{\cdot n}$, the nth power of f with respect to this product, is given by the rule

$$f^{\cdot n}(\sigma) = \sum_{(\sigma_1, \ldots, \sigma_n)} f(\sigma_1) \cdots f(\sigma_n),$$

where the sum ranges over all *ordered* n-tuples $(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 + \cdots + \sigma_n = \sigma$. Thus, if we define a multiplicative function $\gamma_n(f)$ by

$$\gamma_n(f)(\sigma) := \sum_{\{\sigma_1, \ldots, \sigma_n\}} f(\sigma_1) \cdots f(\sigma_n),$$

where the sum ranges over all (unordered) n-element sets $\{\sigma_1, \ldots, \sigma_n\}$ such that $\sigma_1 + \cdots + \sigma_n = \sigma$, then

$$\gamma_n(f)(\sigma) = \frac{1}{n!} f^{\cdot n}(\sigma),$$

and therefore

$$G\gamma_n(f) = (Gf)^n/n!.$$

Let λ be a class and $n \geqslant 1$. The *nth Verschiebung* of λ is the class $\lambda^{\{n\}}$ defined by

$$\lambda_k^{\{n\}} := \begin{cases} \lambda_{k/n} & \text{if } n \mid k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $T_n[\sigma]$ be the set of all small transversals of σ of length n. If $(\pi, \tau) \in T_n[\sigma]$ and $\mathrm{cl}(\sigma/\pi) = \mu$ then $\mathrm{cl}(\sigma) = \mu^{\{n\}}$. An elementary counting argument (cf. [N–R]) shows that if $\mathrm{cl}(\sigma) = \mu^{\{n\}}$ then $|T_n[\sigma]| = (1/n!) \prod_{k \geqslant 1} ((kn)!/k!)^{\mu_k}$.

Given a multiplicative function f and $n \geqslant 1$, define a multiplicative function $f^{\{n\}}$ by the rule

$$f^{\{n\}}(\sigma) := \sum_{(\pi,\tau) \in T_n[\sigma]} f(\sigma/\pi).$$

Then it is clear that

$$f^{\{n\}}(\lambda) = \begin{cases} (1/n!) \prod_{k \geqslant 1} ((kn)!/k!)^{\mu_k} f(\mu) & \text{if } \lambda = \mu^{\{n\}}, \\ 0 & \text{otherwise.} \end{cases}$$

A straightforward computation then shows that

$$\begin{aligned} Gf^{\{n\}}(\mathbf{x}) &= (1/n!)\, Gf(x_n, x_{2n}, \ldots) \\ &= (1/n!)\, F_n Gf, \end{aligned}$$

where F_n is the nth Frobenius operator (cf. Section 1.4).

Define the (*upper*) *class* of a transversal $t = (\sigma, \pi, \tau)$ as the class of the partition $(\sigma \vee \tau)/\tau$:

$$\mathrm{cl}(t) := \mathrm{cl}((\sigma \vee \tau)/\tau).$$

Given a class $\lambda \neq \mathbf{0}$, define

$$T_\lambda[\sigma] := \{t \in T[\sigma] \colon \mathrm{cl}(t) = \lambda\}.$$

Given a multiplicative function f and a class $\lambda \neq \mathbf{0}$, define $f^{[\lambda]}$ by the rule

$$f^{[\lambda]}(\sigma) := \sum_{(\pi,\tau) \in T_\lambda[\sigma]} \prod_{B \in \sigma \vee \tau} f(\sigma_B/\pi_B).$$

Note that by Corollary 2.7 we have

$$f^{[\lambda]}(\sigma) = \sum_{t \in T_\lambda[\sigma]} f([\bar{0}^\sigma, t]).$$

It follows from the definitions that if $\lambda = k\delta_n$ then

$$f^{[\lambda]} = \gamma_k(f^{\{n\}}).$$

Next, observe that for an arbitrary $\lambda \neq \mathbf{0}$ we have $\lambda = \sum_{k \in I} \lambda_k \delta_k$, where $I = \{k \colon \lambda_k \neq 0\}$, and

$$f^{[\lambda]} = \prod_{k \in I} f^{[\lambda_k, \delta_k]}.$$

Therefore:

$$\begin{aligned} Gf^{[\lambda]} &= \prod_{k \in I} Gf^{[\lambda_k \delta_k]} = \prod_{k \in I} (1/\lambda_k!)((F_k Gf)/k!)^{\lambda_k} \\ &= (1/\mathrm{aut}(\lambda)) \prod_{k \geqslant 1} (F_k Gf)^{\lambda_k}. \end{aligned}$$

Finally:

$$\begin{aligned} f * g(\sigma) &= \sum_{t \in T[\sigma]} f([\bar{0}^\sigma, t])\, g([t, \bar{1}^\sigma]) \\ &= \sum_{\lambda \neq \mathbf{0}} \sum_{t \in T_\lambda[\sigma]} f([\bar{0}^\sigma, t])\, g([t, \bar{1}^\sigma]) \\ &= \sum_{\lambda \neq \mathbf{0}} g(\lambda) \sum_{t \in T_\lambda[\sigma]} f([\bar{0}^\sigma, t]) \\ &= \left(\dot{\sum_{\lambda \neq \mathbf{0}}} g(\lambda) f^{[\lambda]} \right)(\sigma), \end{aligned}$$

so

$$\begin{aligned} G(f * g) &= G\left(\dot{\sum_{\lambda \neq \mathbf{0}}} g(\lambda) f^{[\lambda]} \right) \\ &= \sum_{\lambda \neq \mathbf{0}} g(\lambda)\, Gf^{[\lambda]} \\ &= \sum_{\lambda \neq \mathbf{0}} g(\lambda) \prod_{k \geqslant 1} (F_k Gf)^{\lambda_k}/\mathrm{aut}(\lambda) \\ &= Gg[Gf]. \quad \blacksquare \end{aligned}$$

We next compute the Möbius function μ of $B(T)^*$ by combinatorial methods, thus obtaining a combinatorial interpretation (and computation) of the plethystic inverse of the formal power series $\exp \sum_{n \geqslant 1} x_n/n! - 1$, which is the generating function $G\zeta$ of the zeta function of $B(T)^*$. Since μ is multiplicative, it suffices to compute $\mu(\lambda) := \mu(T[\sigma])$ (where $\mathrm{cl}(\sigma) = \lambda$) for all classes $\lambda \neq \mathbf{0}$.

The main tool is the following theorem of Rota (cf. Theorem 2 of [R]). Let P be a poset. A *(dual) closure* on P is a function $x \to \bar{x}$ of P into itself which is order-preserving and such that $\bar{x} \leqslant x$ and $\bar{\bar{x}} = \bar{x}$ for all $x \in P$.

THEOREM 2.12 (Rota). *Let P be a finite poset with minimum $\hat{0}$ and maximum $\hat{1}$, and let $x \to \bar{x}$ be a (dual) closure on P such that $\bar{\hat{1}} < \hat{1}$. Let $a \in P$. Then*

$$\sum_{\{x : \bar{x} = \bar{a}\}} \mu_P(x, \hat{1}) = 0,$$

where μ_p is the Möbius function of P. ∎

First we prove that if $\lambda = (\lambda_1, \lambda_2, ...)$ has more than one non-zero entry, then $\mu(\lambda) = 0$. Let (E, σ) be a partition of class λ. Choose k such that $\lambda_k \neq 0$. Let (E_k, σ_k) be the partition defined by

$$\sigma_k := \{B \in \sigma : |B| = k\},$$

and let C be the difference set

$$C := E - E_k.$$

Define $\rho \in \Pi[E]$ by

$$\rho := \{C, E_k\}.$$

Clearly, $\rho \geqslant \sigma$. By hypothesis, $\rho < \hat{1}_E$.

Define a (dual) closure $t \to \bar{t}$ on $T[\sigma]$ by

$$\bar{t} := \text{the splitting of } t \text{ according to } \rho.$$

Since $\rho < \hat{1}_E$, $\overline{\hat{1}^\sigma} < \hat{1}^\sigma$. Let $a = \bar{0}^\sigma$. Then $\{t \in T[\sigma] : \bar{t} = \bar{a} = \bar{0}^\sigma\} = \{\bar{0}^\sigma\}$. So by Theorem 2.12 we get $\mu(\lambda) = 0$.

We are left with the case in which λ has only one non-zero entry. So let $\lambda = k\delta_n$. We treat the case $k > 1$ first. Let σ be a partition of class $\lambda = (k+1)\,\delta_n$. Then σ consists of $k+1$ blocks $B_0, B_1, ..., B_k$, each of size n.

Define $\rho \in \Pi[E]$ by

$$\rho = \{B_0, B_1 \cup \cdots \cup B_k\}.$$

Clearly, $\sigma < \rho < \hat{1}_E$. Define a dual closure $t \to \bar{t}$ on $T[E]$ as before. Clearly $\bar{1}^\sigma < \bar{1}^\sigma$. Let $a := \bar{0}^\sigma = \bar{a}$.

If $\bar{t} = \bar{0}^\sigma$ and $t \neq \bar{0}^\sigma$ then t has the form $t = (\sigma, \sigma, \tau)$, where τ is such that a representation of t looks like

all blocks $B_1, ..., B_k$ except some block B_j B_j and B_0

so $\mathrm{cl}(t) := \mathrm{cl}((\sigma \vee \tau)/\tau) = k\delta_n$, and the number of such transversals is easily seen to be $k \cdot n!$ Corollary 2.7 and Theorem 2.12 then give us

$$\begin{aligned} 0 &= \sum_{\{t:\, \bar{t} = a\}} \mu(t, \bar{1}^\sigma) = \mu(\bar{0}^\sigma, \bar{1}^\sigma) + \sum_{\{t \neq \bar{0}:\, \bar{t} = \bar{0}\}} \mu(t, \bar{1}^\sigma) \\ &= \mu((k+1)\,\delta_n) + k \cdot n!\,\mu(k\delta_n). \end{aligned}$$

From this recursion (in k) we get

$$\mu(k\delta_n) = (-1)^{k-1}\,(k-1)!\,n!^{k-1}\mu(\delta_n).$$

Finally, we compute $\mu(\delta_n)$ using Hall's theorem (cf. [R, Prop. 6]). Let $\sigma = \hat{1}_E$, $|E| = n$. Let $\bar{1}^\sigma = t_0 > t_1 > \cdots > t_k = \bar{0}^\sigma$ be a k-chain in $T[\sigma]$, where $t_j = (\sigma, \pi_j, \tau_j)$. Then t_j is an n_j-thinning of t_{j-1}, where

$$n_j := \frac{|\tau_j|}{|\tau_{j-1}|} > 1.$$

We say that the chain has type $(n_1, n_2, ..., n_k)$. Observe that the type is an ordered factorization of n into k factors. Using the formula for the number of small transversals of given length of a partition (cf. proof of Theorem 2.11) and the definition of thinning, we obtain that the number of chains in $T[\sigma]$ of type $(n_1, ..., n_k)$ is $n!/n_1! \cdots n_k!$ Hall's theorem then gives

$$\mu(\delta_n) = \sum_{(n_1, ..., n_k)} (-1)^k\, n!/n_1! \cdots n_k!,$$

i.e.,

$$\mu(\delta_n)/n! = \sum_{(n_1, ..., n_k)} (-1)^k/n_1! \cdots n_k!,$$

where the sum ranges over all ordered factorizations of n. Recognizing the right-hand side as the Dirichlet-convolution inverse of the function $n \to 1/n!$, we get the following recursion for the $\mu(\delta_n)$'s:

$$\sum_{ij = n} \mu(\delta_i)/i!\, j! = \delta(n, 1).$$

We can now easily compute $G\mu$ and get

$$G(\mu) = \log \prod_{n \geqslant 1} (1 + x_n)^{\mu(\delta_n)/n!}.$$

2.3. *Morphisms between Partitions*

In this section we define a notion of morphisms between partitions, whose kernels are transversals. The resulting category of partitions has as isomorphisms the usual ones, viz., those described in Section 1.2. In the spirit of [J–R–S], we introduce two sequences of polynomials in $\mathbf{x}$ (indexed by classes) which count morphisms and monomorphisms, prove that they are of "binomial type", and find expressions for the connection constants by summation and differentiation (Möbius inversion) over the poset of transversals, thus obtaining partitional analogues for x^n, $(x)_n$ and the Stirling numbers of first and second kind.

First we must go back to the classical case. If $f: E \to F$ is a function between (finite) sets, the *kernel* of f is the partition $\ker f$ of E whose blocks are the fibers $f^{-1}(y)$ $(y \in F)$ of f. Note that f is injective iff $\ker f = \hat{0}_E$. Also, if $D \subseteq E$, $\ker f|_D = (\ker f)_D$. In general, $\ker g \circ f \geqslant \ker f$.

If $f: E \to F$ and $\pi \leqslant \ker f$, we define a function

$$f/\pi: \pi \to F$$

by

$$(f/\pi)(B) := f(x), \qquad \text{any } x \in B \ (B \in \pi).$$

Observe that $\ker(f/\pi) = (\ker f)/\pi$, that f/π is injective iff $\ker f = \pi$, and that the correspondence $f \to f/\pi$ is a bijection between the set of functions $f: E \to F$ with $\ker f \geqslant \pi$, and the set of functions $\bar{f}: \pi \to F$.

Now let (E, σ) and (R, ρ) be partitions *A morphism* $\mathbf{f}: (E, \sigma) \to (R, \rho)$ is a pair $\mathbf{f} = (f, \pi)$ consisting of

(1) a function $f: E \to R$ such that

- $(\forall B \in \sigma)(f(B) \in \rho)$, i.e., f sends blocks of σ onto blocks of ρ, and
- $(\forall B \in \sigma)(\forall y_1, y_2 \in f(B))$ $(|f^{-1}(y_1) \cap B| = |f^{-1}(y_2) \cap B|)$, i.e.,

for all $B \in \sigma$, $f|_B$ is an n-to-one function for some n;

(2) a partition $\pi \in \Pi[E]$ such that

- $\pi \leqslant \sigma$
- for all $B \in \pi$, $f|_B$ is injective, and
- $(\forall B \in \pi)(f(B) \in \rho)$.

We denote by $\mathrm{Mor}(\sigma, \rho)$ or $\mathrm{Mor}((E, \sigma), (F, \rho))$ the set of morphisms $\mathbf{f}: (E, \sigma) \to (F, \rho)$.

We define the *kernel* $\ker \mathbf{f}$ of $\mathbf{f} = (f, \pi)$ to be the pair of partitions $(\pi, \ker f)$. It follows easily from the definitions that $\ker \mathbf{f} \in T[\sigma]$.

If $D \subseteq E$ is a union of blocks of σ, we define the restriction $\mathbf{f}|_D$ of $\mathbf{f}$ to D by

$$\mathbf{f}|_D := (f|_D, \pi_D) \in \mathrm{Mor}(\sigma_D, \rho).$$

Note that $\ker \mathbf{f}|_D = (\ker \mathbf{f})_D$.

A morphism $\mathbf{f} = (f, \pi)$ is a *monomorphism* if f is injective. Clearly, in such case $\ker \mathbf{f} = \bar{0}^\sigma$. We denote by $\mathrm{Mon}(\sigma, \rho)$ or $\mathrm{Mon}((E, \sigma), (F, \rho))$ the set of monomorphisms.

Next we define the composition of morphisms. Let $\mathbf{f} = (f, \pi_1)$: $(E, \sigma) \to (F, \rho)$ and $\mathbf{g} = (g, \pi_2)$: $(F, \rho) \to (G, \omega)$ be morphisms. Then their *composition* $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ is the pair $\mathbf{h} = (h, \pi)$, where $h = g \circ f$ and $\pi \in \Pi[E]$ is defined in terms of equivalence relations (cf. Section 1.2) by

$$x \sim_\pi y \qquad \text{iff} \quad x \sim_{\pi_1} y \text{ and } f(x) \sim_{\pi_2} f(y).$$

It is straightforward that $\mathbf{h} \in \mathrm{Mor}(\sigma, \omega)$, and that this composition is associative. Also, notice that $\ker(\mathbf{g} \circ \mathbf{f}) \geqslant \ker \mathbf{f}$, which can be seen by observing that, for each $B \in \sigma \vee \ker f$, $(\ker \mathbf{f})_B = \ker \mathbf{f}|_B$ is a thinning of $\ker(g \circ \mathbf{f})|_B = (\ker \mathbf{g} \circ \mathbf{f})_B$, and then using Proposition 2.2(a).

Although the strictly categorical aspects of this theory lie beyond the scope of the present work, we shall make use of the following fact:

PROPOSITION 2.13. *The disjoint union of partitions is the direct sum in the category of partitions and morphisms.*

Proof. Let $\{(E_i, \sigma_i): i \in I\}$ be a finite set of partitions. Let $(E, \sigma) := \sum_{i \in I} (E_i, \sigma_i)$, and for each $i \in I$ let $\mathbf{h} = (h_i, \hat{0}_{E_i})$: $(E_i, \sigma_i) \to (E, \sigma)$ be defined by $h_i(x) = x$ $(x \in E_i)$.

Then, given a partition ρ, the correspondence

$$\psi: \mathbf{f} \to (\mathbf{f}_i = \mathbf{f}|_{E_i}: i \in I)$$

is clearly a bijection

$$\psi: \mathrm{Mor}(\sigma, \rho) \simeq \prod_{i \in I} \mathrm{Mor}(\sigma_i, \rho)$$

such that $\mathbf{f} \circ \mathbf{h}_i = \mathbf{f}_i$. ∎

The author believe that in this category direct products do not exist, an obvious candidate having failed.

PROPOSITION 2.14. *Let* (E, σ), (F, ρ) *be partitions, and let* $t_0 = (\pi_0, \tau_0) \in T[\sigma]$. *Let*

$$\mathrm{Mor}^{t_0}(\sigma, \rho) := \{\mathbf{f} \in \mathrm{Mor}(\sigma, \rho) \colon \ker \mathbf{f} \geqslant t_0\},$$
$$\mathrm{Mor}_{t_0}(\sigma, \rho) := \{\mathbf{f} \in \mathrm{Mor}(\sigma, \rho) \colon \ker \mathbf{f} = t_0\}.$$

Then there is a natural bijection

$$\phi \colon \mathrm{Mor}^{t_0}(\sigma, \rho) \simeq \mathrm{Mor}((\sigma \vee \tau_0)/\tau_0, \rho)$$

whose restriction gives a bijection

$$\mathrm{Mor}_{t_0}(\sigma, \rho) \simeq \mathrm{Mon}((\sigma \vee \tau_0)/\tau_0, \rho).$$

Proof. Given $\mathbf{f} = (f, \pi) \in \mathrm{Mor}^{t_0}(\sigma, \rho)$, define

$$\phi(\mathbf{f}) := (\bar{f}, \bar{\pi}),$$

where

$$\bar{f} := f/\tau_0,$$
$$\bar{\pi} := (\pi \vee \tau_0)/\tau_0. \quad ∎$$

Observe that if $\ker \mathbf{f} = t$ and $\ker \phi(\mathbf{f}) = \bar{t}$ then the correspondence $t \to \bar{t}$ is the natural isomorphism of Corollary 2.7. We define, for $f \in \mathrm{Mor}^{t_0}(\sigma, \rho)$,

$$\mathbf{f}/t_0 := \phi(\mathbf{f}) \in \mathrm{Mor}((\sigma \vee \tau_0)/\tau_0, \rho).$$

COROLLARY 2.15. *There is a natural bijection*

$$\mathrm{Mor}(\sigma, \rho) \simeq \sum_{(\pi, \tau) \in T[\sigma]} \mathrm{Mon}((\sigma \vee \tau)/\tau, \rho).$$

Proof. The bijection is

$$\mathbf{f} = (f, \pi) \to (\ker \mathbf{f}, \mathbf{f}/\ker \mathbf{f}). \quad ∎$$

PROPOSITION 2.16. *Let* $\mathrm{cl}(\sigma) = \lambda$, $\mathrm{cl}(\rho) = \xi$. *Then*

$$\text{(a)} \quad p_\lambda(\xi) := |\mathrm{Mor}(\sigma, \rho)| = \prod_{n \geqslant 1} \left(\sum_{d \mid n} \frac{n!}{(n/d)!} \xi_d \right)^{\lambda_n},$$
$$\text{(b)} \quad q_\lambda(\xi) := |\mathrm{Mon}(\sigma, \rho)| = \prod_{n \geqslant 1} n!^{\lambda_n} (\xi_n)_{\lambda_n},$$

where $(m)_n := m(m-1) \cdots (m-n+1)$ *is the usual lower factorial.*

Proof. Part (b) follows easily from the definitions.

(a) Since $\sigma = \sum_{B \in \sigma} \hat{1}_B$, by Proposition 2.13 we have

$$\operatorname{Mor}(\sigma, \rho) \simeq \prod_{B \in \sigma} \operatorname{Mor}(\hat{1}_B, \rho).$$

Clearly, $\operatorname{Mor}(\hat{1}_B, \rho) = \sum_{C \in \rho} \operatorname{Mor}(\hat{1}_B, \hat{1}_C)$, a disjoint union of sets. If $|B| = n$ and $|C| = d$, then $\operatorname{Mor}(\hat{1}_B, \hat{1}_C) = \varnothing$ unless $d \mid n$. If $d \mid n$, by Corollary 2.15 $|\operatorname{Mor}(\hat{1}_B, \hat{1}_C)| = |T_d[\hat{1}_B]|\, d! = n!/(n/d)!$ The result follows. ∎

PROPOSITION 2.17. *The polynomial sequences* $\{p_\lambda(\mathbf{x})\}$ *and* $\{q_\lambda(\mathbf{x})\}$ *are of "binomial type," i.e., they satisfy*

$$p_\lambda(\mathbf{x}+\mathbf{y}) = \sum_{\mu+\nu=\lambda} \binom{\lambda}{\mu, \nu} p_\mu(\mathbf{x})\, p_\nu(\mathbf{y}),$$

where the sum ranges over all ordered pairs (μ, ν). *Similarly for the* q_λ*'s.*

Proof. It suffices to prove the identity for integral values of $\mathbf{x}$ and $\mathbf{y}$. Let $\operatorname{cl}(\sigma) = \lambda$, $\operatorname{cl}(\rho) = \mathbf{x}$, $\operatorname{cl}(\omega) = \mathbf{y}$. Then each $\mathbf{f} \in \operatorname{Mor}(\sigma, \rho+\omega)$ naturally (and uniquely) decomposes in a pair $(\mathbf{f}_1, \mathbf{f}_2) \in \operatorname{Mor}(\sigma_1, \rho) \times \operatorname{Mor}(\sigma_2, \omega)$, for some pair (σ_1, σ_2) with $\sigma_1 + \sigma_2 = \sigma$. Hence the identity for the p_λ's. Similarly for the q_λ's. ∎

Let $\operatorname{cl}(\sigma) = \lambda$, $\operatorname{cl}(\rho) = \mathbf{x}$. Define functions $f, g \colon T[\sigma] \to K$ by

$$f(t) := |\operatorname{Mor}^t(\sigma, \rho)|,$$
$$g(t) := |\operatorname{Mor}_t(\sigma, \rho)|.$$

Then

$$f(t) = \sum_{s \geqslant t} g(s), \tag{1}$$

and by Möbius inversion we have

$$g(t) = \sum_{s \geqslant t} \mu(t, s) f(s). \tag{2}$$

In particular, by Proposition 2.14 and Corollary 2.15, we get from (1) that

$$\begin{aligned} p_\lambda(\mathbf{x}) = f(\bar{0}^\sigma) &= \sum_{t \in T[\sigma]} g(t) \\ &= \sum_{t \in T[\sigma]} q_{\operatorname{cl}(t)}(\mathbf{x}) \\ &= \sum S(\lambda, \mu)\, q_\nu(\mathbf{x}), \end{aligned}$$

where

$$S(\lambda, \nu) := |T_\nu[\sigma]|.$$

Similarly, from (2) we get

$$\begin{aligned} q_\lambda(\mathbf{x}) = g(\bar{0}^\sigma) &= \sum_{t \in T[\sigma]} \mu(\bar{0}^\sigma, t)\, p_{\mathrm{cl}(t)}(\mathbf{x}) \\ &= \sum_\nu s(\lambda, \nu)\, p_\nu(\mathbf{x}), \end{aligned}$$

where

$$s(\lambda, \nu) := \sum_{t \in T_\nu[\sigma]} \mu(\bar{0}^\sigma, t).$$

As a consequence, since $\{q_\lambda(\mathbf{x})\}$ is clearly a linear basis for $K[\mathbf{x}]$, our computations show that $\{p_\lambda(\mathbf{x})\}$ is also a basis, a fact which *prima facie* is not clear to the author.

III. The Permutational Case

3.1. *Permutations and Compatible Partitions*

Let $S[E]$ be the group of permutations of the (finite) set E. We use the usual multiplicative notation for composition in $S[E]$. If $\alpha \in S[E]$ and $D \subseteq E$ is such that $\alpha(D) = D$, then

$$\alpha_D := \alpha\,|_D \in S[D].$$

We denote by $\bar{\alpha}$ the partition of E induced by the disjoint cycle decomposition of α:

$$\bar{\alpha} := \{B \subseteq E \colon \alpha_B \text{ is cyclic}, B \neq \varnothing\}.$$

Observe that

$$\overline{(\alpha_D)} = (\bar{\alpha})_D =: \bar{\alpha}_D.$$

We define

$$\mathrm{cl}(\alpha) := \mathrm{cl}(\bar{\alpha}).$$

It is well known that two permutations are isomorphic iff they have the same class, and that the number of automorphisms of a permutation of class λ is

$$\mathrm{aut}'(\lambda) := \prod_{n \geqslant 1} n^{\lambda_n} \lambda_n!$$

The *disjoint union* (or *direct sum*) of two permutations (E, α), (F, β) is the permutation

$$(E, \alpha) + (F, \beta) := (E + F, \alpha + \beta),$$

where $\alpha + \beta$ is defined by

$$(\alpha + \beta)_E := \alpha$$
$$(\alpha + \beta)_F := \beta.$$

Observe that

$$\overline{(\alpha + \beta)} = \bar{\alpha} + \bar{\beta}.$$

A partition $\tau \in \Pi[E]$ is compatible with α, if, for $x, y \in E$,

$$x \sim_\tau y \qquad \text{iff} \quad \alpha x \sim_\tau \alpha y.$$

Equivalently:

$$(\forall B \in \tau) \qquad (\alpha(B) \in \tau).$$

We denote by $C[\alpha]$ the set of all partitions compatible with α. In Bergeron's theory of species over permutations (cf. [B]), $C[\alpha]$ plays a role analogous to that played by $T[\sigma]$ in the theory of partitionals of [N–R]. We shall prove in this section that the usual refinement order on $C[\alpha]$ gives a large, reduced incidence bialgebra related to the plethysm of "permutational" formal power series.

Note that if $\alpha(D) = D$ and $\tau \in C[\alpha]$, then $\tau_D \in C[\alpha_D]$. Also, if τ is compatible with α and β, then it is compatible with $\alpha\beta$, and in particular with α^n.

PROPOSITION 3.1. *With the usual refinement order, $C[\alpha]$ is a sublattice of $\Pi[E]$, where $\alpha \in S[E]$.*

Proof. Clearly, $\hat{0}_E$, $\hat{1}_E \in C[\alpha]$. Let π, $\tau \in C[\alpha]$, x, $y \in E$. Then:

$$\begin{aligned} x \sim_{\pi \wedge \tau} y \qquad & \text{iff} \quad x \sim_\pi y \text{ and } x \sim_\tau y \\ & \text{iff } \alpha x \sim_\pi \alpha y \text{ and } \alpha x \sim_\tau \alpha y \\ & \text{iff } \alpha x \sim_{\pi \wedge \tau} \alpha y, \end{aligned}$$

so $\pi \wedge \tau \in C[\alpha]$. That $\pi \vee \tau \in C[\alpha]$ can be shown in a similar fashion using the characterization of $\pi \vee \tau$ in terms of equivalence relations (cf. Section 1.2). ∎

If $\tau \in C[\alpha]$, clearly α induces a permutation

$$\alpha \backslash \tau \in S[\tau]$$

by

$$(\alpha \backslash \tau)(B) := \alpha(B) \qquad (B \in \tau).$$

Observe that

$$\overline{(\alpha \backslash \tau)} = (\bar{\alpha} \vee \tau)/\tau.$$

Let $B \in \tau \in C[\alpha]$, $B \subseteq D \in \bar{\alpha} \vee \tau$. Then τ_D is (the underlying set of) the cycle of $\alpha \backslash \tau$ containing B. The *trace of* α *on* B is the permutation

$$\operatorname{tr}_B \alpha \in S[B]$$

defined by

$$\operatorname{tr}_B \alpha := (\alpha^n)_B =: \alpha^n_B,$$

where $n := |\tau_D|$ is the length of the cycle of $\alpha \backslash \tau$ containing B. Observe that if B' is another block of τ contained in D, then there is a $k \geqslant 1$ such that $\alpha^k(B) = B'$, and therefore α^k is an isomorphism between the permutations $\operatorname{tr}_B \alpha$ and $\operatorname{tr}_{B'} \alpha$. Note that

$$\overline{\operatorname{tr}_B \alpha} = (\bar{\alpha})_B =: \bar{\alpha}_B.$$

We build *representations* for $\tau \in C[\alpha]$ as follows. First assume that α is cyclic, i.e., $\bar{\alpha} = \hat{1}$. Then $\alpha \backslash \tau$ is cyclic, and if $\tau = \{B_1, ..., B_n\}$, then $|B_1| = \cdots = |B_n| =: m$. If

$$\alpha \backslash \tau = (B_1, B_2, ..., B_n), \tag{1}$$

then α has the form

$$\alpha = (a_{11}, ..., a_{n1}, a_{12}, ..., a_{n2}, ..., a_{1m}, ..., a_{nm}),$$

where

$$\operatorname{tr}_{B_i} \alpha = (a_{i1}, a_{i2}, ..., a_{im}).$$

We represent the pair (α, τ) by the matrix $[a_{ij}]$. We obtain a different (but equivalent) representation of (α, τ) for each way of writing $\alpha \backslash \tau$ as in (1).

Next assume α arbitrary, but $\bar{\alpha} \vee \tau = \hat{1}$. Let

$$\alpha \backslash \tau = (B_1, ..., B_n). \tag{2}$$

Again, $|B_1| = \cdots = |B_n| =: m$. Let $\bar{\alpha} = \{C_1, ..., C_r\}$. Then $\tau_{C_k} \in C[\alpha_{C_k}]$, α_{C_k} is cyclic, and

$$\alpha_{C_k} \backslash \tau_{C_k} = (B_1 \cap C_k, ..., B_n \cap C_k). \tag{3}$$

Build a representation $[a_{ij}^k]$ of $(\alpha_{C_k}, \tau_{C_k})$ according to (3). Obtain a representation $[b_{ij}]$ of (α, τ), corresponding to (2), by horizontally juxtaposing the matrices $[a_{ij}^k]$ $(k = 1, ..., r)$, thus obtaining a matrix with n rows, each representing a $\text{tr}_{B_i} \alpha$.

Finally, a representation of an arbitrary pair (α, τ) is a set of matrices $\{M_B: B \in \bar{\alpha} \vee \tau\}$, where M_B is a representation of (α_B, τ_B).

Notice that different representations of (α, τ), viewed as representations of transversals of $\bar{\alpha}$, do *not* define a unique transversal. Even so, one could try to choose representations appropriately to obtain an embedding $C[\alpha] \to T[\bar{\alpha}]$; this, however, is not possible in general: $C[(1, 2, 3, 4)(5, 6, 7, 8)]$ is not embeddable in $T[\{1, 2, 3, 4/5, 6, 7, 8\}]$ in this fashion; nor is it ranked.

Next we define *descending operations* on $\tau \in C[\alpha]$, analogous to those on transversals.

Given $\rho \geqslant \bar{\alpha}$, the *splitting of* τ *according to* ρ is the partition $\tau \wedge \rho$, which is compatible with α, as representations show. The splitting is *elementary* if $|\rho| = |\bar{\alpha} \vee \tau| + 1$, $\bar{\alpha} \leqslant \rho < \bar{\alpha} \vee \tau$.

If $\bar{\alpha} \vee \tau = \hat{1}$ and n is a common divisor of the lengths of the cycles of $\text{tr}_B \alpha$ $(B \in \tau)$, then an *n-thinning* of τ is a partition $\tau' \in C[\alpha]$ corresponding to a representation M' obtained by the following procedure. Choose a representation M of (α, τ). Let $\text{tr}_B \alpha$ $(B \in \tau)$ be the trace represented by the first row of M. Choose $\tau^* \in C[\text{tr}_B \alpha]$ with $\bar{\alpha}_B \vee \tau^* = \hat{1}$ and $|\tau^*| = n$. Choose a representation M^* of $(\text{tr}_B \alpha, \tau^*)$. Finally, construct M' by replacing each entry of M^* by the corresponding column of M.

If n is prime then an n-thinning is an *elementary thinning*.

PROPOSITION 3.2. *Let* $\sigma, \tau \in C[\alpha]$. *Then* $\sigma \leqslant \tau$ *iff there is a sequence of descending operations transforming* τ *into* σ.

Proof. As in Proposition 2.3. ∎

COROLLARY 3.3. *The cover relations in* $C[\alpha]$ *are obtained by elementary descending operations.*

In the sequel, if σ, $\tau \in C[\alpha]$ and $\sigma \leqslant \tau$, then $[\sigma, \tau] := \{\rho \in C[\alpha]: \alpha \leqslant \rho \leqslant \tau\}$.

PROPOSITION 3.4. *If* $\sigma, \tau \in C[\alpha]$, $\sigma \leqslant \tau$, $\bar{\alpha} \vee \tau = \hat{1}$ *and* $B \in \tau$, *then there is a natural poset isomorphism*

$$\phi: [\sigma, \tau] \simeq C[(\text{tr}_B \alpha) \backslash \sigma_B].$$

Proof. $\phi(\rho) := \rho_B/\sigma_B$. ▮

THEOREM 3.5 (Natural factorization of intervals in $C[\alpha]$). *Let $\sigma, \tau \in C[\alpha]$, $\sigma \leqslant \tau$. For each $B \in \bar{\alpha} \vee \tau$, let $B^* \in \tau_B$. Then there is a natural poset isomorphism*

$$[\sigma, \tau] \simeq \prod_{B \in \bar{\alpha} \vee \tau} C[(\mathrm{tr}_{B^*} \alpha)\backslash \sigma_{B^*}].$$

In particular,

$$[\hat{0}, \tau] \simeq \prod_{B \in \bar{\alpha} \vee \tau} C[\mathrm{tr}_{B^*} \alpha],$$

and

$$[\tau, \hat{1}] \simeq C[\alpha\backslash\tau].$$

Proof. Clearly $[\sigma, \tau] \simeq \prod_{B \in \bar{\alpha} \vee \tau} [\sigma_B, \tau_B]$. Now use Proposition 3.4. ▮

3.2. *The Large, Reduced Incidence Bialgebra of Compatible Partitions*

Let C be the (set-theoretic) class of segments of $C[\alpha]$'s. For $[\sigma, \tau] \in C$, using Theorem 3.5, define $M[\sigma, \tau]$ as the multiset

$$M[\sigma, \tau] := \{\mathrm{cl}((\mathrm{tr}_{B^*} \alpha)\backslash \sigma_{B^*}) \colon B \in \bar{\alpha} \vee \tau\},$$

which does not depend on the choice of the B^*'s.

We repeat, *mutatis mutandis*, the construction of Section 2.2, obtaining $B(C) := (K[X], \Delta', \varepsilon)$ with the usual algebra structure.

THEOREM 3.6. *$B(C)$ is a bialgebra, and localizing at x_{δ_1} makes it into a Hopf algebra.* ▮

The multiplicative functions of $B(C)^*$ are determined by their *permutational generating functions*

$$G'f(x) := \sum_{\lambda \neq \mathbf{0}} f(x_\lambda)\, \mathbf{x}^\lambda/\mathrm{aut}'(\lambda).$$

THEOREM 3.7. *The semigroup of multiplicative functions of $B(C)^*$, under convolution, is anti-isomorphic to the semigroup of permutational formal power series in $\mathbf{x}$ without constant term, under plethysm.*

Proof. For $f \in B(C)^*$ and $[\pi, \tau] \in C$, write

$$f([\pi, \tau]) := f(X^{M[\pi,\tau]}).$$

For f multiplicative and α a permutation of class $\lambda \neq \mathbf{0}$, set

$$f(\alpha) := f(\lambda) := f(C[\alpha]).$$

As in the proof of Theorem 2.11, define

$$\begin{aligned}
\left(\sum_i f_i\right)(a) &:= \sum_i f_i(\alpha), \\
f \cdot g(\alpha) &:= \sum_{\alpha_1 + \alpha_2 = \alpha} f(\alpha_1)\, f(\alpha_2), \\
\gamma_n(f)(\alpha) &:= \sum_{\{\alpha_1, \ldots, \alpha_n\}} f(\alpha_1) \cdots f(\alpha_n).
\end{aligned}$$

G' behaves as before for these operations.

Let $C_n[\alpha] := \{\tau \in C[\alpha] : \bar{\alpha} \vee \tau = \hat{1},\ |\tau| = n\}$. Then, if $\mathrm{cl}(\alpha) = \mu^{\{n\}}$, $|C_n[\alpha]| = n^{|\mu| - 1}$, where $|\mu| = \sum_{i \geqslant 1} \mu_i$. Define

$$f^{(n)}(\alpha) := \sum_{\tau \in C_n[\alpha]} f(\mathrm{tr}_B\, \alpha),$$

where the choice of $B \in \tau$ is arbitrary. Then

$$f^{\{n\}}(\lambda) = \begin{cases} n^{|\mu| - 1} f(\mu) & \text{if } \lambda = \mu^{\{n\}} \\ 0 & \text{otherwise,} \end{cases}$$

so $G'f^{\{n\}}(\mathbf{x}) = (1/n)\, F_n G'f(F_n$ as in Section 1.4).

For $\tau \in C[\alpha]$ define

$$\mathrm{cl}'(\tau) := \mathrm{cl}((\bar{\alpha} \vee \tau)/\tau).$$

Given a class $\lambda \neq \mathbf{0}$, define

$$C_\lambda[\alpha] := \{\tau \in C[\alpha] : \mathrm{cl}'(\tau) = \lambda\},$$

and

$$\begin{aligned}
f^{[\lambda]}(\sigma) &:= \sum_{\tau \in C_\lambda[\alpha]} \prod_{B \in \bar{\alpha} \vee \tau} f(\mathrm{tr}_{B^*}\, \alpha) \\
&= \sum_{\tau \in C_\lambda[\alpha]} f([\hat{0}, \tau]).
\end{aligned}$$

Then, as in Theorem 2.11,

$$f^{[\lambda]} = \prod_{n \geqslant 1} \gamma_{\lambda_n}(f^{\{n\}}),$$

and therefore

$$G'f^{[\lambda]} = (1/\mathrm{aut}'(\lambda)) \prod_{k \geqslant 1} (F_k G'f)^{\lambda_k}.$$

The rest of the proof is entirely analogous to that of Theorem 2.11. ∎

The computation of the Möbius function μ' of $B(C)^*$ follows the same lines of that of the partitional case. Using similar dual closures on $C[\alpha]$ one gets that $\mu'(\lambda) = 0$ if λ has more than one non-zero entry, and for $\lambda = (k+1)\,\delta_n$ one gets the recursion

$$0 = \mu'((k+1)\,\delta_n) + kn\mu'(k\delta_n),$$

so

$$\mu'(k\delta_n) = (-1)^{k-1}\,(k-1)!\,n^{k-1}\mu'(\delta_n).$$

Finally, if $\mathrm{cl}(\alpha) = \delta_n$ then $C[\alpha]$ is isomorphic to the poset $\{d\colon d \mid n\}$ under divisibility, so $\mu'(\delta_n) = \mu(n)$, the classical Möbius function.

Thus, one easily computes

$$G'\mu' = \log \prod_{n \geqslant 1} (1 + x_n)^{\mu(n)/n},$$

the plethystic inverse of

$$G'\zeta' = \exp \sum_{n \geqslant 1} x_n/n - 1.$$

3.3. *Morphisms between Permutations*

In this section we carry out, for permutations, the program of Section 2.3.

In the sequel, we shall denote by juxtaposition the composition of functions, and use the conventions at the beginning of Section 2.3.

Let (E, α) and (F, β) be permutations. A *morphism* $f\colon (E, \alpha) \to (F, \beta)$ is a function $f\colon E \to F$ such that

$$f\alpha = \beta f.$$

We denote by $\mathrm{Mor}(\alpha, \beta)$ the set of such morphisms. Observe that the composition of morphisms is a morphism.

PROPOSITION 3.8. *If* $f \in \mathrm{Mor}(\alpha, \beta)$ *then* $\ker f \in C[\alpha]$.

Proof. For x and y in the underlying set of α, we have

$$\begin{aligned} f(x)=f(y) \quad &\text{iff} \quad \beta f(x)=\beta f(y) \\ &\text{iff} \quad f\alpha(x)=f\alpha(y), \end{aligned}$$

i.e.,

$$x \sim_{\ker f} y \qquad \text{iff} \quad \alpha x \sim_{\ker f} \alpha y. \quad \blacksquare$$

We denote by f_D the restriction of $f\colon E \to F$ to $D \subseteq E$. If D is a union of blocks of $\tilde{\alpha}$, then $f_D \in \mathrm{Mor}(\alpha_D, \beta)$ and

$$\ker f_D = (\ker f)_D \in C[\alpha_D].$$

A morphism f is a *monomorphism* if f is injective, i.e., if $\ker f = \hat{0}$. We denote by $\mathrm{Mon}(\alpha, \beta)$ the set of monomorphisms between α and β.

As in Section 2.3, we have:

PROPOSITION 3.9. *The disjoint union of permutations is the direct sum in the category of partitions and morphisms*:

$$\mathrm{Mor}\left(\sum_i \alpha_i, \beta\right) \simeq \prod_i \mathrm{Mor}(\alpha_i, \beta). \quad \blacksquare$$

Recall the definition of f/π of Section 2.3.

PROPOSITION 3.10. *Let (E, α) and (F, β) be permutations. Let $\tau_0 \in C[\alpha]$. Let*

$$\mathrm{Mor}^{\tau_0}(\alpha, \beta) := \{ f \in \mathrm{Mor}(\alpha, \beta) \colon \ker f \geqslant \tau_0 \},$$

and

$$\mathrm{Mor}_{\tau_0}(\alpha, \beta) := \{ f \in \mathrm{Mor}(\alpha, \beta) \colon \ker f = \tau_0 \}.$$

Then the correspondence $f \to f/\tau_0$ gives bijections

$$\begin{aligned} \mathrm{Mor}^{\tau_0}(\alpha, \beta) &\simeq \mathrm{Mor}(\alpha \backslash \tau_0, \beta), \\ \mathrm{Mor}_{\tau_0}(\alpha, \beta) &\simeq \mathrm{Mon}(\alpha \backslash \tau_0, \beta). \quad \blacksquare \end{aligned}$$

COROLLARY 3.11. *There is a natural bijection*

$$\mathrm{Mor}(\alpha, \beta) = \sum_{\tau \in C[\alpha]} \mathrm{Mon}(\alpha \backslash \tau, \beta)$$

given by $f \to (\ker f, f/\ker f)$. $\blacksquare$

PROPOSITION 3.12. *Let* $\mathrm{cl}(\alpha)=\lambda$, $\mathrm{cl}(\beta)=\xi$. *Then*

$$\text{(a)}\quad p'_\lambda(\xi) := |\mathrm{Mor}(\alpha, \beta)| = \prod_{n\geq 1}\Big(\sum_{d|n} d\xi_d\Big)^{\lambda_n}.$$

$$\text{(b)}\quad q'_\lambda(\xi) := |\mathrm{Mon}(\alpha, \beta)| = \prod_{n\geq 1} n^{\lambda_n}(\xi_n)_{\lambda_n}.$$

Proof. As in Proposition 2.16, *mutatis mutandis.* ∎

PROPOSITION 3.13. *The polynomial sequences* $\{p'_\lambda(\mathbf{x})\}$, $\{q'_\lambda(\mathbf{x})\}$ *are of binomial type.* ∎

Finally, as in Section 2.3, we get

$$p'_\lambda(\mathbf{x}) = \sum_\nu S'(\lambda, \nu)\, q_\nu(\mathbf{x}),$$

where

$$S'(\lambda, \nu) := |C_\nu[\alpha]|, \qquad \mathrm{cl}(\alpha)=\lambda,$$

and

$$q'_\lambda(\mathbf{x}) = \sum_\nu s'(\lambda, \nu)\, p'_\nu(\mathbf{x}),$$

where

$$s'(\lambda, \nu) := \sum_{\tau\in C_\nu[\alpha]} \mu'(\hat{0}, \tau), \qquad \mathrm{cl}(\alpha)=\lambda.$$

IV. THE LINEAR PARTITIONAL CASE

4.1. *Linear Partitions and Linear Transversals*

A linear partition of a (finite) set E is a pair $(\sigma, \leqslant)$, where $\sigma\in\Pi[E]$ and $\leqslant$ is a partial order on E consisting of linear orders of each block of σ.

We define $\mathrm{cl}(\sigma, \leqslant) := \mathrm{cl}(\sigma)$. Two linear partitions are isomorphic iff they have the same class, and the number of automorphisms of $(\sigma, \leqslant)$ is $\mathrm{cl}(\sigma)!$

Sums (disjoint unions) and restrictions of linear partitions are defined in the obvious way.

A *linear transversal* of a linear partition $(\sigma, \leqslant)$ is a transversal $(\pi, \tau)\in T[\sigma]$ such that every $B\in\pi$ is a *segment* of $\leqslant$. We denote by $L[(\sigma, \leqslant)]$ (or by $L[\sigma]$ when there is no danger of confusion) the poset of linear transversals of $(\sigma, \leqslant)$, with the order induced from $T[\sigma]$.

PROPOSITION 4.1. *$L[\sigma]$ is a lattice.*

Proof. Since $L[\sigma]$ is finite and $\bar{0}^\sigma$, $\bar{1}^\sigma \in L[\sigma]$, it suffices to see that meets exist.

For $t = (\pi, \tau) \in T[\sigma]$ write

$$\omega(t) := \sigma \vee \tau.$$

Let $t^i = (\sigma, \pi^i, \tau^i) \in L[\sigma]$. Then $t^1 \wedge t^2$ can be constructed as follows. Split t^1 and t^2 according to $\rho := \omega(t^1) \wedge \omega(t^2)$, obtaining $s^1, s^2 \in L[\sigma]$ with $\omega(s^1) = \omega(s^2) = \rho$. Then, for each $B \in \rho$, s^1_B and s^2_B are small linear transversals of σ_B. But the small linear transversals of a linear partition form a lattice, since one can n-thin a small linear transversal in (at most) only one way. So let $u_B = s^1_B \wedge s^2_B$. Then $t^1 \wedge t^2 = \sum_{B \in \rho} u_B$. ∎

We remark that $L[\sigma]$ is not ranked in general.

If $t^i = (\sigma, \pi^i, \tau^i)$ $(i = 1, 2)$ are linear transversals of $(\sigma, \leqslant_\sigma)$ and $t^1 \leqslant t^2$, then, for each $B \in \sigma \vee \tau^2$, one can make $(\sigma \vee \tau^1)_B/(\pi^2 \vee \tau^1)_B$ into a linear partition by defining its partial order $\leqslant_B$ (on the set $(\pi^2 \vee \tau^1)_B$) as follows. Let $C_1, C_2 \in (\pi^2 \vee \tau^1)_B$. Let $m(C_i)$ be the set of minimal elements of C_i under $\leqslant_\sigma$. Define $C_1 \leqslant_B C_2$ iff there is a bijection $f\colon m(C_1) \to m(C_2)$ such that $x \leqslant_\sigma f(x)$ for all $x \in m(C_1)$.

Then, by methods entirely analogous to those of Section 2.1, we can show that

THEOREM 4.2 (Natural factorization of intervals of linear transversals). *There is a natural poset isomorphism*

$$[t^1, t^2] \simeq \prod_{B \in \sigma \vee \tau^2} L[((\sigma \vee \tau^1)_B/(\pi^2 \vee \tau^1)_{B'} \leqslant_B)]. \quad ∎$$

4.2. *The Large, Reduced Incidence Bialgebra of Linear Transversals.*

Carrying out, *mutatis mutandis*, the program of Section 2.2, we obtain $B(L) := (K[X], \Delta'', \varepsilon)$ with the usual algebra structure, and

THEOREM 4.3. *$B(L)$ is a bialgebra, and localizing at x_{δ_1} makes it into a Hopf algebra.* ∎

The multiplicative functions of $B(L)^*$ are determined by their *exponential generating functions*

$$G''f(\mathbf{x}) := \sum_{\lambda \neq \mathbf{0}} f(x_\lambda)\, \mathbf{x}^\lambda/\lambda!$$

THEOREM 4.4. *The semigroup of multiplicative functions of $B(L)^*$, under* *convolution, is anti-isomorphic to the semigroup of exponential formal power series in* $\mathbf{x}$ *without constant term, under plethysm.* ∎

As before, we compute the Möbius function μ'' of $B(L)^*$ to be

$$\mu''(\lambda)=\begin{cases}0 & \text{if } \lambda \text{ has more than one non-zero entry}\\ (-1)^k\, k!\,\mu(n) & \text{if } \lambda=(k+1)\,\delta_n,\ k\geqslant 0,\end{cases}$$

where $\mu(n)$ is the classical Möbius function.

Hence

$$G''\mu''=\log \prod_{n\geqslant 1} (1+x_n)^{\mu(n)},$$

the plethystic inverse of

$$G''\zeta''=\exp \sum_{n\geqslant 1} x_n-1.$$

4.3. *Morphisms between Linear Partitions*

Let $(E, \sigma, \leqslant_1)$ and $(R, \rho, \leqslant_2)$ be linear partitions. A *morphism*

$$\mathbf{f}\colon (E, \sigma, \leqslant_1)\to (R, \rho, \leqslant_2)$$

is a morphism $\mathbf{f}=(f, \pi)\colon (E, \sigma)\to (R, \rho)$ such that for all $B\in\pi$, $f|_B$ is order-preserving.

Then $\ker \mathbf{f}\in L[(\sigma, \leqslant_1)]$ and the composition of linear morphisms is a linear morphism.

Redoing Section 2.3, we get the polynomial sequences of binomial type

$$p''_\lambda(\mathbf{x}):=\prod_{n\geqslant 1}\left(\sum_{d|n} x_d\right)^{\lambda_n}$$

and

$$q''_\lambda(\mathbf{x}):=\prod_{n\geqslant 1} (x_n)_{\lambda_n},$$

counting morphisms and monomorphisms, respectively, and expressions for their connection constants

$$p''_\lambda(\mathbf{x})=\sum_\nu S''(\lambda, \nu)\, q''_\nu(\mathbf{x}),$$

where

$$S''(\lambda, \nu):=|L_\nu[\sigma]|,\qquad \mathrm{cl}(\sigma)=\lambda,$$

and

$$q''_\lambda(\mathbf{x}) = \sum_\nu s''(\lambda, \nu)\, p''_\nu(\mathbf{x}),$$

where

$$s''(\lambda, \nu) := \sum_{t \in L_\nu[\sigma]} \mu''(\bar{0}^\sigma, t), \qquad \mathrm{cl}(\sigma) = \lambda.$$

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