An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter

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Abstract

We consider fractional Brownian motions $B^H_t$ with arbitrary Hurst coefficients $0 < H < 1$ and prove the following results: (i) An integral representation of the fractional white noise as generalized Wiener integral; (ii) an Itô formula for generalized functionals of $B^H_t$; (iii) an analogue of Tanaka’s formula; (iv) a Clark–Ocone formula for Donsker’s delta function of $B^H_t$; (v) an integral representation of the local time of $B^H_t$.

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1. Introduction

A fractional Brownian motion $B^H_t$ with Hurst parameter $0 < H < 1$ is a continuous centered Gaussian process with covariance

$$E[B^H_t B^H_s] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}); \quad t, s \in \mathbb{R}.$$ 

It is well known, that a fractional Brownian motion $B^H_t$ is a semimartingale if and only if $H = \frac{1}{2}$, i.e. in the case of a classical Brownian motion. Hence, Itô’s stochastic integration theory for semimartingales cannot be applied, if $H \neq \frac{1}{2}$.

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During the last years basically two different integration theories for fractional Brownian motions have been developed. The first one is based on the ordinary pathwise product in defining the integral for simple integrands, see Lin (1995). But it turns out, that this integral has rather the properties of the Stratonovich integral than of the Itô integral in the integration theory of a Brownian motion. This leads to essential difficulties in financial modeling, see Rogers (1997).

So a second integration theory based on the Wick product was introduced by Duncan et al. (2000). It has some desirable properties. E.g. the expectation of this integral is zero for a huge class of integrands. This integral has been further developed by Hu and Øksendal (1999) in a fractional white noise setting. However, in this setting the underlying probability space depends on the Hurst parameter of the fractional Brownian motion, i.e. one has to consider different spaces for different parameters. Moreover, $H > \frac{1}{2}$ is assumed in constructing the appropriate spaces.

An essentially equivalent approach is to define the integral with respect to a fractional Brownian motion as divergence operator via the Malliavin calculus. Following this approach all Hurst parameters can be considered (see Alòs and Nualart (2000) for $H > \frac{1}{2}$, Alòs et al. (2000) for $H < \frac{1}{2}$ and more generally Alòs et al. (2001) for Gaussian processes with suitable properties). That approach was originally proposed by Decreusefond and Üstünel (1998).

In this paper we first modify the fractional white noise approach to cover all Hurst parameters (and not only $H > \frac{1}{2}$). More precisely, we construct fractional Brownian motions and fractional white noises for Hurst parameters $0 < H < 1$ on the classical white noise space. There are several advantages in our approach: (i) One can use the same probability space for all Hurst parameters; (ii) The well developed white noise theory can be applied without modifications. In this first part of the paper (Section 2) we also prove an integral representation of the fractional white noise as generalized Wiener integral, which is an extension of the representation of the white noise as Wiener integral of the delta distribution.

In Section 3 we introduce generalized functionals $F(B^H_t)$ of a fractional Brownian motion for tempered distributions $F$ following Kuo (1996). In this section we recall some properties of the $S$-transform and define the fractional Itô integral of a generalized functional of $B^H$ using the Wick product and a Pettis type integral on the space of Hida distributions.

Section 4 contains the main results of the paper. Here we first use white noise techniques, namely the $S$-transform, to prove an Itô formula for fractional Brownian functionals with arbitrary Hurst parameter. Our results and their proofs can be regarded as generalizations of the work by Kubo (1983) for the classical Brownian motion. Note that our Itô formula holds for all Hurst coefficients, whereas most Itô formulas in the literature are only valid for $H > \frac{1}{2}$. But even in this case our Itô formula is more general than those obtained e.g. in Duncan et al. (2000) and Hu and Øksendal (2001) for fractional Brownian functionals. To our best knowledge there is no other Itô formula that holds for all Hurst parameters but the one by Privault (1998) who defines an unusual and rather complicate notion of integrability for $H < \frac{1}{2}$. We refer the reader to Remark 4.7 for a comparison of our results and the known versions of Itô’s formula for fractional Brownian functionals.
As an application we obtain an extension of the Tanaka formula that was also proven by Hu and Øksendal (2001) and Coutin et al. (2001) for the special cases $\frac{1}{2} < H < 1$, and $\frac{1}{3} < H < 1$, respectively. Again our formula holds for all Hurst parameters whereas the techniques in Hu and Øksendal (2001) and Coutin et al. (2001) cannot be used for small Hurst parameters.

Finally we prove two new representations of the local time of $B^H$. One is based on a Clark–Ocone formula for the Donsker delta function of $B^H_t$, that we prove for arbitrary $0 < H < 1$. The other one is another analogue of Tanaka’s formula that holds for $0 < H < \frac{1}{2}$.

Similar results for a geometrical fractional Brownian motion and applications to finance can be found in a forthcoming paper.

2. Construction of the fractional Brownian motion and the fractional white noise on the white noise space

2.1. Construction of the fractional Brownian motion

We assume the underlying probability space $(\Omega, \mathcal{F}, P)$ to be the white noise space. So $\Omega$ is $\mathcal{S}'(\mathbb{R})$, the space of tempered distributions, $\mathcal{F}$ is the $\sigma$-field generated by the open sets in $\mathcal{S}'(\mathbb{R})$ with respect to the weak* topology of $\mathcal{S}'(\mathbb{R})$. Finally, by the Bochner–Minlos theorem the probability measure $P$ is uniquely determined by the property that for all rapidly decreasing functions $f \in \mathcal{S}(\mathbb{R})$

$$\int_{\mathcal{S}'(\mathbb{R})} \exp\{i \langle \omega, f \rangle \} \, dP(\omega) = \exp \left\{ -\frac{1}{2} |f|_0^2 \right\}.$$  

(1)

Here $\langle \cdot, \cdot \rangle$ denotes the dual action and $|\cdot|_0$ is the usual $L^2(\mathbb{R})$-norm. The corresponding inner product is denoted by $\langle \cdot, \cdot \rangle_0$.

From (1) we may conclude that for all finite families $(f_1, \ldots, f_n) \subset \mathcal{S}(\mathbb{R})$ the random vector $(\langle \cdot, f_1 \rangle, \ldots, \langle \cdot, f_n \rangle)$ is a centered Gaussian random vector with covariance matrix $((\langle f_i, f_j \rangle_0)_{i,j}$. The details of the above construction can be found in Hida (1980). Because of the isometry

$$E[\langle \cdot, f \rangle^2] = |f|^2_0; \quad f \in \mathcal{S}(\mathbb{R})$$

we can extend $\langle \cdot, g \rangle$ to $g \in L^2(\mathbb{R})$. Hence, we have for $f, g \in L^2(\mathbb{R})$:

$$E[\langle \cdot, f \rangle \langle \cdot, g \rangle] = (f, g)_0.$$  

(2)

For $a, b \in \mathbb{R}$ we define the indicator function:

$$1(a, b)(t) = \begin{cases} 1, & \text{if } a \leq t < b, \\ -1, & \text{if } b \leq t < a, \\ 0, & \text{otherwise.} \end{cases}$$  

(3)

From (2) and (3) immediately follows that a continuous version of $\langle \cdot, 1(0,t) \rangle$ is a classical Brownian motion $B_t$ on the white noise space. Note, that the existence of a continuous version of $\langle \cdot, 1(0,t) \rangle$ is implied by the Kolmogorov–Centsov theorem,
see Karatzas and Shreve (1991, Theorem 2.8). Consequently, approximating with step functions yields:

$$\langle \cdot, f \rangle = \int_{\mathbb{R}} f(t) \, dB_t,$$

(4)

where $\int_{\mathbb{R}} f(t) \, dB_t$ denotes the classical Wiener integral of a function $f \in L^2(\mathbb{R})$.

We are now going to construct a fractional Brownian motion with arbitrary Hurst parameter $0 < H < 1$:

**Definition 2.1.** A continuous stochastic process $(B^H_t)_{t \in \mathbb{R}}$ is called a fractional Brownian motion with Hurst parameter $H$, if the family $(B^H_t)_{t \in \mathbb{R}}$ is centered Gaussian with

$$E[B^H_t B^H_s] = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}); \quad t, s \in \mathbb{R}. $$

(5)

It is known since the pioneering paper by Mandelbrot and Van Ness (1968), that for $0 < H < 1$, $H \neq \frac{1}{2}$ a fractional Brownian motion is given by a continuous version of the following Wiener integral:

$$B^H_t = \frac{K_H}{\Gamma(H + 1/2)} \int_{\mathbb{R}} [(t-s)^{H-1/2} - (-s)^{H-1/2}] \, dB_s,$$

(6)

where the normalizing constant is given by

$$K_H := \Gamma(H + 1/2) \left( \int_0^\infty ((1+s)^{H-1/2} - s^{H-1/2}) \, ds + \frac{1}{2H} \right)^{-1/2},$$

see Samorodnitsky and Taqqu (1994, p. 320).

In order to obtain a representation of $B^H_t$, $\frac{1}{2} < H < 1$, in terms of the indicator function we use fractional integrals of Weyl’s type. Let $\alpha \in (0,1)$. Then:

$$(I^-_* f)(x) := \frac{1}{\Gamma(x)} \int_{-\infty}^x f(t)(t-x)^{x-1} \, dt = \frac{1}{\Gamma(x)} \int_0^\infty f(x+t)^{x-1} \, dt, \quad \alpha < H \leq 1,$$

(7)

$$(I^+_* f)(x) := \frac{1}{\Gamma(x)} \int_{-\infty}^x f(t)(x-t)^{x-1} \, dt = \frac{1}{\Gamma(x)} \int_0^\infty f(x-t)^{x-1} \, dt, \quad \alpha < H \leq 1,$$

(8)

if the integrals exist for all $x \in \mathbb{R}$. Integration yields for $\frac{1}{2} < H < 1$

$$(I^{H-1/2}_- 1(0,t))(s) = \frac{1}{\Gamma(H + 1/2)} [(t-s)^{H-1/2} - (-s)^{H-1/2}]$$

and hence by (4) a continuous version of $\langle \cdot, K_H I^{H-1/2}_- 1(0,t) \rangle$ is a fractional Brownian motion for Hurst parameter $\frac{1}{2} < H < 1$. Again, the existence of the continuous version follows from the Kolmogorov–Centsov theorem.

To cover the case $0 < H < \frac{1}{2}$ we make use of fractional derivatives of Marchaud’s type. For $\alpha \in (0,1)$ and $\varepsilon > 0$ define

$$(D^\alpha_{\varepsilon, \alpha} f)(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_{\varepsilon}^\infty \frac{f(x) - f(x + t)}{t^{\alpha+1}} \, dt.$$
Then the fractional derivatives of Marchaud’s type are given by

\[(D_{c VT}^\pm f) := \lim_{c SI \to 0^+} (D_{c SI}^\pm f),\]  

if the limit exists almost surely.

By Lemma 3.1 in Pipiras and Taqqu (2000)

\[(t - s)^{H - 1/2} - (-s)^{H - 1/2} = \Gamma(H + 1/2)(D_{-}^{-(H-1/2)}1(0, t))(s)\]

holds. Thus, by (4) we see that for \(0 < H < \frac{1}{2}\) a continuous version of \(\langle \cdot, K_H D_{-}^{-(H-1/2)}1(0, t) \rangle\), that exists by the Kolmogorov–Centsov theorem, is a fractional Brownian motion \(B^H\).

Let us summarize:

**Theorem 2.2.** For \(0 < H < 1\) let the operators \(M_H^\pm\) be defined by

\[M_H^\pm f := \begin{cases} 
K_H D_{\pm}^{-(H-1/2)} f, & 0 < H < \frac{1}{2}, \\
f, & H = \frac{1}{2}, \\
K_H I_{\pm}^{H-1/2} f, & 1/2 < H < 1. 
\end{cases}\]  

Then a fractional Brownian motion is given by a continuous version of \(\langle \cdot, M_H^\pm 1(0, t) \rangle\).

2.2. Some properties of the operators \(M_H^\pm\)

In this section we prove some properties of the operators \(M_H^\pm\) for later use.

**Theorem 2.3.** Let \(H \in (0, 1)\) and \(f \in C^\infty(\mathbb{R})\). Then \(M_H^\pm f\) exist. Moreover, there is a constant \(C_H\) independent of \(f\) such that

\[\max_{x \in \mathbb{R}} |(M_H^\pm f)(x)| \leq C_H \left( \max_{x \in \mathbb{R}} |f(x)| + \max_{x \in \mathbb{R}} |f'(x)| + \|f\|_{L^1(\mathbb{R})} \right).\]  

**Proof.** The case \(H = \frac{1}{2}\) is obvious. Let \(\frac{1}{2} < H < 1\): Then we have for fixed \(x \in \mathbb{R}\) and \(0 < \alpha < \frac{1}{2}\):

\[\Gamma(\alpha)|(I_{\pm}^\alpha f)(x)| \leq \int_{\mathbb{R}} |f(t)| |t - x|^{\alpha - 1} dt\]

\[= \int_{|t-x| < 1} |f(t)| |t - x|^{\alpha - 1} dt + \int_{|t-x| \geq 1} |f(t)| |t - x|^{\alpha - 1} dt\]

\[= I_1 + I_2.\]

The first integral can be estimated as follows:

\[I_1 \leq \max_{x \in \mathbb{R}} |f(x)| \int_{|t-x| < 1} |t - x|^{\alpha - 1} dt = \max_{x \in \mathbb{R}} |f(x)| \int_{-1}^{1} |t|^{\alpha - 1} dt\]

\[= \frac{2}{\alpha} \max_{x \in \mathbb{R}} |f(x)|.\]
For the second integral we obtain:

\[ I_2 \leq \int_{|t-x| \geq 1} |f(t)| \, dt \leq \|f\|_{L^1(\mathbb{R})}. \]

Hence,

\[ \max_{x \in \mathbb{R}} |(M_-^H f)(x)| \leq \frac{2K_H}{\Gamma(H + 1/2)} \left( \max_{x \in \mathbb{R}} |f(x)| + \|f\|_{L^1(\mathbb{R})} \right). \]

Let \( 0 < H < \frac{1}{2} \): Then we have for fixed \( x \in \mathbb{R} \) and \( 0 < \alpha < \frac{1}{2} \):

\[
\frac{\Gamma(1 - \alpha)}{\alpha} \left| (D_\pm^\alpha f)(x) \right|
\leq \int_0^1 \frac{|f(x) - f(x + t)|}{t} \, t^{-\alpha} \, dt + \int_1^\infty \frac{|f(x) - f(x + t)|}{t^{\alpha + 1}} \, dt
\leq \max_{x \in \mathbb{R}} |f'(x)| \int_0^1 t^{-\alpha} \, dt + 2 \max_{x \in \mathbb{R}} |f(x)| \int_1^\infty \frac{1}{t^{1+\alpha}} \, dt
= \frac{1}{1 - \alpha} \max_{x \in \mathbb{R}} |f'(x)| + \frac{2}{\alpha} \max_{x \in \mathbb{R}} |f(x)|.
\]

Thus for a suitable constant \( C_H \),

\[ \max_{x \in \mathbb{R}} |(M_-^H f)(x)| \leq C_H \left( \max_{x \in \mathbb{R}} |f'(x)| + \max_{x \in \mathbb{R}} |f(x)| \right) \]

and the proof is finished. \( \square \)

One can show that \( M_-^H \) and \( M_+^H \) are dual operators in a suitable sense, i.e. for nice functions the following relation holds:

\[
(f, M_-^H g)_0 = (M_+^H f, g)_0. \tag{12}
\]

To be precise, we have:

**Theorem 2.4.** (i) For \( 0 < H < \frac{1}{2} \) the relation (12) holds, if \( M_+^H f \in L^p(\mathbb{R}) \), \( M_-^H g \in L^r(\mathbb{R}) \), \( f \in L^p(\mathbb{R}) \), \( g \in L^r(\mathbb{R}) \) and \( p > 1 \), \( r > 1 \), \( 1/p + 1/r = \frac{3}{2} - H \), \( 1/s = 1/p + H - \frac{1}{2} \), \( 1/t = 1/r + H - \frac{1}{2} \).

(ii) Let \( \frac{1}{2} < H < 1 \). Then (12) holds, if \( f \in L^p(\mathbb{R}) \), \( g \in L^r(\mathbb{R}) \) and \( p > 1 \), \( r > 1 \), \( 1/p + 1/r = \frac{1}{2} + H \).

**Proof.** In view of the definition of \( M_\pm^H \) this is a simple reformulation of Corollary 2 (p. 129) and formula (5.16) in Samko et al. (1993). \( \square \)

A special case is:

**Lemma 2.5.** \((f, M_-^H \mathbf{1}(0, t))_0 = \int_0^t (M_-^H f)(s) \, ds\) for \( f \in \mathcal{D}(\mathbb{R}) \) and \( 0 < H < 1 \).
Proof. The case $H = \frac{1}{2}$ is trivial, because $M_{\pm}^{1/2}$ is the identity map. Let $\frac{1}{2} < H < 1$: Then we have $f \in L^2(\mathbb{R})$ and $1(0, t) \in L^{1/H}(\mathbb{R})$. So the assertion follows from the above theorem. For the case $0 < H < \frac{1}{2}$ note that $f \in L^2(\mathbb{R})$, $M_{-}^{H}1(0, t) \in L^2(\mathbb{R})$ and $1(0, t) \in L^{1/H}(\mathbb{R})$. By the above theorem it remains to show that $M_{+}^{H}f \in L^{1/(1-H)}(\mathbb{R})$.

To this end, notice first that for all $x \neq 0, y$:

$$|f(x) - f(y)| \leq C \left| \frac{1}{x} - \frac{1}{y} \right|. \quad (13)$$

If $x \geq y$ have the same sign, (13) holds, because

$$|f(x) - f(y)| = \left| \int_{y}^{x} f'(\xi) \, d\xi \right| \leq C \int_{y}^{x} \xi^{-2} \, d\xi = C \left| \frac{1}{x} - \frac{1}{y} \right|. \quad (14)$$

If $y < 0$ and $x > 0$, we obtain (13) by the following estimate:

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| \leq C \left( \frac{1}{|x|} + \frac{1}{|y|} \right) = C \left| \frac{1}{x} - \frac{1}{y} \right|. \quad (13)$$

Consequently, a rapidly decreasing function $f$ fulfills the Hölder condition on the real axis (Definition 1.1’ in Samko et al. (1993)). So by Theorem 6.5 and Theorem 6.1 in Samko et al. (1993) $M_{+}^{H}f \in L^{1/(1-H)}(\mathbb{R})$.

Remark 2.6. The above lemma can also be directly verified by use of Fubini’s theorem, some change of variables and—in the case $0 < H < \frac{1}{2}$—a limit argument.

The next theorem is a straightforward consequence of the dominated convergence theorem:

Theorem 2.7. Let $H \in (0, 1)$ and $f \in \mathcal{S}(\mathbb{R})$. Then $M_{+}^{H}f$ is continuous.

An immediate corollary is:

Corollary 2.8. Let $H \in (0, 1)$ and $f \in \mathcal{S}(\mathbb{R})$. Then $(f, M_{-}^{H}1(0, t))_{0}$ is differentiable and $(d/dt)(f, M_{-}^{H}1(0, t))_{0} = M_{+}^{H}f(t)$.

2.3. Hida distributions and fractional white noise

As in the case of a classical Brownian motion ($H = \frac{1}{2}$), a fractional Brownian motion with Hurst parameter $0 < H < 1$ is nowhere differentiable on almost every path, see Mandelbrot and Van Ness (1968). However, we are going to show, that $\mathbb{B}^{H}$ is differentiable as a mapping from $\mathbb{R}$ into a space of stochastic generalized functions, the so called Hida distributions. Moreover, we provide a representation of its derivative as generalized Wiener integral.

Let $(L^2) := L^2(\Omega, \mathcal{G}, P)$, where $\mathcal{G}$ is the $\sigma$-field generated by $\langle \cdot, f \rangle_{f \in L^2(\mathbb{R})}$. By the Wiener–Itô theorem (see Nualart, 1995, p.12) every $\Phi \in (L^2)$ can be uniquely expanded into a series of multiple Wiener integrals:

$$\Phi = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in \hat{L}^2(\mathbb{R}^n). \quad (14)$$
Here \( \hat{L}^2(\mathbb{R}^n) \) denotes the subspace of symmetric functions in \( L^2(\mathbb{R}^n) \). The above decomposition is called the Wiener chaos of \( \Phi \). Moreover,

\[
\|\Phi\|_0^2 := E[\Phi^2] = \sum_{n=0}^{\infty} n! |f_n|^2.
\]

Consider now the operator \( A := -\frac{d^2}{dx^2} + x^2 + 1 \) and define its second quantization operator \( \mathcal{N}UL(A) \) in terms of the Wiener chaos by

\[
\mathcal{N}UL(A)\Phi = \sum_{n=0}^{\infty} I_n(A^\otimes n f_n).
\]

Note that both operators, \( A \) and \( \mathcal{N}UL(A) \), are densely defined in \( L^2(\mathbb{R}) \) and \( \mathcal{L}^2(\mathbb{S}) \), respectively, that they are invertible and the inverse operators are bounded on \( L^2(\mathbb{R}) \) and \( \mathcal{L}^2(\mathbb{S}) \), respectively.

For \( p \in \mathbb{Z} \) and \( \Phi \) in the domain of \( \mathcal{N}UL(A) \phi \) let

\[
\|\Phi\|_p := \|\mathcal{N}UL(A)^\phi \Phi\|_0.
\]

If \( p \geq 0 \), define \((S)_p := \{\Phi \in (L^2); \mathcal{N}UL(A)^\phi \Phi \text{ exists and } \mathcal{N}UL(A)^\phi \Phi \in (L^2)\}\) and endow \((S)_p \) with the norm \( \|\cdot\|_p \).

If \( p < 0 \), let \((S)_{-p} \) be the completion of \((L^2) \) with respect to the \( \|\cdot\|_p \)-norm.

**Definition 2.9.** (i) The projective limit of the spaces \((S)_p, p \in \mathbb{N}\), is called the space of stochastic test functions and is denoted by \((S)\).

(ii) The inductive limit of the spaces \((S)_{-p}, p \in \mathbb{N}\), is said to be the space of Hida distributions, \((S)^*\).

As the notation suggests, \((S)^*\) is the dual of \((S)\). We only sketch the proof: For \( p > 0 \) \((S)_{-p} \) is the dual space of \((S)_p \), see Kuo (1996, p.21). The assertion then follows, because \((S)\) is a countably Hilbert space (Kuo, 1996, p. 21, 28–29), see Gel’fand and Vilenkin (1964, p.59).

The dual action is denoted by \( \langle\langle \Phi, \eta \rangle\rangle \) for \( \Phi \in (S)^*, \eta \in (S) \). If \( \Phi \in (L^2) \), then

\[
\langle\langle \Phi, \eta \rangle\rangle = E[\Phi \cdot \eta].
\]

**Definition 2.10.** (i) Let \( I \subset \mathbb{R} \) be an interval. A mapping \( X : I \rightarrow (S)^* \) is called a stochastic distribution process.

(ii) A stochastic distribution process \( X \) is said to be differentiable, if the limit \( \lim_{h \to 0} h^{-1}(X_{t+h} - X_t) \) exists in \((S)^*\).

**Remark 2.11.** Convergence in \((S)^*\) means convergence in the inductive limit topology, i.e. convergence in \((S)_{-p}\) for some \( p \in \mathbb{N} \). By Kuo (1996, p.10), this is equivalent to convergence in the weak* topology.

Before we can prove, that for \( 0 < H < 1 \) the stochastic distribution process \( B^H \) is differentiable, we need to recall some properties of the Hermite functions:
Definition 2.12. (i) The $n$th Hermite polynomial $(n=0, 1, \ldots)$ is defined as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$ 

(ii) The $n$th Hermite function $(n=0, 1, \ldots)$ is defined as

$$\tilde{\xi}_n(x) = \pi^{-1/4} (2^n n!)^{-1/2} e^{-\frac{1}{2} x^2} H_n(x).$$

The following properties of the Hermite functions are useful:

Theorem 2.13. (i) $(\tilde{\xi}_n)_{n \in \mathbb{N}_0}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$.

(ii) There is a constant $K > 0$ such that for $n \geq 1$

$$\max_{x \in \mathbb{R}} |\tilde{\xi}_n(x)| \leq Kn^{-1/2}.$$ 

(iii) There is a constant $K > 0$ such that

$$\|\tilde{\xi}_n\|_{L^1(\mathbb{R})} \leq Kn^{1/4}.$$ 

The proof can be found in Hille and Phillips (1957). Combining this with Theorem 2.3 yields:

Lemma 2.14. Let $H \in (0, 1)$. Then there is a constant $C_H > 0$ such that

$$\max_{x \in \mathbb{R}} |(M^H \tilde{\xi}_n)(x)| \leq C_H (n + 1)^{5/12}.$$ 

Proof. In view of the relation

$$\tilde{\xi}_n'(x) = \sqrt{n} \frac{n}{2} \tilde{\xi}_{n-1}(x) - \sqrt{n + 1} \frac{1}{2} \tilde{\xi}_{n+1}(x)$$

the assertion directly follows from the above estimates for the Hermite functions and Theorem 2.3. \(\square\)

Lemma 2.15. Let $H \in (0, 1)$. Then $M^H \mathbf{1}(0, \cdot) : \mathbb{R} \to \mathcal{S}'(\mathbb{R})$ is differentiable and $(d/dt) M^H \mathbf{1}(0, t) = \sum_{k=0}^{\infty} (M^H \tilde{\xi}_k)(t) \tilde{\xi}_k$ (limit in $\mathcal{S}'(\mathbb{R})$).

Remark 2.16. Recall that one can reconstruct $\mathcal{S}'(\mathbb{R})$ as an inductive limit. Define a family of norms on $L^2(\mathbb{R})$ for $p \in \mathbb{N}$ by

$$|f|_{-p}^2 := |A^{-p} f|_0^2 = \sum_{k=0}^{\infty} (2k + 2)^{-2p} (f, \tilde{\xi}_k)^2.$$ 

The last equation follows from the fact, that $\tilde{\xi}_k$ is an eigenfunction of $A$ with eigenvalue $(2k + 2)$.

Let now $\mathcal{S}_{-p}(\mathbb{R})$ be the completion of $L^2(\mathbb{R})$ with respect to $| \cdot |_{-p}, (p \in \mathbb{N})$. Then $\mathcal{S}'(\mathbb{R})$ is the inductive limit of $\mathcal{S}_{-p}(\mathbb{R})$. See Kuo (1996, p. 17–18), for more details. Moreover note, that convergence in the inductive limit topology coincides with both the convergence in the strong and the weak* topology of $\mathcal{S}'(\mathbb{R})$. 
Proof of Lemma 2.15. As the Hermite functions constitute an orthonormal basis of $L^2(\mathbb{R})$ we can use Fourier coefficients and Lemma 2.5 to obtain:

$$M^H_1(0, t) = \sum_{k=0}^{\infty} (M^H_1(0, t), \xi_k)_0 \xi_k = \sum_{k=0}^{\infty} \int_0^t (M^H_+ \xi_k)(s) ds \xi_k.$$  \hfill (17)

Hence,

$$\left| \frac{M^H_1(0, t + h) - M^H_1(0, t)}{h} - \sum_{k=0}^{\infty} (M^H_+ \xi_k)(t) \xi_k \right|^2 = \sum_{k=0}^{\infty} \left( \frac{1}{h} \int_t^{t+h} (M^H_+ \xi_k)(s) ds - (M^H_+ \xi_k)(t) \right)^2 \xi_k.$$

By Lemma 2.14 the right hand side converges uniformly in $h$. Thus,

$$\lim_{h \to 0} \left| \frac{M^H_1(0, t + h) - M^H_1(0, t)}{h} - \sum_{k=0}^{\infty} (M^H_+ \xi_k)(t) \xi_k \right|^2 = \sum_{k=0}^{\infty} (2k + 2)^{-2} \left( \frac{1}{h} \int_t^{t+h} (M^H_+ \xi_k)(s) ds - (M^H_+ \xi_k)(t) \right)^2.$$

Because of the identity $B^H_t = \langle \cdot, M^H_1(0, t) \rangle$ the preceding lemma might suggest that $(d/dt)B^H_t = \langle \cdot, \sum_{k=0}^{\infty} (M^H_+ \xi_k)(t) \xi_k \rangle$. But the integrand of this Wiener integral is not an element of $L^2(\mathbb{R})$, but a tempered distribution. So we need to extend the Wiener integral to tempered distributions.

Let $f \in L^2(\mathbb{R})$, $p \in \mathbb{N}$. Then using (4) and the isometry (2) of the Wiener integral we get the following isometry:

$$\| \langle \cdot, f \rangle \|_{L^p} = \| I_1(f) \|_{L^p} = \| I_1(A^{-p} f) \|_0 = |A^{-p} f|_0 = |f|_{L^p}.$$

So the Wiener integral can be extended to $f \in \mathcal{S}_p(\mathbb{R})$ such that the isometry (18) holds, and consequently to $f \in \mathcal{S}'(\mathbb{R})$. Note that this extended Wiener integral is a Hida distribution and need not to be a random variable.

The following theorem allows us to calculate the derivative of $B^H$:

**Theorem 2.17.** Let $I \subset \mathbb{R}$ be an interval and let $F: I \to \mathcal{S}'(\mathbb{R})$ be differentiable. Then $\langle \cdot, F(t) \rangle$ is a differentiable stochastic distribution process and $(d/dt)\langle \cdot, F(t) \rangle = \langle \cdot, (d/dt)F(t) \rangle$. 

Proof. Let $h_n$ be a sequence that tends to zero. By the assumption, there is a $p \in \mathbb{N}$ such that
\[
\lim_{n \to \infty} \left| h_n^{-1}(F(t + h_n) - F(t)) - \frac{d}{dt} F(t) \right|^{-p} = 0.
\]
So the assertion follows from the linearity of the Wiener integral and the isometry (18). □

Combining this theorem with Lemma 2.15 we see that $B^H_t$ is differentiable for $0 < H < 1$ and
\[
\frac{d}{dt} B^H_t = \langle \cdot, \sum_{k=0}^{\infty} (M^H_+ \xi_k)(t) \xi_k \rangle.
\]
For $t \in \mathbb{R}$ we define the distribution
\[
\langle \delta_t \circ M^H_+, f \rangle := (M^H_+ f)(t).
\]
Then:
\[
\sum_{k=0}^{\infty} (M^H_+ \xi_k)(t) \xi_k - \delta_t \circ M^H_+ \right|_{-1}^2
= \sum_{n=0}^{\infty} (2n + 2)^{-2} \left( \sum_{k=0}^{\infty} (M^H_+ \xi_k)(t)(\xi_k, \xi_n) - \langle \delta_t \circ M^H_+, \xi_n \rangle \right)^2 = 0.
\]
Hence,
\[
\frac{d}{dt} B^H_t = \langle \cdot, \delta_t \circ M^H_+ \rangle.
\]

Definition 2.18. Let $0 < H < 1$. Then the derivative of $B^H$ in $(S)^*$
\[
W^H_t = \langle \cdot, \delta_t \circ M^H_+ \rangle
\]
is called the fractional white noise.

Remark 2.19. The above representation of the fractional white noise is a generalization of the well known representation $W^{1/2} = \langle \cdot, \delta_t \rangle$ for the white noise, see Kuo (1996, p. 21).

Remark 2.20. There is a related construction of fractional Brownian motions and fractional white noises on the white noise space due to Elliott and van der Hoek (2001). It is based on Lindstrøm’s representation of fractional Brownian fields (Lindstrøm, 1993). In this construction a fractional Brownian motion is given by a continuous version of $c_H \langle \cdot, M^H_1(0,t) - M^H_1(0,t) \rangle$. There is one drawback of this construction: In this setting the fractional Brownian motion with Hurst parameter $H \neq \frac{1}{2}$ is not adapted to the filtration generated by the driving Brownian motion $\langle \cdot, 1(0,t) \rangle$. So one
has to consider several filtrations depending on \( H \). As our construction is based on the Mandelbrot/Van Ness representation (Mandelbrot and Van Ness, 1968) the filtrations of all fractional Brownian motions coincide with the filtration generated by \( \langle \cdot, 1(0,t) \rangle \), see Rogers (1997).

3. Generalized functionals of a fractional Brownian motion

3.1. Definition of \( F(B_t^H) \)

Before we introduce generalized functionals of a fractional Brownian motion, let us recall that a tempered distribution \( F \) is said to be of function type, if there is a locally integrable function \( f \) such that

\[
\langle F, \phi \rangle = \int_{\mathbb{R}} f(x) \phi(x) \, dx, \quad \phi \in \mathcal{S}(\mathbb{R}).
\]

In that case we identify the function \( f \) and the distribution \( F \).

**Theorem 3.1.** Let \( 0 < H < 1, F \in \mathcal{S}'(\mathbb{R}) \) and \( t > 0 \). Then

\[
\frac{1}{\sqrt{2\pi t^H}} \sum_{n=0}^{\infty} \left( \frac{n!}{2^n} \right)^{-1} t^{-2Hn} \langle F, g_{t,H,n} \rangle J_n((M_H^1(0,t))^\otimes^n) \tag{19}
\]

is a Hida distribution, where

\[
g_{t,H,n}(x) := (\sqrt{2})^{-n} t^{Hn} \zeta_n(x/(\sqrt{2}t^H)) \exp \left\{ -\frac{x^2}{4t^{2H}} \right\} \tag{20}
\]

Moreover, if \( F \) is of function type, (19) coincides with \( F(B_t^H) \). We denote (19) by \( F(B_t^H) \).

**Proof.** This is a special case of the considerations in Kuo (1996, p. 61–64) for \( f = M_H^1(0,t) \). \( \Box \)

**Definition 3.2.** We call \( F(B_t^H) \) defined in the above theorem a generalized functional of \( B_t^H \).

We now give an alternative proof that \( F(B_t^H), \ t > 0 \), is a Hida distribution. In that way we establish an estimate that proves useful later.

**Theorem 3.3.** Let \( 0 < H < 1, t > 0, p \in \mathbb{N} \) and \( F \in \mathcal{S}_{-p}(\mathbb{R}) \). Then there is a constant \( C_p \) independent of \( F \) and \( t \) such that

\[
\| F(B_t^H) \|_{-p}^2 \leq \max \{ t^{-4Hp}, t^{4Hp} \} t^{-H} C_p |F|_{-p}^2. \tag{21}
\]
Proof. By the definition of $F(B_t^H)$ and the $\| \cdot \|_p$-norm we get:

$$
\| F(B_t^H) \|_p^2 = \frac{1}{2\pi t^{2H}} \sum_{n=0}^{\infty} (n!)^{-1} t^{-4Hn} \langle F, g_{t,H,n} \rangle^2 \| (A^{-p})(M^H 1(0,t))^{\otimes n} \|_0^2 
$$

$$
= \frac{1}{2\sqrt{\pi} t^{2H}} \sum_{n=0}^{\infty} t^{-2Hn} \left\langle F, \xi_n \left( \frac{\cdot}{\sqrt{2t^H}} \right) \exp \left\{ -\frac{1}{4} \frac{\cdot^2}{t^{2H}} \right\} \right\rangle^2 
$$

$$
\times \| A^{-p}(M^H 1(0,t))^{\otimes n} \|_0^{2n}.
$$

As the operator $A^{-1}$ has operator norm $\frac{1}{2}$ and the operator $A$ is symmetric we obtain:

$$
\| F(B_t^H) \|_p^2 \leq \frac{1}{2\sqrt{\pi} t^{2H}} \sum_{n=0}^{\infty} 2^{-2np} |F|_p^2 
$$

$$
\times \left\| A^p \left( \xi_n \left( \frac{\cdot}{\sqrt{2t^H}} \right) \exp \left\{ -\frac{1}{4} \frac{\cdot^2}{t^{2H}} \right\} \right) \right\|_0^2.
$$

(22)

Note that the identity $\| M^H 1(0,t) \|_0^2 = t^{2H}$ used in the above estimate follows from the isometry of the Wiener integral and Theorem 2.2.

In order to estimate the last factor on the right hand side we first calculate $Af_n$, where

$$
f_n(x) := \xi_n \left( \frac{\cdot}{\sqrt{2t^H}} \right) \exp \left\{ -\frac{x^2}{4t^{2H}} \right\}.
$$

We use that $\xi_n$ is an eigenfunction of $A$ with eigenvalue $(2n - 2)$ and the following equalities:

$$
\xi_n'(x) = \sqrt{\frac{n}{2}} \xi_{n-1}(x) - \sqrt{\frac{n+1}{2}} \xi_{n+1}(x),
$$

$$
x\xi_n(x) = \sqrt{\frac{n}{2}} \xi_{n-1}(x) + \sqrt{\frac{n+1}{2}} \xi_{n+1}(x).
$$

After tedious calculations one gets:

$$
Af_n = f_n(x) + 2t^{2H} (n + 1/2) f_n + t^{2H} \sqrt{n(n-1)} f_{n-2}
$$

$$
+ (t^{2H} - t^{-2H}) \sqrt{(n+2)(n+1)} f_{n+2},
$$

(23)

where we used the convention $f_{-2}(x) := f_{-1}(x) := 0$. For the rest of the proof let $C_p$ be a constant depending only on $p$ and $H$ that may differ for different applications.
Iterating (23) yields the following estimate:

\[ |(A^p f_n)(x)| \leq \max\{ t^{-2H_p}, t^{2H_p} \} C_p^2 (n + 2)^p \sum_{k=(n-2p)\vee 0}^{n+2p} |f_k(x)| \]

\[ \leq \max\{ t^{-2H_p}, t^{2H_p} \} C_p^2 (n + 2)^p e^{-x^2/(4t^{2H})} \]

\[ \times \sum_{k=(n-2p)\vee 0}^{n+2p} \max_{y \in \mathbb{R}} |\xi_k(y)| \]

\[ \leq \max\{ t^{-2H_p}, t^{2H_p} \} C_p^2 (n + 2)^p e^{-x^2/(4t^{2H})} \sum_{k=(n-2p)\vee 0}^{n+2p} 1 \]

\[ \leq \max\{ t^{-2H_p}, t^{2H_p} \} C_p^2 (n + 2)^p e^{-x^2/(4t^{2H})}. \]

Consequently,

\[ |A^p \left( \xi_n \left( \frac{\cdot}{\sqrt{2t^{H}}} \right) \exp \left( \frac{-1}{4 \cdot t^{2H}} \right) \right) |^2_0 \leq \max\{ t^{-4H_p}, t^{4H_p} \} C_p(n + 2)^2 p \sqrt{2\pi t^{2H}}. \]

Combining this with (32) we obtain (21), because \( \sum_{n=0}^{\infty} (n + 2)^{2p} 2^{-2np} < \infty \) for \( p \geq 1 \).

### 3.2. The S-transform

The S-transform is a fundamental tool of the white noise analysis.

**Definition 3.4.** For \( \Phi \in (S)^* \) the S-transform is defined by

\[ S\Phi(\eta) := \langle \langle \Phi, :e^{(\cdot, \eta)} : \rangle \rangle; \quad \eta \in \mathcal{S}(\mathbb{R}). \]

Here the Wick exponential of \( \langle \cdot, \eta \rangle \) is given by \( :e^{(\cdot, \eta)} := \sum_{n=0}^{\infty} (n!)^{-1} I_n(\eta^{\otimes n}). \)

**Remark 3.5.** The S-transform is well defined, because the Wick exponential of the Wiener integral of a rapidly decreasing function is a stochastic test function, see Kuo (1996, p. 37).

We now state some properties of the S-transform:

**Theorem 3.6.** (i) The S-transform is injective, i.e. \( S\Phi(\eta) = S\Psi(\eta) \) for all \( \eta \in \mathcal{S}(\mathbb{R}) \) implies \( \Phi = \Psi. \)

(ii) For \( \Phi, \Psi \in (S)^* \) there is a unique element \( \Phi \diamond \Psi \in (S)^* \) such that for all \( \eta \in \mathcal{S}(\mathbb{R}) \)

\[ S(\Phi \diamond \Psi)(\eta) = S\Phi(\eta) \cdot S\Psi(\eta). \]

It is called the Wick product of \( \Phi \) and \( \Psi. \)

(iii) Let a stochastic distribution process \( X : I \rightarrow (S)^* \) be differentiable. Then

\[ S((d/dt)X_t)(\eta) = (d/dt)(SX_t(\eta)). \]

Proofs of (i) and (ii) can be found in Kuo (1996, p. 39 resp. p. 92). (iii) is obvious by the definition.
Using this result we can calculate the S-transforms of the fractional Brownian motion and the fractional white noise:

**Theorem 3.7.** Let $0 < H < 1$. Then:

(i) $SB^H_t(\eta) = (\eta, M^H_t \mathbf{1}(0,t))_0$,

(ii) $SW^H_t(\eta) = (M^+_H \eta)(t)$.

**Proof.** (i) By (16) and the polarization of (15) we get:

$$SB^H_t(\eta) = \langle \langle B^H_t, e^{\langle \cdot, \eta \rangle} \rangle \rangle = E[B^H_t, e^{\langle \cdot, \eta \rangle}] = (\eta, M^H_t \mathbf{1}(0,t))_0.$$ 

(ii) immediately follows from Corollary 2.8 and Theorem 3.6(iii), because $W^H_t$ is the derivative of $B^H_t$. □

We also need some information about the S-transform of generalized functionals of $B^H$:

**Theorem 3.8.** Let $0 < H < 1$, $p \in \mathbb{N}$, $F \in \mathcal{S}_p(\mathbb{R})$ and $t > 0$. Then:

$$S(F(B^H_t))(\eta) = \left\{ F, \exp \left\{ - \frac{1}{2t^{2H}} \left( \cdot - \int_0^t (M^+_H \eta)(s) \, ds \right)^2 \right\} \right\}$$

and there is a constant $C_p$ independent of $F$, $t$ and $\eta$ such that

$$|S(F(B^H_t))(\eta)|^2 \leq \max \{ t^{-AHp}, t^{AHp} \} t^{-H} C_p \| F \|_{-p}^2 \exp\{ |A^p \eta|_0^2 \}. \quad (24)$$

**Proof.** The S-transform can be derived from Theorem 7.3 in Kuo (1996) (consider the special case $f = M^H \mathbf{1}(0,t)$) and Lemma 2.5. (24) is implied by (21) in the following way:

$$|S(F(B^H_t))(\eta)|^2 \leq \| F(B^H_t) \|_{-p}^2 \| : e^{\langle \cdot, \eta \rangle} : \|_p^2$$

$$\leq \max \{ t^{-AHp}, t^{AHp} \} t^{-H} C_p \| F \|_{-p}^2 \| : e^{\langle \cdot, \eta \rangle} : \|_p^2.$$

From Theorem 5.7 in Kuo (1996) we have:

$$\| : e^{\langle \cdot, \eta \rangle} : \|_p = \exp \left\{ \frac{1}{2} |A^p \eta|_0^2 \right\},$$

which yields (24). □

### 3.3. White noise integration

Let $X : I \to (S)^*$ be a stochastic distribution process. Integrability of $X$ can be defined in terms of the S-transform:

**Definition 3.9.** A stochastic distribution process $X : I \to (S)^*$ is integrable, if

(i) $SX(\eta)$ is measurable for all $\eta \in \mathcal{F}(\mathbb{R})$;
(ii) \( SX(\eta) \in L^1(I) \) for all \( \eta \in \mathcal{S}(\mathbb{R}) \);

(iii) \( \int_I SXt(\eta) \, dt \) is the \( \mathcal{S} \)-transform of a Hida distribution \( \Phi \).

In this case \( \Phi \) is unique by Theorem 3.6(i). It is called the \textit{white noise integral of} \( X \) and is denoted by \( \int_I X_t \, dt \).

The following criterion for integrability is taken from Kuo (1996, Theorem 13.5):

Theorem 3.10. Let \( X : I \to (S)^* \) be a stochastic distribution process satisfying:

(1) \( SX(\eta) \) is measurable for all \( \eta \in \mathcal{S}(\mathbb{R}) \);

(2) There is an \( p \in \mathbb{N} \) and a nonnegative function \( L \in L^1(I) \) such that for all \( \eta \in \mathcal{S}(\mathbb{R}) \)

\[
|SXt(\eta)| \leq L(t) \exp \left\{ \frac{1}{2} |A^p \eta|^2 \right\}.
\]

Then \( X \) is integrable and for \( q \) sufficiently large

\[
\int_I \|X_t\|_{-q} \, dt \leq \sqrt{2} \|L\|_{L^1(I)}.
\]

So we obtain the following theorem:

Theorem 3.11. Let \( p \in \mathbb{N}, 0 < a \leq b \) and \( F : [a,b] \to \mathcal{S}_{-p}(\mathbb{R}) \) be continuous. Then \( F(t,B^H_t) \) is integrable over \([a,b]\) for \( 0 < H < 1 \).

Proof. By (24) and the continuity of \( F \) we have for \( t \in [a,b] \) :

\[
|S(F(t,B^H_t))| \leq \max \{ t^{-2H}r^2, t^2r \} t^{-H/2}c_p \max_{s \in [a,b]} |F(s)|_{-p} \exp \left\{ \frac{1}{2} |A^p \eta|^2 \right\}.
\]

As the measurability of \( S(F(t,B^H_t)) \) is obvious, Theorem 3.10 applies.

Moreover,

Theorem 3.12. Let \( p \in \mathbb{N}, 0 < a \leq b \) and \( F : [a,b] \to \mathcal{S}_{-p}(\mathbb{R}) \) be continuous. Then \( F(t,B^H_t) \cdot W^H_t \) is integrable over \([a,b]\) for \( 0 < H < 1 \).

Proof. Recall that by Theorem 3.6, (ii), and Theorem 3.7:

\[
S(F(t,B^H_t) \cdot W^H_t)(\eta) = SF(t,B^H_t)(\eta) \cdot (M^H_t \eta)(t).
\]

As \( M^H_t \eta \) is continuous, the \( \mathcal{S} \)-transform is measurable.

By Kuo (1996, p. 92), the Wick product is a continuous mapping, i.e. for \( q \) sufficiently large:

\[
\|F(t,B^H_t) \cdot W^H_t\|_{-q} \leq \|F(t,B^H_t)\|_{-p} \cdot \|W^H_t\|_{-1} \leq C_p \max \{ t^{-2H}r^2, t^2r \} t^{-H/2} \max_{s \in [a,b]} |F(s)|_{-p} \left( \sum_{n=0}^{\infty} (2n+2)^{-2}(n+1)^{5/6} \right)^{1/2}.
\]
where we made use of (21) and Lemma 2.14. As the above series is convergent we get for a suitable constant $\tilde{C}_p$ as in the proof of (24):

$$|S(F(t, B^H_t) \diamond W^H_t)(\eta)| \leq \tilde{C}_p \max \{t^{-2Hp}, t^{2Hp}\} t^{-H/2} \max_{s \in [a, b]} |F(s)|_{-p} \exp \left\{ \frac{1}{2} |A^q \eta|_0^2 \right\}.$$ 

Again the assertion follows from Theorem 3.10. $\square$

We shall use the notation:

$$\int_a^b F(t, B^H_t) \, dB^H_t := \int_a^b F(t, B^H_t) \diamond W^H_t \, dt \quad (26)$$

and call this integral the *fractional Itô integral* of $F(t, B^H_t)$. This is motivated by the fact that in the classical case of a Brownian motion ($H = \frac{1}{2}$) this integral coincides with the Itô integral, if the Itô integral exists. See Holden et al. (1996, chapter 2.5) for a proof.

4. The Itô formula

4.1. The main result

In this section we prove the following Itô formula for generalized functionals of a fractional Brownian motion:

**Theorem 4.1.** Let $p \in \mathbb{N}$, $0 < a \leq b$ and $F \in C^1([a, b], \mathcal{S}_{-p}(\mathbb{R}))$ such that $(\hat{\partial}/\hat{\partial} x) F$, $(\hat{\partial}^2/\hat{\partial} x^2) F : [a, b] \to \mathcal{S}_{-p}(\mathbb{R})$ continuous and assume $0 < H < 1$. Then in $(S)^*$ the following equation holds:

$$F(b, B^H_b) - F(a, B^H_a) = \int_a^b \frac{\hat{\partial}}{\hat{\partial} t} F(t, B^H_t) \, dt + \int_a^b \frac{\hat{\partial}}{\hat{\partial} x} F(t, B^H_t) \, dB^H_t$$

$$+ H \int_a^b t^{2H-1} \frac{\hat{\partial}^2}{\hat{\partial} x^2} F(t, B^H_t) \, dt. \quad (27)$$

**Remark 4.2.** The derivatives of $F$ in $x$-direction are generalized derivatives of the distribution $F(t)$.

**Proof.** First, all integrals on the right hand side exist by Theorems 3.11 and 3.12. So in view of Theorem 3.6(i), it is sufficient to prove that both sides of (27) have the same $S$-transform.
Let \( g(t,x) := (1/\sqrt{2\pi t}) \exp \left\{ -\frac{x^2}{2t} \right\} \) be the heat kernel. Then:

\[
\frac{d}{dt} S(F(t,B_t^{H}))(\eta) = \frac{d}{dt} \left\langle F(t), g \left( t^{2H}, \cdot - \int_0^t (M_+^{H} \eta)(s) \, ds \right) \right\rangle 
= \left\langle \frac{\partial}{\partial t} F(t), g \left( t^{2H}, \cdot - \int_0^t (M_+^{H} \eta)(s) \, ds \right) \right\rangle 
+ 2Ht^{2H-1} \left\langle F(t), \frac{\partial}{\partial t} g \left( t^{2H}, \cdot - \int_0^t (M_+^{H} \eta)(s) \, ds \right) \right\rangle 
- (M_+^{H} \eta)(t) \left\langle F(t), \frac{\partial}{\partial x} g \left( t^{2H}, \cdot - \int_0^t (M_+^{H} \eta)(s) \, ds \right) \right\rangle 
=: (1) + (2) + (3).
\]

By Theorem 3.8:

\[(1) = S \left( \frac{\partial}{\partial t} F(t,B_t^{H}) \right)(\eta).\]

As the heat kernel fulfills \((\partial/\partial t)g = \frac{1}{2}(\partial^2/\partial x^2)g\) we obtain:

\[(2) = Ht^{2H-1} \left\langle F(t), \frac{\partial^2}{\partial x^2} g \left( t^{2H}, \cdot - \int_0^t (M_+^{H} \eta)(s) \, ds \right) \right\rangle 
= Ht^{2H-1} \left\langle \frac{\partial^2}{\partial x^2} F(t), g \left( t^{2H}, \cdot - \int_0^t (M_+^{H} \eta)(s) \, ds \right) \right\rangle 
= Ht^{2H-1} \left( \frac{\partial^2}{\partial x^2} F(t,B_t^{H}) \right)(\eta).
\]

Finally, by Theorems 3.7 and 3.6, (ii):

\[(3) = (SW_t^{H})(\eta) \left\langle \frac{\partial}{\partial x} F(t), g \left( t^{2H}, \cdot - \int_0^t (M_+^{H} \eta)(s) \, ds \right) \right\rangle 
= (SW_t^{H})(\eta) \cdot S \left( \frac{\partial}{\partial x} F(t,B_t^{H}) \right)(\eta) = S \left( \frac{\partial}{\partial x} F(t,B_t^{H}) \right)(W_t^{H})(\eta).
\]

Combining the above equalities we get:

\[
\frac{d}{dt} S(F(t,B_t^{H}))(\eta) = S \left( \frac{\partial}{\partial t} F(t,B_t^{H}) \right)(\eta) + S \left( \frac{\partial}{\partial x} F(t,B_t^{H}) \right)(W_t^{H})(\eta) 
+ Ht^{2H-1} \left( \frac{\partial^2}{\partial x^2} F(t,B_t^{H}) \right)(\eta).
\]

We can now integrate this equality over \([a,b]\) and recall the definitions of the white noise integral and the fractional Itô integral to conclude that both sides of (27) have the same \(S\)-transforms. \(\square\)
If $F$ is constant in $t$ we get the following corollary, which was proven in the case of a Brownian motion by Kubo (1983):

**Corollary 4.3.** Let $0 < H < 1$, $0 < a \leq b$ and $F \in \mathcal{S}'(\mathbb{R})$. Then in $(S)^*$:

$$
F(B^H_t) - F(B^H_a) = \int_a^b F'(B^H_t) \, dB^H_t + H \int_a^b t^{2H-1} F''(B^H_t) \, dt.
$$

### 4.2. The case $a = 0$

If the distribution $F$ is of function type, we can define $F(B^H_0) := F(0)$. In this section we state conditions that ensure the Itô formula to hold in the case $a = 0$.

**Theorem 4.4.** Let $0 < H < 1$, $0 < b$. Assume $F : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at $0$ and of polynomial growth. Moreover, assume that the first distributional derivative of $F$ is of function type. Then in $(L^2)$:

$$
F(B^H_t) - F(0) = \int_0^b F'(B^H_t) \, dB^H_t + H \int_0^b t^{2H-1} F''(B^H_t) \, dt.
$$  \hspace{1cm} (28)

**Remark 4.5.** The assumptions on $F$ are fulfilled, if $F$ is absolutely continuous and of polynomial growth or convex and of polynomial growth.

**Proof.** Step 1: $\lim_{t \rightarrow 0^+} F(B^H_t) = F(0)$ in $(S)^*$:

As $F$ is of polynomial growth, there is an $p \in \mathbb{N}$ and a constant $K$ such that for all $x \in \mathbb{R}$

$$
|F(x)| \leq K(1 + |x|^p).
$$

Hence a simple computation of the moments of $B^H_t$ yields:

$$
\left\|F(B^H_t)\right\|_0^2 \leq 2K^2(1 + E[(B^H_t)^2]) = 2K^2 \left(1 + \frac{(2p)!}{2^p p!} |t|^{2pH}\right).
$$  \hspace{1cm} (29)

The right hand side is uniformly bounded in $t$ for $0 \leq t \leq b$. So in view of Theorem 8.6 in Kuo (1996) it is sufficient to prove convergence of the $S$-transforms. By Theorem 3.1 and change of variable $s = t^H$ we have:

$$
\lim_{t \rightarrow 0^+} S(F(B^H_t))(\eta) = \lim_{s \rightarrow 0^+} \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} F(x) \exp \left\{-\frac{1}{2s^2} \left(x - \int_0^{s^{1/H}} (M^H_{\eta})(u) \, du\right)^2\right\} \, dx.
$$

Because of the growth condition on $F$ and the continuity at 0 we can apply Theorem 2 in Widder (1944) to see that

$$
\lim_{t \rightarrow 0^+} S(F(B^H_t))(\eta) = F(0) = S(F(0))(\eta).
$$
Step 2: Convergence of the fractional Itô integral:

We are first going to establish the following estimate: There is a constant $C$ depending on $F$ and $H$ such that for all $0 < t \leq b$

$$
\|F'(B_t^H)\|_{-1} \leq Ct^{-H}.
$$

(30)

By assumption the first distribution derivative $F'$ of $F$ is of function type. Moreover, $F'(x) \exp\{-x^2/(4b^{2H})\} \in L^1(\mathbb{R})$ because $F' \in \mathcal{S}'(\mathbb{R})$. Consequently for $0 \leq t \leq b$:

$$
\left\langle F', \frac{\tilde{z}_n}{\sqrt{2t^H}} \exp\left\{-\frac{x^2}{4t^{2H}}\right\}\right\rangle \leq \max_{y \in \mathbb{R}} |\tilde{z}_n(y)|^2 \left( \int_{\mathbb{R}} |F'(x)| \exp\left\{-\frac{x^2}{4b^{2H}}\right\} dx \right)^2 \leq \tilde{C},
$$

where $\tilde{C}$ is a constant depending on $F$ and $H$, but not on $n$ (see the estimates for the Hermite functions). (30) now follows as in the proof of (21). Following the proof of Theorem 3.12 we get for $q$ sufficiently large and a suitable constant $\tilde{C}$ independent of $t$:

$$
\left| S(F'(B_t^H) \diamond W_t^H)(\eta) \right| \leq \tilde{C}t^{-H} \exp\left\{ \frac{1}{2} |A^q\eta|_0^2 \right\}.
$$

Thus, by Theorem 3.10 $\int_0^b F'(B_t^H) dB_t^H$ exists and in $(S)^*$:

$$
\lim_{a \to 0^+} \int_a^b F'(B_t^H) dB_t^H = \int_0^b F'(B_t^H) dB_t^H.
$$

Step 3: Proof of the assertion: By the growth condition $F \in \mathcal{S}'(\mathbb{R})$ and so Corollary 4.3 yields for $0 < a < b$:

$$
F(B_b^H) - F(B_a^H) - \int_a^b F'(B_t^H) dB_t^H = H \int_a^b t^{2H-1} F''(B_t^H) dt.
$$

By steps 1 and 2 the left hand side converges in $(S)^*$ as $a > 0$ tends to zero. Hence, the right hand side converges, too, and by the definition of the white noise integral we get in $(S)^*$

$$
\lim_{a \to 0^+} H \int_a^b t^{2H-1} F''(B_t^H) dt = H \int_0^b t^{2H-1} F''(B_t^H) dt.
$$

Combining this with steps 1 and 2 we see that the asserted equation holds in $(S)^*$. But the left hand side is in $(L^2)$ by (29) and the proof is finished. 

Remark 4.6. In the above theorem the identity holds in $(L^2)$ in the sense that both members are in $(L^2)$. To ensure that all terms are elements of $(L^2)$ one can impose an additional assumption: If $F''(B_t^H) \in (L^2)$ and

$$
\int_0^b t^{2H-1} \|F''(B_t^H)\|_0^2 dt < \infty,
$$

(31)

then the last integral on the right hand side is an $(L^2)$-valued Bochner integral. Thus, all terms in (28) are in $(L^2)$ under this additional assumption. One can easily check,
that (31) holds, if $F''$ is of function type and of polynomial growth. In that case the last integral in (28) coincides with the pathwise integral.

Moreover, the proof of Theorem 4.1 can easily be modified to hold in the following situation: $F \in C^1_b(R)$ and

$$\max \left\{ \left| F(t,x) \right|, \left| \frac{\partial}{\partial t} F(t,x) \right|, \left| \frac{\partial}{\partial x} F(t,x) \right|, \left| \frac{\partial^2}{\partial x^2} F(t,x) \right| \right\} \leq c \exp\{\lambda x^2\} \quad (32)$$

for some $\lambda < \frac{1}{4} b^{-2H}$. Under this assumptions the Itô formula in the form of Theorem 4.1 is valid and all terms are elements of $(L^2)$. However, note that $F$ need not to be a tempered distribution in that situation. More details can be found in a forthcoming paper.

**Remark 4.7.** Several versions of Itô’s formula for functionals of a fractional Brownian motion can be found in the literature, see Alòs et al. (2001), Carmona et al. (2000), Coutin et al. (2001), Decreusefond and Üstünel (1998), Duncan et al. (2000), Hu and Øksendal (2001), Privault (1998). We now compare our Itô formula with the most general of those known results.

To our best knowledge the only Itô formula that holds for arbitrary Hurst coefficient was proven by Privault (1998). But his definition of the integral with respect to a fractional Brownian motion is rather complicated and unusual for $H < \frac{1}{2}$. In that case it is defined as a limit of stochastic integrals with respect to sequences of other processes that fulfill suitable conditions and converge to a fractional Brownian motion in a suitable sense. It is not quite clear under what conditions $F'(BH_t)$ is integrable in that sense, which is, of course, supposed in Privault’s Itô formula. Moreover Privault assumes $F \in C^2_b$, i.e. $F$ bounded, twice classically differentiable and with bounded derivatives up to order 2.

Using the Malliavin calculus approach Alòs et al. (2001) proved the Itô formula (28) for $H > \frac{1}{4}$ if $F$ is twice differentiable in the classical sense and fulfills the growth condition (compare with (32)):

$$\max \{ \left| F(x) \right|, \left| F'(x) \right|, \left| F''(x) \right| \} \leq c \exp\{\lambda x^2\} \quad (33)$$

for some $\lambda < \frac{1}{4} T^{-2H}$, fixed $T \in \mathbb{R}$ and $b \leq T$. There is some price to be paid for the weaker growth condition: (i) The definition of the integral with respect to $B^H_t$ depends on $T$; (ii) The Itô formula by Alòs et al. (2001) requires classical differentiability.

For $H > \frac{1}{6}$ there is a related formula by Carmona et al. (2000). Here $F \in C^2_b$ is supposed. But the formula looks much more complicated than (28).

There are two versions of Itô’s formula for fractional Brownian functionals that assume differentiability in the sense of distributions only: The result by Hu and Øksendal (2001) requires $H > \frac{1}{2}$, $F$ convex and of polynomial growth. Coutin et al. (2001) assume $H > \frac{1}{3}$, $F$ convex and $F'$ bounded.

If we apply Theorem 4.4 to $|x-a|$ we get the following generalization of the Tanaka formula that was also proven in Hu and Øksendal (2001) in the case $\frac{1}{2} < H < 1$ and in Coutin et al. (2001) for $\frac{1}{3} < H < 1$ by different methods, that cannot be extended to the general case:
Corollary 4.8. Let $0 < H < 1$, $a \in \mathbb{R}$ and $T > 0$. Then in $(L^2)$:

$$|B_t^H - a| = |a| + \int_0^T \text{sign}(B_t^H - a) dB_t^H + 2H \int_0^T t^{2H-1} \delta_a(B_t^H) dt.$$  

Note, that by Hu and Øksendal (2001)

$$\int_0^T t^{2H-1} \delta_a(B_t^H) dt \in (L^2).$$

Thus, all terms in the above corollary are elements of $(L^2)$.

4.3. Integral representation for Donsker’s delta function

In this section we prove the following Clark–Ocone type integral representation of Donsker’s delta function $\delta_a(B_t^H)$:

Theorem 4.9. Let $0 < H < 1$, $a \in \mathbb{R}$ and $T > 0$. Then in $(S)^*$:

$$\delta_a(B_T^H) = \frac{1}{\sqrt{2\pi T^H}} \exp \left\{ -\frac{a^2}{2T^H} \right\} + \int_0^T \frac{\partial}{\partial x} (V(t, B_t^H)) dB_t^H,$$

where

$$V(t, x) = \frac{1}{\sqrt{2\pi (T^{2H} - t^{2H})}} \exp \left\{ -\frac{(x - a)^2}{2(T^{2H} - t^{2H})} \right\}.$$  

The proof uses the fundamental solution of the heat equation. It is in the spirit of Ma et al. (1994) where Clark–Ocone formulas for functionals of a Brownian motion have been obtained via a PDE approach.

Proof. Let

$$u(t) := \frac{1}{\sqrt{4\pi t}} \exp \left\{ -\frac{(t - a)^2}{4t} \right\}; \quad t > 0$$

and $u(0) := \delta_a$. Then it is well known (see e.g. Taylor (1996, p. 217)) that $u \in C^\infty([0, \infty), \mathscr{S}'(\mathbb{R}))$ and $u$ solves the heat equation

$$\frac{\partial}{\partial t} u(t) - \frac{\partial^2}{\partial x^2} u(t) = 0; \quad u(0) = \delta_a.$$  

As $(\partial/\partial t) u(0) = \delta_a''$ and $\delta_a^{(n)} \in \mathscr{S}_p(\mathbb{R})$ holds for all $p \geq n/2 + 1$ we see, that $u \in C^1([0, \infty), \mathscr{S}'_{-2}(\mathbb{R}))$. Moreover, $(\partial/\partial x) u(t), (\partial^2/\partial x^2) u(t) \in C^{1}([0, \infty), \mathscr{S}_{-2}(\mathbb{R}))$.

Define $s(T, t) := \frac{1}{2}(T^{2H} - t^{2H})$. Then $V(t) = u(s(T, t))$. Hence, the assumptions of Theorem 4.1 are fulfilled for $v$ and $v$ solves for $0 \leq t \leq T$

$$\frac{\partial}{\partial t} v(t) + H T^{2H-1} \frac{\partial^2}{\partial x^2} v(t) = 0; \quad u(T) = \delta_a.$$  

Applying Theorem 4.1 we obtain for arbitrary $\epsilon > 0$:

$$
\delta_\epsilon(B^H_t) - v(\epsilon, B^H_t) = \int_\epsilon^T \frac{\partial}{\partial s} v(t, B^H_t) \, dt + \int_\epsilon^T \frac{\partial}{\partial x} v(t, B^H_t) \, dB^H_t
$$

$$
H \int_\epsilon^T t^{2H-1} \frac{\partial^2}{\partial x^2} v(t, B^H_t) \, dt
$$

$$
= \int_\epsilon^T \frac{\partial}{\partial x} v(t, B^H_t) \, dB^H_t.
$$

As $v(\epsilon, B^H_t)$ converges to $v(0,0)$ in $(L^2)$ as $0 < \epsilon$ tends to zero, the assertion can easily be obtained by letting $\epsilon \to 0+$. $\square$

4.4. Applications to the local time

The local time is the amount of time that the fractional Brownian motion spends at a point $a$:

**Definition 4.10.** The local time of a fractional Brownian motion $B^H_t$ at a point $a \in \mathbb{R}$ up to time $\tau > 0$ is defined by the property

$$
l^H_\tau(a) = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \lambda(\{s \in [0, \tau]; B^H_s \in (a - \epsilon, a + \epsilon)\}).
$$

Here $\lambda$ denotes the Lebesgue measure and the limit is to be taken in $(S)^*$. Using Fubini’s theorem one can check (see Hu and Øksendal, 2000, Proposition 4.9 for the details), that the local time of a fractional Brownian motion is given by

$$
l^H_\tau(a) = \int_0^\tau \delta_\epsilon(B^H_s) \, ds. \quad (34)
$$

Moreover, the nondeterminism property of the fractional Brownian motion (Berman, 1991) and the following representation (Kuo, 1996, p. 251)

$$
\delta_\epsilon(B^H_t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i u (B^H_t - a)} \, du
$$

imply, that $l^H_\tau(a) \in (L^2)$. Again the details can be found in Hu and Øksendal (2000).

In the case of a Brownian motion the Tanaka formula provides a representation of the local time. But for the case $H \neq \frac{1}{2}$ the analogous formula (Corollary 4.8) only admits a representation for a weighted local time. We shall now give two new representations of the local time of a fractional Brownian motion:

**Theorem 4.11.** Let $0 < H < 1$, $H \neq \frac{1}{2}$. Then in $(L^2)$:

$$
l^H_\tau(a) = \frac{1}{2H} \lim_{\epsilon \to 0^+} \left[ (2H - 1) \int_\epsilon^\tau t^{-2H} |B^H_t - a| \, dt
$$

$$
- \int_\epsilon^\tau t^{-2H} \text{sign}(B^H_t - a) \, dB^H_t + \epsilon^{-2H} |B^H_\tau - a| - \epsilon^{1-2H} |B^H_\epsilon - a| \right].
$$
Proof. Applying Theorem 4.1 to $F(t,x) := t^{1-2H}|x-a|$ implies:

$$2H \int_0^\tau \delta_a(B^H_s) \, ds = \tau^{1-2H}|B^H_\tau - a| - \tau^{1-2H}|B^H_0 - a| + (2H - 1) \int_0^\tau t^{-2H}|B^H_t - a| \, dt$$

$$- \int_0^\tau t^{1-2H} \text{sign}(B^H_t - a) \, dB^H_t.$$

But by the considerations at the beginning of this section the left hand side converges in $(L^2)$ to $2H l_H^H(a)$ as $\varepsilon \to 0^+$. \qed

In the case $0 < H < \frac{1}{2}$ one can check that all summands on the right hand side converge in $(S)^*$ as $\varepsilon \to 0^+$. Thus, we can obtain another analogue of Tanaka’s formula for $0 < H < \frac{1}{2}$:

**Corollary 4.12.** Let $0 < H < \frac{1}{2}$. Then in $(L^2)$:

$$2H l_H^H(a) = \tau^{1-2H}|B^H_\tau - a| + (2H - 1) \int_0^\tau t^{-2H}|B^H_t - a| \, dt$$

$$- \int_0^\tau t^{1-2H} \text{sign}(B^H_t - a) \, dB^H_t.$$

The following representation holds for all Hurst parameters:

**Theorem 4.13.** Let $0 < H < 1$. Then in $(L^2)$:

$$l_H^H(a) = \frac{1}{\sqrt{2\pi}} \int_0^\tau u^{-H} \exp \left\{ -\frac{a^2}{2u^{2H}} \right\} \, du + \int_0^\tau \int_0^u \frac{\partial}{\partial x} v^{(T)}(t,B^H_t) \, dB^H_t \, du,$$

where

$$v^{(T)}(t,x) = \frac{1}{\sqrt{2\pi(T^{2H} - t^{2H})}} \exp \left\{ -\frac{(x-a)^2}{2(T^{2H} - t^{2H})} \right\}.$$

**Proof.** Integrate the integral representation of Donsker’s delta function (Theorem 4.9) over $[0,\tau]$ and use the results at the beginning of this section. \qed

5. Conclusion

Although several versions of Itô’s formula for functionals of a fractional Brownian motion can be found in the literature (see Alôs et al. (2001), Carmona et al. (2000), Coutin et al. (2001), Decreusefond and Üstünel (1998), Duncan et al. (2000), Hu and Øksendal (2001), Privault (1998)), all these formulas need very restrictive assumptions, see Remark 4.7. In particular, none of these formulas holds for the case $H < \frac{1}{4}$ under assumptions that can be easily checked.
In this paper we used white noise techniques to prove an Itô formula for generalized functionals of a fractional Brownian motion that holds for arbitrary Hurst coefficients. As we worked with tempered distributions and distributional derivatives, we did not have to impose smoothness conditions. Moreover we could replace the boundedness conditions, which are supposed in most papers, by polynomial growth. Due to the weak assumptions of our Itô formula it was possible to prove new extensions of the Tanaka formula and new applications to the local time.

References

Alós, E., Mazet, O., Nualart, D., 2000. Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than \( \frac{1}{2} \). Stochastic Process. Appl. 86, 121–139.


