J. Math. Anal. Appl. 348 (2008) 834–842

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Uncountable products of determined groups need not be determined

Salvador Hernández [a,1], Sergio Macario [a,*,1], F. Javier Trigos-Arrieta [b,2]

[a] Universitat Jaume I, Departamento de Matemáticas, Campus de Riu Sec, 12071 Castellón, Spain
[b] Department of Mathematics, California State University, Bakersfield, 9011 Stockdale Highway, Bakersfield, CA 93311-1099, USA

ARTICLE INFO

Article history:
Received 1 February 2008
Available online 5 August 2008
Submitted by B. Cascales

Keywords:
Dual group
Pontryagin–van Kampen duality
Außenhofer–Chasco theorem
Compact-open topology
Dense subgroup
Determined group
Metrizable group
Reflexive group
Compact group
Rosenthal compact
Determined locally convex spaces
Direct sum
Direct product

ABSTRACT

If H is a dense subgroup of G, we say that H determines G if their groups of characters are topologically isomorphic when equipped with the compact open topology. If every dense subgroup of G determines G, then we say that G is determined. The importance of this property is justified by the recent generalizations of Pontryagin–van Kampen duality to wider classes of topological Abelian groups. Among other results, we show (a) $\bigoplus_{i\in I}\mathbb{R}$ determines the product $\prod_{i\in I}\mathbb{R}$ if and only if I is countable, (b) a compact group is determined if and only if its weight is countable. These answer questions of Comfort, Raczkowski and the third listed author. Generalizations of the above results are also given.

1. Introduction and motivation

If (G,t) is an Abelian topological group, with underlying group G and topology t, a *character of* G is a t-continuous group homomorphism from G into the unit circle $\mathbb{T}$, the latter equipped with the usual product as composition law and with the topology inherited from the complex plane. Thus $\widehat{G}$, *the character group of* G, is defined by

$$\widehat{G} := \{h : G \to \mathbb{T} \mid h \text{ is a character}\},$$

with group operation defined pointwise:

$$(h_1h_2)(x) := h_1(x)h_2(x), \quad \forall x \in G.$$

The topology on $\widehat{G}$ of *uniform convergence on the compact sets*, denoted by $\tau_c(G)$ (τ_c when there is no possibility of confusion and also known as the compact-open topology) is the topology whose basic open sets are of the form

* Corresponding author.
E-mail addresses: hernande@mat.uji.es (S. Hernández), macario@mat.uji.es (S. Macario), jtrigos@csub.edu (F.J. Trigos-Arrieta).

[1] The first two authors acknowledge partial financial support by the Spanish Ministry of Science (including FEDER funds), grant MTM2008-04599/MTM; and the Generalitat Valenciana, grant GV/2007/012.
[2] The third listed author is pleased to acknowledge hospitality and support from the Department of Mathematics of the Universitat Jaume I in November 2007.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.07.065

$$(K, O) := \{h \in \widehat{G}:\ h[K] \subseteq O\},$$

where $K \subseteq G$ is compact and $O \subseteq \mathbb{T}$ is open. It follows that $(\widehat{G}, \tau_c)$ is an Abelian topological group.

We say that G satisfies *Pontryagin–van Kampen duality* (P-duality in short) if the evaluation map

$$\Omega : G \to \big(\widehat{(\widehat{G}, \tau_c)}, \tau_c\big),$$

defined by

$$\Omega(g)(h) := h(g), \quad \forall g \in G,$$

is a surjective topological isomorphism. The celebrated theorem of *Pontryagin–van Kampen* states that every locally compact Abelian group satisfies P-duality. If G satisfies P-duality, we say that G is P-*reflexive,* but if Ω above is only onto, then we say that G is P-*semireflexive.* The class of P-reflexive groups contains also some non-locally compact groups since it is closed under arbitrary products, as proved by Kaplan [12]. In fact, the problem of obtaining wider classes of groups satisfying P-duality has received a lot of attention in the last years. For example, groups satisfying P-duality have been recently characterized in [9]. For a characterization of semireflexive groups, see Theorem 1.1 at the end of this section.

Recall that if H is a dense subgroup of G, then H *determines* G if the topological groups $\widehat{H}$ and $\widehat{G}$ are topologically isomorphic, when equipped with the compact open topology. If every dense subgroup of G determines G, then we say that G is *determined*. Obviously, groups without proper dense subgroups are trivially determined [6]. On the other hand, Aussenhofer [1] and Chasco [3] proved independently that metric groups are determined (for metrizable locally convex spaces, this result is due to Banaszczyk [2, 17.4]). Comfort, Raczkowski and the third-listed author [6] proved that determined groups have very poor permanence properties. For example, (a) "natural" dense subgroups determine the product under special circumstances; to be more precise, it is known that a direct sum determines the product if the latter satisfies a somewhat artificial property [6, (3.7)]; (b) a group containing an open determined subgroup does not need to be determined; (c) products of $\mathfrak{c}$-many or more non-trivial compact metric groups are not determined; furthermore, a compact group of weight equal or larger than $\mu :=$ the smallest cardinal of any set of the reals with positive outer measure, is not determined. So, under CH, any compact group of weight $\aleph_1$ or larger is not determined. The question then is what happens for cardinals κ satisfying $\aleph_1 \leqslant \kappa < \mu$. Another elusive problem which is still open, is whether the product of two determined groups is determined.

Applying results from [4], one shows that *compactly generated* LCA groups of weight κ are determined if, and only if, compact groups of weight κ are determined. A not compactly generated LCA group can be determined and not metric, as witnessed in 3.4(iii) of [6]; such groups do not have proper dense subgroups. Examples of not compactly generated, determined, LCA groups with proper dense subgroups that are not metric are constructed in [4] where permanence properties—or lack thereof—of determined groups are studied.

In this article, we prove the following:

(1) The direct sum of countably many determined groups determines the product (Theorem 3.2).
(2) The direct sum of real lines determines the product if and only if the product is countable (Theorem 4.5).
(3) The direct sum of copies of integers determines the product if and only if the product is countable (Theorem 4.8).
(4) A compact group is determined if and only if it is metrizable (Corollary 5.11).

These results solve Questions 874–876 in [5], 7.2, 7.5 and 7.6 of [6] and 3.10 of [17].

A topological vector space equipped with the sum as operation is *a priori* an Abelian topological group. As such, it is natural to ask whether it is P-reflexive or at least P-semireflexive. The first attempt in doing so was done by Smith [16] who proved that real Banach spaces and real reflexive locally convex spaces are P-reflexive. Some other authors have made contributions to the Smith's program and, as a consequence of their results, we get the following characterization of P-semireflexivity for locally convex spaces (LCS) (see [7,9–11]).

Theorem 1.1. *For every locally convex vector space E the following conditions are equivalent*:

(1) *E is P-semireflexive*;
(2) *the closed convex hull of every compact subset of E is compact*;
(3) *the closed convex hull of every compact subset of E is weakly compact.*

2. Notation and preliminaries

We consider only topologies that are Hausdorff and completely regular. To avoid unpleasant pathologies, we restrict ourselves to groups with sufficiently many characters, i.e., to the class MAPA. We try to follow the notation of the treatise [13], regarding locally convex spaces: E denotes a linear space over the real numbers $\mathbb{R}$, and t denotes a Hausdorff topology on E such that $E[t]$ is a locally convex vector space. E' denotes the (real) vector space of all continuous linear functions from $E[t]$ to $\mathbb{R}$. The symbol σ denotes the weak topologies on E and E'. The symbol $\tau_c(E)$ (τ_c when there is no possibility of confusion) denotes the locally convex topology on E' with zero neighborhoods given by the polars of the t-compact subsets of E: If $K \subseteq E$, its polar K^0 is defined as $\{f \in E'\colon\ |f(x)| \leqslant 1,\ \forall x \in K\}$.

The symbol (G,t) denotes the Abelian group G, equipped with the group topology t. Thus if $E[t]$ denotes a locally convex space, then (E,t) denotes the topological group obtained by considering E as a group under the sum operation.

The extension of the theory of locally convex spaces to topological Abelian groups presents difficulties. However many notions of the theory of locally convex spaces are still preserved in the passage to topological Abelian groups if the role of $\mathbb{R}$ in functionals is played by the unit circle $\mathbb{T}$, i.e., the range of characters. To simplify matters, $\mathbb{T}$ will be identified with the interval $[-1/2, 1/2)$ equipped with the canonical quotient topology of $\mathbb{R}/\mathbb{Z}$. Thus, given a topological Abelian group (G,t) with dual $X := \widehat{(G,t)}$, for any subset A of G, we define $U(A,\epsilon) = \{\chi \in X\colon\ |\chi(g)| \leqslant \epsilon,\ \forall g \in A\}$ and $A^0 := U(A, 1/4)$. Assuming that we are considering the *dual pair* (G,X), for any subset L of X, we define $L^0 := \{g \in G\colon\ |\chi(g)| \leqslant 1/4,\ \forall \chi \in L\}$. This set operator behaves in many aspects like the polar operator in vector spaces. For instance, it is easily checked that $A^{000} = A^0$ for any $A \subseteq G$. Given an arbitrary subset A of G, we define the *quasi convex hull* of A, denoted $co(A)$, as the set A^{00}. A set A is said to be *quasi convex* when it coincides with its quasi convex hull. These definitions also apply to subsets L of X. The topological group (G,t) is said to be *locally quasi convex* when there is a neighborhood base of the identity consisting of quasi convex sets. Considering the dual pair (G,X) again, the symbols $\tau_c(G)$ and $\sigma(G)$, or simply τ_c and σ if confusion is impossible, denote the topologies of the uniform convergence on compact and finite sets, respectively. As in the locally convex space case, it is readily verified that σ is the weakest locally quasi convex topology on G whose dual group is X. Notice that (G,σ) is always a precompact group, i.e., its completion is a compact group. We denote by $b(G,t)$ the completion of $(G,\sigma(G,X))$. The group $b(G,t)$, called *the Bohr compactification* of (G,t), is always compact. Sometimes it is useful to consider the dual pair (bG,X). Thus, we will denote by $co_{bG}(A)$ the quasi convex hull of A in bG.

Now consider the space $E[t]$ as the group (E,t). It is a theorem of Smith [16] that E' can be identified, as a group, with the dual group X of (E,t), through the map $f \mapsto \pi \circ f$ where $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the natural projection. Waterhouse has shown that (E', τ_c) and (X, τ_c) are topologically isomorphic groups [18, Theorem 2]. Since (E',σ) is obtained from $E'[\sigma]$ and (X,σ) is precompact (as a topological subgroup of $\mathbb{T}^E$), it follows that (E',σ) and (X,σ) are never topologically isomorphic. Therefore we must be careful whether we are using σ on E' or in X.

The following characterization of determining groups is found in [6].

Proposition 2.1. *Let G be a topological group and let D be a dense subgroup of G. The following assertions are equivalent*:

(1) *D determines G.*
(2) *For every compact subset K in G there is a compact subset E in D such that $E^0 \subseteq K^0$.*

Next corollary follows directly from this proposition.

Corollary 2.2. *Let G be a topological group and let D be a dense subgroup of G. The following assertions are equivalent*:

(1) *D determines G.*
(2) *For every compact subset K in G there is a compact subset C in D such that $K \subseteq co_G(C)$.*

As a consequence of the results above, it follows that P-reflexive groups do not determine the groups where they are embedded (see [6, Theorem 5.2]).

Corollary 2.3. *Let G be a locally quasi-convex group and let D be any P-semireflexive, dense, subgroup of G. Then D does not determine G.*

Proof. Since D is P-semireflexive, for every compact subset C of D it holds that $co(C) = co_{bD}(C)$ (see [9, Theorem 3]). On the other hand, since D is dense in G, it follows that bG is a group compactification of D and $co(C) = co_{bG}(C)$. Hence, the quasi-convex hull in G of any compact subset of D is contained in D as well. By Corollary 2.2, we obtain that D does not determine G. □

Let κ be a cardinal such that $\mathrm{cf}(\kappa) > \omega$. Identifying the cardinal κ, as a set, to the initial ordinal of cardinality κ, the symbol X_κ denotes a topological space with the following properties: $X_\kappa = \kappa \cup \{\infty\}$, all points of κ are isolated, and the neighborhoods of ∞ are of the form $E \cup \{\infty\}$, where $|X_\kappa \setminus E| < \kappa$. It is readily seen that X_κ is a P-space. We assume, without loss of generality, that $C_p(X_\kappa)$ is a subspace of $\mathbb{R}^\kappa$ from here on.

From [11, Example 3.11] we have

Corollary 2.4. *There exist cardinal numbers $\kappa > \mathfrak{c}$ such that $C_p(X_\kappa)$ is P-reflexive. As a consequence it follows that $C_p(X_\kappa)$ does not determine $\mathbb{R}^\kappa$.*

Since $\bigoplus_\kappa \mathbb{R} \subseteq C_p(X_\kappa)$, we obtain that there exist $\kappa > \mathfrak{c}$ such that $\bigoplus_\kappa \mathbb{R}$ does not determine $\mathbb{R}^\kappa$. In Theorem 4.5 below, we will extend this result to $\kappa = \omega_1$.

The following lemma, whose proof is left to the reader, will be used in the sequel.

Lemma 2.5. *Let G be a topological Abelian group. If $\bigoplus_\kappa G$ does not determine the product G^κ, then $\bigoplus_\beta G$ does not determine G^β, for all $\beta > \kappa$.*

3. Countable products

In [6] the authors ask for the following question: If $\{G_i\colon i \in I\}$ is a set of topological Abelian groups and D_i is a determining subgroup of G_i, must $\bigoplus_{i\in I} D_i$ determine $\prod_{i\in I} G_i$? In this section, we show that the answer is positive for countable products. Firstly, we need the following auxiliary result.

Lemma 3.1. *Let j_n be the natural injection of D_n into the product $\prod_{n<\omega} D_n$. If C_n is a compact set of D_n for every $n < \omega$ then $C = \bigcup_{n<\omega} j_n(C_n) \cup \{0\}$ is a compact subset of the group $\bigoplus_{n<\omega} D_n$, equipped with the product topology.*

Proof. Let U be a basic neighborhood of the identity in $\bigoplus_{n<\omega} D_n$. Then $U = \prod_{i<\omega} U_i$ with U_i a neighborhood of the identity in D_i, for every $i < \omega$, and $U_i \neq D_i$ only for a finite number of indices i, say J.

Now we can write

$$C = \bigcup_{i\in J} j_i(C_i) \cup \bigcup_{i\notin J} j_i(C_i) \cup \{0\} \subseteq \bigcup_{i\in J} j_i(C_i) \cup U.$$

So, being $j_i(C_i)$ a compact subset and J a finite set, it follows that every covering of C by open sets admits a finite subcovering. This shows that C is a compact subset of $\bigoplus_{n<\omega} D_n$. □

Theorem 3.2. *Let $\{G_n\colon n < \omega\}$ be a family of topological Abelian groups where every G_n has a determining subgroup D_n. Then, the direct sum $\bigoplus_{n<\omega} D_n$ determines the product $\prod_{n<\omega} G_n$.*

Proof. Let K be a compact subset of the product $\prod_{n<\omega} G_n$. For our purposes, we can suppose that $K = \prod_{n<\omega} K_n$ with K_n a compact subset of G_n. We need to find a compact subset E of the direct sum such that $E^0 \subseteq K^0$.

Since each subgroup D_n determines G_n we have that there exists a compact subset C_n in D_n such that $U(C_n, \frac{1}{4\cdot 2^n}) \subseteq U(K_n, \frac{1}{4\cdot 2^n})$.

Put $E_n = C_n \cup 2C_n \cup \cdots \cup 2^n C_n$ and $E = \bigcup_{n<\omega} j_n(E_n)$. By Lemma 3.1, E is compact in $\bigoplus_{n<\omega} D_n$. We have the following trivial facts:

- $E_n^0 \subseteq U(C_n, \frac{1}{4\cdot 2^n}) \subseteq U(K_n, \frac{1}{4\cdot 2^n})$.
- If $\chi \in (j_n(E_n))^0$, then $\chi \circ j_n \in E_n^0$.
- $E^0 = (\bigcup_{n<\omega} j_n(E_n))^0 = \bigcap_{n<\omega} (j_n(E_n))^0$.

Now, take $\chi \in E^0$ and $(x_n) \in K = \prod_{n<\omega} K_n$. We can identify χ with an element $(\chi_n) \in \bigoplus_{n<\omega} \widehat{G_n}$ with only a finite number of coordinates different from identity. Let P_χ be the set of its non-trivial coordinates. Then

$$\left|\chi(x_n)\right| = \left|\sum_{n\in P_\chi} \chi_n(x_n)\right| = \left|\sum_{n\in P_\chi} \chi\bigl(j_n(x_n)\bigr)\right| \leqslant \sum_{n\in P_\chi} \left|(\chi \circ j_n)(x_n)\right| \leqslant \sum_{n\in P_\chi} \frac{1}{2^n}\frac{1}{4} \leqslant \frac{1}{4}. \qquad \square$$

4. Locally convex spaces

The notions introduced in the previous sections have also meaning in the context of locally convex vector spaces. Thus, if E is a locally convex vector space and F is a dense subspace, we say that F determines E, as locally convex space, when the *vector* topologies $\tau_c(E)$ and $\tau_c(F)$ coincide on the dual space E'. Taking into account the isomorphisms of the groups (E', τ_c) and $(\widehat{E}, \tau_c)$, we deduce the following equivalence.

Proposition 4.1. *If E is a locally convex space and F a dense subspace, the following conditions are equivalent*:

(1) *F determines E as locally convex space*;
(2) *F determines E as topological group.*

Moreover, for locally convex spaces, the Hahn–Banach theorem allows us to give a stronger version of Corollary 2.2.

Corollary 4.2. *Let E be a locally convex space and let F be a dense subspace of E. The following assertions are equivalent*:

(1) *F determines E.*
(2) *For every compact subset K of E there is a compact subset C of F such that $K^{00} \subseteq C^{00}$ (the bipolar being taken in E).*
(3) *For every compact subset K of E there is a compact subset C of F such that K is contained in the closed convex hull of C in E.*

Proof. The equivalence between (1) and (2) is trivial. The equivalence between (2) and (3) follows from the fact that C^{00} is just the closed absolutely convex cover of C and it coincides with the closed convex cover of the circled cover of C (see [13, §16, (2)]). □

If K is compact, we denote by $M(K)$ the space of regular Borel measures on K, that is, the dual space of the Banach space $C(K)$. The canonical pairing $C(K) \times M(K) \to \mathbb{R}$ will be denoted by $\langle \cdot, \cdot \rangle$. Thus $\langle f, \mu \rangle = \int_K f\,d\mu$. Let $C_\omega(K)$ be the Banach space $C(K)$ equipped with the weak topology. We consider the weak* topology on $M(K)$. The closed convex hull of K in $M(K)$ is the space $P(K)$ of all probability measures on K. The space $P(K)$ is compact and has the following universal property: if Φ is a map of K to an LCS E such that the closed convex hull L of $\Phi(K)$ is compact, then the integral $\Psi(\mu) = \int_K \Phi\,d\mu \in E$ is defined for every $\mu \in M(K)$, and the linear map $\Psi : M(K) \to E$ extends Φ and sends $P(K)$ onto L.

Next we are going to apply the considerations above to the case $E = \mathbb{R}^{X_\kappa}$ and $F = C_p(X_\kappa)$, with $\kappa = \omega_1$. We denote by $C_p^0(X_\kappa)$ the subspace of $C_p(X_\kappa)$ consisting of the functions f such that $f(\infty) = 0$.

Proposition 4.3. *Let K be a compact subset of $C_p^0(X_\kappa)$ such that all measures in $P(K)$ have separable support. If $\kappa = \omega_1$, then $co_E(K) \subseteq C_p(X_\kappa)$.*

Proof. By the Hahn–Banach Theorem, it will suffice to prove that the closed convex hull of K in $C_p(X_\kappa)$ is compact. Let $j : K \to C_p(X_\kappa)$ be the canonical inclusion and let $j^* : X_\kappa \to C(K)$ be its adjoint map defined by $j^*(x)(f) = f(x)$ for all $f \in K$ and $x \in X_\kappa$. According to [11, Theorem 3.7], we must verify that the function $x \mapsto \langle j^*(x), \mu \rangle$ is continuous on X_κ for every $\mu \in P(K)$. Moreover, since K is contained in $C_p^0(X_\kappa)$, it follows that $j^*(\infty)$ is equal to zero throughout K. Thus, all we have to do is to show that the net $\{\langle j^*(x), \mu \rangle\}_{x \in \kappa}$ converges to zero. Set $V_x = \{f \in K\colon\ j^*(x)(f) = f(x) \neq 0\}$ for all $x \in \kappa$. We have a collection of open subsets $\{V_x\colon\ x \in \kappa\}$ in K of cardinality $\aleph_1$. Observe that, since X_κ is a P-space and $K \subseteq C_p(X_\kappa)$, for every $f \in K$ there is a countable subset S_f of κ such that $f(x) = 0$ for all $x \in \kappa \setminus S_f$. That is to say, every f in K belongs to countably many subsets V_x at most, namely, those V_x for which $x \in S_f$. Thus, if N is a countable dense subset of $\operatorname{supp}\mu$, it follows that N intersects countably many subsets V_x at most. So, there is a countable subset S of κ such that $\operatorname{supp}\mu \cap V_x = \emptyset$ if $x \notin S$. Hence $j^*(x)_{|\operatorname{supp}\mu} = 0$ for all $x \in \kappa \setminus S$. As a consequence $\langle j^*(x), \mu \rangle = 0$ for all $x \in \kappa \setminus S$. This completes the proof. □

Remember that a compact Hausdorff space K is a *Rosenthal compact* if K is homeomorphic to a subspace of $\mathbf{B}_1(M)$ (the space of real-valued functions of the first Baire class) for some Polish space M (see [14]). In [8, Proposition 8], Godefroy proves that, if K is Rosenthal compact and μ is a Radon measure on K, then the support of μ is separable. As a consequence we obtain:

Lemma 4.4. *Let K be a space homeomorphic to a compact subspace of $\bigoplus_\kappa \mathbb{R}$ and let μ be a Radon measure on K. If $\kappa \leqslant \mathfrak{c}$, then the support of μ is separable.*

Proof. Let D be a subset of a Polish space M with $|D| = \kappa$. We have $\bigoplus_\kappa \mathbb{R} \cong \bigoplus_D \mathbb{R} \subseteq \bigoplus_M \mathbb{R} \subseteq \mathbf{B}_1(M)$. Thus K is a Rosenthal compact and the proof follows from Godefroy's result mentioned above. □

Theorem 4.5. *The subgroup $\bigoplus_\kappa \mathbb{R}$ determines the group $\mathbb{R}^\kappa$ if and only if $\kappa \leqslant \omega$.*

Proof. We only verify the necessity: First, observe that by Lemma 2.5, we may assume, without loss of generality, that $\kappa = \omega_1$ and that the subgroup $\bigoplus_\kappa \mathbb{R}$ is a dense subgroup of $C_p^0(X_\kappa) \subseteq \mathbb{R}^\kappa$. That is to say, κ is the discrete dense subspace of the Lindelöf space X_κ defined above. By Lemma 4.4, compact subsets of $\bigoplus_\kappa \mathbb{R}$ accept only Radon measures with separable support. Therefore, Proposition 4.3 implies that the closed convex hull of compact subsets of $\bigoplus_\kappa \mathbb{R}$ are always included in $C_p(X_\kappa)$ and, as consequence, they may never contain $[-1, 1]^D$. An application of Corollary 4.2 completes the proof. □

Compare with the fact [6, (3.12)] that $\bigoplus_\kappa \mathbb{T}$ always determines the group $\mathbb{T}^\kappa$.

Corollary 4.6. *If E is a locally convex space and $\kappa \geqslant \omega_1$, then the direct sum $\bigoplus_\kappa E$ does not determine the product E^κ.*

Proof. By Lemma 2.5 it suffices to assume that $\kappa = \omega_1$. Fix an element $e \in E$ and consider the topological isomorphism $f : \langle e \rangle \to \mathbb{R}$. By the Hahn–Banach Theorem this map can be extended to a continuous linear map $\overline{f}$ from E to $\mathbb{R}$. We define now the continuous linear map $\tilde{f}: E^{\omega_1} \to \mathbb{R}^{\omega_1}$ by doing

$$\tilde{f}\big((x_i)_{i\in I}\big) = \big(\overline{f}(x_i)\big)_{i\in I}.$$

Claim. *If K is a compact subset of $\mathbb{R}^{\omega_1}$ there exists a compact subset L of E^{ω_1} such that $\tilde{f}(L) = K$.*

We can assume that $K = \prod_{i\in\omega_1} K_i$, with each K_i a compact subset in $\mathbb{R}$. Then, there exists a compact subset L_i in E with $\overline{f}(L_i) = K_i$. Take the compact subset $L = \prod_{i\in\omega_1} L_i$.

Suppose that $\bigoplus_{\omega_1} E$ determines the product E^{ω_1}. Take a compact subset K in $\mathbb{R}^{\omega_1}$ and determine a compact subset L in E^{ω_1} with $\tilde{f}(L) = K$. There exists a compact subset M in $\bigoplus_{\omega_1} E$ such that $M^\circ \subseteq L^\circ$. Then

$$\tilde{f}(M)^\circ = \big(\tilde{f}^t\big)^{-1}\big(M^\circ\big) \subseteq \big(\tilde{f}^t\big)^{-1}\big(L^\circ\big) = \tilde{f}(L)^\circ = K^\circ$$

with $\tilde{f}^t$ the transpose map. So $\mathbb{R}^{\omega_1}$ should be determined by $\bigoplus_{\omega_1} \mathbb{R}$, which is not true. □

Compare with [6, 3.12]. As a matter of fact, in [6, Corollary 3.10] it is established that the direct sum $\oplus_{i\in I} D_i$ of determining subgroups D_i of the groups G_i determines the product $\prod_{i\in I} G_i$ when the groups G_i have the cofinally zero property. The theorem above shows that this property may not be omitted in general.

As a consequence of this theorem we can also obtain that if $\kappa \geqslant \omega_1$, then $\mathbb{Z}^\kappa$ is not determined by $\bigoplus_\kappa \mathbb{Z}$. We need the following lemma that can be easily verified.

Lemma 4.7. *Let H be a subgroup of a topological group G. Then, for every $K \subseteq H$ we have*

$$co_H(K) \subseteq co_G(K).$$

Theorem 4.8. *If $\kappa \geqslant \omega_1$, then the subgroup $\bigoplus_\kappa \mathbb{Z}$ does not determine the group $\mathbb{Z}^\kappa$.*

Proof. Using Lemma 2.5 again, it suffices to do the proof for $\kappa = \omega_1$. So, let κ and X_κ be as in Theorem 4.5. Then, taking into account that $\bigoplus_\kappa \mathbb{Z}$ is a subgroup of $\bigoplus_\kappa \mathbb{R}$ and applying lemma above we obtain that, for every compact subset K of $\bigoplus_\kappa \mathbb{Z}$, $co_{\mathbb{Z}^\kappa}(K) \subseteq co_{\mathbb{R}^\kappa}(K)$. But K is also a compact subset of $\bigoplus_\kappa \mathbb{R}$ so, applying Proposition 4.3, we have that $co_{\mathbb{R}^\kappa}(K) \subseteq C_p^0(X_\kappa)$. So, the quasi convex hull of compact subsets of $\bigoplus_\kappa \mathbb{Z}$ may never contain compact subsets of the form $[n,m]^\kappa$. Corollary 2.2 applies to complete the proof. □

Using the ideas of Corollary 4.6 and the previous results about $\mathbb{R}^{\omega_1}$ and $\mathbb{Z}^{\omega_1}$, we can obtain:

Corollary 4.9. *Let H be any of the groups $\mathbb{R}$ or $\mathbb{Z}$ and let G be a group such that there exists a continuous projection onto the group H with the property that every compact subset K of H is just the image of a compact subset L from G. If $\kappa \geqslant \omega_1$, then the direct sum $\bigoplus_\kappa G$ does not determine the product G^κ.*

The last two results have been achieved in [4] by different methods.

5. Compact groups

The main result of this section is a characterization of the compact groups that are determined. In order to do this, we prove that the groups $\mathbb{T}^{\omega_1}$ and $\mathbb{Z}(p)^{\omega_1}$ are not determined (here, p is prime and $\mathbb{Z}(p) := \mathbb{Z}/p\mathbb{Z}$). To show that $\mathbb{T}^{\omega_1}$ is not determined, we prove that its (dense) subgroup $\mathbb{Z}(p^\infty)^{\omega_1}$ is not determined either. To do so, we use the fact that $\mathbb{Z}(p^\infty)^{\omega_1}$ determines $\mathbb{T}^{\omega_1}$ [6, (3.10)], i.e., this subgroup contains a compact subset K whose polar K° in $\widehat{\mathbb{T}^{\omega_1}} = \bigoplus_{\omega_1} \mathbb{Z}$ is $\{0\}$. Hence, the group

$$\Big(\bigoplus_{\omega_1} \mathbb{Z}, \tau_c\big(\mathbb{Z}(p^\infty)^{\omega_1}\big)\Big)$$

is equipped with the discrete topology. Remember also that the dual group of $\mathbb{Z}(p)^{\omega_1}$ is $\bigoplus_{\omega_1} \mathbb{Z}(p)$, equipped with the discrete topology. From now on we take $G = F^{\omega_1}$, where F is either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p)$. The group G is separable and, therefore, has a countable dense subgroup, say H. The question is whether such H determines G. It is proven here that H never determines G. Being countable, any compact subspace of a countable subgroup must be metric by a theorem of Smirnov [15].

Lemma 5.1. *If $f: G \to H$ is an onto and continuous homomorphism of topological groups, and $K \subseteq G$, then $K^\circ = \{0\}$ implies $f[K]^\circ = \{0\}$.*

Proof. If $\phi \in f[K]^\circ \setminus \{0\}$, consider $\phi \circ f \in \widehat{G}$. Since there is $y \in H$ with $\phi(y) \neq 0$, if $x \in G$ satisfies $f(x) = y$, then $\phi \circ f(x) = \phi(y) \neq 0$ which implies $\phi \circ f \neq 0$. Hence, for every $z \in K$ we have $|\phi(f(z))| \leqslant 1/4$ or, equivalently, $|(\phi \circ f)(z)| \leqslant 1/4$. Hence $0 \neq (\phi \circ f) \in K^\circ$. This completes the proof. □

Corollary 5.2. *If G is a compact determined group, then every quotient of G is determined as well.*

Corollary 5.3. *If G is any of the groups $\mathbb{Z}(p)^{\omega_1}$ or $\mathbb{Z}(p^\infty)^{\omega_1}$ and F is a finite subset of G whose elements are all torsion, then F° is never* $\{0\}$.

Proof. Let H be the (finite) subgroup generated by F. Then $F^\circ \supseteq H^\perp \supsetneqq \{0\}$. □

Let K be a compact space. Then one can inductively define the (Cantor) α-derivative of K as follows. Set $K^{(0)} = K$. If α is an ordinal and $K^{(\beta)}$ has been defined for each $\beta < \alpha$, then

$$K^{(\alpha)} := \begin{cases} \{x \in K\colon \exists \langle x_n \rangle \subseteq K^{(\beta)}[x_n \to x]\} & \text{if } \alpha = \beta + 1, \\ \bigcap_{\beta<\alpha} K^{(\beta)} & \text{otherwise.} \end{cases}$$

Notice that each derivative is compact, and $\beta < \alpha$ implies $K^{(\beta)} \supseteq K^{(\alpha)}$.

When K is scattered (for instance, if K is countable) there is a minimal $\gamma < \omega_1$ with $K^{(\gamma)}$ finite. We call such γ the *degree* of K.

Lemma 5.4. *If K is a metrizable compact space, the degree of K is γ, and $f : K \to f[K]$ is continuous, then the degree of $f[K]$ is less than or equal to γ. In fact, if $\alpha < \omega_1$ then*

$$\big(f[K]\big)^{(\alpha)} \subseteq f\big[K^{(\alpha)}\big].$$

Moreover, if $f^{-1}(y)$ is finite for all $y \in f[K]$, we have $(f[K])^{(\alpha)} = f[K^{(\alpha)}]$.

Proof. We will verify that the following diagrams hold:

$$\begin{array}{ccccc} K^{(\beta)} & \xleftarrow{\ \supseteq\ } & K^{(\beta+1)} & & \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \\ f[K^{(\beta)}] & \xleftarrow{\ \supseteq\ } & f[K^{(\beta+1)}] & \xleftarrow{\ \supseteq\ } & (f[K])^{(\beta+1)} \end{array}$$

or, if α is a limit ordinal and $\beta < \alpha$, then

$$\begin{array}{ccccc} K^{(\beta)} & \xleftarrow{\ \supseteq\ } & K^{(\alpha)} & & \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \\ f[K^{(\beta)}] & \xleftarrow{\ \supseteq\ } & f[K^{(\alpha)}] & \xleftarrow{\ \supseteq\ } & (f[K])^{(\alpha)} \end{array}$$

Clearly, we must show only that (a) $f[K^{(\beta+1)}] \supseteq (f[K])^{(\beta+1)}$ and (b) $f[K^{(\alpha)}] \supseteq (f[K])^{(\alpha)}$. We use induction.

To prove (a), assume first that $f[K^{(\beta)}] \supseteq (f[K])^{(\beta)}$ and let $y \in (f[K])^{(\beta+1)}$. Then there is $\langle y_n \rangle \subseteq (f[K])^{(\beta)} \subseteq f[K^{(\beta)}]$ such that $y_n \to y$. Choose $x_n \in K^{(\beta)}$ with $f(x_n) = y_n$. Then there is a subsequence $\langle x_{n_j} \rangle$ of $\langle x_n \rangle$ that converges, say to x necessarily in $K^{(\beta+1)}$, and one sees that $f(x) = y$, hence (a) holds.

For (b), let $y \in (f[K])^{(\alpha)}$. By induction hypothesis, $y \in (f[K])^{(\beta)} \subseteq f[K^{(\beta)}]$ for each $\beta < \alpha$. Choose $x_\beta \in K^{(\beta)}$ with $f(x_\beta) = y$. If the set $X := \{x_\beta\colon \beta < \alpha\}$ is finite, find $x \in K^{(\beta)}$ for all $\beta < \alpha$ (as they are nested). If X is infinite, find $x \in X^{(1)}$. In either case, $x \in K^{(\alpha)}$, and one easily proves that $y = f(x)$. As for the case in which $f^{-1}(y)$ is finite for all $y \in f(K)$, it suffices to observe that $f^{-1}(F)$ is finite for all finite subset $F \subseteq f(K)$. □

Suppose that I denotes an index set of cardinality $\aleph_1$. In the rest of this paper we are going to deduce repeatedly that a certain property holds for an uncountable subset of I. Therefore, for the sake of simplicity, given a subset S of a product group $G = F^I$, we will say that S satisfies a property $\mathcal{P}$ *for uncountably many indices* when there is an uncountable subset J of I such that $\pi_J(S)$ satisfies $\mathcal{P}$ in F^J. Here on, the symbol π_J denotes the canonical projection from F^I onto F^J.

The easy verification of next lemma is omitted.

Lemma 5.5. *Let G be any of the groups $\mathbb{Z}(p)^{\omega_1}$ or $\mathbb{Z}(p^\infty)^{\omega_1}$ and let K be a compact subset of G whose degree is γ. Suppose that the subgroup generated by $K^{(\gamma)}$, say H, is finite and let $f : G \to G/H$ be the canonical quotient map. Then $f(K)^{(\gamma)} = \{0\}$.*

Given a group of the form $G = F^I$, we denote by $\Delta_I(F)$ the subset of the product group G consisting of the elements in the diagonal; that is $\Delta_I(F) = \{(x_i)\colon x_i = x_j \text{ for all } i, j \text{ in } I\}$.

Lemma 5.6. *Let S be any finite subset of $G = \mathbb{Z}(p^\infty)^I$, where $|I| = \aleph_1$. Then there is $k < \omega$ such that $S \subseteq \Delta_I(\mathbb{Z}(p^k))$ holds for uncountably many indices in I.*

Proof. Since $|I| = \aleph_1$ and S is finite, there is an uncountable subset $J \subseteq I$ such that $\pi_J(S)$ consists only of torsion elements. That is to say, there is $k < \omega$ such that $\pi_J(S) \subseteq \mathbb{Z}(p^k)^J$. Repeating the same argument again, we obtain an uncountable subset $L \subseteq J$ such that $\pi_L(S) \subseteq \Delta_L(\mathbb{Z}(p^k))$. This completes the proof. □

Lemma 5.7. *Let F be either $\mathbb{Z}(p)$ or $\mathbb{Z}(p^\infty)$, considered as subgroups of $\mathbb{T}$, and let K be a countable compact subspace, whose degree is less than or equal to γ, of the group $G = F^I$, with $|I| = \aleph_1$. If there is an ordinal $\alpha \leqslant \gamma$ such that either $K^{(\alpha)} = \emptyset$ or there is some positive real number $\delta < \min(\frac{\pi}{4}, \frac{1}{p})$ with $K^{(\alpha)} \subset (-\delta, \delta)^I$ for uncountably many indices, then $K^\circ \neq \{0\}$.*

Proof. We use induction on the ordinal α. Given an arbitrary compact subset K of G, if K satisfies the assertion for $\alpha = 0$, then there is $0 < \delta < \min(\frac{\pi}{4}, \frac{1}{p})$ and an uncountable subset A of I such that $\pi_A(K) \subseteq (-\delta, \delta)^A$. It is easily verified that, for the interval $(-\delta, \delta) \subseteq \mathbb{T}$, we have $(-\delta, \delta)^\circ \neq \{0\}$ in $\mathbb{Z}$. Thus, it suffices to observe that the subsets of the form $\prod V_i$, with $V_{i_0} = (-\delta, \delta)^\circ$ for some index i_0 and $V_i = \{0\}$ if $i \neq i_0$, are all contained in K°.

Thus, assume that $K^\circ \neq \{0\}$ for every countable compact subset K of G such that $K^{(\beta)} \subseteq (-\delta, \delta)^I$ for uncountably many indices, for some $\beta < \alpha$ and let K be a countable compact subset of G such that $K^{(\alpha)} \subseteq (-\delta, \delta)^I$ for uncountably many indices. Furthermore, we may assume that $\pi_J(K)^{(\beta)} \neq \emptyset$ for each subset $J \subseteq I$ with $|I \setminus J| < \aleph_1$ and each $\beta < \alpha$. Otherwise, we would have $\pi_J(K)^{(\rho)} = \emptyset$ for some $\rho < \alpha$ and the hypothesis of induction would apply to $\pi_J(K)$. Then use Lemma 5.1.

Suppose firstly that $\alpha = \beta + 1$ and $K^{(\alpha)} \neq \emptyset$. Passing to an appropriate uncountable subset of I, if it were necessary, we may assume WLOG that $K^{(\alpha)} \subseteq (-\delta, \delta)^I$. Hence $\pi_i(K^{(\alpha)}) \subseteq (-\delta, \delta)$ for all $i \in I$. By the continuity of the projections, for every $i \in I$, there is a finite subset $S_i \subseteq K$ such that $\pi_i(K^{(\beta)} \setminus S_i) \subseteq (-\delta, \delta)$. Now, since K is countable, there must be an uncountable subset $J \subseteq I$ and a finite subset $S \subseteq K$ such that $\pi_i(K^{(\beta)} \setminus S) \subseteq (-\delta, \delta)$ for all $i \in J$. Thus

$$\pi_J(K)^{(\beta)} \subseteq \pi_J\bigl(K^{(\beta)}\bigr) \subseteq \pi_J(S) \cup (-\delta, \delta)^J.$$

If we now suppose that $K^{(\alpha)} = \emptyset$, then it follows that $K^{(\beta)}$ is a finite subset, say S, of G. Hence, in either case, applying Lemma 5.6, there are $k < \omega$ and an uncountable subset $L \subseteq J$ such that $\pi_L(S) \subseteq \Delta_L(\mathbb{Z}(p^k)) \subseteq F^L$. Since the group G is topologically isomorphic to F^L, we may assume WLOG that $L = I$. Therefore we have $S \subseteq \Delta_I(\mathbb{Z}(p^k)) \subseteq G$ and $K^{(\beta)} \subseteq S \cup (-\delta, \delta)^I$. Set $H = \Delta_I(\mathbb{Z}(p^k))$, which is a finite subgroup of G containing S. We now consider the canonical quotient map $\pi : G \to G/H$ that we are going to realize as follows. Fix an index $i_0 \in I$ and define $\phi : G \to G$ as $\phi(x_i) = (y_i)$, where $y_{i_0} = x_{i_0}^{p^k}$ and $y_i = x_{i_0}^{-1} x_i$ if $i \neq i_0$. It is readily seen that $\ker\phi = H$. Therefore ϕ can be seen as a realization of π. Take $J = I \setminus \{i_0\}$ and define $C = (\pi_J \circ \phi)(K)$. Observe that, since $I \setminus J$ is finite, we have $\pi_J(K)^{(\beta)} \neq \emptyset$. Thus $C^{(\beta)} = \pi_J(\phi(K))^{(\beta)} \neq \emptyset$ because $\ker\phi$ is finite. Therefore, we have that $(\pi_J \circ \phi)(S) = \{0\}$ and ϕ is the identity on the subgroup of G consisting of the elements whose i_0-coordinate is 0. From this fact, it follows that $C^{(\beta)} \subseteq \pi_J(K^{(\beta)}) \subseteq (-\delta, \delta)^J$. It suffices now to apply the hypothesis of induction to C and use Lemma 5.1.

Finally, suppose that $\alpha = \sup\{\beta\colon \beta < \alpha\}$ is a limit ordinal and let K be a countable compact subset of G such that $K^{(\alpha)} \subseteq (-\delta, \delta)^I$ for uncountable many indices. Again, we may assume WLOG that $K^{(\alpha)} \subseteq (-\delta, \delta)^I$. Thus $\pi_i(K^{(\alpha)}) \subseteq (-\delta, \delta)$ for all $i \in I$. Since $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$, it follows that for every $i \in I$ there is $\beta_i < \alpha$ such that $\pi_i(K^{(\beta_i)}) \subseteq (-\delta, \delta)$. Again, because α is countable, there must be an uncountable subset $J \subseteq I$ and an ordinal $\beta < \alpha$ such that $\pi_i(K^{(\beta)}) \subseteq (-\delta, \delta)$ for all $i \in J$. Thus $\pi_J(K^{(\beta)}) \subseteq (-\delta, \delta)^J$. As a consequence, $\pi_J(K)^{(\rho)} \subseteq (-\delta, \delta)^J$ for some $\rho < \omega$. It suffices now to apply the hypothesis of induction to $\pi_J(K)$ and use Lemma 5.1 again. This completes the proof. □

Theorem 5.8. *Let F be either $Z(p)$ or $Z(p^\infty)$ and let K be a countable compact subspace of the group $G = F^I$, with $|I| = \aleph_1$. Then there is $0 \neq \phi \in \widehat{G}$ with $\phi(x) = 0$ for all $x \in K$.*

Proof. We will show that no countable compact subset K of G satisfies $K^\circ = \{0\}$. The proof is on induction on the degree of K. It is clear that $K^\circ \neq \{0\}$ if K is finite. Hence, suppose that $\alpha < \omega_1$ is given such that no K with degree less than α satisfies $K^\circ = \{0\}$. We proceed to show the same if the degree of K is α.

By hypothesis, we have that $S = K^{(\alpha)}$ is a finite subset of G. Applying Lemma 5.6 again, there is $k < \omega$ and an uncountable subset $J \subseteq I$ such that $\pi_J(S) \subseteq \Delta_J(\mathbb{Z}(p^k)) \subseteq F^J$. Replacing I by J, if it were necessary, we may assume that $S \subseteq \Delta_I(\mathbb{Z}(p^k)) \subseteq G$. Set $H = \Delta_I(\mathbb{Z}(p^k))$ and let $\pi : G \to G/H$ be the canonical quotient map. Since $S \subseteq H$ and $\ker\pi$ is finite, we have that $\pi(K)^{(\alpha)} = \{0\}$. Now, observing that G/H is topologically isomorphic to G, we may assume WLOG that K is a countable compact subset of G with $K^{(\alpha)} = \{0\}$. Therefore, we apply Lemma 5.7 and the proof is done. □

Three applications of Lemma 5.1provide:

Corollary 5.9. *If G is either $\mathbb{Z}(p^\infty)^\kappa$ or $\mathbb{Z}(p)^\kappa$ and $\omega_1 \leqslant \kappa$, then no countable subgroup determines G.*

Of course, the above result is vacuously true if $\kappa > \mathfrak{c}$.

Corollary 5.10. *If $\omega_1 \leqslant \kappa$, then $\mathbb{Z}(p)^\kappa$ and $\mathbb{T}^\kappa$ are not determined.*

Next follows the main result of this section, which solves Questions 874–876 in [5], Question 7.5 in [6] and also Question 3.10 in [17].

Corollary 5.11. *Compact Abelian groups of uncountable weight are not determined.*

Proof. Assume that G is a compact determined group and let κ denote the weight of the group. Using Corollary 5.2 and taking into account that, by Theorem 4.15 of [6], either $\mathbb{T}^\kappa$ or $\mathbb{Z}(p)^\kappa$ is a quotient of G, it follows that one of these groups must be determined. In either case, Corollary 5.10 yields $\kappa \leqslant \omega$, which completes the proof. □

By Corollary 5.9, no countable (dense) subgroup of $(\mathbb{Z}(p))^\kappa$ or of the dense subgroup $(\mathbb{Z}(p^\infty))^\kappa$ of $\mathbb{T}^\kappa$ can determine $(\mathbb{Z}(p))^\kappa$ or $(\mathbb{Z}(p^\infty))^\kappa$ if $\kappa > \omega$. Nevertheless, it remains unclear which topological properties a dense subgroup must satisfy to be determining. This suggests the following questions:

Question 5.12.

(i) *Is there a compact non-metric separable group G with a countable dense subgroup H such that H determines G?*
(ii) *What if $G = \mathbb{T}^\kappa$?*
(iii) *Does there exist (in ZFC) a non-metrizable compact Abelian G group such that every G_δ-dense subgroup D of G determines G?*

Note added in proof

Question 5.12(i) has been answered negatively in http://uk.arxiv.org/abs/0807.3846.

Acknowledgments

We want to thank Professors S. Ardanza, W.W. Comfort, M.J. Chasco, B. Givens, E. Martín Peinador, and V. Tarieladze for several suggestions and comments that have helped us to improve this paper. We are also very grateful to the referee as we have benefitted from his/her helpful report.

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