The Quaternion Formalism for Möbius Groups in Four or Fewer Dimensions*

J. B. Wilker

Physical Sciences Division
Scarborough Campus
University of Toronto
1265 Military Trail
Scarborough, Ontario M1C 1A4, Canada

Submitted by Richard A. Brualdi

ABSTRACT

The Möbius group of $\mathbb{R}^N \cup \{\infty\}$ defines $N$-dimensional inversive geometry. This geometry can serve as an alternative to projective geometry in providing a common foundation for spherical Euclidean and hyperbolic geometry. Accordingly the Möbius group plays an important role in geometry and topology. The modern emphasis on low-dimensional topology makes it timely to discuss a useful quaternion formalism for the Möbius groups in four or fewer dimensions. The present account is self-contained.

It begins with the representation of quaternions by $2 \times 2$ matrices of complex numbers. It discusses $2 \times 2$ matrices of quaternions and how a suitably normalized subgroup of these matrices, extended by a certain involution related to sense reversal, is $2$-1 homomorphic to the Möbius group acting on $\mathbb{R}^4 \cup \{\infty\}$. It provides details of this action and the relation of this action to various models of the classical geometries. In higher dimensions $N \geq 5$, the best description of the Möbius group is probably by means of $(N + 2) \times (N + 2)$ Lorentz matrices. In the lower dimensions covered by the quaternion formalism, this alternative Lorentz formalism is a source of interesting homomorphisms. A sampling of these homomorphisms is computed explicitly both for intrinsic interest and for an illustration of the ease with which one can handle the quaternion formalism.

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1. INTRODUCTION

The Möbius group (≡ conformal group) of \( \mathbb{R}^N \cup \{\infty\} \) is the group generated by reflections in \((N - 1)\)-flats and inversions in \((N - 1)\)-spheres. Any element of this group can be written as the product of \(N + 2\) or fewer generators and is equal to a Euclidean similarity or the product of an inversion and a Euclidean isometry. This description shows that it is easy to express Möbius transformations in terms of standard Cartesian coordinates \([1]\). But the resulting formulae are not convenient for many applications, and it is generally better to adopt \((N + 2)\)-component inversive coordinates. In terms of these coordinates (which will be described later) Möbius transformations are expressed by linear transformations which preserve the Lorentz bilinear form

\[
U \ast V = U_1 V_1 + U_2 V_2 + \cdots + U_{N+1} V_{N+1} - U_{N+2} V_{N+2}
\]

and the sign of \(U_{N+2}\) on the cone \(U \ast U = 0\) \([7]\). Thus the composition of \(N\)-dimensional Möbius transformations can be reduced to the multiplication of \((N + 2) \times (N + 2)\) Lorentz matrices.

When \(N = 2\) we obtain a more familiar description of Möbius transformations by adopting a complex coordinate \(z\). Then products of two or four inversions can be written as homographies

\[
z \rightarrow \frac{a_1 z + b_1}{c_1 z + d_1}, \quad a_1 d_1 - b_1 c_1 = 1
\]

and represented by the symbols

\[
\pm H_1 = \pm \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},
\]

while single inversions or products of three can be written as antihomographies

\[
z \rightarrow \frac{a_2 \bar{z} + b_2}{c_2 \bar{z} + d_2}, \quad a_2 d_2 - b_2 c_2 = -1
\]
and represented by the symbols

\[ \pm H_2^\# = \pm \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}^\#. \]

These complex symbols afford a convenient shorthand for the 4 × 4 Lorentz matrices supplied by the general theory. The product of Möbius transformations with symbols \( M_1^\#, M_2^\#, M_3, M_4^\# \) has the symbol

\[ M_4^\#M_3M_2^\#M_1^\# = \left( M_4M_3\overline{M_2}M_1 \right)^\#, \]

where \( \overline{M} \) is the matrix whose entries are the complex conjugates of those in \( M \). Also, the conjugacy class of a transformation with symbol \( M \) or \( M^\# \) can be determined from trace \( M \) or trace(\( M^\# \))^2, and interesting subgroups can be specified by the form of the symbols which occur in them [7, 8].

The purpose of this paper is to investigate the analogous description of Möbius transformations in dimension \( N = 4 \) which is obtained by adopting a quaternion coordinate \( Q \). It turns out that products of two, four, or six inversions can be written

\[ Q \to (AQ + B)(CQ + D)^{-1}, \]

while single inversions and products of three or five inversions can be written

\[ Q \to (A\overline{Q} + B)(C\overline{Q} + D)^{-1}. \]

After suitable normalization we can represent these by the symbols

\[ \pm \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad \pm \begin{pmatrix} A & B \\ C & D \end{pmatrix}^\# \]

respectively. Then the product of Möbius transformations with symbols \( M_1^\#, M_2^\#, M_3, M_4^\# \) has the symbol

\[ M_4^\#M_3M_2^\#M_1^\# = \left( M_4M_3\overline{M_2}M_1 \right)^\#. \]

This is strongly reminiscent of the complex formula, but we shall see that the matrix mapping \( M \to M' \) is considerably more complicated than mere quaternion conjugation of the entries.
Two applications of quaternions in connection with Möbius transformations are well known. We shall mention them here and include details with our discussion of other applications.

Möbius transformations of $\mathbb{R}^2 \cup \{\infty\}$ have a Poincaré extension to Möbius transformations of $\mathbb{R}^3 \cup \{\infty\}$, and these act as isometries of hyperbolic 3-space modeled in the half space $\{(x, y, t) = (z, t) : t > 0\}$. If the original Möbius transformation is given by the complex formula

$$z \mapsto \frac{az + b}{cz + d}, \quad ad - bc = 1$$

and if we write $Q = z + tj$, then as shown in [3, pp. 58, 59], the extension is given by the quaternion formula

$$Q \mapsto (aQ + b)(cQ + d)^{-1}.$$

In another connection, the points of $\mathbb{R}^3$ can be represented by pure quaternions $q = q_1i + q_2j + q_3k$. Then rotation through $2\psi$ about an axis $a$ can be expressed in terms of the unit quaternion

$$A = \cos \psi + \sin \psi a$$

by the formula [5]

$$q \mapsto Aq\bar{A}.$$ 

One dimension higher, the points of $\mathbb{R}^4$ can be represented by arbitrary quaternions $Q$, and any proper orthogonal transformation can be expressed using a suitable pair of unit quaternions by the formula [5]

$$Q \mapsto \Lambda Q\bar{\Lambda}.$$ 

These applications come later; we shall begin with a discussion of the quaternions themselves and of $2 \times 2$ matrices of quaternions. The reader who enjoys this account will also want to consult Ahlfors [2] for a representation of $N$-dimensional Möbius transformations by $2 \times 2$ matrices of Clifford numbers. The $2 \times 2$ matrices of quaternions which occur in his account serve to describe products of an even number of inversions among 3-dimensional Möbius transformations.
2. COMPLEX NUMBERS AND QUATERNIONS

Complex numbers can be regarded as a field extension of the reals, as a real vector space with multiplication, or as a special set of real matrices:

\[ z = x + yi \sim (x, y) \sim \begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \]

Depending on the point of view adopted, the various fundamental properties of complex numbers are more accessible or less so. For example, since the rule for multiplying matrices is the standard one, the matrix approach shows instantly that complex multiplication is associative and distributive, but it requires computation to verify that it is commutative and that the given system is closed under multiplication. It is not our purpose to give an elementary discussion of the fundamental properties of complex numbers, but only to note certain aspects of the dictionary between these representations, namely,

\[ \bar{z} = x - yi \sim \begin{pmatrix} x & y \\ -y & x \end{pmatrix}', \]

\[ |z|^2 = z\bar{z} = x^2 + y^2 = \det\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \]

\[ z^{-1} = \frac{1}{|z|^2} z\bar{z} \sim \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1}. \]

To reconcile the first two of these formulae with the standard expression for the inverse of a matrix we note that the cofactor matrix of

\[ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \]

is \( \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \) itself.

In order to obtain a considerable simplification later, we wish to advocate a flexible notation in which \( z \) denotes a vector or a field element depending on the immediate context. Thus we allow ourselves to write

\[ z_1 \cdot z_2 = x_1 x_2 + y_1 y_2 = \text{Re} \ z_1 \bar{z}_2 = \frac{1}{2}(z_1 \bar{z}_2 + z_2 \bar{z}_1). \]
In the same spirit we derive the formula for reflection in the line through the origin with unit normal \( n \) as follows:

\[
z \to z - 2(n \cdot z)n = z - (n \overline{z} + z \overline{n})n = -n \overline{z}n.
\]

The most familiar way to introduce quaternions is by writing

\[
Q = Q_0 + Q_1 i + Q_2 j + Q_3 k \sim (Q_0, Q_1, Q_2, Q_3),
\]

where \( i^2 = j^2 = k^2 = ijk = -1 \). But it is not at all obvious from these rules that the quaternions constitute a skew field, so we quickly indicate the equivalent formulations in terms of complex vectors and a special set of complex matrices:

\[
Q = (Q_0 + Q_1 i) + (Q_2 + Q_3 i)j = u + vj \sim \begin{pmatrix} u & v \\ -v & u \end{pmatrix}.
\]

The rule for handling \( j \) is that \( j^2 = -1 \) and \( vj = j \overline{v} \), so we are explicitly stating that multiplication is not commutative. The other field properties are fairly easily verified from the complex matrices, and so we do not dwell on their derivation. Instead we emphasize several aspects of the dictionary between these representations.

First we note that

\[
\overline{Q} = Q_0 - Q_1 i - Q_2 j - Q_3 k = \overline{u} - vj \sim \begin{pmatrix} u & v \\ -v & u \end{pmatrix}^*,
\]

where \( * \) denotes the Hermitian conjugate, i.e. the transpose of the matrix of complex conjugates. It follows from familiar matrix rules that \( \overline{AB} = \overline{BA} \). The fact that \( \overline{AB} \neq \overline{A} \overline{B} \) in general is the main reason our quaternion story is more complicated than the complex one.

We also note that

\[
|Q|^2 = \overline{Q}Q = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2 = |u|^2 + |v|^2 = \det \begin{pmatrix} u & v \\ -v & u \end{pmatrix}
\]

and

\[
Q^{-1} = \frac{1}{|Q|^2} \overline{Q} \sim \begin{pmatrix} u & v \\ -v & u \end{pmatrix}^{-1}.
\]
We define the scalar and vector parts of \( Q \) to be

\[
S(Q) = Q_0 = \frac{1}{2}(Q + \overline{Q}) = \frac{1}{2} \text{trace}
\begin{pmatrix}
    u & v \\
    -v & u 
\end{pmatrix}
\]

and

\[
V(Q) = Q_1 i + Q_2 j + Q_3 k = \frac{1}{2}(Q - \overline{Q}).
\]

We refer to the individual components of the vector part as

\[
V_1(Q) = Q_1, \quad V_2(Q) = Q_2, \quad V_3(Q) = Q_3
\]

when necessary.

As an extension of the flexible notation advocated for complex numbers, we accept that \( V(Q) \) is a vector in 3-space and \( Q \) itself a vector in 4-space. This leads to formulae such as

\[
PQ = S(P)S(Q) - V(P) \cdot V(Q) + S(P)V(Q) + S(Q)V(P) + V(Q) \times V(P),
\]

which is perhaps most memorable for pure quaternions \( p \) and \( q \) defined by \( S(p) = S(q) = 0 \), for then it reads

\[
pq = -p \cdot q + p \times q.
\]

With 4-vectors we have

\[
P \cdot Q = P_0Q_0 + P_1Q_1 + P_2Q_2 + P_3Q_3 = S(PQ) = \frac{1}{2}(PQ + Q\overline{P}).
\]

Then reflection in the 3-flat through the origin with unit normal \( N \) is given by

\[
Q \rightarrow Q - 2(N \cdot Q)N = Q - (N\overline{Q} + Q\overline{N})N = -N\overline{Q}N,
\]

exactly as for complex numbers.

Two further remarks are in order. The formula for the product \( PQ \) shows us that \( PQ = QP \) for all \( Q \) if and only if \( V(P) = 0 \), i.e. if and only if \( P \) is real. The fact that \( P\overline{Q} = \overline{Q\overline{P}} \) and \( P\overline{P} = |P|^2 > 0 \) allows us to prove that the
norm satisfies $|PQ| = |P||Q|$. This also follows directly from the matrix representation via determinants.

3. MATRICES OF QUATERNIONS

We shall have quite a lot to say about $2 \times 2$ matrices of quaternions

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$  

Even though quaternions do not commute, the product of two such matrices is well defined,

$$M_2 M_1 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} A_2 A_1 + B_2 C_1 & A_2 B_1 + B_2 D_1 \\ C_2 A_1 + D_2 C_1 & C_2 B_1 + D_2 D_1 \end{pmatrix},$$

and an important question is whether $M_1$ admits an inverse $M_2$ such that $M_2 M_1 = I$. The theory of determinants is not immediately available, but if we temporarily adopt the point of view that a quaternion is a $2 \times 2$ matrix of complex numbers, then we are dealing with partitioned $4 \times 4$ matrices of complex numbers and for these the theory of determinants is available.

We make two observations which would apply if $M$ were a $2n \times 2n$ matrix over an arbitrary field, partitioned into four $n \times n$ matrices. First, if $C = 0$ then

$$\det M = \det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A \det D = \det AD.$$  

Second, if $A$ is invertible, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

is just a collection of elementary row operations done simultaneously, and so, using our first observation, we find

$$\det M = \det (AD - ACA^{-1}B).$$
On the other hand, if \( A = 0 \) we find

\[
\det\begin{pmatrix} 0 & B \\ C & D \end{pmatrix} = (-1)^n \det\begin{pmatrix} C & D \\ 0 & B \end{pmatrix} = (-1)^n \det CB
\]

\[
= (-1)^{n^2+n} \det(-CB) = \det(-CB).
\]

With quaternions, \( A \) is either invertible or 0, and so the second observation can be strengthened to the unrestricted formula

\[
\det M = \det(AD - ACA^{-1}B)
\]

provided we interpret conjugation by 0 as the identity. (This is a slightly outrageous mnemonic, but at least 0 does commute with all quaternions.) Since \( AD - ACA^{-1}B \) is a quaternion, \( \det(AD - ACA^{-1}B) \) is the square of its norm, and hence \( M \) is invertible if and only if \( AD - ACA^{-1}B \neq 0 \).

The calculation which we have just performed can be varied, perhaps using column operations instead, to produce a total of eight forms for the answer. We define

\[
L_{11} = DA - DBD^{-1}C, \quad L_{12} = BDB^{-1}A - BC,
\]

\[
L_{21} = CAC^{-1}D - CB, \quad L_{22} = AD - ACA^{-1}B,
\]

\[
R_{11} = AD - BD^{-1}CD, \quad R_{12} = DB^{-1}AB - CB,
\]

\[
R_{21} = AC^{-1}DC - BC, \quad R_{22} = DA - CA^{-1}BA
\]

and obtain

\[
\det M = |L_{ij}|^2 = |R_{ij}|^2 = \Delta^2, \quad i, j = 1, 2.
\]

Although these eight associated quaternions have the same norm \( \Delta \), it is quite possible for them to be mutually distinct, as indeed they are for the matrix

\[
\begin{pmatrix} 1 + i & j \\ -k & j + k \end{pmatrix}.
\]
For future reference we note that

$$\Delta^2 = (AD - ACA^{-1}B)(\bar{DA} - \bar{B}A^{-1}\bar{C}\bar{A})$$

$$= |AD|^2 + |BC|^2 - 2S(A\bar{DB}\bar{A}^{-1}\bar{C}\bar{A})$$

$$= |AD|^2 + |BC|^2 - 2S(\bar{AC}D\bar{B}).$$

In deriving this formula we have used the fact that $S(XY) = S(YX)$, which follows either from the explicit formula for a product or from the formula for $S$ as a trace. We have also used the fact that real numbers such as $|A|^2$ commute with arbitrary quaternions. The reader may wonder if $\Delta^2$ is also given by

$$|AD - BC|^2 = |AD|^2 + |BC|^2 - 2S(AD\bar{C}\bar{B}),$$

but this is definitely not the case, as we see by considering the matrix

$$\begin{pmatrix} 1 & i \\ k & j \end{pmatrix}.$$

It follows from the formula $\det M = \Delta^2$ that $M$ is invertible if any one (and hence all) of the eight associated quaternions is different from 0. Granting that $M$ is invertible as a $4 \times 4$ matrix of complex numbers, one might wonder if the inverse has the proper shape to be regarded as a quaternion matrix. We answer this question in the affirmative by giving an explicit formula for the inverse.

**Theorem 1.** Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a $2 \times 2$ matrix of quaternions with any one of the eight associated quaternions $L_{ij}$ or $R_{ij}$, $i, j = 1, 2$, different from 0. Then $M$ is invertible and

$$M^{-1} = \begin{pmatrix} L_{11}^{-1}D & -L_{12}^{-1}B \\ -L_{21}^{-1}C & L_{22}^{-1}A \end{pmatrix} = \begin{pmatrix} DR_{11}^{-1} & -BR_{12}^{-1} \\ -CR_{21}^{-1} & AR_{22}^{-1} \end{pmatrix}.$$
Proof. The formula involving $L$'s can be checked by computing $M^{-1}M$. Then the formula involving $R$'s can be obtained by matching entries. For example, the $(1, 1)$ entry in $M^{-1}M$ is

$$(DA - DBD^{-1}C)^{-1}DA - (BDB^{-1}A - BC)^{-1}BC$$

$$= (1 - A^{-1}BD^{-1}C)^{-1}A^{-1}D^{-1}DA - (BDB^{-1}A - BC)^{-1}BC$$

$$= (1 - A^{-1}BD^{-1}C)^{-1}(BDB^{-1}A)^{-1}(BDB^{-1}A)$$

$$- (BDB^{-1}A - BC)^{-1}BC$$

$$= (BDB^{-1}A - BDB^{-1}AA^{-1}BD^{-1}C)^{-1}(BDB^{-1}A)$$

$$- (BDB^{-1}A - BC)^{-1}BC$$

$$= (BDB^{-1}A - BC)^{-1}(BDB^{-1}A - BC) = 1.$$  

The equation $L_{11}^{-1}D = DR_{11}^{-1}$ yields

$$R_{11} = D^{-1}(DA - DBD^{-1}C)D = AD - BD^{-1}CD.$$  

We note that if the entries in $M$ commute, the $L$'s and $R$'s all reduce to $AD - BC$ and our expressions for $M^{-1}$ simplify to the familiar formula. As noncommutative analogue of the fact that the determinant of a product is equal to the product of the determinants of its factors, we can use the formula $(M_2M_1)^{-1} = M_1^{-1}M_2^{-1}$ to read off expressions which determine the various $L$'s (or $R$'s) of a product in terms of the $L$'s (or $R$'s) and the entries of its factors.

4. FRACTIONAL TRANSFORMATIONS

We let $\mathbb{R}^4 \cup \{\infty\}$ denote the one-point compactification of $\mathbb{R}^4$ and consider the transformations of this space defined on most points by the formula

$$Q \to (AQ + B)(CQ + D)^{-1}.$$
where the coefficient matrix

\[ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]

is assumed to be invertible. We complete the definition by declaring that if \( C = 0 \) then \( \infty \) stays fixed, while if \( C \neq 0 \) then \( -C^{-1}D \to \infty \to AC^{-1} \). Since quaternion arithmetic and the quaternion norm are both continuous, these transformations are continuous as well.

If the transformation

\[ Q \to (A_1Q + B_1)(C_1Q + D_1)^{-1} \]

is followed by the transformation

\[ Q \to (A_2Q + B_2)(C_2Q + D_2)^{-1}, \]

then the product transformation is

\[ Q \to \left[ A_2(A_1Q + B_1)(C_1Q + D_1)^{-1} + B_2 \right] \]

\[ \times \left[ C_2(A_1Q + B_1)(C_1Q + D_1)^{-1} + D_2 \right]^{-1} \]

\[ = \left\{ A_2(A_1Q + B_1) + B_2(C_1Q + D_1) \right\} (C_1Q + D_1)^{-1} \]

\[ \times \left\{ C_2(A_1Q + B_1) + D_2(C_1Q + D_1) \right\} (C_1Q + D_1)^{-1} \]

\[ = \left[ (A_2A_1 + B_2C_1)Q + (A_2B_1 + B_2D_1) \right] \]

\[ \times \left[ (C_2A_1 + D_2C_1)Q + (C_2B_1 + D_2D_1) \right]^{-1}. \]

In spite of noncommutativity, the coefficient matrix of the product is simply the product \( M_2M_1 \) of the coefficient matrices \( M_1 \) and \( M_2 \) of the factors. Since the transformation corresponding to \( I \) is the identity, \( M^{-1}M = I \) shows the transformation with coefficient matrix \( M \) is injective, and \( MM^{-1} = I \) shows that it is surjective. Our transformations are bijections, and we are dealing with a group of homeomorphisms of \( \mathbb{R}^4 \cup \{\infty\} \).
If $M_1$ and $M_2$ represent the same transformation, $M_2M_1^{-1}$ represents the identity. If

$$Q \equiv (AQ + B)(CQ + D)^{-1}$$

then

$$QCQ + QD \equiv AQ + B.$$ 

By considering $Q = 0$ and $Q \to \infty$ we deduce $B = C = 0$; then putting $Q = 1$, we find $A = D$, and noting that $AQ = QA$ for all $Q$, we discover that $A$ is real. Thus two matrices $M_1$ and $M_2$ represent the same transformation if and only if $M_2 = \lambda M_1$ for some real number $\lambda \neq 0$. We can choose $\lambda$ so that the eight associated quaternions have common norm $\Delta = 1$ and thereby reduce the ambiguity in our coefficient matrices to a factor of $\pm 1$. Since the square of the common norm actually represents the determinant of a $4 \times 4$ complex matrix, this normalization is preserved under products. Note that if the matrix itself has complex entries, the normalization is only $|AD - BC| = 1$ and not the more familiar $AD - BC = 1$. More details about this are available in Section 10. We summarize the developments of the present section in

**Theorem 2.** There is a 2-1 homomorphism from the group of $2 \times 2$ quaternion matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $\Delta = 1$ to the group of homeomorphisms of $\mathbb{R}^4 \cup \{\infty\}$ of the form

$$Q \to (AQ + B)(CQ + D)^{-1}.$$

5. **QuatERNion Conjugation**

The product of reflections in the 3-flats through the origin with normals $i$, $j$ and $k$ is given by the formula $Q \to \overline{Q}$. When we adjoin this transformation to our group, we generate products of the form

$$Q \to (A\overline{Q} + B)(C\overline{Q} + D)^{-1}.$$
and also conjugates of the form

\[ Q \rightarrow (A\overline{Q} + B)(C\overline{Q} + D)^{-1} \]

\[ = (C\overline{Q} + D)^{-1} (A\overline{Q} + B) \]

\[ = (Q\overline{C} + \overline{D})^{-1} (Q\overline{A} + \overline{B}). \]

The products described above are genuinely new transformations, but the conjugates are equal to transformations which are already present in our group. We shall prove this second assertion by determining explicitly how to rewrite these conjugates in the standard form

\[ Q \rightarrow (A'Q + B')(C'Q + D')^{-1}. \]

Consider the transformation given by

\[ Q \rightarrow P = (QG + H)^{-1}(QE + F). \]

We can solve for \( Q \) in terms of \( P \) by writing

\[ QGP + HP = QE + F \]

or

\[ Q(GP - E) = -HP + F; \]

hence

\[ Q = (HP - F)(-GP + E)^{-1}. \]

Since the transformation with which we began is just the inverse of this one, our earlier work shows that it can be written in standard form using the coefficient matrix

\[ \begin{pmatrix} H & -F \\ -G & E \end{pmatrix}^{-1}. \]
A more convenient formula for this matrix is

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
E & F \\
C & H
\end{pmatrix}^{-1}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}^{-1}^{-1},
\]

and if we apply this formula to the conjugate discussed in the last paragraph, we find that

\[
\begin{pmatrix}
A' & B' \\
C' & D'
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}'
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}^{-1}^{-1}.
\]

In other words, if the original transformation has coefficient matrix \( M \), then the transformation conjugated by \( Q \to Q' \) has coefficient matrix

\[
M' = \left[ KM^* K^{-1} \right]^{-1},
\]

where \( M^* \) denotes the transpose of the matrix whose entries are the quaternion conjugates of those in \( M \), and

\[
K = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

The notation \( M \to M^* \) reminds us of Hermitian conjugation, and there are two senses in which this is appropriate. First, since the quaternion conjugate of a complex number is just its complex conjugate, the operation reduces to Hermitian conjugation on matrices with complex entries. Second, if the mapping \( M \to M^* \) is interpreted in terms of \( 4 \times 4 \) matrices of complex numbers, it really is Hermitian conjugation. This fully justifies the notation and confirms that \( (MN)^* = N^*M^* \), \( (M^*)^{-1} = (M^{-1})^* \), etc. Note that neither transposition nor conjugation separately enjoys this property of having a definite meaning independent of the interpretation of \( M \).

A number of facts about the mapping \( M \to M' \) are now clear. Since the matrix \( K \) above is real orthogonal, \( K^* = K^t = K^{-1} \), and an easy computation shows that the mapping \( M \to M' \) is an involutory automorphism of the group of invertible matrices. Moreover, if \( M \) is normalized so that its eight associated quaternions have common norm \( \Delta = 1 \) (that is, so that its incarnation as a \( 4 \times 4 \) matrix of complex numbers has determinant \( \Delta^2 = 1 \)), then inspection of the formula for \( M' \) shows that it too has this property. Finally, if
$M$ is a matrix of complex numbers, computation shows that

$$M' = \frac{1}{\det M} M.$$  

This means that the restriction of the quaternion formalism to complex matrices has an extra factor $-1$ associated with the sense-reversing transformations that was omitted from the complex formalism described in the introduction. The utility of this hidden $-1$ will emerge in Section 7.

We summarize our results to date in

**Theorem 3.** The group of $2 \times 2$ quaternion matrices $M$ which satisfy the normalization condition $\Delta = 1$ can be extended by an involution $\#$ satisfying $\#M\# = M' = (KM*K^{-1})^{-1}$. There is a 2-1 homomorphism from the extended group to the group of Möbius transformations of $\mathbb{R}^4 \cup \{\infty\}$ given by

$$\pm \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow Q \rightarrow (AQ + B)(CQ + D)^{-1},$$

$$\pm \begin{pmatrix} A & B \# \\ C & D \end{pmatrix} \longrightarrow Q \rightarrow (A\overline{Q} + B)(C\overline{Q} + D)^{-1}.$$  

The statement of Theorem 3 anticipates our identification of the homeomorphisms described by the formulae of Theorem 3 and our simultaneous proof that the homeomorphisms of the second type are really distinct from those of the first type.

6. **The Möbius Group of $\mathbb{R}^4 \cup \{\infty\}**

We proceed to identify the homeomorphisms of Theorem 3 with the elements in the Möbius group of $\mathbb{R}^4 \cup \{\infty\}$. This group can be defined as the group generated by reflections in 3-flats and inversions in 3-spheres. Alternatively it can be described as the set of transformations of $\mathbb{R}^4 \cup \{\infty\}$ which preserve cross ratio [7]. By using both of these definitions we obtain our result quickly and with a line of argument that generates other useful information as well.
We have already seen that reflection $\bar{L}_1$ in the 3-flat $L_1$ through the origin with unit normal $N$ is given by

$$Q \rightarrow -N\bar{Q}N.$$ 

Reflection in the 3-flat $L_2$ parallel to $L_1$ and a distance $d$ from $L_1$ in direction $N$ is given by

$$\bar{L}_2 = \bar{L}_2(\bar{L}_1)^2 = (\bar{L}_2 \bar{L}_1)\bar{L}_1$$

Since $\bar{L}_2 \bar{L}_1$ is the translation $Q \rightarrow Q + 2dN$, $\bar{L}_2$ can be written

$$Q \rightarrow -N\bar{Q}N + 2dN$$

$$= (N\bar{Q} - 2d)(0\bar{Q} - \bar{N})^{-1},$$

which is a fractional transformation with the symbol

$$\begin{pmatrix}
N & -2d \\
0 & -\bar{N}
\end{pmatrix}^*$$

having all eight associated quaternions equal to $-1$.

Inversion in the 3-sphere $|Q - A| = r$ maps $Q$ to $Q'$, say, and is determined by the fact that $Q' - A$ is a positive multiple of $Q - A$ and satisfies $|Q' - A||Q - A| = r^2$. It follows that

$$Q' - A = \frac{r^2}{|Q - A|} \cdot \frac{Q - A}{|Q - A|} = r^2(\bar{Q} - \bar{A})^{-1},$$

and hence the transformation is given by

$$Q \rightarrow \left[A\bar{Q} + (r^2 - |A|^2)\right](\bar{Q} - \bar{A})^{-1}$$

$$= \left(\frac{1}{r}A\bar{Q} + \frac{r^2 - |A|^2}{r}\right)\left(\frac{1}{r}\bar{Q} - \frac{1}{r}\bar{A}\right)^{-1}.$$
This is a fractional transformation with the symbol

$$\left( \begin{array}{cc} \frac{1}{r^2 - |A|^2} & r^2 - |A|^2 \\ r & 1 \\ \frac{1}{r} & -\frac{1}{r} \end{array} \right),$$

which again has all eight associated quaternions equal to $-1$.

Since the generators of the Möbius group are of the form

$$Q \rightarrow (A\overline{Q} + B)(C\overline{Q} + D)^{-1},$$

products of an even number of generators are of the form

$$Q \rightarrow (AQ + B)(CQ + D)^{-1}$$

and arbitrary products of an odd number of generators are of the form

$$Q \rightarrow (A\overline{Q} + B)(C\overline{Q} + D)^{-1}.$$

Thus the Möbius group of $\mathbb{R}^4 \cup \{\infty\}$ is a subgroup of the group of homeomorphisms of Theorem 3, and the form of the Möbius transformation determines whether it is orientation-preserving (product of an even number of generators) or orientation-reversing (product of an odd number of generators).

We have already mentioned that the transformation $Q \rightarrow \overline{Q}$ is a product of three reflections and hence a Möbius transformation. Thus to complete our identification of the Möbius group with the homeomorphisms of Theorem 3, we have only to show that arbitrary homeomorphisms of the form

$$Q \rightarrow (AQ + B)(CQ + D)^{-1}$$

are Möbius transformations. We do this by demonstrating that they preserve the cross ratio

$$(|Q_1 - Q_2||Q_3 - Q_4|)(|Q_1 - Q_3||Q_2 - Q_4|)^{-1}$$.
of arbitrary sets of four distinct points in $\mathbb{R}^4 \cup \{\infty\}$. The special case when one of the original points or one of the image points is equal to $\infty$ is covered by a standard continuity argument on the 4-sphere in $\mathbb{R}^5$. We therefore restrict attention to the general case when the points and their images all lie in $\mathbb{R}^4$.

**Lemma 1.** If $Q_1$ and $Q_2$ are transported to $Q'_1$ and $Q'_2$ by the transformation $Q \rightarrow (AQ + B)(CQ + D)^{-1}$ with $\Delta = |AD - ACA^{-1}R|$, then

$$|Q'_2 - Q'_1| = \frac{\Delta}{|CQ_1 + D||CQ_2 + D|}|Q_2 - Q_1|.$$

**Proof.** If $C = 0$,

$$|Q'_2 - Q'_1| = |AQ_2D^{-1} + BD^{-1} - AQ_1D^{-1} - BD^{-1}|$$

$$= |A||Q_2 - Q_1||D^{-1}|$$

$$= \frac{\Delta}{|D|^2}|Q_2 - Q_1|,$$

which is of the required form.

If $C \neq 0$,

$$|Q'_2 - Q'_1| = \left|\left(\begin{array}{c} AQ_2 + B \\ CQ_2 + D \end{array}\right)^{-1} - \left(\begin{array}{c} AQ_1 + B \\ CQ_1 + D \end{array}\right)^{-1}\right|$$

$$= \left|\left(\begin{array}{c} AQ_2 + B \\ CQ_2 + D \end{array}\right) - AC^{-1}(CQ_2 + D)\right|\left(\begin{array}{c} AQ_2 + B \\ CQ_2 + D \end{array}\right)^{-1}$$

$$= |B - AC^{-1}D|\left|\left(\begin{array}{c} CQ_2 + D \\ CQ_1 + D \end{array}\right)^{-1} - \left(\begin{array}{c} CQ_1 + D \\ CQ_1 + D \end{array}\right)^{-1}\right|$$

$$= |B - AC^{-1}D||CQ_1 + D|^{-1}$$

$$\times \left|\left(\begin{array}{c} CQ_1 + D \\ CQ_1 + D \end{array}\right) - \left(\begin{array}{c} CQ_2 + D \\ CQ_2 + D \end{array}\right)^{-1}\right|$$

$$= \frac{|AC^{-1}Dc - BC|}{|CQ_1 + D||CQ_2 + D|}|Q_2 - Q_1|$$

$$= \frac{\Delta}{|CQ_1 + D||CQ_2 + D|}|Q_2 - Q_1|,$$

as required. \[\blacksquare\]
Corollary. The transformations $Q \to (AQ + B)(CQ + D)^{-1}$ preserve cross ratio and are therefore equal to Möbius transformations.

Proof. The result is immediate if we compute the cross ratio of the images using the formula of the lemma.

Möbius transformations are conformal and therefore represented locally by a similarity. The lemma allows us to calculate the magnification factor of this local similarity.

Corollary. The local magnification factors for $Q \to (AQ + B)(CQ + D)^{-1}$ and $Q \to (A\overline{Q} + B)(C\overline{Q} + D)^{-1}$ with $\Delta = |AD - ACA^{-1}B|$ are $\Delta|CQ + D|^{-2}$ and $\Delta|C\overline{Q} + D|^{-2}$ respectively.

Proof. For the orientation-preserving case we let $Q_1$ and $Q_2 \to Q$ in the formula of the lemma. For the orientation-reversing case we precompose with the isometry $Q \to \overline{Q}$ and use the fact that magnification factors are multiplicative.

Note that the magnification factor for an arbitrary segment is the geometric mean of the magnification factors which attach to its ends.

7. FORMULAE FROM $\mathbb{R}^N \cup \{\infty\}$

Our basic attitude towards the quaternion formalism being developed here is that it provides a useful shorthand for the real $6 \times 6$ Lorentz matrices which the general theory offers. Accordingly we draw freely from the general theory [7] in our effort to sharpen the efficiency of the quaternion formalism. It is now appropriate to quote detailed formulae for the $(N + 2)$-component inversive coordinates used to describe the inversive geometry of $\mathbb{R}^N \cup \{\infty\}$:

<table>
<thead>
<tr>
<th>Description</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point $x \in \mathbb{R}^N$</td>
<td>$X = \left( x, \frac{1}{2}(|x|^2 - 1), \frac{1}{2}(|x|^2 + 1) \right)$</td>
</tr>
<tr>
<td>Point $\infty$</td>
<td>$X = (0, 1, 1)$</td>
</tr>
<tr>
<td>Half space $n \cdot x \geq d$ $[n \cdot n = 1]$</td>
<td>$C = (n, d, d)$</td>
</tr>
<tr>
<td>Proper $N$-ball $|x - a| \leq r$</td>
<td>$C = \left( \frac{1}{r}a, \frac{1}{2r}(|a|^2 - r^2 - 1), \frac{1}{2r}(|a|^2 - r^2 + 1) \right)$</td>
</tr>
<tr>
<td>Improper $N$-ball $|x - a| \geq r$</td>
<td>$C = \left( \frac{1}{-r}a, \frac{1}{-2r}(|a|^2 - r^2 - 1), \frac{1}{-2r}(|a|^2 - r^2 + 1) \right)$</td>
</tr>
</tbody>
</table>
Point coordinates are regarded as positive homogeneous, and in connection with the Lorentz bilinear form mentioned in the introduction they satisfy \( X \cdot X = 0 \), \( X_{N+2} > 0 \). We refer to half spaces and proper and improper \( N \)-balls as caps, because stereographic projection lifts them to spherical caps on the \( N \)-sphere in \( \mathbb{R}^{N+1} \). Cap coordinates satisfy \( C \cdot C = 1 \) and name the set of points \( X \) such that \( C \cdot X > 0 \). The complement of \( C \) is \(-C\), and reflection (inversion) in the common boundary of \( C \) and \(-C\) is given by the linear transformation

\[
U \rightarrow U - 2(C \cdot U)C.
\]

If \( C \) and \( D \) are both caps, then \( C \cdot D \) has a quantitative geometric interpretation: if the boundaries of \( C \) and \( D \) meet, then \( C \cdot D = \cos \psi \), where \( \psi \) is the angle between inward-pointing normals at any common point; if the boundaries of \( C \) and \( D \) do not meet but are separated by an inversive distance \( \delta \), then \( C \cdot D = \cosh \delta \) or \(-\cosh \delta \) as the caps are nested or not.

By comparing the 6-vector for a half space (4-ball) with the symbols for reflection (inversion) in its boundary, we see that the linear correspondence

\[
C = (C_1, C_2, C_3, C_4, C_5, C_6) = (Q, C_5, C_6)
\]

maps complementary cap vectors \( C \) and \(-C\) to the two symbols for their common boundary involution. Henceforth we shall refer to this reflection or inversion by the generic term inversion.

Notice that the symbol \( Q \) represents a vector in the expression for \( C \) and a quaternion in the expression for \( M_C^\# \). Our flexible notation has been designed to allow this freedom. Notice also that the eight quaternions associated with \( M_C^\# \) are each equal to

\[
-\overline{Q}Q - C_5^2 + C_6^2 = -C \cdot C = -1,
\]

just as expected.

**Lemma 2.** If \( M_1^\# \) and \( M_2^\# \) are symbols for inversions in the boundaries of caps \( C_1 \) and \( C_2 \), then \( M_i^\# = -\overline{M}_i \), \( i = 1, 2 \), and

\[
M_1 M_2' + M_2 M_1' = -2(C_1 \cdot C_2) I.
\]
Proof. Both matrices $M$ are of the form

$$M = \begin{pmatrix} Q & R \\ S & -\overline{Q} \end{pmatrix}$$

with $R$ and $S$ real and all eight associated quaternions equal to $-1$. It follows that

$$M' = (KM^*K^{-1})^{-1} = K\begin{pmatrix} \overline{Q} & R \\ S & -\overline{Q} \end{pmatrix}^* K^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} Q & S \\ R & -\overline{Q} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \overline{Q} & -R \\ -S & Q \end{pmatrix} = -M.$$

Then

$$M_1M'_2 + M_2M'_1 = \begin{pmatrix} Q_1 & R_1 \\ S_1 & -\overline{Q}_1 \end{pmatrix} \begin{pmatrix} -\overline{Q}_2 & -R_2 \\ -S_2 & Q_2 \end{pmatrix} + \begin{pmatrix} Q_2 & R_2 \\ S_2 & -\overline{Q}_2 \end{pmatrix} \begin{pmatrix} -\overline{Q}_1 & -R_1 \\ -S_1 & Q_1 \end{pmatrix}$$

$$= \begin{pmatrix} -Q_1\overline{Q}_2 - R_1S_2 & -R_2Q_1 + R_1Q_2 \\ -S_1\overline{Q}_2 + S_2\overline{Q}_1 & -\overline{Q}_1Q_2 - S_1R_2 \end{pmatrix} + \begin{pmatrix} -Q_2\overline{Q}_1 - R_2S_1 & -R_1Q_2 + R_2Q_1 \\ -S_2\overline{Q}_1 + S_1\overline{Q}_2 & -\overline{Q}_2Q_1 - S_2R_1 \end{pmatrix}$$

$$= \begin{pmatrix} Q_1\overline{Q}_2 + Q_2\overline{Q}_1 & R_1S_2 - S_1R_2 \end{pmatrix} I.$$

Since both vectors $C$ are of the form

$$C = (Q, -\frac{1}{2}(R + S), -\frac{1}{2}(R - S)),$$
we have

\[ 2C_1 \ast C_2 = 2Q_1 \cdot Q_2 + 2 \left( \frac{R_1 + S_1}{2} \right) \left( \frac{R_2 + S_2}{2} \right) - 2 \left( \frac{R_1 - S_1}{2} \right) \left( \frac{R_2 - S_2}{2} \right) \]

\[ = Q_1 \overline{Q}_2 + Q_2 \overline{Q}_1 + R_1 S_2 + S_1 R_2 \]

and the result follows.

\[ \text{COROLLARY. If } M^\# \text{ is the symbol for inversion in the boundary of a cap, then} \]

(a) \((M^\#)^2 = -I\) and
(b) \(M^\# = -M^{-1}\).

\[ \text{Proof. If we put } C_1 = C_2 \text{ in the lemma, we find } MM' = -I, \text{ and this proves both results.} \]

\[ \text{COROLLARY. If } M_1^\# \text{ and } M_2^\# \text{ are the symbols for inversions in the boundaries of perpendicular caps } C_1 \text{ and } C_2, \text{ then} \]

(a) \(M_1 M_2' = -M_2 M_1'\) and
(b) \(M_1' M_2 = -M_2' M_1\).

\[ \text{Proof. We obtain (a) by putting } C_1 \ast C_2 = 0 \text{ in the lemma. Then we obtain (b) by using } M' = -M^{-1}. \]

\[ \text{COROLLARY. The symbol for a product of } n \text{ inversions in mutually perpendicular caps satisfies} \]

\[ \left( M_n^\# \cdots M_2^\# M_1^\# \right)^2 = \begin{cases} -I & \text{if } n = 1, 2, 5, \\ I & \text{if } n = 3, 4. \end{cases} \]

\[ \text{Proof. We use (a) and (b) of the previous corollary and then (b) of the first corollary to show that} \]

\[ M_n^\# M_{n-1}^\# \cdots M_2^\# M_1^\# M_n M_{n-1} \cdots M_2 M_1' \]

\[ = M_n M_{n-1}' \cdots M_2 M_1' \]

\[ = (-1)^{n-1} M_n M_{n-1}' (M_{n-1} M_{n-2}' \cdots M_2 M_1') \]

\[ = (-1)^n (M_{n-1}^\# \cdots M_2^\# M_1^\#)^2. \]
The result then follows by a finite induction starting with (a) of the first corollary.

**Lemma 3.** Suppose inversion in the boundary of cap C maps cap D to cap D'. Then

\[ M_C \# M_D \# M_C = M_D'. \]

*Proof.* We compute the expression on the left using in succession Lemma 2, part (b) of the first corollary of Lemma 2, the linearity of the mapping \( C \rightarrow M_C \), and the formula for inversion:

\[ M_C \# M_D \# M_C = M_C M_D' M_C \]

\[ = [-M_D M_C' - 2(C * D) I] M_C' \]

\[ = [M_D - 2(C * D) M_C] \]

\[ = M_D' - 2(C * D) M_C = M_D'. \]

In the complex formalism described in the introduction, the analogue of the conjugation equation \( M_C \# M_D \# M_C = M_D' \) has a curious extra factor of \(-1\). See [8] for details. Our remarks preceding the proof of Theorem 3 now show that this factor can be removed by changing the formalism from \( \# M \# = \overline{M} \) to \( \# M \# = M' = (1/\det M) \overline{M} \). It is probably not worthwhile to make this alteration when the complex formalism is used on its own, but it is interesting that our investigation of the quaternion formalism has brought this possible modification to light.

**Theorem 4.** Suppose \( h \) is a Möbius transformation which maps the cap \( D \) to the cap \( h(D) \). Then

(a) if \( h \) has symbol \( H \), then \( H M_D \# H^{-1} = M_{h(D)} \# \);

(b) if \( h \) has symbol \( H' \), then \( -H' M_D \# (H^{-1})' = M_{h(D)} \# \).

*Proof.* The symbol for \( h \) can be written as a product of symbols for inversions,

\[ M_n \# M_{n-1} \cdots M_2 M_1, \]
and then by repeated applications of Lemma 3

\[(M_n^# M_{n-1}^# \cdots M_1^#) M_D^# (M_1^# \cdots M_{n-1}^# M_n^#) = M_{h(D)}^#.\]

On the other hand, repeated applications of part (a) of the first corollary to Lemma 2 yield

\[(M_n^# M_{n-1}^# \cdots M_1^#)(M_1^# \cdots M_{n-1}^# M_n^#) = (-1)^n I.\]

This means that if \(n\) is even, so the first factor is \(H\), then the second factor must be \(H^{-1}\), while if \(n\) is odd, so the first factor is \(H^\#\), then the second factor must be \([-H^{-1}]^\#\).

8. THE LINEAR ACTION ON \(\mathbb{R}^6\)

We have been led to associate cap vectors in \(\mathbb{R}^6\) with the symbols for their boundary inversion:

\[C = (C_1, C_2, C_3, C_4, C_5, C_6) = (Q, -\frac{1}{2}(R + S), -\frac{1}{2}(R - S))\]

\[\Leftrightarrow M_C^\# = \begin{pmatrix} Q & R \\ S & -Q \end{pmatrix}^\#.\]

Now we modify this to a bijection between \(\mathbb{R}^6\) and a 6-dimensional subspace of the quaternion matrices, which we shall refer to as the coordinate matrices:

\[U = (U_1, U_2, U_3, U_4, U_5, U_6) = (Q, -\frac{1}{2}(R + S), -\frac{1}{2}(R - S))\]

\[\Leftrightarrow M_U = \begin{pmatrix} Q & R \\ S & -Q \end{pmatrix}.\]

For a cap vector \(C\), \(M_C\) is just \(M_C^\#\) without the \#, but in general coordinate matrices will not be invertible. For example a point \(Q \in \mathbb{R}^4\) has coordinate
vector and coordinate matrix

\[
(Q, \frac{1}{2}(Q\overline{Q} - 1), \frac{1}{2}(Q\overline{Q} + 1)) \leftrightarrow M_Q = \begin{pmatrix}
Q & -Q\overline{Q} \\
1 & -\overline{Q}
\end{pmatrix}.
\]

**Theorem 5.** Transformations \( Q \rightarrow (AQ + B)(CQ + D)^{-1} \) and \( Q \rightarrow (A\overline{Q} + B)(C\overline{Q} + D)^{-1} \) normalized to \( \Delta = 1 \) act on \( \mathbb{R}^6 \) via coordinate matrices by

\[
M_U \rightarrow \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} M_U \begin{pmatrix}
\overline{D} & -\overline{B} \\
-\overline{C} & \overline{A}
\end{pmatrix}
\]

and

\[
M_U \rightarrow \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \overline{M_U} \begin{pmatrix}
\overline{D} & -\overline{B} \\
-\overline{C} & \overline{A}
\end{pmatrix}
\]

respectively.

**Proof.** Since every vector in \( \mathbb{R}^6 \) is a real linear combination of cap vectors, it is enough to check the formulae on these vectors. But for cap vectors Theorem 4(a) gives

\[
M_{h(D)} = HM_D(H^{-1})',
\]

and the first formula follows because

\[
(H^{-1})' = \begin{pmatrix}
\overline{D} & -\overline{B} \\
-\overline{C} & \overline{A}
\end{pmatrix}.
\]

In the orientation-reversing case Theorem 4(b) gives

\[
M_{h(D)} = -HM_D'(H^{-1})',
\]

and the second formula follows because

\[
(H^{-1})' = \begin{pmatrix}
\overline{D} & -\overline{B} \\
-\overline{C} & \overline{A}
\end{pmatrix}
\]

and \( M_D' = -\overline{M}_D \).
Note that the second formula also follows because \( Q \rightarrow (A\overline{Q} + B) \) \((C\overline{Q} + D)^{-1}\) is just the composite of \( Q \rightarrow \overline{Q} \) and \( Q \rightarrow (A\overline{Q} + B)(C\overline{Q} + D)^{-1}\). The effect of \( Q \rightarrow \overline{Q} \) on point coordinate matrices \( M_Q \) is obviously \( M_Q \rightarrow \overline{M}_Q \), and since these span \( \mathbb{R}^6 \), they determine the action.

A second point of interest related to the matrices \( M_Q \) concerns their behavior under the action of \( h(Q) = (AQ + B)(CQ + D)^{-1} \). If we work out the details, we find

\[
\left( \begin{array}{cc}
A & B \\
C & D
\end{array} \right) M_Q \left( \begin{array}{cc}
\bar{D} & -\bar{B} \\
-\bar{C} & \bar{A}
\end{array} \right) = |CQ + D|^2 M_{h(Q)} = \frac{1}{|h'(Q)|} M_{h(Q)},
\]

where \( |h'(Q)| \) is the local magnification factor of \( h \) at \( Q \) as determined in Lemma 1.

As an easy corollary of Theorem 5 we have

**Theorem 6.** The 2-1 homomorphism from \( 2 \times 2 \) quaternion matrices satisfying \( \Delta = 1 \) to \( 6 \times 6 \) real proper orthochronous Lorentz matrices is given explicitly by mapping

\[
\pm \left( \begin{array}{cc}
A & B \\
C & D
\end{array} \right)
\]

to

\[
\left( \begin{array}{cccc}
A\overline{D} + B\overline{C} & A\overline{D} - B\overline{C} & A\overline{j}D - B\overline{j}C & A\overline{k}D - B\overline{k}C \\
S(A\overline{D} - C\overline{D}) & S(A\overline{D} - C\overline{D}) & S(A\overline{j}D - C\overline{j}D) & S(A\overline{k}D - C\overline{k}D) \\
S(A\overline{D} + C\overline{D}) & S(A\overline{j}D + C\overline{j}D) & S(A\overline{k}D + C\overline{k}D) & S(A\overline{k}D + C\overline{k}D)
\end{array} \right)
\]

where the first four entries in each column are given by the components of the quaternion indicated, and \((\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)\) is an abbreviation for the sum

\[
\frac{1}{2}(\epsilon_1 |A|^2 + \epsilon_2 |B|^2 + \epsilon_3 |C|^2 + \epsilon_4 |D|^2).
\]

(See also the announcement of Hellegouarch [6].)
Proof. The standard basis for \( \mathbb{R}^6 \) is represented by the coordinate matrices

\[
M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad M_3 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix},
\]

\[
M_4 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

For a sample we calculate the third column \( U^t \) in the Lorentz matrix. First

\[
M_U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \begin{pmatrix} \bar{D} & -\bar{B} \\ -\bar{C} & \bar{A} \end{pmatrix}
\]

\[
= \begin{pmatrix} A\bar{D} - Bj\bar{C} & -Aj\bar{B} + Bj\bar{A} \\ Cj\bar{D} - Dj\bar{C} & -Cj\bar{B} + Dj\bar{A} \end{pmatrix}
\]

\[
= \begin{pmatrix} A\bar{D} - Bj\bar{C} & -2S(Aj\bar{B}) \\ 2S(Cj\bar{D}) & -(A\bar{D} - Bj\bar{C}) \end{pmatrix};
\]

then

\[
U = \left( A\bar{D} - Bj\bar{C}, S(Aj\bar{B} - Cj\bar{D}), S(Aj\bar{B} + Cj\bar{D}) \right).
\]

We are continuing to make use of our flexible notation in order to streamline these calculations.

9. MODELS OF VARIOUS GEOMETRIES

In \( N \) dimensions the Möbius group \( M_N \) acts on \( \Pi^N = \mathbb{R}^N \cup \{ \infty \} \) or (after conjugation by stereographic projection) on the unit \( N \)-sphere \( \Sigma^N \subset \mathbb{R}^{N+1} \). There are various connections with conformal models of the classical geometries and the groups belonging to these geometries.
9.1. Hyperbolic \((N + 1)\)-Space

First of all, the unit ball

\[
B^{N+1} = \{ x \in \mathbb{R}^{N+1} : \|x\| < 1 \}
\]

bounded by \(\Sigma^N\) can be equipped with the hyperbolic metric

\[
 dh = \frac{2 \, ds}{1 - \|x\|^2},
\]

and then it serves as a conformal model of hyperbolic \((N + 1)\)-space. The elements of \(M_N\) have a unique Poincaré extension from \(\Sigma^N\) to the elements of \(M_{N+1}\) which act on \(\Pi^{N+1}\) and stabilize the ball \(B^{N+1}\). In this way the group \(M_N\) acts as the full group of isometries of hyperbolic \((N + 1)\)-space.

An element of \(M_N\) is represented by an \((N + 2)\times(N + 2)\) Lorentz matrix \(L = (l_{ij})\) and can be assigned a norm \(\|L\| = (\Sigma l_{ij}^2)^{1/2}\). This norm is related to the hyperbolic distance \(\rho\) through which the Poincaré extension of \(L\) moves the center \(0\) of \(B^{N+1}\). The exact formula is [4]

\[
\|L\|^2 = \|I\|^2 + 4 \sinh^2 \rho.
\]

The proof of this formula involves the intermediate stage

\[
\|L\|^2 = N + 2 + 4(l_{N+2,N+2}^2 - 1),
\]

so that really the key fact is that

\[
l_{N+2,N+2} = \cosh \rho.
\]

But now the formula of Theorem 6 shows us that when \(N = 4\) there is a neat connection with our quaternion matrices.

**Theorem 7.** If the Möbius transformation \(Q \to (AQ + B)(CQ + D)^{-1}\) or \(Q \to (AQ + B)(C\overline{Q} + D)^{-1}\) with \(\Delta = 1\) acts through its Poincaré extension as an isometry of the hyperbolic 5-space modeled in the ball \(B^5\), it moves the center \(0\) of \(B^5\) through a hyperbolic distance \(\rho\) given by the formula

\[
\|M\|^2 = |A|^2 + |B|^2 + |C|^2 + |D|^2 = 2 \cosh \rho.
\]
9.2. Spherical N-Space

The stabilizer of the center of $B^{N+1}$ acts on all of $\mathbb{R}^{N+1}$ as the orthogonal group $O(N+1)$. We can recognize the elements of the subgroup $O(N+1) \subset M_N$ as the transformations of minimal norm $\sqrt{N+2}$. Alternatively, the elements of $O(N+1)$ are the isometries of $\Sigma^N$ with its usual metric, and they can be recognized because they commute with the standard antipodal map. In point of fact all the fixed-point-free involutions of $\Sigma^N$ in $M_N$ are conjugate to the standard antipodal map. This means that subgroups conjugate to $O(N+1)$ in $M_N$ can be described either as the stabilizer of a point in $B^{N+1}$ or as the centralizer of a fixed-point-free involution of $\Sigma^N$.

Returning to our quaternion description of $M_4$, we observe that the antipodal map of the 4-sphere is given by $Q \rightarrow -\overline{Q}^{-1}$, and hence the transformation $Q \rightarrow \overline{Q}$ belongs to the orthogonal group $O(5)$. This means that the transformations $Q \rightarrow (A\overline{Q} + B)(C\overline{Q} + D)^{-1}$ and $Q \rightarrow (AQ + B)(CQ + D)^{-1}$ are either both in $O(5)$ or both not in $O(5)$.

**Theorem 8.** The following conditions are equivalent for the transformation

$$Q \rightarrow (AQ + B)(CQ + D)^{-1}$$

with symbol

$$M = \begin{pmatrix} \Lambda & B \\ C & D \end{pmatrix}$$

satisfying $\Delta = 1$ to be in $O(5)$:

(i) $\|M\|^2 = |A|^2 + |B|^2 + |C|^2 + |D|^2 = 2$,

(ii) $M^* = M^{-1}$,

(iii) $|A|^2 = |D|^2$, $|B|^2 = |C|^2 = 1 - |A|^2$, $A\overline{C} + B\overline{D} = \overline{AB} + \overline{CD} = 0$.

**Proof.** The elements of $O(5)$ stabilize the center 0 of $B^5$, and by Theorem 7 this is equivalent to condition (i).
The elements of $O(5)$ commute with the antipodal map $Q \to -\bar{Q}^{-1}$, which carries the symbol

$$A^\# = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^\#.$$ 

Our transformation commutes with the antipodal map if and only if

$$A^\#M = \pm MA^\#, \quad \text{hence} \quad A^\#M(A^{-1})^\# = \pm M.$$

To clear up the ambiguity of sign we note that $M$ must factor as a product of inversions in great 3-spheres, and by Theorem 4 these satisfy

$$-A^\#M_D^\#(A^{-1})^\# = M_D^\#, = M_D^* = -M_D^*.$$

The appropriate criterion is therefore

$$A^\#M(A^{-1})^\# = M \quad \text{or} \quad AM'A^{-1} = M.$$

We recognize $A = K^{-1}$, so $M' = A^{-1}M^{-1}A$ and our criterion reduces to condition (ii).

The equivalence of conditions (ii) and (iii) follows from the fact that

$$MM^* = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{pmatrix} = \begin{pmatrix} |A|^2 + |B|^2 & \bar{A}C + \bar{B}D \\ \bar{A}C + \bar{B}D & |C|^2 + |D|^2 \end{pmatrix}$$

and

$$M^*M = \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} |A|^2 + |C|^2 & \bar{A}B + \bar{C}D \\ \bar{A}B + \bar{C}D & |B|^2 + |D|^2 \end{pmatrix}.$$
equivalence of conditions (i) and (iii). This proof uses the formula for $\Delta^2$
which we developed in Section 3.

\[
|AC + BD|^2 + |\overline{AB} + \overline{CD}|^2 + \frac{1}{2}(|A|^2 - |D|^2)^2 + \frac{1}{2}(|B|^2 - |C|^2)^2
\]

\[
= (AC + B\overline{D})(C\overline{A} + D\overline{B}) + (\overline{AB} + \overline{CD})(\overline{BA} + \overline{DC})
\]

\[
+ \frac{1}{2}(|A|^2 + |D|^2)^2 + \frac{1}{2}(|B|^2 + |C|^2)^2 - 2|A|^2|D|^2 - 2|B|^2|C|^2
\]

\[
= |A|^2|C|^2 + |B|^2|D|^2 + 2S(ACDB)
\]

\[
+ |A|^2|B|^2 + |C|^2|D|^2 + 2S(ACDB)
\]

\[
+ \frac{1}{2}(|A|^2 + |D|^2)^2 + \frac{1}{2}(|B|^2 + |C|^2)^2
\]

\[
- 2|A|^2|D|^2 - 2|B|^2|C|^2
\]

\[
= \frac{1}{2}(|A|^2 + |D|^2)^2 + (|A|^2 + |D|^2)(|B|^2 + |C|^2)
\]

\[
+ \frac{1}{2}(|B|^2 + |C|^2)^2 - 2\Delta^2
\]

\[
= \frac{1}{2}(|A|^2 + |B|^2 + |C|^2 + |D|^2)^2 - 2.
\]

**Theorem 9.** There is a 2-1 homomorphism from the quaternion matrices

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

satisfying $\Delta = 1$ and $|A|^2 + |B|^2 + |C|^2 + |D|^2 = 2$ onto the real proper orthogonal $5 \times 5$ matrices given by mapping

\[
\pm \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
to

\[
\begin{pmatrix}
A\bar{D} + B\bar{C} & A\bar{a}D - B\bar{b}C & A\bar{j}D - B\bar{j}C & A\bar{k}D - B\bar{k}C & A\bar{c} - B\bar{d}
\end{pmatrix}
\]

\[
S(\bar{A}D - C\bar{D}) \quad S(\bar{A}B - C\bar{D}) \quad S(\bar{A}jD - C\bar{j}D) \quad S(\bar{A}k\bar{D} - C\bar{k}\bar{D})
\]

The same notational conventions apply as in Theorem 6.

Proof. When \( M \) represents an orthogonal transformation, the matrix of Theorem 6 must reduce to a diagonal 1 and the orthogonal matrix above. For interest’s sake we verify directly that conditions (i) and (iii) of Theorem 8 reduce the last row and column of the matrix of Theorem 6 as required. It is immediate that \((+,-,+,+)=1 \) and \( A\bar{C} + B\bar{D} = (+,+,-,-) = (+,-,+,-) = 0 \). The vanishing of the other four entries follows from \( AB + CD = 0 \) because, for example,

\[
S(\bar{A}B + C\bar{D}) = S((\bar{B}A + \bar{D}C)i)
\]

\[
= -V_{1}(\bar{B}A + \bar{D}C)
\]

\[
- V_{1}(\bar{A}B + \bar{C}D).
\]

9.3. Euclidean N-Space

Turning from spherical geometry to Euclidean, we recall that the stabilizer of \( 00 \) represents the Euclidean similarities. Thus the similarities of \( R^{4} \) are given by the quaternion transformations with \( C = 0 \), namely

\[
Q \rightarrow AQD^{-1} + BD^{-1},
\]

\[
A \rightarrow A\bar{Q}D^{-1} + BD^{-1}.
\]

By Lemma 1 or directly we see that the magnification factor for these transformations is \(|AD^{-1}|\). The isometries of \( \mathbb{R}^{N} \) are the similarities with magnification factor equal to 1. Thus if we maintain the normalization \( \Delta = |AD| = 1 \), the isometries of \( R^{4} \) constitute the subgroup with \( C = 0 \) and \(|A| = |D| = 1 \). The isometries which stabilize the origin also satisfy \( B = 0 \),
and we obtain a representation of $O(4)$ by the transformations

$$Q \rightarrow AQ\overline{D} \quad \text{and} \quad Q \rightarrow A\overline{Q}\overline{D}.$$ 

In a moment we shall derive this result a second way, but this seems the appropriate place to state

**Theorem 10.** There is a 2-1 homomorphism from the quaternion matrices

$$M = \begin{pmatrix} A & O \\ O & D \end{pmatrix}$$

satisfying $|A| = |D| = 1$ onto the real proper orthogonal $4 \times 4$ matrices given by

$$\pm \begin{pmatrix} A & O \\ O & D \end{pmatrix} \rightarrow \begin{pmatrix} | & | & | & | \\ A\overline{D} & Ai\overline{D} & Aj\overline{D} & Ak\overline{D} \end{pmatrix}.$$ 

The same notational convention applies as in Theorems 6 and 9.

**Proof.** This result can be derived either by specializing Theorem 9 or by applying the formula $Q \rightarrow AQ\overline{D}$ to the basis $1, i, j, k$.

### 9.4. Hyperbolic N-Space

An arbitrary $N$-ball or half space in $\mathbb{R}^N \cup \{\infty\}$ can serve as a Poincaré (≡ conformal) model of hyperbolic $N$-space. The isometries of this model are given by the restrictions of the Möbius transformations in $M_N$ which stabilize it. If $H$ is an arbitrary 4-ball or half space in $\mathbb{R}^4 \cup \{\infty\}$, then by Theorem 5, $Q \rightarrow (AQ + B(CQ + D)^{-1}$ stabilizes $H$ if and only if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} \overline{D} \\ -\overline{C} \end{bmatrix} = \begin{bmatrix} \overline{-B} \\ \overline{A} \end{bmatrix} = M_H.$$
and \( Q \to (A\overline{Q} + B)(C\overline{Q} + D)^{-1} \) stabilizes \( H \) if and only if
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
\overline{D} & -\overline{B} \\
-\overline{C} & \overline{A}
\end{pmatrix}
= M_H.
\]

There are two very natural choices for \( H \)—the unit ball
\[ H_0 = (0, -1, 0) \quad \text{with} \quad M_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
and the half spaces
\[ H_N = (N, 0, 0) \quad \text{with} \quad M_N = \begin{pmatrix} N & 0 \\ 0 & -\overline{N} \end{pmatrix}. \]

In the case of the unit ball, the direct isometries satisfy
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\overline{D} & -\overline{B} \\
-\overline{C} & \overline{A}
\end{pmatrix}
= \begin{pmatrix}
-A\overline{C} + B\overline{D} \\
-|C|^2 + |D|^2
\end{pmatrix}
= \begin{pmatrix}
|A|^2 - |B|^2 \\
-(-A\overline{C} + B\overline{D})
\end{pmatrix}
= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
or
\[ |A|^2 - |B|^2 = |D|^2 - |C|^2 = 1 \quad \text{and} \quad A\overline{C} = B\overline{D}. \]

The opposite isometries are just these composed with \( Q \to \overline{Q} \). Among the direct isometries the ones which stabilize 0 are those with \( 0 = (A0 + B)(C0 + D)^{-1} \), hence \( B = 0 \). But this implies in turn \( |A| = 1, C = 0, |D| = 1, \) and we recover the representation of \( O(4) \) which we had a moment ago.

In the case of the half space \( H_N \), the direct isometries satisfy
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
N & 0 \\
0 & -\overline{N}
\end{pmatrix}
\begin{pmatrix}
\overline{D} & -\overline{B} \\
-\overline{C} & \overline{A}
\end{pmatrix}
= \begin{pmatrix}
A\overline{D} + B\overline{N}\overline{C} \\
2S(C\overline{N}\overline{D})
\end{pmatrix}
= \begin{pmatrix}
2S(AN\overline{B}) \\
-(-A\overline{D} + B\overline{N}\overline{C})
\end{pmatrix}
= \begin{pmatrix} N & 0 \\ 0 & -\overline{N} \end{pmatrix},
\]
or

\[ S(ANB) = S(CND) = 0 \quad \text{and} \quad AND + BNC = N, \]

and the opposite isometries satisfy

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
N & 0 \\
0 & -N
\end{pmatrix}
\begin{pmatrix}
\overline{D} & -\overline{B} \\
-\overline{C} & \overline{A}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
A\overline{N}\overline{D} + BNC & -2S(ANB) \\
2S(C\overline{N}\overline{D}) & -(AND + BNC)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
N & 0 \\
0 & -\overline{N}
\end{pmatrix},
\]

or

\[ S(ANB) = S(CND) = 0 \quad \text{and} \quad A\overline{N}\overline{D} + BNC = N. \]

10. INTRODUCTION REVISITED

The applications of quaternions mentioned in the introduction can now be viewed from our more advanced perspective.

In the remarks preceding Theorem 10 and again in Section 9.4 we saw that pairs of unit quaternions \( A \) and \( D \) with \( |A| = |D| = 1 \) can be used to represent the direct isometries of \( \mathbb{R}^4 \) in the form

\[ Q \rightarrow AQD. \]

Theorem 10 itself reminds us that the action of this transformation on the basis vectors \( 1, i, j, k \) of \( \mathbb{R}^4 \) is \( 1 \rightarrow A\overline{D}, i \rightarrow Ai\overline{D}, j \rightarrow Aj\overline{D}, k \rightarrow Ak\overline{D} \). The isometries of the \( \mathbb{R}^3 \) spanned by \( i, j, k \) appear as the subgroup that stabilizes \( 1 \). Since \( |D| = |A| = 1 \), the condition \( A\overline{D} = 1 \) means simply that \( D = A \) and we get our representation of the isometries of \( \mathbb{R}^3 \) by the mapping

\[ q \rightarrow Aq\overline{A} \]

on pure quaternions.
Theorem 6 gives insight into the action of a $2 \times 2$ quaternion matrix

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
$$

whose entries all happen to be complex numbers. Since complex numbers commute, the eight associated quaternions all reduce to $ad - bc$. The standard quaternion normalization assures that the complex number $\lambda = ad - bc$ has norm 1, but not that it actually equal to 1. We shall use Theorem 6 to get an insight into the action of this transformation and the way it specializes when we do have $\lambda = 1$.

Two nice pieces of algebra related to the simplification of the matrix of Theorem 6 are that

$$
ajd - bj\bar{c} = (ad - bc)j = \lambda j \in \text{span}\{j, k\}
$$

and

$$
S(ajb - bj\bar{c}) = S((ab - cd)j) = 0.
$$

With these simplifications and other similar ones the $6 \times 6$ real matrix becomes

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
ad + b\bar{c} & (ad - b\bar{c})i & a\bar{c} - b\bar{d} & a\bar{c} + b\bar{d} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
S(ab - cd) & S((ab - cd)i) & 0 & 0 & (+-+-) & (+-+-) \\
S(ab + cd) & S((ab + cd)i) & 0 & 0 & (+---) & (+---)
\end{pmatrix}
$$

The 2-space span$\{1, i\}$ is invariant, and the pencil of 3-spaces containing this 2-space, which includes the 3-spaces with normals $j$ and $k$, is turned in a uniform way about it. If $\lambda = 1$, these 3-spaces remain fixed, so that each of them is mapped into itself. Any one of them, say the one with normal $k$, furnishes a Poincaré half-space model of hyperbolic 3-space with absolute equal to $\mathbb{C}$. The Poincaré extension of the mapping $z \rightarrow (az + b)/(cz + d)$
from the absolute to this hyperbolic 3-space is naturally given by

\[ z + tj \rightarrow [a(z + tj) + b][c(z + tj) + d]^{-1}, \]

as claimed in the introduction.

Finally, we can use the remarks above to see that the famous homomorphism of \( SL(2, \mathbb{C}) \) onto the \( 4 \times 4 \) orthochronous Lorentz matrices must be the restriction of the homomorphism of Theorem 6 contracted so that its image consists of real \( 4 \times 4 \) matrices:

\[
\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

\[
\rightarrow \begin{pmatrix} a\bar{d} + b\bar{c} & (a\bar{d} - b\bar{c})i \\ a\bar{c} - b\bar{d} & a\bar{c} + b\bar{d} \end{pmatrix}.
\]

This formula is in agreement with similar ones developed by other means. (See [8] and references cited there.)

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