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On the L-Functions Associated with Certain Exponential Sums*

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By use of p-adic analytic methods, we study the L-functions associated to certain exponential sums defined over a finite field. Estimates for the degree of this L-function as rational function are obtained. In an "asymptotic" sense, these estimates are shown to be best possible. Precise determination of the degree is computed in the one-variable case.

Let $\mathbf{F}_q$ be the finite field of characteristic p with $q = p^a$ elements. Let $f(X) \in \mathbf{F}_q[X_1, ..., X_n]$. The associated exponential sum over $\mathbf{F}_{q^m}$ is defined by

$$\mathscr{S}_m(f) = \sum_{x \in (\mathbf{F}_{q^m})^n} \exp \frac{2\pi i}{p} \operatorname{Tr}_m f(x), \tag{0.1}$$

where $\operatorname{Tr}_m: \mathbf{F}_{q^m} \to \mathbf{F}_p$ is the absolute trace. The associated L-function is defined by

$$\mathscr{L}(f, t) = \exp\left(\sum_{m=1}^{\infty} \frac{\mathscr{S}_m(f)\, t^m}{m}\right). \tag{0.2}$$

As shown in [1], $\mathscr{L}(f, t)$ is an Artin L-function associated to the Galois group $\mathbf{Z}/p\mathbf{Z}$ of the Artin–Schreier covering

$$Y^p - Y = f(X) \tag{0.3}$$

of affine n-space $\mathbf{A}^n_{\mathbf{F}_q}$, and the character $e_p = \exp(2\pi i/p)$ of the Galois group. In Dwork's theory, the exponential sum

$$S_m(f) = \sum_{x \in (\mathbf{F}^*_{q^m})^n} e_p(\operatorname{Tr}_m f(x)) \tag{0.4}$$

and its associated L-function, $L(f, t)$, are more accessible. Let $A \subseteq \{1, 2, ..., n\}$

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be any subset (including the empty set); let $n(A)$ = cardinality of A; let f_A denote the polynomial in $n - n(A)$ variables obtained from f by putting $x_i = 0$ for $i \in A$. Then, the relations between the two types of exponential sums above are given by the evident combinatorial identities:

$$\begin{aligned} &\text{(a)} \quad \mathscr{S}_m(f) = \sum_A S_m(f_A); \\ &\text{(b)} \quad S_m(f) = \sum_A (-1)^{n(A)} \mathscr{S}_m(f_A). \end{aligned} \tag{0.5}$$

Consequently,

$$\begin{aligned} &\text{(a)} \quad \mathscr{L}(f, t) = \prod_A L(f_A, t); \\ &\text{(b)} \quad L(f, t) = \prod_A \mathscr{L}(f_A, t)^{(-1)^{n(A)}}. \end{aligned} \tag{0.6}$$

It is well known [1, 6, 8] that both of these type L-functions are rational functions with coefficients in **Q** and algebraic integers as reciprocal zeros and poles. In [1], Bombieri proves some general results on the degree of the rational function $L(f, t)$ where degree (rational function) = degree (numerator) − degree (denominator). He also derives estimates for the p-adic values of the reciprocal zeros and poles of this function.

In Section 1 of the present article, we extend Bombieri's results to the case $f(X) \in \mathbf{F}_q[X_1, \ldots, X_n, 1/(X_1 \cdots X_n)]$, i.e., f is an arbitrary regular function on the variety in question, namely, the complement in $\mathbf{A}^n_{\mathbf{F}_q}$ of the hypersurface with equation $X_1 \cdots X_n = 0$ consisting of the coordinate axes. The best known examples of such sums are the Kloosterman and hyper-Kloosterman exponential sums. This paper is based on an understanding of the role of the weight function (1.2) which already appeared in our work on the hyper-Kloosterman sum [10]. The key ingredients in Bombieri's argument are (1) the use of Dwork's theory to obtain good p-adic estimates for the entries of the matrix of the Frobenius map, and (2) a "Jensen-type" formula for the relation between the growth of a (p-adic) analytic function and the distribution of its zeros. We follow this approach faithfully. We thank B. Dwork for bringing Bombieri's work to our attention.

In Section 2, we indicate that the estimates of theroem 1.23(iv) are asymptotically best possible (i.e., for fixed n and d, and p large enough, there exists $f^{(d)} \in \mathbf{F}_q[x_1, \ldots, x_n, 1/(X_1 \cdots X)]$ (Section 2b, below) such that equality holds on the right in (Theorem 1.23(iv)). In Section 2a, we generalize the argument of [2], and determine the precise degree of the L-function associated with a Laurent polynomial in one variable (again with a restriction on the size of p).

1. Let Ω be an algebraically closed, complete field under a non-Archimedean valuation extending that of Q_p, normalized so that $\operatorname{ord}(p) = 1$.

(We will occasionally also use the normalization $\text{ord}_q = (1/a)\text{ord}$). Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n$, and denote

$$s(\alpha) = \max\{0, -\alpha_1, \ldots, -\alpha_n\}. \tag{1.1}$$

Define

$$w(\alpha) = \sum_{i=1}^{n} \alpha_i + (n+1)\, s(\alpha). \tag{1.2}$$

Note that

$$w(\alpha + \beta) \leqslant w(\alpha) + w(\beta). \tag{1.3}$$

Let $b, c \in \mathbf{R}$, $b > 0$, and define

$$L(b, c) = \left\{ \sum_{\alpha \in \mathbf{Z}^n} A(\alpha)\, X^\alpha \mid A(\alpha) \in \Omega, \text{ ord } A(\alpha) \geqslant c + b \cdot w(\alpha) \right\}. \tag{1.4}$$

Let $L(b) = \bigcup_{c \in \mathbf{R}} L(b, c)$. The elements of $L(b)$ converge on

$$G(b) = \left\{ x = (x_1, \ldots, x_n) \in \Omega^n \mid -b < \text{ord } x_i, \sum_{i=1}^{n} \text{ord } x_i < b \right\}. \tag{1.5}$$

It follows that multiplication of $L(b)$ by a fixed element of $L(b)$ is a well-defined endomorphism of $L(b)$. Let

$$E(X) = \exp\left(\sum_{i=0}^{\infty} \frac{X^{p^i}}{p^i} \right) \tag{1.6}$$

be the Artin–Hasse exponential series, and let $\gamma \in \Omega$ be a root of $\sum_{i=0}^{\infty} (X^{p^i}/p^i) = 0$ with $\text{ord } \gamma = 1/(p-1)$. Then

$$\Theta(X) = E(\gamma X) = \sum_{m=0}^{\infty} B_m X^m \tag{1.7}$$

is a "splitting function" in the terminology of [3, 4], and the coefficients satisfy

$$\text{ord } B_m \geqslant \frac{m}{p-1}. \tag{1.8}$$

Let $\bar{f}(X) = \sum \bar{A}_\alpha X^\alpha \in \mathbf{F}_q[X_1, \ldots, X_n, 1/(X_1 \cdots X_n)]$ and $d = \max_\alpha w(\alpha)$ in which α runs over indices such that $\bar{A}_\alpha \neq 0$. We say $\bar{f}(X)$ has weight d. Let $f(X) = \sum A_\alpha X^\alpha$ where A_α is the Teichmuller lifting of $\bar{A}_\alpha$ in K_a, the unramified extension of Q_p in Ω of degree a. Hence $A_\alpha{}^q = A_\alpha$. If ζ is a primitive pth root of 1, it is well-known that $Q_p(\gamma) = Q_p(\zeta)$. It follows that if we define

$$F(X) = \prod_\alpha \Theta(A_\alpha X^\alpha), \tag{1.9}$$

then

$$F(X) \in L\left(\frac{1}{(p-1)\,d}, 0\right) \tag{1.10}$$

and has coefficients in $K_a(\zeta)$.

Let σ be the Frobenius automorphism in $\mathrm{Gal}(K_a \mid Q_p)$, and denote by σ as well its extension to $\Omega_0 = K_a(\zeta)$ obtained by setting $\sigma(\zeta) = \zeta$. Define

$$F_0(X) = \prod_{j=0}^{a-1} \sigma^j F(X^{p^j}), \tag{1.11}$$

then $F_0(X) \in L(p/q(p-1)d, 0)$. Define a linear map ψ: $L(b, c) \to L(pb, c)$ by

$$\psi\colon \sum c_\alpha X^\alpha \to \sum c_{p\alpha} X^\alpha. \tag{1.12}$$

We also denote by $F_0(X)$ the endomorphism of $L(p/q(p-1)d)$ defined by multiplication by $F_0(X)$. Let α be the following composite map:

$$L\left(\frac{p}{(p-1)d}\right) \xrightarrow{i} L\left(\frac{p}{q(p-1)d}\right) \xrightarrow{F_0(X)} L\left(\frac{p}{q(p-1)d}\right) \xrightarrow{\psi^a} L\left(\frac{p}{(p-1)d}\right). \tag{1.13}$$

Then $\alpha = \psi^a \circ F_0(X)$ is a completely continuous endomorphism of $L(p/(p-1)d)$ in the sense of Serre [9].

Viewing α as an Ω_0 endomorphism of $L(p/(p-1)d)$, Dwork's trace formula [3] yields

$$(q^m - 1)^n \,\mathrm{Tr}(\alpha^m) = S_m(f). \tag{1.14}$$

For a completely continuous endomorphism α, $\mathrm{Tr}(\alpha^m)$ and $\det(I - \alpha t)$ are independent of choice of orthonormal basis, and are related by

$$\det(I - \alpha t) = \exp\left(-\sum_{m=1}^{\infty} \frac{\mathrm{Tr}(\alpha^m)\, t^m}{m}\right). \tag{1.15}$$

Let δ denote the operator

$$\delta\colon g(t) \to \frac{g(t)}{g(qt)}. \tag{1.16}$$

It is a topological group automorphism of $1 + t\Omega_0[[t]]$ with inverse operator

$$\delta^{-1}: g(t) \to \prod_{i=0}^{\infty} g(q^i t). \tag{1.17}$$

Hence, (1.14) is equivalent by (1.15) to

$$L(f, t)^{(-1)^{n+1}} = \det(I - \alpha t)^{\delta^n}. \tag{1.18}$$

If $\Omega_1 = Q_p(\zeta)$, then the following analog of [1, Lemma 2] holds.

LEMMA 1.19. *Let* $\det(I - \alpha t) = \sum_{m=0}^{\infty} a_m t^m$. *Then* $a_m \in \Omega_1$, *for all* m, *and the convex closure in the real* (x, y) *plane of the points* $(m, \mathrm{ord}_q(a_m))$ *is contained in the convex closure of the points*

$$\left(\sum_{j=0}^{m} g(j), \frac{1}{d} \sum_{j=0}^{m} jg(j)\right), \qquad m = 0, 1,...,$$

where in terms of binomial coefficients

$$g(j) = \binom{n+j}{m} - \binom{j-1}{n}.$$

(Convention: $\binom{i}{j} = 0$, if $j > i$).

Proof. As in [5] and [1], sharper estimates for a_m may be obtained by viewing α as an Ω_1-linear map. Let α_0 be the Ω_1-linear endomorphism of $L_0(p/q(p-1)d)$ defined by

$$\alpha_0 = \psi \circ \sigma^{-1} \circ F(X). \tag{1.20}$$

By the argument of [5, Sect. 7], the convex closure of the points $(m, \mathrm{ord}_q(a_m))$ is obtained from the Newton polygon of $\det_{\Omega_1}(I - \alpha_0 t)$ by multiplying the abscissas and ordinates of its vertices by the factor $1/a$.

To complete the proof of the lemma, we compute the matrix of α_0 with respect to the basis $\{\xi_i X^\alpha; \alpha \in \mathbf{Z}^n, 1 \leqslant i \leqslant a\}$ in which $\{\xi_i\}_{i=1}^{a}$ is an integral basis of Ω_0 over Ω_1 with the property that

$$\mathrm{ord}\left(\sum_{i=1}^{a} \beta_i \xi_i\right) = \min_{1 \leqslant i \leqslant a} \{\mathrm{ord}\, \beta_i\}.$$

If $u, v \in \mathbf{Z}^n$; $i, j \in \{1, 2,..., a\}$, $\alpha = (G_{u,i;v,j})$, then by a standard computation [1, 7, 10],

$$\mathrm{ord}\, G_{u,i;v,j} \geqslant \frac{w(pu - v)}{(p-1)\,d}.$$

Hence, by (1.3),

$$\operatorname{ord} G_{u,i;v,j} \geqslant \frac{pw(u) - w(v)}{(p-1)\,d}. \tag{1.21}$$

It follows as in [1] that the Newton polygon of $\det_{\Omega_1}(I - \alpha_0 t)$ is contained in the convex closure of the points

$$\left(m, \frac{1}{d} \min \sum_{l=1}^{m} w(u_l)\right),$$

where the minimum is taken over all sets of m distinct pairs (u_l, i_l). The map $\lambda\colon \mathbf{Z}^n \to \mathbf{Z}_+^{n+1}$ defined by

$$\lambda(\alpha) = (\alpha_1 + s(\alpha),\ldots, \alpha_n + s(\alpha), s(\alpha)) \tag{1.22}$$

is easily seen to be one-to-one and onto those elements of $\mathbf{Z}_+^{n+1}$ at least one coordinate of which is zero. Furthermore let $\beta = (\beta_1, \ldots, \beta_{n+1}) \in \mathbf{Z}^{n+1}$ and denote $|\beta| = \sum_{i=1}^{n+1} \beta_i$, then $w(\alpha) = |\lambda(\alpha)|$ for $\alpha \in \mathbf{Z}^n$. It follows that the number of n-tuples $\alpha \in \mathbf{Z}^n$ with $w(\alpha) = j$ is precisely $g(j)$. By [5, Sect. 7], the Newton polygon of $\det_{\Omega_1}(I - \alpha_0 t)$ is contained in the convex closure of the points

$$\left(a \sum_{j=0}^{m} g(j), \frac{a}{d} \sum_{j=0}^{m} jg(j)\right).$$

This completes the proof of (1.19).

THEOREM 1.23 (cf. [1, Theorem 1]). (i) *There exist algebraic integers* ω_h, η_j *for* $1 \leqslant h \leqslant r$, $1 \leqslant j \leqslant s$, *depending on* f *but not on* m, *different from* 0, *such that* $\omega_h \neq \eta_j$ *for every* h, j, *and such that*

$$S_m(f) = (-1)^n \left(\sum_{h=1}^{r} \omega_h{}^m - \sum_{j=1}^{s} \eta_j{}^m\right);$$

(ii) $s \leqslant r$, *and if* $s \geqslant 1$, *then every* η_j *is of the type* $\eta_j = q^{a_j}\omega_j$ *where the* a_j *are nonzero integers*;

(iii) *if* $a_j < 0$, *there is some* j' *with* $\eta_j = \eta_{j'}$ *and* $a_{j'} > 0$;

(iv) *we have the inequality*

$$0 \leqslant r - s \leqslant (n+1)\,d^n,$$

where $d =$ *weight of* f.

Proof. We are concerned with (iv), the proofs of (i), (ii) and (iii) are identical here to those given in [1]. Let $L(f, t)^{(-1)^{n+1}} = p(t)/q(t)$, where $p(t) = \prod_{h=1}^{r} (1 - \omega_h t), q(t) = \prod_{j=1}^{s} (1 - \eta_j t)$. Denote $D_1(t) = p(t)^{\delta^{-n}}, D_2(t) = q(t)^{\delta^{-n}}$. Then (1.18) yields

$$\det(I - \alpha t) = D_1(t)/D_2(t).$$

Therefore, if $c(m) = \binom{n+m-1}{m}$, the formulas [1, Lemma 3 and Corollary] imply

$$\sum_{h=1}^{r} \sum{}' (x - \operatorname{ord}_q(q^m \omega_h))\, c(m) - \sum_{j=1}^{s} \sum{}' (x - \operatorname{ord}_q(q^m \eta_j))\, c(m)$$
$$\leqslant d^{-1} \sum_{i \leqslant dx} (dx - i)\, g(i), \tag{1.24}$$

where $\sum'$ denotes a sum over those m for which the summands are positive. Hence

$$(r - s) \frac{x^{n+1}}{(n + 1)!} + O(x^n) \leqslant (n + 1)\, d^n \frac{x^{n+1}}{(n + 1)!} + O(x^n).$$

Dividing by x^{n+1} and letting $x \to \infty$ completes the proof.

Exactly as in [1], the following two theorems may now be proved.

THEOREM 1.25 (cf. [1, Theorem 2]). *Suppose that $S_m(f)$ has $r = (n + 1)\, d^n$, $s = 0$. Then we have*

$$\sum_{h=1}^{r} \operatorname{ord}_q(\omega_h) \geqslant \frac{n(n + 1)\, d^n}{2}.$$

THEOREM 1.26 (cf. [1, Theorem 3]). *Let $S_m(f)$ be an exponential sum where $f(X) \in \mathbf{F}_q[X_1, ..., X_n, 1/(X_1 \cdots X_n)]$ has weight d. Let ω_h, η_j be the characteristic roots of the sum $S_m(f)$ as defined in* (1.23). *Then precisely one of the ω_h, say ω_1, is a unit. The other roots ω_j, η_j satisfy*

$$\operatorname{ord}_q(\omega_h) \geqslant \frac{1}{d},$$

$$\operatorname{ord}_q(\eta_j) \geqslant 1 + \frac{1}{d},$$

except for at most n roots η_j which must all be equal to $q\omega_1$.

Observe that in the case of the hyper-Kloosterman sum [11] (assuming $p \geqslant n+3$), the ω_h can be arranged so that

$$\operatorname{ord}_q \omega_h = h - 1.$$

Hence equality in (1.25) holds in this case.

2. EXAMPLES

a. *Exponential Sums in One Variable*

We assume $\bar{f}(X) \in \mathbf{F}_q[X, 1/X]$. Let $f(X) = \sum_{i=d'}^{d} a_i X^i \in K_a[X, 1/X]$ be the lifting of $\bar{f}$ by Teichmuller units ($a_i^q = a_i$). Assume $d' < 0 < d$, and $(d', p) = (d, p) = 1$. Replacing X by $1/X$ if necessary, we may assume that $d \geqslant -d'$. By definition (1.2), $f(X)$ has weight $2d$. By constructing p-adic cohomology spaces for this example, we will prove the following:

THEOREM 2.1. *Let $\bar{f}(X) \in \mathbf{F}_q[X, 1/X]$, with $(d, p) = (d', p) = 1$ as above. Assume also that p satisfies*

$$\frac{(p-1)^2}{p} > \frac{d}{|d'|}.$$

Then $L(\bar{f}, t) \in \mathbf{Z}[t]$ is a polynomial of degree $d - d'$.

This will follow from [4, Theorems 4.2 and 4.3] once (2.6) below is established.

We follow the well-known argument of [4, Sect. 3]. For purposes of constructing Dwork-type cohomology spaces, it is more convenient to use the "splitting function"

$$\Theta_1(X) = \exp \pi(X - X^p) \tag{2.2}$$

(where $\pi \in \Omega$, $\pi^{p-1} = -p$) rather than $\Theta(X)$ (1.7). It is known [4, Sect. 4] that

$$\Theta_1(X) \in L(b', 0), \tag{2.3}$$

where $b' = (p-1)/p^2$. Therefore

$$F(X) = \prod_{i=d'}^{d} \Theta_1(a_i X^i) \in L(b'/d, 0). \tag{2.4}$$

Let V be the vector space spanned over Ω by the monomials $\{X^i\}_{i=d'}^{d}$. Let $E = X(d/dX)$, and define

$$D = E + H, \tag{2.5}$$

where $H = \pi X f'(X)$. Let b be arbitrary in $\mathbf{R}_+$ subject to the condition that if $e = db - 1/(p-1)$ and $e' = -d'b - 1/(p-1)$, then $e' > 0$. We will use a series of lemmas to establish the following theorem.

THEOREM 2.6. *If* $e' > 0$,

$$L(b) = V \oplus DL(b).$$

Let $V(b, c) = L(b, c) \cap V$.

LEMMA 2.7. *Let* $\gamma = \max(0, d + d' - 2)$. *Then*

$$V(b, c) + HL(b, c + e) \subseteq L(b, c) \subseteq V(b, c - \gamma b) + HL(b, c + e').$$

Consequently,

$$L(b) = V + HL(b).$$

Proof. Note that if $\xi = \sum_{j=-\infty}^{+\infty} B_j X^j \in L(b, c)$ and

$$\pi i a_i X^i \xi = \sum_{j=-\infty}^{+\infty} C_j X^j$$

then $C_j = \pi i a_i B_{j-i}$ and

$$\operatorname{ord} C_j \geqslant \frac{1}{p-1} + c + |j - i| b.$$

Therefore

$$\pi i a_i X^i \xi \in L\left(b, c - \left(|i| b - \frac{1}{p-1}\right)\right) \subseteq L(b, c - e)$$

proving $HL(b, c + e) \subseteq L(b, c)$. To show the second inclusion, we divide by H. A routine inductive argument shows that

$$X^n = \sum_{i=d'}^{d-1} A_i^{(n)} X^i + \frac{1}{\pi} H \left(\sum_{j=0}^{n-d} B_j^{(n)} X^j \right) \qquad (\text{for} \quad n \geqslant d), \tag{2.8}$$

$$X^m = \sum_{i=d'}^{d-1} A_i^{(m)} X^i + \frac{1}{\pi} H \left(\sum_{j=0}^{d'-m} B_j^{(m)} X^{-j} \right) \qquad (\text{for} \quad m < d'), \tag{2.9}$$

where in both (2.8) and (2.9), ord $A_i^{(j)} \geqslant 0$, and ord $B_i^{(j)} \geqslant 0$. It follows easily that if $\xi \in L(b, c)$, division by H via (2.8) and (2.9) yields the second inclusion.

LEMMA 2.10. *$V \cap HL(b) = (0)$.*

Proof. Suppose $H\xi = v$, for $\xi \in L(b)$, $v \in V$. Then $X^{-d'}H\xi = X^{-d'}v$ is a polynomial of degree $d - d' - 1$. The Newton polygon of the factor $(1/\pi)\, X^{-d'}H$ shows that all of its $d - d'$ roots are units. The elements of $L(b)$ are easily seen to converge on the annulus $-b < \text{ord}\, x < b$. It follows that $X^{-d'}H\xi$ has at least $d - d'$ zeros (counting multiplicities). But $X^{-d'}v$ is a polynomial in X of degree $d - d' - 1$. This contradiction proves the lemma.

LEMMA 2.11. *If $e' > 0$*

$$V(b, c) + DL(b, c + e) \subseteq L(b, c) \subseteq V(b, c - \gamma b) + DL(b, c + e').$$

Hence

$$L(b) = V + DL(b).$$

Proof. The inclusion on the left is an obvious consequence of (2.7) and the assumption that e', and therefore e, is positive. Let $\xi \in L(b, c)$. Assume $\xi^{(r)} \in L(b, c + (r - 1)\, e')$, $\zeta^{(r)} \in V(b, c + (r - 1)\, e' - \gamma b)$, $\eta^{(r)} \in L(b, c + re')$ (where $\gamma = \max(0, d + d' - 2)$, as in (2.7)) have been defined satisfying

$$\xi^{(r)} = \zeta^{(r)} + H\eta^{(r)}. \tag{2.12}$$

For $r = 1$, we may take $\xi^{(1)} = \xi$, and $\zeta^{(1)}$ and $\eta^{(1)}$ as given by Lemma 2.7 satisfying (2.12). Assuming (2.12) for arbitrary r, define

$$\xi^{(r+1)} = \xi^{(r)} - \zeta^{(r)} - D\eta^{(r)} = -E\eta^{(r)} \tag{2.13}$$

so that $\xi^{(r+1)} \in L(b, c + re')$. Let $\zeta^{(r+1)} \in V(b, c + re' - \gamma b)$, $\eta^{(r+1)} \in L(b, c + (r + 1)\, e')$ be determined by Lemma 2.7 so that (2.12) holds for $r + 1$. Summing (2.13) from 1 to R yields

$$\xi^{(R+1)} = \xi^{(1)} - \sum_{r=1}^{R} \zeta^{(r)} - D\left(\sum_{r=1}^{R} \eta^{(r)}\right).$$

Since $\xi^{(r)}$, $\zeta^{(r)}$, $\eta^{(r)}$ all tend to 0 as $r \to \infty$

$$\xi = \sum_{r=1}^{\infty} \zeta^{(r)} + D\left(\sum_{r=1}^{\infty} \eta^{(r)}\right),$$

completing the proof of the lemma.

LEMMA 2.14. *Let $e' > 0$. If $\xi \in L(b)$, $H\xi \in L(b, c)$, then $\xi \in L(b, c + e')$.*

Proof. As in [7] we solve formally for ξ when $H\xi = \eta$. Let $\xi = \sum A_i X^i$, let $\eta = \sum B_i X^i \in L(b, c)$. If Δ is the index shift $i \to i + 1$, then

$$\left(\Delta^{-d} + \sum_{i=d'}^{d-1} \frac{ia_i}{da_d} \Delta^{-i}\right) A = \frac{1}{\pi da_d} B.$$

Therefore,

$$A = \left(I + \sum_{i=1} g_i \Delta^i\right) \frac{\Delta^d}{\pi da_d} B \tag{2.15}$$

for some constants g_i, ord $g_i \geqslant 0$. Similarly we also obtain,

$$A = \left(I + \sum_{i=1}^{\infty} h_i \Delta^{-i}\right) \frac{\Delta^{d'}}{\pi d' a_{d'}} B \tag{2.16}$$

with constants h_i, ord $h_i \geqslant 0$. As a consequence of (2.15),

$$A_r = \frac{1}{\pi da_d} \sum_{i=0} g_i B_{r+d+i} \,.$$

Hence for $r \geqslant 0$,

$$\text{ord } A_r \geqslant c - \frac{1}{p-1} + (r + d)\, b. \tag{2.17}$$

Similarly (2.16) leads to the estimate for $r < 0$,

$$\text{ord } A_r \geqslant c - \frac{1}{p-1} - (r + d')\, b. \tag{2.18}$$

The two estimates (2.17) and (2.18) complete the proof of the lemma.

LEMMA 2.19. *Let $e' > 0$. If $\xi \in L(b)$, $D\xi \in L(b, c)$, then $\xi \in L(b, c + e')$.*

Proof. Let $\xi \in L(b, \rho)$ for some ρ so chosen that $\xi \notin L(b, \rho + e')$. Then

$$H\xi = D\xi - E\xi \in L(b, c) + L(b, \rho) = L(b, l),$$

where $l = \min(\rho, c)$. If $H\xi \in L(b, \rho)$ then (2.14) contradicts the choice of ρ. Therefore $l = c$ and $H\xi \in L(b, c)$. By (2.14) once again, $\xi \in L(b, c + e')$.

LEMMA 2.20. *If $e' > 0$,*

$$DL(b) \cap V = \{0\},$$

Proof. Let $\xi \in DL(b) \cap V$, $\xi \in L(b, c)$, and $\xi = D\eta$, where η may be taken by Lemma 2.19 in $L(b, c + e')$. Then,

$$\xi - H\eta = E\eta \in L(b, c + e').$$

By Lemma 2.7, there exist $\zeta \in V(b, c + e' - \gamma b)$, $\omega \in L(b, c + 2e')$ such that

$$E\eta = \zeta + H\omega.$$

By Lemma 2.10, $\zeta = \xi$ and $\omega = -\eta$. Therefore, $\eta \in L(b, c + 2e')$. By an obvious inductive argument $\eta \in L(b, c + re')$ for all $r \geqslant 0$. Therefore $\eta = 0$ and $\xi = 0$.

Theorem 2.6 is now a consequence of Lemmas 2.11 and 2.20. To complete the proof of Theorem 2.1, we observe that we may define $\alpha = \psi^a \circ \tilde{F}_0$ as in (1.13) where $\tilde{F}_0$ is defined by (1.11) the only change being that Θ is replaced by Θ_1 in (1.9). Then α is an endomorphism of $L(pb'/d)$ where $b' = (p - 1)/p^2$. Therefore $e' > 0$, is satisfied for primes p,

$$\frac{(p-1)^2}{p} > \frac{d}{|d'|}. \tag{2.21}$$

The proof of Theorem 2.1 then follows as in Theorems 4.2 and 4.3 of [4], and the fact that $\ker D \cap L(b) = \{0\}$.

b. *Generalized Hyper-Kloosterman Sums*

Let $a \in \mathbf{F}_q$ and define

$$f_a^{(d)}(X) = \sum_{i=1}^{n} X_i^d + \frac{a}{(X_1 \cdots X_n)^d}.$$

For $d = 1$, the resulting exponential sum is the hyper-Kloosterman sum in n variables [2, 10, 11]. We will show in a subsequent article that for sufficiently large p, p-adic cohomology spaces can be constructed for this example having a basis of $(n + 1)\, d^n$ elements. This will prove that the inequality of (iv) in Theorem 1.23 is best possible (at least in an asymptotic sense): for any n, any d, and for large enough p, $f_a^{(d)}$ has the property that $L(f_a^{(d)}, t)^{(-1)^{n+1}}$ is a polynomial in $\mathbf{Z}[t]$ of precise degree $(n + 1)\, d^n$. It is our belief that the hypotheses that p be "sufficiently large" (here as well as in Theorem 2.1) are not required. The only essential hypothesis should be the requirement that $(d, p) = (d', p) = 1$.

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