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Note

Fourier–Motzkin Elimination Extension to Integer Programming Problems

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This paper describes how the Fourier–Motzkin Elimination Method, which can be used for solving Linear Programming Problems, can be extended to deal with Integer Programming Problems. The extension derives from a known decision procedure for the formal theory of a fragment of arithmetic which excludes multiplication.

INTRODUCTION

The purpose of this paper is to show how the Fourier–Motzkin Elimination Method for solving simultaneous linear inequalities, and in particular Linear Programming problems, can be naturally extended to deal with Integer Programming (IP) problems. A description of Fourier–Motzkin Elimination is given by Dantzig (3).

As far as the author is aware nobody has realized before that the Fourier–Motzkin elimination method is the same as a decision procedure of Langford (6) for the formal logical Theory of Dense Linear Order. A rather similar but more complicated decision theory has been produced by Presburger (8) for a fragment of arithmetic which excludes multiplication. Presburger's procedure has already been applied to IP problems by Lee (7).

The purpose of this paper is to demonstrate Presburger's procedure as a generalization of Fourier–Motzkin Elimination. Hopefully this unification may lead to a greater understanding of the relationship between Linear Programming and Integer Programming as well as increase the awareness of Mathematical Programmers to some results in logic.

Before describing the detailed procedures it should be pointed out that Fourier–Motzkin Elimination has been extended in other ways to deal with Integer Programming Problems. Bradley (1) applies Fourier–Motzkin Elimination to a relaxed version of the Integer Programming Problem, and Cabot (2) applies Fourier–Motzkin Elimination to the solution of Knapsack Problems. A possible way of applying Fourier–Motzkin Elimination to Integer Programming Problems is to use the dual of the associated Linear Programming Problem. This has been recently done by Dantzig and Eaves (4).

The general integer programming problem can be considered as one of maximizing or minimizing a linear function $\sum_j c_j x_j$ subject to linear (in)equalities. In the subsequent description the nonnegativity inequalities $x_j \geqslant 0$ will be considered as being explicitly stated. It is also necessary to consider the problem as one of optimizing a new single variable z with the added equality constraint $z - \sum_j c_j x_j = 0$.

Eliminating Integer Variables between (In)Equalities

When the Fourier–Motzkin method is applied to linear programming problems all the (continuous) variables are eliminated, in turn between the (in)equalities in which they occur. Finally one is left with a series of (in)equalities involving only one variable representing the value of the objective function. The optimal value for this variable can then be determined immediately. Eliminating *integer* variables between (in)equalities is more complicated. We will demonstrate the difficulty and its resolution by means of an example before describing the general method.

Suppose we wished to eliminate x between the following two inequalities:

$$3x + 5y \leqslant 19, \tag{1}$$

$$2x + y \geqslant 3. \tag{2}$$

Multiplying (1) by 2 and (2) by 3 we obtain the inequalities:

$$-3y + 9 \leqslant 6x \leqslant -10y + 38. \tag{3}$$

If x is taken from the continuum of real or rational numbers there is no difficulty. The expression (3) is equivalent to:

$$-3y + 9 \leqslant -10y + 38, \tag{4}$$

giving

$$7y \leqslant 29. \tag{5}$$

If, however, x is an integer valued variable the import of (3) is that a multiple of 6 lies between the left-hand and right-hand expressions in (3). By means of a *congruence* relation it is still possible to give an equivalent statement to (3) without the variable x as:

$$7y \leqslant 29 \,.\, y \equiv 1 \pmod 2 \vee 7y \leqslant 26 \,.\, y \equiv 0 \pmod 2. \qquad (6)$$

("." stands for the logical conjunction "and"; "v" stands for the logical disjunction "or".)

At any stage in the calculation after a number of variables have been eliminated the problem can be expressed in the form

$$R_1 \vee R_2 \vee \cdots \vee R_n \qquad (7)$$

where each R_i is a conjunction of (in)equality and congruence relations. Each such conjunction must be considered separately in the subsequent calculation.

Suppose we wish to eliminate a new variable x from one of the conjunctions R_i. R_i will be made up of (in)equalities and congruences of the following types involving x.

$$(E) \quad p_1x = s.$$
$$(L) \quad p_2x \leqslant t.$$
$$(G) \quad p_3x \geqslant u.$$
$$(M) \quad p_4x \equiv v \pmod k.$$

p_i and k are positive integers. s, t, u, and v are expressions involving constants and the variables not yet eliminated. By use of the Chinese Remainder Theorem it is always possible to either show the congruences involving x to be incompatible (and hence R_i may be ignored as contradictory) or to combine them into a single congruence. The set (M) can therefore be considered as containing at most one congruence.

In order to eliminate x from the above relations it is convenient to distinguish a number of cases.

Case (a), (E) *contains at least one member.* In this case one of these equations can be used to substitute out x in each of the other relations. The equation and each relation in turn would be multiplied by suitable amounts in order to give x the same coefficient (p) in each before sub-substituting out the term in x. For example if x were to be eliminated from a relation (L) we would obtain the relations

$$q_1s \leqslant q_2t,$$
$$s \equiv 0 \pmod{p_1}$$

where $q_1p_1 = q_2p_2$ and q_1, p_2 are integers.

Case (b), *(E) contains no members. (L) or (G) or both contain no members.* In this case the inequalities containing x can be removed and the congruence replaced by

$$v \equiv 0 \quad (\mathrm{mod}(p_4, k)),$$

where (p_4, k) is the greatest common divisor (g.c.d.) of p_4 and k. This g.c.d. may be 1 allowing the congruence to be ignored.

Case (c), *(E) contains no members. (L) and (G) both contain at least one member.* In this case each (L) inequality must be combined with each (G) inequality taking into account the congruence relation. The variable x can then be eliminated in the following stages:

The (L) inequality and (G) inequality can be combined to give

$$u' \leqslant px \leqslant t',$$
$$px \equiv v' \quad (\mathrm{mod}\ k'),$$

where $p = q_2p_2 = q_3p_3 = q_4p_4$, $u' = q_3u$, $t' = q_2t$, $v' = q_4v$, $k' = q_4k$.

These relations can be reexpressed as

$$u' \leqslant y \leqslant t',$$
$$y \equiv v' \quad (\mathrm{mod}\ k'),$$
$$y \equiv 0 \quad (\mathrm{mod}\ p).$$

The last two congruences can either be shown to be incompatible or combined into a single congruence $y \equiv w \pmod q$ by the Chinese Remainder Theorem. Then it is possible to rewrite the above relations as:

$$u'' \leqslant y' \leqslant t'',$$
$$y' \equiv 0 \quad (\mathrm{mod}\ q)$$

where $y' = y - w$, $u'' = u' - w$, $t'' = t' - w$, q is the least common multiple of p and k'. w depends upon v', k', and p. The import of the above pair of relations is that a multiple of q lies between u'' and t''. This relation can be expressed without reference to y' by

$$u'' \leqslant t'' . u'' \equiv 0 \ (\mathrm{mod}\ q) \vee u'' + 1$$
$$\leqslant t'' . u'' + 1 \equiv 0 \ (\mathrm{mod}\ q) \vee \cdots \vee u'' + (q - 1)$$
$$\leqslant t'' . u'' + (q - 1) \equiv 0 \ (\mathrm{mod}\ q).$$

In this way the original variable x is eliminated. For many practical examples there may be no congruence involving the variables to be eliminated. If this happens the number of expressions in the above disjunction can be usually greatly reduced as happened with the elimination of x between the two inequalities (1) and (2) above.

After eliminating all variables apart from z in all the conjunctions R_i and the conjunctions resulting from them we will be left with a disjunction of conjunctions in (in)equality and congruence relations involving z. From these the optimum value of z can be obtained (or the problem seen to be unbounded or infeasible). By "backtracking" through the variables in the reverse order to their elimination the values of them resulting in this optimal solution can be determined.

Comments on the Algorithm

The computational difficulties in solving an integer programming problem by the above algorithm will clearly depend very critically on the size of coefficients in the problem. For large coefficients the number of inequalities and congruences generated could be enormous. It is conceivable, however, that the method could prove practical for restricted classes of IP problem.

The algorithm can be used as a method of generating all solutions to an IP problem. It can also be used to obtain the solution in terms of the right-hand-side coefficients in an analogous fashion to the way the Fourier–Motzkin method does this for LP problems, Kohler [5] describes how this may be done for Fourier–Motzkin elimination.

The algorithm does not require the feasible region of the IP to be bounded as do many other algorithms.

References

1. G. H. Bradley, An algorithm for integer linear programming: A combined algebraic and enumeration appsach, *Operations Res.* **21** No. 1, 1973.
2. A. V. Cabot, An enumeration algorithm for knapsack problems, *Operations Res.* **18** No. 2 (1970).
3. G. B. Dantzig, "Linear Programming and Extensions," Princeton Univ. Press, Princeton, New Jersey, 1963.
4. G. B. Dantzig and B. C. Eaves, Fourier–Motzkin elimination and its dual, *J. Combinatorial Theory* **14** No. 3 (1973), 288–297.
5. D. A. Kohler, Projections of Convex Polyhedral sets, Operations Research Center, University of California, Berkeley, August, 1967.

6. C. H. LANGFORD, Some theorems on deducibility, *Ann. of Math. I*, **28** (1927), pp. 16–40, **28** (1927), pp. 459–471.
7. R. D. LEE, An application of mathematical logic to the integer linear programming problem, *Notre Dame J. Formal Logic* **23**, No. 2 (1972).
8. M. PRESBURGER, "Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen in welchem die Addition als einzige Operation hervortritt, *C.-R. I Congres des Math. des Pays Slaves, Warsaw* (1930), pp. 92–101.