ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Advances in Mathematics 226 (2011) 4373–4428

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Cocycles over interval exchange transformations and multivalued Hamiltonian flows

Jean-Pierre Conze [a], Krzysztof Frączek [b,c,*,1]

[a] *Université de Rennes I, IRMAR, CNRS UMR 6625, Campus de Beaulieu, 35042 Rennes Cedex, France*
[b] *Faculty of Mathematics and Computer Science, N. Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland*
[c] *Institute of Mathematics, Polish Academy of Science, ul. Śniadeckich 8, 00-956 Warszawa, Poland*

Received 21 March 2010; accepted 22 November 2010

Available online 23 December 2010

Communicated by R. Daniel Mauldin

Abstract

We consider interval exchange transformations of periodic type and construct different classes of ergodic cocycles of dimension $\geqslant 1$ over this special class of IETs. Then using Poincaré sections we apply this construction to obtain the recurrence and ergodicity for some smooth flows on non-compact manifolds which are extensions of multivalued Hamiltonian flows on compact surfaces.

MSC: 37A40; 37C40

Keywords: Interval exchange transformation; Cocycle; Multivalued Hamiltonian flow; Infinite invariant measure; Ergodicity

Contents

* Corresponding author at: Faculty of Mathematics and Computer Science, N. Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland.

E-mail addresses: Jean-Pierre.Conze@univ-rennes1.fr (J.-P. Conze), fraczek@mat.umk.pl (K. Frączek).

[1] Research partially supported by MNiSzW grant N N201 384834 and Marie Curie "Transfer of Knowledge" program, project MTKD-CT-2005-030042 (TODEQ).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2010.11.014

1. Introduction

In this paper we investigate the conservativity (recurrence) and ergodicity of dynamical systems given by skew products of ergodic interval exchange transformations (abbreviated as IETs) and some piecewise smooth cocycles taking values in locally compact abelian groups.

Let G be a locally compact abelian second countable group. We denote by 0 its identity element, by $\mathcal{B}_G$ its σ-algebra of Borel sets and by m_G its Haar measure. Recall that, for each ergodic automorphism $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ of a standard Borel probability space, each measurable function $\varphi:X\to G$ defines a *skew product* automorphism T_φ which preserves the σ-finite measure $\mu\times m_G$:

$$\begin{aligned}T_\varphi:(X\times G,\mathcal{B}\times\mathcal{B}_G,\mu\times m_G)&\to(X\times G,\mathcal{B}\times\mathcal{B}_G,\mu\times m_G),\\ T_\varphi(x,g)&=\big(Tx,g+\varphi(x)\big).\end{aligned}\tag{1.1}$$

We will mostly consider groups G which are closed subgroups of $\mathbb{R}^\ell$, for $\ell\geqslant 1$. The function $\varphi:X\to G$ determines also a *cocycle* $\varphi^{(\cdot)}:\mathbb{Z}\times X\to G$ for the $\mathbb{Z}$-action given by T by the formula

$$\varphi^{(n)}(x)=\begin{cases}\varphi(x)+\varphi(Tx)+\dots+\varphi(T^{n-1}x) & \text{if } n>0,\\ 0 & \text{if } n=0,\\ -(\varphi(T^nx)+\varphi(T^{n+1}x)+\dots+\varphi(T^{-1}x)) & \text{if } n<0.\end{cases}$$

Then $T_\varphi^n(x,g) = (T^n x, g + \varphi^{(n)}(x))$ for every $n \in \mathbb{Z}$. The cocycle $\varphi^{(\cdot)}$ can be viewed as a *"stationary" walk* in G over the dynamical system $(X, \mathcal{B}, \mu, T)$. For simplicity, the expression "cocycle φ" refers to the cocycle $\varphi^{(\cdot)}$ generated by φ over the dynamical system $(X, \mathcal{B}, \mu, T)$. We say that the cocycle φ is *recurrent* if for each neighborhood $V \subset G$ of the identity element and for a.e. $x \in X$, $\varphi^{(n)}(x) \in V$ for infinitely many $n \geqslant 0$.

Recall that the skew product T_φ has a trivial Hopf decomposition, that is T_φ is either conservative or totally dissipative (see [32] and [1]). Moreover, the skew product T_φ is conservative if and only if the cocycle φ is recurrent. The skew product system $(X \times G, \mu \times m_G, T_\varphi)$ is called *ergodic* if for any set $A \in \mathcal{B} \times \mathcal{B}_G$ which is invariant, i.e. such that $\mu \times m_G(T_\varphi^{-1} A \triangle A) = 0$, either $\mu \times m_G(A) = 0$ or $\mu \times m_G(A^c) = 0$, where A^c denotes the complement of A. Then we call the cocycle $\varphi^{(\cdot)}$ ergodic as well. Recall that if $(X, \mathcal{B}, \mu)$ is a standard space then any ergodic cocycle is automatically recurrent.

Skew products over IETs appear in a natural way in the study of the billiard flows in the plane with $\mathbb{Z}^2$ periodically distributed obstacles (wind-tree models). For instance when the obstacles are rectangles, such flows can be modeled as special flows built over such skew products. The recurrence and ergodicity of these models are among main open questions. More generally, skew products over IETs appear when studying linear flows on $\mathbb{Z}^\ell$-covers of translation surfaces (see a recent paper by P. Hubert and B. Weiss [20]). Another motivation to study skew products over IETs comes from some extensions of area-preserving flows on closed surfaces, defined as couplings of such flows with the geodesic flow on $\mathbb{R}^\ell$ (see [11] and Section 6 for details). All these dynamical systems have a special representation over a skew product of an IET and piecewise smooth cocycle. Therefore, the recurrence and ergodicity of these systems are equivalent to the recurrence and ergodicity of the corresponding skew products.

The literature on skew products over an irrational rotation on the circle is vast, and several classes of ergodic cocycles with values in $\mathbb{R}$ or $\mathbb{R}^\ell$ are known in that case; see [3,7,11,14,15,26, 29,30] and [32] for some classes of ergodic $\mathbb{R}$-valued cocycles (smooth or piecewise absolutely continuous non-continuous or piecewise smooth with logarithmic singularities), [27] for ergodic cocycles in $\mathbb{R}^2$, [19] for examples of ergodic cocycles with values in a nilpotent group, [9] for ergodic cocycles in $\mathbb{Z}^2$ associated to special directional rectangular billiard flows in the plane. But very little is known about the ergodicity of cocycles over IETs.

In this paper we will deal with so-called interval exchange transformations of bounded or periodic type (see [33]) which are analogues of rotations on the circle by irrational numbers with bounded partial quotients (abbreviated as bpq) or quadratic irrationals, respectively. The aim of this paper is to provide different classes of recurrent and ergodic cocycles over IETs in these special classes. This is done in Sections 3, 4, and 5. We emphasize that typically proofs of the ergodicity of skew products over irrational rotations are based on so-called Denjoy–Koksma type inequalities along the sequence of denominators which break down for cocycles of bounded variation (abbreviated as BV cocycles) over IETs (see [41]). For BV cocycles φ over IETs of bounded (periodic) type we bypass this problem in different ways.

In Section 3 the ergodicity problem is studied in the class of piecewise linear cocycles with non-zero sum of jumps and their products. Here we combine the recurrence of the cocycle with the control of the growth of slope to localize so-called essential values of φ.

In Section 4 we deal with $\mathbb{Z}^k$-valued cocycles which are constant on exchanged intervals. We prove that such a $\mathbb{Z}$-valued cocycle φ is ergodic if the vector composed from the values of φ is a fixed vector of the periodic matrix of the IET. Moreover, we give a condition that guarantees the ergodicity of $\mathbb{Z}^k$-valued cocycles as well.

In Section 5 piecewise constant cocycles have supplementary discontinuity points inside exchanged intervals. Adapting the idea of corrections of piecewise smooth cocycles introduced by Marmi, Moussa and Yoccoz in [28], we correct φ by a function constant on exchanged intervals and then we prove a Denjoy–Koksma type inequality for the corrected cocycle. Since the supplementary discontinuities survive the procedure of correction, they are used to localize essential values of the corrected cocycle.

In Sections 6 and 7 we present smooth models for recurrent and ergodic systems based on the previous sections. Following [11] we deal with couplings of multivalued Hamiltonian flows on surfaces M and the geodesic flow on $\mathbb{R}^\ell$, $\ell \geqslant 1$. More precisely, we will consider coupled differential equations on $M \times \mathbb{R}^\ell$ of the form

$$\dot{x} = X(x), \qquad \dot{y} = f(x) \quad \text{for } (x, y) \in M \times \mathbb{R}^\ell, \tag{1.2}$$

where $X : M \to TM$ is a multivalued Hamiltonian vector field and $f : M \to \mathbb{R}^\ell$ is a smooth function. If f vanishes at zeros of X then the flow associated to (1.2) has a Poincaré section for which the first-return map is isomorphic to the skew product of an IET and a BV cocycle. This allows us to prove a sufficient condition for recurrence and ergodicity (see Section 6) whenever the IET is of periodic type.

The whole theory developed in the paper is illustrated by examples in Section 7 (see also Appendix B).

In Appendices A and C proofs of the needed results on the growth of cocycles of bounded variation are given. They are mainly adapted from [28].

2. Preliminaries

2.1. Interval exchange transformations

In this subsection, we recall standard facts on IETs with the presentation and notation from [37,38] and [39]. Let $\mathcal{A}$ be a d-element alphabet and let $\pi = (\pi_0, \pi_1)$ be a pair of bijections $\pi_\varepsilon : \mathcal{A} \to \{1, \ldots, d\}$ for $\varepsilon = 0, 1$. Denote by $\mathcal{S}^0_{\mathcal{A}}$ the subset of irreducible pairs, i.e. such that $\pi_1 \circ \pi_0^{-1}\{1, \ldots, k\} \neq \{1, \ldots, k\}$ for $1 \leqslant k < d$. We will denote by π_d^{sym} any pair (π_0, π_1) such that $\pi_1 \circ \pi_0^{-1}(j) = d + 1 - j$ for $1 \leqslant j \leqslant d$.

Let us consider $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}_+^{\mathcal{A}}$, where $\mathbb{R}_+ = (0, +\infty)$. Set

$$|\lambda| = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha, \qquad I = \bigl[0, |\lambda|\bigr)$$

and

$$I_\alpha = [l_\alpha, r_\alpha), \quad \text{where } l_\alpha = \sum_{\pi_0(\beta) < \pi_0(\alpha)} \lambda_\beta, \ r_\alpha = \sum_{\pi_0(\beta) \leqslant \pi_0(\alpha)} \lambda_\beta.$$

Then $|I_\alpha| = \lambda_\alpha$. Denote by Ω_π the matrix $[\Omega_{\alpha\beta}]_{\alpha, \beta \in \mathcal{A}}$ given by

$$\Omega_{\alpha\beta} = \begin{cases} +1 & \text{if } \pi_1(\alpha) > \pi_1(\beta) \text{ and } \pi_0(\alpha) < \pi_0(\beta), \\ -1 & \text{if } \pi_1(\alpha) < \pi_1(\beta) \text{ and } \pi_0(\alpha) > \pi_0(\beta), \\ 0 & \text{in all other cases.} \end{cases}$$

This defines an antisymmetric form on $\mathbb{R}^{\mathcal{A}}$.

Given $(\pi,\lambda)\in\mathcal{S}^0_{\mathcal{A}}\times\mathbb{R}^{\mathcal{A}}_+$, let $T_{(\pi,\lambda)}:[0,|\lambda|)\to[0,|\lambda|)$ stand for the *interval exchange transformation* (IET) on d intervals I_α, $\alpha\in\mathcal{A}$, which are rearranged according to the permutation $\pi_1\circ\pi_0^{-1}$, i.e. $T_{(\pi,\lambda)}x=x+w_\alpha$ for $x\in I_\alpha$, where $w=\Omega_\pi\lambda$, that is $w_\alpha=\sum_{\beta\in\mathcal{A}}\Omega_{\alpha\beta}\lambda_\beta$.

Note that for every $\alpha\in\mathcal{A}$ with $\pi_0(\alpha)\neq 1$ there exists $\beta\in\mathcal{A}$ such that $\pi_0(\beta)\neq d$ and $l_\alpha=r_\beta$. It follows that

$$\bigl\{l_\alpha\colon \alpha\in\mathcal{A},\ \pi_0(\alpha)\neq 1\bigr\}=\bigl\{r_\alpha\colon \alpha\in\mathcal{A},\ \pi_0(\alpha)\neq d\bigr\}. \tag{2.1}$$

By $\hat{T}_{(\pi,\lambda)}:(0,|I|]\to(0,|I|]$ denote the exchange of the intervals $\hat{I}_\alpha=(l_\alpha,r_\alpha]$, $\alpha\in\mathcal{A}$, i.e. $T_{(\pi,\lambda)}x=x+w_\alpha$ for $x\in\hat{I}_\alpha$. Note that for every $\alpha\in\mathcal{A}$ with $\pi_1(\alpha)\neq 1$ there exists $\beta\in\mathcal{A}$ such that $\pi_1(\beta)\neq d$ and $T_{(\pi,\lambda)}l_\alpha=\hat{T}_{(\pi,\lambda)}r_\beta$. It follows that

$$\bigl\{T_{(\pi,\lambda)}l_\alpha\colon \alpha\in\mathcal{A},\ \pi_1(\alpha)\neq 1\bigr\}=\bigl\{\hat{T}_{(\pi,\lambda)}r_\alpha\colon \alpha\in\mathcal{A},\ \pi_1(\alpha)\neq d\bigr\}. \tag{2.2}$$

A pair (π,λ) satisfies the *Keane condition* if $T^m_{(\pi,\lambda)}l_\alpha\neq l_\beta$ for all $m\geqslant 1$ and for all $\alpha,\beta\in\mathcal{A}$ with $\pi_0(\beta)\neq 1$.

Let $T=T_{(\pi,\lambda)}$, $(\pi,\lambda)\in\mathcal{S}^0_{\mathcal{A}}\times\mathbb{R}^{\mathcal{A}}_+$, be an IET satisfying the Keane condition. Then $\lambda_{\pi_0^{-1}(d)}\neq\lambda_{\pi_1^{-1}(d)}$. Let

$$\tilde{I}=\bigl[0,\max(l_{\pi_0^{-1}(d)},l_{\pi_1^{-1}(d)})\bigr)$$

and denote by $\mathcal{R}(T)=\tilde{T}:\tilde{I}\to\tilde{I}$ the first-return map of T to the interval $\tilde{I}$. Set

$$\varepsilon=\varepsilon(\pi,\lambda):=\begin{cases}0 & \text{if } \lambda_{\pi_0^{-1}(d)}>\lambda_{\pi_1^{-1}(d)},\\ 1 & \text{if } \lambda_{\pi_0^{-1}(d)}<\lambda_{\pi_1^{-1}(d)}.\end{cases}$$

Let us consider the pair $\tilde{\pi}=(\tilde{\pi}_0,\tilde{\pi}_1)\in\mathcal{S}^0_{\mathcal{A}}$, where

$$\tilde{\pi}_\varepsilon(\alpha)=\pi_\varepsilon(\alpha)\quad\text{for all }\alpha\in\mathcal{A}\quad\text{and}$$
$$\tilde{\pi}_{1-\varepsilon}(\alpha)=\begin{cases}\pi_{1-\varepsilon}(\alpha) & \text{if } \pi_{1-\varepsilon}(\alpha)\leqslant\pi_{1-\varepsilon}\circ\pi_\varepsilon^{-1}(d),\\ \pi_{1-\varepsilon}(\alpha)+1 & \text{if } \pi_{1-\varepsilon}\circ\pi_\varepsilon^{-1}(d)<\pi_{1-\varepsilon}(\alpha)<d,\\ \pi_{1-\varepsilon}\circ\pi_\varepsilon^{-1}(d)+1 & \text{if } \pi_{1-\varepsilon}(\alpha)=d.\end{cases}$$

As it was shown by Rauzy in [31], $\tilde{T}$ is also an IET on d-intervals

$$\tilde{T}=T_{(\tilde{\pi},\tilde{\lambda})}\quad\text{with } \tilde{\lambda}=\Theta^{-1}(\pi,\lambda)\lambda,$$

and

$$\Theta(T)=\Theta(\pi,\lambda)=Id+E_{\pi_\varepsilon^{-1}(d)\pi_{1-\varepsilon}^{-1}(d)}\in\mathrm{SL}\bigl(\mathbb{Z}^{\mathcal{A}}\bigr),$$

where Id is the identity matrix and $(E_{\alpha\beta})_{\alpha'\beta'}=\delta_{\alpha\alpha'}\delta_{\beta\beta'}$, using the Kronecker delta notation. Moreover,

$$\Theta^t(\pi,\lambda)\Omega_\pi\Theta(\pi,\lambda)=\Omega_{\tilde{\pi}}. \tag{2.3}$$

It follows that $\ker \Omega_\pi = \Theta(\pi,\lambda)\ker \Omega_{\tilde\pi}$. Since $\Omega_\pi^t = -\Omega_\pi$, thus taking $H_\pi = \Omega_\pi(\mathbb{R}^{\mathcal{A}}) = (\ker\Omega_\pi)^\perp$, we get $H_{\tilde\pi} = \Theta^t(\pi,\lambda)H_\pi$. Recall that for every $(\pi,\lambda)\in \mathcal{S}^0_{\mathcal{A}}\times\mathbb{R}^{\mathcal{A}}_+$ and $h\in H_\pi\cap\mathbb{R}^{\mathcal{A}}_+$ we associate a translation surface $S(\pi,\lambda,h)$ obtained (in the zippered rectangles construction introduced by Veech in [35]) by gluing rectangles of horizontal sides I_α and vertical sides of height h_α. Then the genus g and the number of singularities κ of $S(\pi,\lambda,h)$ depend only on π and satisfy the following relations: $\dim H_\pi = 2g$ and $\dim\ker\Omega_\pi = \kappa - 1$. For more details we refer the reader to [38].

The IET $\tilde T$ fulfills the Keane condition as well. Therefore we can iterate the renormalization procedure and generate a sequence of IETs $(T^{(n)})_{n\geqslant 0}$, where $T^{(n)} = \mathcal{R}^n(T)$ for $n\geqslant 0$. Denote by $\pi^{(n)} = (\pi_0^{(n)},\pi_1^{(n)})\in\mathcal{S}^0_{\mathcal{A}}$ the pair and by $\lambda^{(n)} = (\lambda^{(n)}_\alpha)_{\alpha\in\mathcal{A}}$ the vector which determines $T^{(n)}$. Then $T^{(n)}$ is the first-return map of T to the interval $I^{(n)} = [0,|\lambda^{(n)}|)$ and

$$\lambda = \Theta^{(n)}(T)\lambda^{(n)}\quad \text{with } \Theta^{(n)}(T) = \Theta(T)\cdot\Theta\big(T^{(1)}\big)\cdot\ldots\cdot\Theta\big(T^{(n-1)}\big).$$

2.2. *IETs of periodic type*

Definition 1. (See [33].) An IET T is of *periodic type* if there exists $p>0$ (called a *period of T*) such that $\Theta(T^{(n+p)}) = \Theta(T^{(n)})$ for every $n\geqslant 0$ and $\Theta^{(p)}(T)$ (called a *periodic matrix of T* and denoted by A in what follows) has strictly positive entries.

Remark 2.1. Suppose that $T = T_{(\pi,\lambda)}$ is of periodic type. It follows that

$$\lambda = \Theta^{(pn)}(T)\lambda^{(pn)} = \Theta^{(p)}(T)^n\lambda^{(pn)}\in\Theta^{(p)}(T)^n\mathbb{R}^{\mathcal{A}}_+,$$

and hence λ belongs to $\bigcap_{n\geqslant 0}\Theta^{(p)}(T)^n\mathbb{R}^{\mathcal{A}}_+$ which by [35] it is a one-dimensional convex cone. Therefore λ is a positive right Perron–Frobenius eigenvector of the matrix $\Theta^{(p)}(T)$. Since the set $\mathcal{S}^0_{\mathcal{A}}$ is finite, by taking a multiple of the period p if necessary, we can assume that $\pi^{(p)} = \pi$. It follows that $(\pi^{(p)},\lambda^{(p)}/|\lambda^{(p)}|) = (\pi,\lambda/|\lambda|)$ and $\rho := |\lambda|/|\lambda^{(p)}|$ is the Perron–Frobenius eigenvector of the matrix $\Theta^{(p)}(T)$. Recall that similar arguments to those above show that every IET of periodic type is uniquely ergodic (see [34]).

An explicit construction of IETs of periodic type has been presented in [33].

Let $T = T_{(\pi,\lambda)}$ be an IET of periodic type and p be its period such that $\pi^{(p)} = \pi$. Let $A = \Theta^{(p)}(T)$. By (2.3),

$$A^t\Omega_\pi A = \Omega_{\pi^{(p)}} = \Omega_\pi\quad\text{and hence}\quad \ker\Omega_\pi = A\ker\Omega_\pi\quad\text{and}\quad H_\pi = A^tH_\pi.$$

By taking a multiple of the period p if necessary, we can assume that $A|_{\ker\Omega_\pi} = Id$ (see Appendix C for details). Denote by $Sp(A)$ the collection of complex eigenvalues of A, counted with multiplicities. Let us consider the collection of Lyapunov exponents $\log|\rho|$, $\rho\in Sp(A)$. It consists of the numbers

$$\theta_1 > \theta_2\geqslant\theta_3\geqslant\cdots\geqslant\theta_g\geqslant 0 = \cdots = 0\geqslant -\theta_g\geqslant\cdots\geqslant-\theta_3\geqslant-\theta_2 > -\theta_1,$$

where $2g = \dim H_\pi$ and 0 occurs with the multiplicity $\kappa - 1 = \dim\ker\Omega_\pi$ (see e.g. [40] and [41]). Moreover, $\rho_1 := \exp\theta_1$ is the Perron–Frobenius eigenvalue of A. We will use sometimes the symbol $\theta_i(T)$ instead of θ_i to emphasize that it is associated to T.

Definition 2. An IET of periodic type $T_{(\pi,\lambda)}$ is said to have *non-degenerated spectrum* if $\theta_g > 0$.

Let $T : I \to I$ be an arbitrary IET satisfying the Keane condition. Each finite subset $D \subset I$ determines a partition $\mathcal{P}(D)$ of I into left-closed and right-open intervals. Denote by $\min \mathcal{P}(D)$ and $\max \mathcal{P}(D)$ the length of the shortest and the longest interval of the partition $\mathcal{P}(D)$ respectively. For every $n \geqslant 0$ let $\mathcal{P}_n(T)$ stand for the partition given by the subset $\{T^{-k} l_\alpha\colon \alpha \in \mathcal{A},\ 0 \leqslant k < n\}$. Then T^n is a translation on each interval of the partition $\mathcal{P}_n(T)$.

Definition 3. An IET T satisfying the Keane condition is said to be of *bounded type* if there exists $c \geqslant 1$ such that for every $n \geqslant 1$ we have

$$\frac{1}{cn} \leqslant \min \mathcal{P}_n(T) \leqslant \max \mathcal{P}_n(T) \leqslant \frac{c}{n}. \tag{2.4}$$

Proposition 2.2. *(See [5, Theorem 1.7].) Every IET of bounded type is uniquely ergodic.*

The following result shows that the discontinuities for iterations of IETs of periodic type are well distributed.

Proposition 2.3. *(See [25].) Every IET of periodic type is of bounded type.*

2.3. Growth of BV cocycles

The recurrence of a cocycle φ with values in $\mathbb{R}^\ell$ is related to the growth of $\varphi^{(n)}$ when n tends to ∞.

For an irrational rotation $T : x \to x + \alpha \bmod 1$ (this can be viewed as an exchange of 2 intervals), when φ has a bounded variation, the growth of $\varphi^{(n)}$ is controlled by the Denjoy–Koksma inequality: if φ is a zero mean function on $X = \mathbb{R}/\mathbb{Z}$ with bounded variation $\operatorname{Var}\varphi$ and (q_n) are the denominators (of the convergents) given by the continued fraction expansion of α, then the following inequality holds:

$$\left|\sum_{j=0}^{q_n-1} \varphi(x + j\alpha)\right| \leqslant \operatorname{Var}\varphi \quad \text{for all } x \in X. \tag{2.5}$$

This inequality implies the recurrence of the cocycle $\varphi^{(\cdot)}$ (see [32]) and if α has bounded partial quotients (see [24])

$$\sum_{j=0}^{n-1} \varphi(x + j\alpha) = O(\log n) \quad \text{uniformly in } x \in X. \tag{2.6}$$

It is much more difficult to get a precise upper bound for the growth of a cocycle over an IET. The following theorem (proved in Appendix A) gives for an IET of periodic type a control on the growth of a BV cocycle in terms of the Lyapunov exponents of the matrix A.

Theorem 2.4. *Suppose that $T_{(\pi,\lambda)} : I \to I$ is an interval exchange transformation of periodic type, $0 \leqslant \theta_2 < \theta_1$ are the two largest Lyapunov exponents, and M is the maximal size of Jordan blocks in the Jordan decomposition of its periodic matrix A. Then there exists $C > 0$ such that*

$$\left\|\varphi^{(n)}\right\|_{\sup} \leqslant C \cdot \log^{M+1} n \cdot n^{\theta_2/\theta_1} \cdot \operatorname{Var}\varphi$$

for every function $\varphi : I \to \mathbb{R}$ of bounded variation with zero mean and for each natural n.

When $\theta_2(T)/\theta_1(T)$ is small, this inequality will help us to prove the recurrence of φ. In Appendix B we will give examples with arbitrary small values of this ratio.

2.4. Recurrence, essential values, and ergodicity of cocycles

In this subsection we recall some general facts about cocycles. For relevant background material concerning skew products and infinite measure-preserving dynamical systems, we refer the reader to [32] and [1].

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism of a standard Borel probability space and G be a locally compact abelian group. Denote by $\overline{G}$ the one point compactification of the group G. An element $g \in \overline{G}$ is said to be an *essential value* of φ, if for every open neighborhood V_g of g in $\overline{G}$ and any set $B \in \mathcal{B}$, $\mu(B) > 0$, there exists $n \in \mathbb{Z}$ such that

$$\mu\big(B \cap T^{-n}B \cap \big\{x \in X\colon \varphi^{(n)}(x) \in V_g\big\}\big) > 0. \tag{2.7}$$

The set of essential values of φ will be denoted by $\overline{E}(\varphi)$. The set of finite essential values $E(\varphi) := G \cap \overline{E}(\varphi)$ is a closed subgroup of G.

Two cocycles $\varphi, \psi : X \to G$ are called *cohomologous* for T if there exists a measurable function $g : X \to G$ such that $\varphi = \psi + g - g \circ T$. The corresponding skew products T_φ and T_ψ are then measure-theoretically isomorphic. A cocycle $\varphi : X \to G$ is a *coboundary* if it is cohomologous to the zero cocycle.

If φ and ψ are cohomologous then $\overline{E}(\varphi) = \overline{E}(\psi)$. Moreover, φ is a coboundary if and only if $\overline{E}(\varphi) = \{0\}$.

A cocycle $\varphi : X \to G$ is recurrent (as defined in the introduction) if and only if, for each open neighborhood V_0 of 0, (2.7) holds for some $n \neq 0$. This is equivalent to the conservativity of the skew product T_φ (cf. [32]). In the particular case $G = R^\ell$ and $\varphi : X \to \mathbb{R}^\ell$ integrable we have: the recurrence of φ implies $\int_X \varphi\, d\mu = 0$; moreover, for $\ell = 1$ the zero mean condition is sufficient for recurrence when T is ergodic.

The group $E(\varphi)$ coincides with the group of *periods* of T_φ-invariant functions, i.e. the set of all $g_0 \in G$ such that, if $f : X \times G \to \mathbb{R}$ is a T_φ-invariant measurable function, then $f(x, g + g_0) = f(x, g)$ $\mu \times m_G$-a.e. In particular, T_φ is ergodic if and only if $E(\varphi) = G$.

A simple sufficient condition of recurrence is the following:

Proposition 2.5. *(See [6, Corollary 1.2].) If $\varphi : X \to \mathbb{R}^\ell$ is a square integrable cocycle for an ergodic automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ and $\|\varphi^{(n)}\|_{L^2(\mu)} = o(n^{1/\ell})$, then it is recurrent.*

By Proposition 2.5 and Theorem 2.4, we have the following.

Corollary 2.6. *If $T: I \to I$ is an IET of periodic type such that $\theta_2(T)/\theta_1(T) < 1/\ell$ for an integer $\ell \geqslant 1$, then every cocycle $\varphi: I \to \mathbb{R}^\ell$ of bounded variation with zero mean is recurrent. If, for $j = 1, \dots, \ell$, $T_j: I_j \to I_j$ are interval exchange transformations of periodic type such that $\theta_2(T_j)/\theta_1(T_j) < 1/\ell$ and the product automorphism $T_1 \times \dots \times T_\ell$ is ergodic, then every "product" cocycle $\varphi = (\varphi_1, \dots, \varphi_\ell): I_1 \times \dots \times I_\ell \to \mathbb{R}^\ell$ of bounded variation with zero mean over $T_1 \times \dots \times T_\ell$ is recurrent.*

In the remainder of this section we will present tools useful for proving the ergodicity of cocycles. Let (X, d) be a compact metric space. Let $\mathcal{B}$ stand for the σ-algebra of Borel sets and let μ be a probability Borel measure on X. By χ_B we will denote the indicator function of a set B. Suppose that $T: (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic measure-preserving automorphism and there exist an increasing sequence of natural numbers (q_n) and a sequence of Borel sets (C_n) such that

$$\mu(C_n) \to \alpha > 0, \qquad \mu\big(C_n \triangle T^{-1} C_n\big) \to 0 \quad \text{and} \quad \sup_{x \in C_n} d\big(x, T^{q_n} x\big) \to 0.$$

Assume that $G \subset \mathbb{R}^\ell$, $\ell \geqslant 1$, is a closed subgroup. Let $\varphi: X \to G$ be a Borel integrable cocycle for T with zero mean. Suppose that the sequence $(\int_{C_n} |\varphi^{(q_n)}(x)|\, d\mu(x))_{n \geqslant 1}$ is bounded. As the set of distributions

$$\big\{\mu(C_n)^{-1}\big(\varphi^{(q_n)}\big|_{C_n}\big)_*(\mu|_{C_n}) \colon n \in \mathbb{N}\big\}$$

is uniformly tight, by passing to a subsequence if necessary, we can assume that they converge weakly to a probability Borel measure P on G.

Lemma 2.7. *(Cf. [26].) The topological support of the measure P is included in the group $E(\varphi)$ of essential values of the cocycle φ.*

Proof. Suppose that $g \in \operatorname{supp}(P)$. Let V_g be an open neighborhood of g. Let $\psi: G \to [0, 1]$ be a continuous function such that $\psi(g) = 1$ and $\psi(h) = 0$ for $h \in G \setminus V_g$. Thus $\int_G \psi(g)\, dP(g) > 0$. By Lemma 5 in [16], for every $B \in \mathcal{B}$ with $\mu(B) > 0$ we have

$$\begin{aligned} \mu\big(B \cap T^{-q_n} B \cap \big(\varphi^{(q_n)} \in V_g\big)\big) &\geqslant \int_{C_n} \psi\big(\varphi^{(q_n)}(x)\big)\chi_B(x)\chi_B\big(T^{q_n} x\big)\, d\mu(x) \\ &\to \alpha \int_X \int_G \psi(g)\chi_B(x)\, dP(g)\, d\mu(x) \\ &= \alpha\mu(B) \int_G \psi(g)\, dP(g) > 0, \end{aligned}$$

and hence $g \in E(\varphi)$. $\square$

Corollary 2.8. *(See also [8].) If $\varphi^{(q_n)}(x) = g_n$ for all $x \in C_n$ and $g_n \to g$, then $g \in E(\varphi)$.*

Proposition 2.9. *(See [32, Proposition 3.8].) Let $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be an ergodic automorphism and let $\varphi:X\to G$ be a cocycle for T. If $K\subset G$ is a compact set such that $K\cap E(\varphi)=\emptyset$, then there exists $B\in\mathcal{B}$ such that $\mu(B)>0$ and*

$$\mu\big(B\cap T^{-n}B\cap\big(\varphi^{(n)}\in K\big)\big)=0\quad\textit{for every } n\in\mathbb{Z}.$$

Lemma 2.10. *Let $K\subset G$ be a compact set. If for every $B\in\mathcal{B}$ with $\mu(B)>0$ and every neighborhood $V_0\subset G$ of zero there exists $n\in\mathbb{Z}$ such that*

$$\mu\big(B\cap T^{-n}B\cap\big(\varphi^{(n)}\in K+V_0\big)\big)>0,$$

then $K\cap E(\varphi)\neq\emptyset$. In particular, when $K=\{g,-g\}$, where g is an element of G, then $g\in E(\varphi)$.

Proof. Suppose that $K\cap E(\varphi)=\emptyset$. Since K is compact and $E(\varphi)$ is closed, there exists a neighborhood V_0 of zero such that $\overline{V_0}$ is compact and $(K+\overline{V_0})\cap E(\varphi)=\emptyset$. As $K+\overline{V_0}$ is also compact, by Proposition 2.9, there exists $B\in\mathcal{B}$ such that $\mu(B)>0$ and

$$\mu\big(B\cap T^{-n}B\cap\big(\varphi^{(n)}\in(K+\overline{V_0})\big)\big)=0\quad\text{for every } n\in\mathbb{Z},$$

contrary to assumption. The last claim is clear. □

Consider the quotient cocycle $\varphi^*:X\to G/E(\varphi)$ given by $\varphi^*(x)=\varphi(x)+E(\varphi)$. Then $E(\varphi^*)=\{0\}$. The cocycle φ is called *regular* if $\overline{E}(\varphi^*)=\{0\}$ and *non-regular* if $\overline{E}(\varphi^*)=\{0,\infty\}$. Recall that if φ is regular then it is cohomologous to a cocycle $\psi:X\to E(\varphi)$ such that $E(\psi)=E(\varphi)$.

Lemma 2.11. *If H is a closed subgroup of $E(\varphi)$ such that the quotient cocycle $\varphi_H:X\to G/H$, $\varphi_H(x)=\varphi(x)+H$ is ergodic, then $\varphi:X\to G$ is ergodic as well.*

Proof. Let $f(x,g)$ be a measurable T_φ-invariant function. Then, since $H\subset E(\varphi)$, f is H-invariant. Since φ_H is ergodic, f is constant. □

3. Ergodicity of piecewise linear cocycles

We will consider cocycles φ for the dynamical systems defined by IETs acting on the interval I endowed with the Lebesgue measure denoted by μ.

Notation. We denote by $\mathrm{BV}(\bigsqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)})$ the space of functions $\varphi:I^{(k)}\to\mathbb{R}$ such that the restriction $\varphi:I_\alpha^{(k)}\to\mathbb{R}$ is of bounded variation for every $\alpha\in\mathcal{A}$, and by $\mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)})$ the subspace of functions in $\mathrm{BV}(\bigsqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)})$ with zero mean. We adopt the notation from [28]. The space $\mathrm{BV}(\bigsqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)})$ is equipped with the norm $\|\varphi\|_{\mathrm{BV}}=\|\varphi\|_{\sup}+\operatorname{Var}\varphi$, where

$$\operatorname{Var}\varphi:=\sum_{\alpha\in\mathcal{A}}\operatorname{Var}\varphi|_{I_\alpha^{(k)}}.$$

For $\varphi\in\mathrm{BV}(\bigsqcup_{\alpha\in\mathcal{A}}I_\alpha)$ and $x\in I$, $\varphi_+(x)$ and $\varphi_-(x)$ will denote the right-handed and left-handed limit of φ at x respectively. We denote by $\mathrm{BV}^1(\bigsqcup_{\alpha\in\mathcal{A}}I_\alpha)$ the space of functions

$\varphi: I \to \mathbb{R}$ which are absolutely continuous on each I_α, $\alpha \in \mathcal{A}$, and such that $\varphi' \in \mathrm{BV}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$. For $\varphi \in \mathrm{BV}^1(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ let

$$s(\varphi) = \int_I \varphi'(x)\,dx = \sum_{\alpha\in\mathcal{A}} \bigl(\varphi_-(r_\alpha) - \varphi_+(l_\alpha)\bigr).$$

We denote by $\mathrm{BV}^1_*(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the subspace of functions $\varphi \in \mathrm{BV}^1(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ for which $s(\varphi) = 0$, and by $\mathrm{PL}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the set of piecewise linear cocycles with a constant derivative equal to s a.e., i.e. $\varphi(x) = sx + c_\alpha$ for $x \in I_\alpha$.

Proposition 3.1. *(See [28].) If an IET $T: I \to I$ satisfies a Roth type condition (defined in* [28]*), then each cocycle $\varphi \in \mathrm{BV}^1_*(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ for T is cohomologous to a cocycle which is constant on each interval I_α, $\alpha \in \mathcal{A}$. Moreover, the set of IETs satisfying this Roth type condition has full measure and contains all IETs of periodic type.*

As a consequence of Proposition 3.1 we have the following.

Lemma 3.2. *If $T: I \to I$ is of periodic type, then each cocycle $\varphi \in \mathrm{BV}^1(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is cohomologous to a cocycle $\varphi_{pl} \in \mathrm{PL}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $s(\varphi_{pl}) = s(\varphi)$.*

3.1. Piecewise linear cocycles

We will focus on the case where the slope $s(\varphi)$ of a piecewise linear cocycle φ is non-zero. We begin by a preliminary result whose general version will be proved later (see Theorems 3.4 and 3.9 for $\ell = 1$).

Theorem 3.3. *Let $T: I \to I$ be an IET of bounded type. If $\varphi \in \mathrm{PL}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is a piecewise linear cocycle with zero mean and $s(\varphi) \neq 0$, then the skew product T_φ is ergodic.*

Now we consider cocycles taking values in $\mathbb{R}^\ell$, $\ell \geqslant 1$. Suppose that $\varphi: I \to \mathbb{R}^\ell$ is a piecewise linear cocycle with zero mean (i.e. each coordinate function is piecewise linear with zero mean) such that the slopes vector $s(\varphi) \in \mathbb{R}^\ell$ (the vector of slopes of coordinate functions) is non-zero. Then, by an appropriate choice of basis of $\mathbb{R}^\ell$, we obtain $s(\varphi_1) = 1$ and $s(\varphi_2) = 0$, where $\varphi = (\varphi_1, \varphi_2)$ and $\varphi_1: I \to \mathbb{R}$, $\varphi_2: I \to \mathbb{R}^{\ell-1}$. Thus φ_2 is piecewise constant and, under the assumptions of Corollary 2.6, the ergodicity of φ_2 implies the ergodicity of φ. The ergodicity of piecewise constant cocycles will be studied in Sections 4 and 5.

Theorem 3.4. *Suppose that $T: I \to I$ is an IET of periodic type such that $\theta_2(T)/\theta_1(T) < 1/\ell$. Let $\varphi_1 \in \mathrm{PL}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R})$, $\varphi_2 \in \mathrm{PL}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^{\ell-1})$ be piecewise linear cocycles with zero mean such that $s(\varphi_1) \neq 0$ and $s(\varphi_2) = 0$. If the cocycle $\varphi_2: I \to \mathbb{R}^{\ell-1}$ is ergodic, then the cocycle $\varphi = (\varphi_1, \varphi_2): I \to \mathbb{R}^\ell$ is ergodic as well.*

Proof. Without loss of generality we can assume that $s(\varphi_1) = 1$. By Proposition 2.3, there exists $c \geqslant 1$ satisfying (2.4). It suffices to show that for every $0 < a < \frac{1}{4c}$, the pair $(a, 0)$ belongs to $E(\varphi_1, \varphi_2)$. Indeed this implies that $\mathbb{R} \times \{0\} \subset E(\varphi_1, \varphi_2)$, and since the cocycle φ_2 is ergodic, by Lemma 2.11, it follows that $(\varphi_1, \varphi_2): I \to \mathbb{R}^\ell$ is ergodic as well.

Fix $0 < a < \frac{1}{4c}$. By a Lebesgue density point argument, for every measurable $B \subset I$ with $\mu(B) > 0$ and every $\varepsilon \in (0, a/2)$, there are $B' \subset B$ with $\mu(B') > 0$ and $n_0 \geqslant 1$ such that for $n \geqslant n_0$,

$$\mu\left(\left(x - \frac{c}{n}, x + \frac{c}{n}\right) \setminus B\right) < \frac{\varepsilon}{n} \quad \text{for every } x \in B'. \tag{3.1}$$

Since $\theta_2(T)/\theta_1(T) < 1/\ell$, by Corollary 2.6, (φ_1, φ_2) is recurrent, and hence there exists $n \geqslant n_0$ such that

$$\mu\big(B' \cap T^{-n}B' \cap \big(\big|\varphi_1^{(n)}\big| < \varepsilon\big) \cap \big(\big\|\varphi_2^{(n)}\big\| < \varepsilon\big)\big) > 0.$$

Let $x_0 \in I$ be such that $x_0, T^n x_0 \in B'$, $|\varphi_1^{(n)}(x_0)| < \varepsilon$ and $\|\varphi_2^{(n)}(x_0)\| < \varepsilon$. Denote by $J(x_0) \subset I$ the interval of the partition $\mathcal{P}_n(T)$ which contains x_0. Then $\varphi_1^{(n)}$ is a linear function on $J(x_0)$ with slope n. Since $2\varepsilon < a < 1/(2c) - 2\varepsilon$ and $|J(x_0)| > 1/(cn)$ (by (2.4)), there exists y_0 such that $(y_0 - \varepsilon/n, y_0 + \varepsilon/n) \subset J(x_0)$ and

$$\big|\varphi_1^{(n)}(y)\big| \in a + (-\varepsilon, \varepsilon) \quad \text{for all } y \in (y_0 - \varepsilon/n, y_0 + \varepsilon/n).$$

Since $\varphi_2^{(n)}$ is constant on $J(x_0)$, we have

$$\big\|\varphi_2^{(n)}(x)\big\| < \varepsilon \quad \text{for all } x \in (y_0 - \varepsilon/n, y_0 + \varepsilon/n).$$

Therefore

$$\begin{aligned}&\mu\big(B \cap T^{-n}B \cap \big(\varphi_1^{(n)} \in \{-a, a\} + (-\varepsilon, \varepsilon)\big) \cap \big(\varphi_2^{(n)} \in (-\varepsilon, \varepsilon)^{\ell-1}\big)\big) \\ &\quad \geqslant \mu\big((y_0 - \varepsilon/n, y_0 + \varepsilon/n) \cap B \cap T^{-n}B\big).\end{aligned} \tag{3.2}$$

By (2.4) we have $|J(x_0)| < c/n$, and hence $J(x_0) \subset (x_0 - c/n, x_0 + c/n)$. Moreover, $T^n J(x_0)$ is an interval such that $|T^n J(x_0)| = |J(x_0)| < c/n$, so that

$$T^n J(x_0) \subset \left(T^n x_0 - \frac{c}{n}, T^n x_0 + \frac{c}{n}\right).$$

Since $x_0, T^n x_0 \in B'$, by (3.1), $\mu(J(x_0) \setminus B) < \varepsilon/n$ and $\mu(T^n J(x_0) \setminus B) < \varepsilon/n$. Therefore, $\mu(J(x_0) \setminus (B \cap T^{-n}B)) < 2\varepsilon/n$, and hence

$$\mu\big((y_0 - \varepsilon/n, y_0 + \varepsilon/n) \setminus \big(B \cap T^{-n}B\big)\big) < 2\varepsilon/n.$$

Thus

$$\mu\big((y_0 - \varepsilon/n, y_0 + \varepsilon/n) \cap B \cap T^{-n}B\big) > 0.$$

In view of (3.2), it follows that

$$\mu\big(B \cap T^{-n}B \cap \big(\varphi_1^{(n)} \in \{-a, a\} + (-\varepsilon, \varepsilon)\big) \cap \big(\varphi_2^{(n)} \in (-\varepsilon, \varepsilon)^{\ell-1}\big)\big) > 0.$$

By Lemma 2.10, we conclude that $(a, 0) \in E(\varphi_1, \varphi_2)$, which completes the proof. $\square$

Remark 3.5. Note that if $\ell = 1$ then we need only the assumption that T is of bounded type (T is ergodic by Proposition 2.2). In this case the recurrence of φ follows directly from the zero mean assumption, this yields Theorem 3.3.

3.2. Product cocycles

The method used in the proof of Theorem 3.3 allows us to prove the ergodicity for Cartesian products of certain skew products. As an illustration of it, first we apply this method for cocycles taking values in $\mathbb{Z}$ over irrational rotations on the circle. This will give a class of ergodic $\mathbb{Z}^2$-cocycles for 2-dimensional rotations.

Let $T(x, y) = (x + \alpha_1, y + \alpha_2)$ be an ergodic rotation on the torus $\mathbb{T}^2$ and φ be a zero mean function on $\mathbb{T}^2$ of the form $\varphi(x, y) = (\varphi_1(x), \varphi_2(y))$ with φ_1 and φ_2 BV functions. If α_1 and α_2 have bounded partial quotients, then (2.6) implies $\|\varphi^{(n)}\|_{\sup} = O(\log n)$, and therefore, by Proposition 2.5, the cocycle φ is recurrent.

We will deal with $\varphi : \mathbb{T}^2 \to \mathbb{Z}^2$ for which φ_1, φ_2 are step functions in one variable with values in $\mathbb{Z}$. It defines a recurrent $\mathbb{Z}^2$-cocycle for a 2-dimensional ergodic rotation. A question is then the ergodicity (with respect to the measure $\mu \times m$ the product of the uniform measure on $\mathbb{T}^2$ by the counting measure on $\mathbb{Z}^2$) of the skew product

$$T_\varphi : \mathbb{T}^2 \times \mathbb{Z}^2 \to \mathbb{T}^2 \times \mathbb{Z}^2, \qquad T_\varphi(x, y, \bar{n}) = \big(x + \alpha_1, y + \alpha_2, \bar{n} + \varphi(x, y)\big).$$

For $i = 1, 2$, we denote by $D_i \subset \mathbb{T}$ the finite set of discontinuities of φ_i and by $J_i \subset \mathbb{Z}$ the corresponding set of jumps of the functions φ_i. We will show the ergodicity of φ when discontinuities of φ_1, φ_2 are rational and the jumps of φ generate $\mathbb{Z}^2$, i.e. $(J_1 \times \{0\}) \cup (\{0\} \times J_2)$ generates $\mathbb{Z}^2$. The simplest example of such function is $\varphi(x, y) = (2 \cdot \chi_{[0,\frac{1}{2})}(x) - 1, 2 \cdot \chi_{[0,\frac{1}{2})}(y) - 1)$.

Theorem 3.6. *Let α_1 and α_2 be two rationally independent irrational numbers with bpq, and let $\varphi(x, y) = (\varphi_1(x), \varphi_2(y))$ be a function on the torus with step function components $\varphi_i : \mathbb{T} \to \mathbb{Z}$, $i = 1, 2$, such that $D_1, D_2 \subset \mathbb{Q}$ and the sets of the jumps $J_1 \times \{0\}$, $\{0\} \times J_2$ generate $\mathbb{Z}^2$. Then the system $(\mathbb{T}^2 \times \mathbb{Z}^2, \mu \times m, T_\varphi)$ is ergodic.*

Proof. We have seen that the cocycle $\varphi^{(n)}$ is recurrent. We prove that the group of its finite essential values is $\mathbb{Z}^2$.

Let $n \in \mathbb{N}$ be fixed and let $(\gamma^i_{n,k})_{k=1,\ldots,d_i n}$ be the ordered set of the $d_i n$ discontinuities of $\varphi_i^{(n)}$ in $[0, 1)$ (where $d_i := \#D_i$). In the sequence $(q^i_k)_{k\geqslant 0}$ of denominators of α_i, choose $r_i(n) \geqslant 0$ so that $q^i_{r_i(n)} \leqslant n < q^i_{r_i(n)+1}$. We write simply $q^i_{r_i}$ for $q^i_{r_i(n)}$. As α_i has bpq, the ratio $q^i_{r_i(n)+1}/q^i_{r_i(n)}$ is bounded, as a function of $n \in \mathbb{N}$ and $i = 1, 2$, by a constant $C \geqslant 1$ independent from n.

Since α_i has bpq and the discontinuity points of φ_i are rational, the distances between consecutive discontinuities of $\varphi^{(n)}$ are of the same order: there are two positive constants c_1, c_2 such that

$$\frac{c_1}{n} \leqslant \gamma^i_{n,k+1} - \gamma^i_{n,k} \leqslant \frac{c_2}{n}, \quad k = 1, \ldots, d_i n, \ i = 1, 2. \tag{3.3}$$

Recall that, for each $t \in D_i$ and each $0 \leqslant \ell < q_{r_i}$, there is (mod 1) a point $t - k\alpha_i$, $0 \leqslant k < q_{r_i}$, in each interval $[t + \ell/q^i_{r_i}, t + (\ell + 1)/q^i_{r_i}]$. Therefore,

$$\begin{aligned}&\text{in each interval of length greater than } 2/q_r^i \text{ and for each } t \in D_i,\\ &\text{there is at least one discontinuity of } \varphi_i^{(n)} \text{ of the form } t - k\alpha_i,\ 0 \leqslant k < n.\end{aligned} \tag{3.4}$$

For $x \in \mathbb{T}$, consider the interval $[\gamma^i_{n,k}, \gamma^i_{n,k+1})$ which contains x and denote it by $I^i_n(x)$. The intervals $[\gamma^i_{n,k+\ell}, \gamma^i_{n,k+\ell+1})$, where $k+\ell$ is taken mod $d_i n$, are denoted by $I^i_{n,\ell}(x)$. This gives two collections of rectangles

$$R^n_{k,\ell}(x,y) := I^1_{n,k}(x) \times I^2_{n,\ell}(y) \quad \text{and} \quad \tilde{R}^n_{k,\ell}(x,y) := T^n R^n_{k,\ell}\big(T^{-n}(x,y)\big)$$

for each $(x,y) \in \mathbb{T}^2$. By (3.3), we have

$$\mu\big(R^n_{k,\ell}(x,y)\big), \mu\big(\tilde{R}^n_{k,\ell}(x,y)\big) \in \left[\frac{c_1^2}{n^2}, \frac{c_2^2}{n^2}\right]. \tag{3.5}$$

Let M be a natural number such that $c_1 M/C > 1$. Then, by (3.3), the length of $\bigcup_{k=-M}^{M} I^i_{n,k}(x)$ is greater than $2/q^i_{r_i}$, $i = 1,2$. Let $\delta > 0$ be such that $\delta c^2(2M+1)^2 < 1/2$ with $c = c_2^2/c_1^2$. Set

$$R^n_M(x,y) := \bigcup_{k=-M}^{M} \bigcup_{\ell=-M}^{M} R^n_{k,\ell}(x,y),$$

$$\tilde{R}^n_M(x,y) := T^n R^n_M\big(T^{-n}(x,y)\big) = \bigcup_{k=-M}^{M} \bigcup_{\ell=-M}^{M} \tilde{R}^n_{k,\ell}(x,y).$$

In view of (3.3),

$$\frac{\text{length}(R^n_M(x,y))}{\text{width}(R^n_M(x,y))}, \frac{\text{length}(\tilde{R}^n_M(x,y))}{\text{width}(\tilde{R}^n_M(x,y))} \in \left[\frac{c_1}{c_2}, \frac{c_2}{c_1}\right]. \tag{3.6}$$

The cocycle $\varphi^{(n)}$ has a constant value on each rectangle $R^n_{k,\ell}(x,y)$ and the difference between its value on $R^n_{k+1,\ell}(x,y)$ and $R^n_{k,\ell}(x,y)$ (resp. $R^n_{k,\ell+1}(x,y)$ and $R^n_{k,\ell}(x,y)$) belongs to $J_1 \times \{0\}$ (resp. $\{0\} \times J_2$). Denote by $\kappa^n_{k,\ell}(x,y)$ the value of $\varphi^{(n)}$ on $R^n_{k,\ell}(x,y)$. Since the length of $R^n_M(x,y)$ is greater than $2/q^1_{r_1}$ and the width of $R^n_M(x,y)$ is greater than $2/q^2_{r_2}$, in view of (3.4), we have

$$\begin{aligned}\big\{\kappa^n_{k+1,\ell}(x,y) - \kappa^n_{k,\ell}(x,y)\colon -M \leqslant k < M\big\} &= J_1 \times \{0\},\\ \big\{\kappa^n_{k,\ell+1}(x,y) - \kappa^n_{k,\ell}(x,y)\colon -M \leqslant l < M\big\} &= \{0\} \times J_2.\end{aligned} \tag{3.7}$$

Let

$$K := \overbrace{\big(\big(J_1 \cup \{0\}\big) + \cdots + \big(J_1 \cup \{0\}\big)\big)}^{M} \times \overbrace{\big(\big(J_2 \cup \{0\}\big) + \cdots + \big(J_2 \cup \{0\}\big)\big)}^{M}.$$

Let K_1 be the subset of elements of K which are not essential values of φ, and suppose $K_1 \neq \emptyset$. By Proposition 2.9, there exists $B \subset \mathbb{T}^2$ such that $\mu(B) > 0$ and

$$\mu\big(B \cap T^{-n} B \cap \big(\varphi^{(n)} \in K_1\big)\big) = 0 \quad \text{for every } n \in \mathbb{Z}. \tag{3.8}$$

Since the areas of $R^n_M(x, y)$, $\tilde{R}^n_M(x, y)$ tend to 0 as $n \to \infty$ and the rectangles satisfy (3.6), by a Lebesgue density point argument, there is a Borel subset B' of B of positive measure and there is $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ and $(x, y) \in B'$:

$$\mu\big(B \cap R^n_M(x, y)\big) \geqslant (1 - \delta)\mu\big(R^n_M(x, y)\big), \qquad \mu\big(B \cap \tilde{R}^n_M(x, y)\big) \geqslant (1 - \delta)\mu\big(\tilde{R}^n_M(x, y)\big).$$

By (3.5), the areas of the small rectangles being comparable, and hence

$$\mu\big(R^n_M(x, y)\big) \leqslant (2M + 1)^2 c^2 \mu\big(R^n_{k,\ell}(x, y)\big) \quad \text{for all } k, \ell \in [-M, M].$$

Therefore, by the choice of δ, for each $(x, y) \in B'$ we have

$$\begin{aligned}
\mu\big(B \cap R^n_{k,\ell}(x, y)\big) &\geqslant \mu\big(R^n_{k,\ell}(x, y)\big) - \mu\big(B^c \cap R^n_{k,\ell}(x, y)\big) \\
&\geqslant \mu\big(R^n_{k,\ell}(x, y)\big) - \mu\big(B^c \cap R^n_M(x, y)\big) \geqslant \mu\big(R^n_{k,\ell}(x, y)\big) - \delta\mu\big(R^n_M(x, y)\big) \\
&\geqslant \mu\big(R^n_{k,\ell}(x, y)\big) - \delta(2M + 1)^2 c^2 \mu\big(R^n_{k,\ell}(x, y)\big) > \frac{1}{2}\mu\big(R^n_{k,\ell}(x, y)\big).
\end{aligned} \tag{3.9}$$

Similarly, if $T^n(x, y) \in B'$, then $\mu(B \cap \tilde{R}^n_{k,\ell}(T^n(x, y))) > \frac{1}{2}\mu(\tilde{R}^n_{k,\ell}(T^n(x, y)))$. As $\tilde{R}^n_{k,\ell}(T^n(x, y)) = T^n R^n_{k,\ell}(x, y)$, we have

$$\mu\big(T^{-n} B \cap R^n_{k,\ell}(x, y)\big) > \frac{1}{2}\mu\big(R^n_{k,\ell}(x, y)\big). \tag{3.10}$$

Combining (3.9) with (3.10) gives

$$\mu\big(B \cap T^{-n} B \cap R^n_{k,\ell}(x, y)\big) > 0, \quad \forall k, \ell \in [-M, M]. \tag{3.11}$$

Since φ is recurrent, there is $n > n_0$ such that

$$\mu\big(B' \cap T^{-n} B' \cap \big\{\varphi^{(n)}(\cdot) = (0, 0)\big\}\big) > 0.$$

If $(x, y) \in B' \cap T^{-n} B' \cap \{\varphi^{(n)}(\cdot) = (0, 0)\}$, then $\varphi^{(n)}$ is equal to $(0, 0)$ on $R^n_{0,0}(x, y)$. Moreover, on each rectangle $R^n_{k,\ell}(x, y)$, $k, \ell \in [-M, M]$, the cocycle $\varphi^{(n)}$ is constant and is equal to $\kappa_{k,\ell}(x, y) \in K$. In view of (3.11), it follows that

$$\mu\big(B \cap T^{-n} B \cap \big\{\varphi^{(n)}(\cdot) = \kappa_{k,\ell}(x, y)\big\}\big) > 0, \quad \forall k, \ell \in [-M, M].$$

Therefore, by (3.8) and the definition of K_1, $\kappa_{k,l}(x, y) \notin K_1$, and so it belongs to $E(\varphi)$ for all $k, \ell \in [-M, M]$. In view of (3.7), it follows that $J_1 \times \{0\}, \{0\} \times J_2 \subset E(\varphi)$, and hence $E(\varphi) = \mathbb{Z}^2$. $\square$

Remark 3.7. The ergodicity of T_φ can be proven also for the more general case where α_i has bpq and $(D_i - D_i) \setminus \{0\} \subset (\mathbb{Q} + \mathbb{Q}\alpha_i) \setminus (\mathbb{Z} + \mathbb{Z}\alpha_i)$ for $i = 1, 2$. To extend the result of Theorem 3.6, we use that the discontinuities of the cocycle are "well distributed" (the condition (3.3)) which is a consequence of Lemma 2.3 in [18].

Now by a similar method we show the ergodicity of Cartesian products of skew products that appeared in Theorem 3.3. We need an elementary algebraic result:

Remark 3.8. Let R be a real $m \times k$-matrix. Then the subgroup $R(\mathbb{Z}^k)$ is dense in $\mathbb{R}^m$ if and only if

$$\forall a \in \mathbb{R}^m, \quad R^t(a) \in \mathbb{Z}^k \quad \Rightarrow \quad a = 0. \tag{3.12}$$

For instance, if $R = [r_{ij}]$ is an $m \times (m+1)$-matrix such that $r_{ij} = \pm\delta_{ij}$ for $1 \leqslant i, j \leqslant m$ and $1, r_{1\,m+1}, \dots, r_{m\,m+1}$ are independent over $\mathbb{Q}$, then (3.12) holds.

Theorem 3.9. *Let $T_j : I_j \to I_j$ be an interval exchange transformation of periodic type such that $\theta_2(T_j)/\theta_1(T_j) < 1/\ell$ for $j = 1, \dots, \ell$. Suppose that the Cartesian product $T_1 \times \dots \times T_\ell$ is ergodic. If $\varphi_j \in \mathrm{PL}(\bigsqcup_{\alpha\in\mathcal{A}_|}(I_j)_\alpha)$ is a piecewise linear cocycle with zero mean and $s(\varphi_j) \neq 0$ for $j = 1, \dots, \ell$, then the Cartesian product $(T_1)_{\varphi_1} \times \dots \times (T_\ell)_{\varphi_\ell}$ is ergodic.*

Proof. Since $T_1, \dots, T_\ell$ have periodic type, by Proposition 2.3, there exists $c > 0$ such that

$$\frac{1}{cn} \leqslant \min \mathcal{P}_n(T_j) \leqslant \max \mathcal{P}_n(T_j) \leqslant \frac{c}{n} \quad \text{for all } j = 1, \dots, \ell \text{ and } n > 0. \tag{3.13}$$

Let $\bar{I} = I_1 \times \dots \times I_\ell$, $\bar{T} = T_1 \times \dots \times T_\ell$ and let $\bar{\varphi} : \bar{I} \to \mathbb{R}^\ell$ be given by

$$\bar{\varphi}(x_1, \dots, x_\ell) = \big(\varphi_1(x_1), \dots, \varphi_\ell(x_\ell)\big).$$

Then $(T_1)_{\varphi_1} \times \dots \times (T_\ell)_{\varphi_\ell} = \bar{T}_{\bar{\varphi}}$. Denote by $\bar{\mu}$ the Lebesgue measure on $\bar{I}$. Without loss of generality we can assume that $s(\varphi_j) = 1$ for $j = 1, \dots, \ell$. By Corollary 2.6, the cocycle $\bar{\varphi}$ for $\bar{T}$ is recurrent.

To prove the result, it suffices to show that, for every $r = (r_1, \dots, r_\ell) \in [0, \frac{1}{4c})^\ell$, the set $E(\bar{\varphi})$ has non-trivial intersection with

$$\big\{s \bullet r := (s_1 r_1, \dots, s_\ell r_\ell) \colon s = (s_1, \dots, s_\ell) \in \{-1, 1\}^\ell\big\}.$$

Indeed, for a fixed rational $0 < q < \frac{1}{4c}$, let us consider a collection of vectors $r^{(i)} = (r_{1i}, \dots, r_{\ell i}) \in [0, 1/(4c))^\ell$, $1 \leqslant i \leqslant \ell + 1$, such that $r_{ij} = q\delta_{ij}$ for all $1 \leqslant i, j \leqslant \ell$ and $1, r_{1\ell+1}, \dots, r_{\ell\ell+1}$ are independent over $\mathbb{Q}$. By Remark 3.8, for any choice $s^{(i)} \in \{-1, 1\}^\ell$, $1 \leqslant i \leqslant \ell + 1$, the subgroup generated by vectors $s^{(i)} \bullet r^{(i)}$, $1 \leqslant i \leqslant \ell + 1$, is dense in $\mathbb{R}^\ell$. Since $E(\bar{\varphi}) \subset \mathbb{R}^\ell$ is a closed subgroup and for every $1 \leqslant i \leqslant \ell + 1$ there exists $s^{(i)} \in \{-1, 1\}^\ell$ such that $s^{(i)} \bullet r^{(i)} \in E(\varphi)$, it follows that $E(\bar{\varphi}) = \mathbb{R}^\ell$, and hence $\bar{T}_{\bar{\varphi}}$ is ergodic.

Fix $r = (r_1, \dots, r_\ell) \in [0, \frac{1}{4c})^\ell$. We have to show that for every measurable set $B \subset \bar{I}$ with $\bar{\mu}(B) > 0$ and $0 < \varepsilon < 1/c$ there exists $n > 0$ such that the set of all $\bar{x} = (x_1, \dots, x_\ell) \in B$ such that

$$\left(T_1^n x_1, \dots, T_\ell^n x_\ell\right) \in B, \qquad \varphi_j^{(n)}(x_j) \in \{-r_j, r_j\} + (-\varepsilon, \varepsilon) \quad \text{for } 1 \leqslant j \leqslant \ell$$

has positive $\bar{\mu}$ measure. By a Lebesgue density point argument, there exist $B' \subset B$ and $n_0 \geqslant 1$ such that $\bar{\mu}(B') > 0$ and for every $(x_1, \dots, x_\ell) \in B'$ and $n \geqslant n_0$ we have

$$\bar{\mu}\left(\prod_{j=1}^{\ell}\left(x_j - \frac{c}{n}, x_j + \frac{c}{n}\right) \setminus B\right) < \frac{\varepsilon}{4(2n)^\ell}. \tag{3.14}$$

Since $\bar{\varphi}$ (as a cocycle for $\bar{T}$) is recurrent, there exists $n \geqslant n_0$ such that

$$\bar{\mu}\left(B' \cap \bar{T}^{-n} B' \cap \left(\bar{\varphi}^{(n)} \in (-\varepsilon/2, \varepsilon/2)^\ell\right)\right) > 0.$$

Next choose $x^0 = (x_1^0, \dots, x_\ell^0) \in B'$ so that $(T_1^n x_1^0, \dots, T_\ell^n x_\ell^0) \in B'$, $|\varphi_j^{(n)}(x_j^0)| < \varepsilon/2$ for $1 \leqslant j \leqslant \ell$. For each $1 \leqslant j \leqslant \ell$ denote by $J_{j,n}(x_j^0) \subset I_j$ the interval of the partition $\mathcal{P}_n(T_j)$ such that $x_j^0 \in J_{j,n}(x_j^0)$. By assumption, $\varphi_j^{(n)}$ is continuous on every interval of $\mathcal{P}^n(T_j)$. Therefore, for every $1 \leqslant j \leqslant \ell$, the function $\varphi_j^{(n)}$ is continuous on $J_{j,n}(x_j^0)$, and hence $\varphi_j^{(n)}(x) = nx + d_{n,j}$ for $x \in J_{j,n}(x_j^0)$. In view of (3.13), $\frac{1}{cn} < |J_{j,n}(x_j^0)| < \frac{c}{n}$, and hence $J_{j,n}(x_j^0) \subset (x_j^0 - c/n, x_j^0 + c/n)$ for every $1 \leqslant j \leqslant \ell$. Moreover, $T_j^n J_{j,n}(x_j^0)$ is an interval such that $|T_j^n J_{j,n}(x_j^0)| = |J_{j,n}(x_j^0)| < c/n$, so

$$T_j^n J_{j,n}\left(x_j^0\right) \subset \left(T_j^n x_j^0 - \frac{c}{n}, T_j^n x_j^0 + \frac{c}{n}\right). \tag{3.15}$$

Since $|\varphi_j^{(n)}(x_j^0)| < \varepsilon/2$, $\varphi_j^{(n)}$ is linear on $J_{j,n}(x_j^0)$ with slope n and $0 \leqslant r_j < \frac{1}{4c} < \frac{1}{2c} - \frac{\varepsilon}{4}$, we can find $(y_j^0 - \varepsilon/(4n), y_j^0 + \varepsilon/(4n)) \subset J_{j,n}(x_j^0)$ such that

$$\left|\varphi_j^{(n)}(x)\right| \in r_j + (-\varepsilon, \varepsilon) \quad \text{for all } x \in \left(y_j^0 - \varepsilon/(4n), y_j^0 + \varepsilon/(4n)\right). \tag{3.16}$$

Let $y^0 = (y_1^0, \dots, y_\ell^0) \in \prod_{j=1}^{\ell} J_{j,n}(x_j^0)$. Since

$$\prod_{j=1}^{\ell}\left(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\right) \subset \prod_{j=1}^{\ell} J_{j,n}\left(x_j^0\right) \subset \prod_{j=1}^{\ell}\left(x_j^0 - \frac{c}{n}, x_j^0 + \frac{c}{n}\right),$$

$x^0 \in B'$ and $n \geqslant n_0$, by (3.14), we have

$$\bar{\mu}\left(\prod_{j=1}^{\ell}\left(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\right) \setminus B\right) < \frac{\varepsilon}{4(2n)^\ell}.$$

Moreover, by (3.15),

$$\bar{T}^n \prod_{j=1}^{\ell}\left(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\right) \subset \prod_{j=1}^{\ell} T_j^n J_{j,n}\big(x_j^0\big) \subset \prod_{j=1}^{\ell}\left(T_j^n x_j^0 - \frac{c}{n}, T_j^n x_j^0 + \frac{c}{n}\right).$$

Since $(T_1^n x_1^0, \ldots, T_\ell^n x_\ell^0) \in B'$ and $n \geqslant n_0$, by (3.14), it follows that

$$\bar{\mu}\left(\prod_{j=1}^{\ell}\left(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\right) \setminus \bar{T}^{-n} B\right) = \bar{\mu}\left(\bar{T}^n \prod_{j=1}^{\ell}\left(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\right) \setminus B\right) < \frac{\varepsilon}{4(2n)^{\ell}}.$$

Hence

$$\bar{\mu}\left(\prod_{j=1}^{\ell}\left(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\right) \cap \big(B \cap \bar{T}^{-n} B\big)\right) > \frac{\varepsilon}{2(2n)^{\ell}} > 0.$$

By (3.16),

$$\bar{\varphi}^{(n)}(x) \in \prod_{j=1}^{\ell}\big(\{-r_j, r_j\} + (-\varepsilon, \varepsilon)\big) \quad \text{if } x \in \prod_{j=1}^{\ell}\left(y_j^0 - \frac{\varepsilon}{4n}, y_j^0 + \frac{\varepsilon}{4n}\right).$$

Thus

$$\bar{\mu}\left(B \cap \bar{T}^{-n} B \cap \left(\bar{\varphi}^{(n)} \in \prod_{j=1}^{\ell}\big(\{-r_j, r_j\} + (-\varepsilon, \varepsilon)\big)\right)\right) > \frac{\varepsilon}{2(2n)^{\ell}} > 0.$$

By Lemma 2.10, it follows that $(\prod_{j=1}^{\ell}\{-r_j, r_j\}) \cap E(\bar{\varphi}) \neq \emptyset$. This completes the proof. $\square$

4. Ergodicity of certain step cocycles

In this section we apply Corollary 2.8 to prove the ergodicity of step cocycles over IETs of periodic type.

4.1. Step cocycles

Let $T: I \to I$ be an arbitrary IET satisfying the Keane condition. Suppose that $(n_k)_{k \geqslant 0}$ is an increasing sequence of natural numbers such that $n_0 = 0$ and the matrix

$$Z(k+1) = \Theta\big(T^{(n_k)}\big) \cdot \Theta\big(T^{(n_k+1)}\big) \cdot \ldots \cdot \Theta\big(T^{(n_{k+1}-1)}\big)$$

has positive entries for each $k \geqslant 0$. In what follows, we denote by $(\pi^{(k)}, \lambda^{(k)})$ the pair defining $T^{(n_k)}$. By abuse of notation, we continue to write $T^{(k)}$ for $T^{(n_k)}$ with the same change of notation for $I^{(\cdot)}$, $\pi^{(\cdot)}$ and $\lambda^{(\cdot)}$. We then have

$$\lambda^{(k)} = Z(k+1)\lambda^{(k+1)}.$$

We adapt the notation from [28]. For each $0 \leqslant k \leqslant l$ let

$$Q(k,l) = Z(k+1) \cdot Z(k+2) \cdot \ldots \cdot Z(l).$$

Then

$$\lambda^{(k)} = Q(k,l)\lambda^{(l)}.$$

We will write $Q(l)$ for $Q(0,l)$. By definition, $T^{(l)} : I^{(l)} \to I^{(l)}$ is the first-return map of $T^{(k)} : I^{(k)} \to I^{(k)}$ to the interval $I^{(l)} \subset I^{(k)}$. Moreover, $Q_{\alpha\beta}(k,l)$ is the time spent by any point of $I_\beta^{(l)}$ in $I_\alpha^{(k)}$ until it returns to $I^{(l)}$. It follows that

$$Q_\beta(k,l) := \sum_{\alpha\in\mathcal{A}} Q_{\alpha\beta}(k,l)$$

is the first-return time of points of $I_\beta^{(l)}$ to $I^{(l)}$.

Suppose that $T = T_{(\pi,\lambda)}$ is of periodic type and p is a period such that $\pi^{(p)} = \pi$. Let $A = \Theta^{(p)}(T)$. Considering the sequence $(n_k)_{k\geqslant 0}$, $n_k = pk$ we get $Z(l) = A$ and $Q(k,l) = A^{l-k}$ for all $0 \leqslant k \leqslant l$.

The norm of a vector is defined as the largest absolute value of the coefficients. We set $B_\beta := \sum_{\alpha\in\mathcal{A}} |B_{\alpha\beta}|$ and $\|B\| := \max_{\beta\in\mathcal{A}} B_\beta$ for $B = [B_{\alpha\beta}]_{\alpha,\beta\in\mathcal{A}}$. Following [36], for every matrix $B = [B_{\alpha\beta}]_{\alpha,\beta\in\mathcal{A}}$ with positive entries, we set

$$\nu(B) = \max_{\alpha,\beta,\gamma\in\mathcal{A}} \frac{B_{\alpha\beta}}{B_{\alpha\gamma}}.$$

Then

$$\sum_{\alpha\in\mathcal{A}} B_{\alpha\beta} \leqslant \nu(B) \sum_{\alpha\in\mathcal{A}} B_{\alpha\gamma} \quad \text{for all } \beta,\gamma\in\mathcal{A} \text{ and } \nu(CB) \leqslant \nu(B), \tag{4.1}$$

for any non-negative matrix C whose all rows are non-zero. It follows that $\nu(B^m) \leqslant \nu(B)$, and hence

$$\|B^m\| = \max_{\beta\in\mathcal{A}} \sum_{\alpha\in\mathcal{A}} B^m_{\alpha\beta} \leqslant \nu(B) \min_{\beta\in\mathcal{A}} \sum_{\alpha\in\mathcal{A}} B^m_{\alpha\beta}. \tag{4.2}$$

Denote by $\Gamma^{(k)}$ the space of functions $\varphi : I^{(k)} \to \mathbb{R}$ constant on each interval $I_\alpha^{(k)}$, $\alpha\in\mathcal{A}$, and denote by $\Gamma_0^{(k)}$ the subspace of functions with zero mean. Every function $\varphi = \sum_{\alpha\in\mathcal{A}} h_\alpha \chi_{I_\alpha^{(k)}}$ in $\Gamma^{(k)}$ can be identified with the vector $h = (h_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$. Moreover,

$$\varphi^{(Q_\alpha(k,l))}(x) = \big(Q(k,l)^t h\big)_\alpha \quad \text{for every } x \in I_\alpha^{(l)},\ \alpha\in\mathcal{A}. \tag{4.3}$$

The induced IET $T^{(n)} : I^{(n)} \to I^{(n)}$ determines a partition of I into disjoint towers $H_\alpha^{(n)}$, $\alpha\in\mathcal{A}$, where

$$H_\alpha^{(n)} = \big\{T^k I_\alpha^{(n)}:\ 0 \leqslant k < h_\alpha^{(n)} := Q_\alpha(n)\big\}.$$

Denote by $h_{\max}^{(n)}$ and $h_{\min}^{(n)}$ the height of the highest and the lowest tower, respectively.

Assume that $(n_k)_{k\geqslant 0}$ is chosen so that $I^{(n+1)} \subset I^{(n)}_{\alpha_1}$, where $\pi^{(n)}_0(\alpha_1) = 1$. For every $\alpha \in \mathcal{A}$ denote by $C^{(n)}_\alpha$ the tower $\{T^i I^{(n+1)}_\alpha\colon 0 \leqslant i < h^{(n)}_{\alpha_1}\}$.

Lemma 4.1. *For every $\alpha \in \mathcal{A}$ we have*

$$\mu\big(C^{(n)}_\alpha \triangle T C^{(n)}_\alpha\big) \to 0 \quad \text{and} \quad \sup_{x\in C^{(n)}_\alpha} \big|T^{h^{(n+1)}_\alpha} x - x\big| \to 0 \quad \text{as } n \to +\infty. \tag{4.4}$$

If $\varphi = \sum_{\beta\in\mathcal{A}} v_\beta \chi_{I^{(0)}_\beta}$ for some $v = (v_\beta)_{\beta\in\mathcal{A}} \in \Gamma^{(0)}_0$, then

$$\varphi^{(h^{(n+1)}_\alpha)}(x) = \big(Q(n+1)^t v\big)_\alpha \quad \text{for all } x \in C^{(n)}_\alpha. \tag{4.5}$$

If additionally T is of periodic type then

$$\liminf_{n\to\infty} \mu\big(C^{(n)}_\alpha\big) > 0 \tag{4.6}$$

and

$$\varphi^{(h^{(n+1)}_\alpha)}(x) = \big(\big(A^t\big)^{n+1} v\big)_\alpha \quad \text{for all } x \in C^{(n)}_\alpha. \tag{4.7}$$

Proof. Since $C^{(n)}_\alpha \triangle T C^{(n)}_\alpha \subset T^{h^{(n)}_{\alpha_1}} I^{(n+1)}_\alpha \cup I^{(n+1)}_\alpha$, we have

$$\mu\big(C^{(n)}_\alpha \triangle T C^{(n)}_\alpha\big) \leqslant 2\mu\big(I^{(n+1)}_\alpha\big) \to 0 \quad \text{as } n \to +\infty.$$

Suppose that $x \in T^i I^{(n+1)}_\alpha$ for some $0 \leqslant i < h^{(n)}_{\alpha_1}$. Then

$$T^{h^{(n+1)}_\alpha} x \in T^i T^{h^{(n+1)}_\alpha} I^{(n+1)}_\alpha \subset T^i I^{(n+1)} \subset T^i I^{(n)}_{\alpha_1}.$$

It follows that

$$x, T^{h^{(n+1)}_\alpha} x \in T^i I^{(n)}_{\alpha_1} \subset I_\beta \quad \text{for some } \beta \in \mathcal{A}. \tag{4.8}$$

Therefore

$$\big|x - T^{h^{(n+1)}_\alpha} x\big| \leqslant \big|I^{(n)}_{\alpha_1}\big| \quad \text{for all } x \in C^{(n)}_\alpha.$$

Next, by (4.3), $\varphi^{(h^{(n+1)}_\alpha)}(x) = (Q(n+1)^t v)_\alpha$ for every $x \in I^{(n+1)}_\alpha$. Moreover, if $x \in C^{(n)}_\alpha$, say $x = T^i x_0$ with $x_0 \in I^{(n+1)}_\alpha$ and $0 \leqslant i < h^{(n)}_{\alpha_1}$, then

$$\varphi^{(h^{(n+1)}_\alpha)}\big(T^i x_0\big) - \varphi^{(h^{(n+1)}_\alpha)}(x_0) = \sum_{0\leqslant j<i} \big(\varphi\big(T^{h^{(n+1)}_\alpha} T^j x_0\big) - \varphi\big(T^j x_0\big)\big).$$

By (4.8), $\varphi(T^{h^{(n+1)}_\alpha} T^j x_0) = \varphi(T^j x_0)$ for every $0 \leqslant j < h^{(n)}_{\alpha_1}$, and hence

$$\varphi^{(h^{(n+1)}_\alpha)}(x) = \varphi^{(h^{(n+1)}_\alpha)}(x_0) = \big(Q(n+1)^t v\big)_\alpha \quad \text{for all } x \in C^{(n)}_\alpha.$$

Assume that $T = T_{(\pi,\lambda)}$ is of periodic type and A is its periodic matrix. Denote by ρ_1 the Perron–Frobenius eigenvalue of A. Then there exists $C > 0$ such that $\frac{1}{C}\rho_1^n \leqslant \|A^n\| \leqslant C\rho_1^n$. Since $h_{\max}^{(n)} = \|A^n\| = \max_{\alpha\in\mathcal{A}}(A^n)_\alpha$ and $h_{\min}^{(n)} = \min_{\alpha\in\mathcal{A}}(A^n)_\alpha$, by (4.2), it follows that

$$\frac{1}{C\nu(A)}\rho_1^n \leqslant h_{\min}^{(n)} < h_{\max}^{(n)} \leqslant C\rho_1^n. \tag{4.9}$$

As $|I_\alpha^{(n+1)}| = \rho_1^{-(n+1)}|I_\alpha^{(0)}|$, we have

$$\mu\big(C_\alpha^{(n)}\big) = \big|I_\alpha^{(n+1)}\big|h_{\alpha_1}^{(n)} \geqslant \big|I_\alpha^{(0)}\big|h_{\min}^{(n)}/\rho_1^{n+1} \geqslant \frac{|I_\alpha^{(0)}|}{C\nu(A)\rho_1} > 0.$$

By taking a multiple of the period of T if necessary, we have $I^{(n+1)} \subset I_{\alpha_1}^{(n)}$ for every natural n, and hence

$$\varphi^{(h_\alpha^{(n+1)})}(x) = \big(Q(n+1)^t v\big)_\alpha = \big(\big(A^t\big)^{n+1} v\big)_\alpha \quad \text{for all } x \in C_\alpha^{(n)}. \qquad \square$$

4.2. Ergodic cocycles in case $\kappa > 1$

Assume that $T = T_{(\pi,\lambda)}$ is of periodic type and $\kappa = \ker \Omega_\pi + 1 > 1$. Then the linear space $\ker \Omega_\pi$ is not trivial. As we already mentioned A is the identity on $\ker \Omega_\pi$. Let

$$F(T) = \big\{v \in \mathbb{R}^{\mathcal{A}}:\ A^t v = v\big\}.$$

Then $F(T)$ is a linear subspace with $\dim F(T) = k \geqslant \kappa - 1$. Since

$$\langle v, \lambda\rangle = \big\langle A^t v, \lambda\big\rangle = \langle v, A\lambda\rangle = \rho_1\langle v, \lambda\rangle \quad \text{for each } v \in F(T),$$

we have $F(T) \subset \Gamma_0^{(0)}$. Since A has integer entries, we can choose a basis of the linear space $F(T)$ such that each element of the basis belongs to $\mathbb{Z}^{\mathcal{A}}$. It follows that $\mathbb{Z}^{\mathcal{A}} \cap F(T)$ is a free abelian group of $\mathbb{Z}^{\mathcal{A}}$ whose rank is at least k. Note that $\operatorname{rank}(\mathbb{Z}^{\mathcal{A}} \cap F(T)) = k$. Indeed, choose a subset $\mathcal{A}' \subset \mathcal{A}$ such that $\#\mathcal{A}' = k$ and the standard projection $p : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}'}$ restricted to $F(T)$ establishes a linear isomorphism. Since $p : \mathbb{Z}^{\mathcal{A}} \cap F(T) \to \mathbb{Z}^{\mathcal{A}'}$ is an injective group homomorphism,

$$k \leqslant \operatorname{rank}\big(\mathbb{Z}^{\mathcal{A}} \cap F(T)\big) = \operatorname{rank}\big(p\big(\mathbb{Z}^{\mathcal{A}} \cap F(T)\big)\big) \leqslant \operatorname{rank}\big(\mathbb{Z}^{\mathcal{A}'}\big) = k.$$

Lemma 4.2. *Let $v_i = (v_{i\alpha})_{\alpha\in\mathcal{A}}$, $1 \leqslant i \leqslant k$, be a basis of the group $\mathbb{Z}^{\mathcal{A}} \cap F(T)$. Then the collection of vectors $w_\alpha = (v_{i\alpha})_{i=1}^k \in \mathbb{Z}^k$, $\alpha \in \mathcal{A}$, generates the group $\mathbb{Z}^k$.*

Proof. Denote by $W \subset \mathbb{Z}^k$ the group generated by w_α, $\alpha \in \mathcal{A}$. Since the rank of the matrix $[v_{i\alpha}]_{\alpha\in\mathcal{A},1\leqslant i\leqslant k}$ equals k, the rank of the group W is k. Let $b_j = (b_{lj})_{l=1}^k \in \mathbb{Z}^k$, $1 \leqslant j \leqslant k$, be a basis of W. Thus the determinant of the matrix $B = [b_{lj}]_{1\leqslant j,l\leqslant k}$ is non-zero. Let $M = [m_{\alpha j}]_{\alpha\in\mathcal{A},1\leqslant j\leqslant k}$ stand for the integer matrix such that $w_\alpha = \sum_{j=1}^k m_{\alpha j} b_j$. Denote by $m_j \in \mathbb{Z}^{\mathcal{A}}$, $1 \leqslant j \leqslant k$, the columns of the matrix M. Then $v_l = \sum_{j=1}^k b_{lj} m_j$ for $1 \leqslant l \leqslant k$. Since $\det B \neq 0$, the vectors m_j, $1 \leqslant j \leqslant k$, belong to the linear space generated by v_i, $1 \leqslant i \leqslant k$, that is to $F(T)$. Therefore, $m_j \in \mathbb{Z}^{\mathcal{A}} \cap F(T)$ for $1 \leqslant j \leqslant k$. Since v_i, $1 \leqslant i \leqslant k$, generate $\mathbb{Z}^{\mathcal{A}} \cap F(T)$, it follows that $|\det B| = 1$, and hence $W = \mathbb{Z}^k$. $\square$

Theorem 4.3. *Let $v_i = (v_{i\alpha})_{\alpha\in\mathcal{A}}$, $1 \leqslant i \leqslant k$, be a basis of the group $\mathbb{Z}^{\mathcal{A}} \cap F(T)$. Then the cocycle $\varphi : I \to \mathbb{Z}^k$ given by $\varphi = (\varphi_1, \dots, \varphi_k)$ with $\varphi_i = \sum_{\alpha\in\mathcal{A}} v_{i\alpha}\chi_{I_\alpha}$ for $i = 1, \dots, k$ is ergodic.*

If R is a $(k-1) \times k$-real matrix satisfying (3.12)*, then the cocycle $\tilde{\varphi} : I \to \mathbb{R}^{k-1}$ given by $\tilde{\varphi}(x) = R\varphi(x)$, which is constant over the exchanged intervals, is ergodic.*

Proof. By (4.7) and the definition of $F(T)$, for every $\alpha \in \mathcal{A}$ we have

$$\varphi^{(h_\alpha^{(n+1)})}(x) = \left(\left(\left(A^t\right)^{n+1} v_1\right)_\alpha, \dots, \left(\left(A^t\right)^{n+1} v_k\right)_\alpha\right) = \left((v_1)_\alpha, \dots, (v_k)_\alpha\right) = w_\alpha$$

for $x \in C_\alpha^{(n)}$. In view of Lemma 4.1, we can apply Corollary 2.8. Thus $w_\alpha \in E(\varphi)$ for all $\alpha \in \mathcal{A}$. Since $E(\varphi)$ is a group, by Lemma 4.2, we obtain $E(\varphi) = \mathbb{Z}^k$.

It is easy to show that $RE(\varphi) \subset E(R\varphi)$. Since $E(\varphi) = \mathbb{Z}^k$ and $E(R\varphi)$ is closed, by Remark 3.8, we obtain $E(\tilde{\varphi}) = E(R\varphi) \supset \overline{R\mathbb{Z}^k} = \mathbb{R}^{k-1}$. $\square$

Notice that Remark 3.8 indicates how to construct matrices R satisfying (3.12).

Finally remark that the method of construction of ergodic cocycles presented in this section fails when π gives a symmetric permutation, i.e. $\pi_1 \circ \pi_0^{-1}(j) = d + 1 - j$ with even d, for example when an IET is a rotation. Then $\kappa = 1$ and the space $F(T)$ is trivial.

5. Ergodicity of corrected cocycles

In this section, using a method from [28], we present a procedure of correction of functions in $\mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(0)})$ by piecewise constant functions (in $\Gamma_0^{(0)}$) in order to obtain a better control on the growth of Birkhoff sums. This will allow us to prove the ergodicity of some corrected cocycles.

5.1. Rauzy–Veech induction for cocycles

For every cocycle $\varphi : I^{(k)} \to \mathbb{R}$ for the IET $T^{(k)} : I^{(k)} \to I^{(k)}$ and $l > k$ denote by $S(k,l)\varphi : I^{(l)} \to \mathbb{R}$ the renormalized cocycle for $T^{(l)}$ given by

$$S(k,l)\varphi(x) = \sum_{0 \leqslant i < Q_\beta(k,l)} \varphi\left(\left(T^{(k)}\right)^i x\right) \quad \text{for } x \in I_\beta^{(l)}.$$

We will write $S(k)$ for $S(0,k)$. Note that $S(k,l)$ maps $\mathrm{BV}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ into $\mathrm{BV}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)})$ and

$$\operatorname{Var} S(k,l)\varphi \leqslant \operatorname{Var}\varphi, \tag{5.1}$$

$$\left\|S(k,l)\varphi\right\|_{\sup} \leqslant \left\|Q(k,l)\right\| \|\varphi\|_{\sup}, \tag{5.2}$$

$$\int_{I^{(l)}} S(k,l)\varphi(x)\,dx = \int_{I^{(k)}} \varphi(x)\,dx \tag{5.3}$$

for all $\varphi \in \mathrm{BV}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$. In view of (5.3), $S(k,l)$ maps $\mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ into $\mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)})$.

Recall that $\Gamma^{(k)}$ is the space of functions $\varphi : I^{(k)} \to \mathbb{R}$ which are constant on each interval $I_\alpha^{(k)}$, $\alpha \in \mathcal{A}$, and $\Gamma_0^{(k)}$ is the subspace of functions with zero mean. Then

$$S(k,l)\Gamma^{(k)} = \Gamma^{(l)} \quad \text{and} \quad S(k,l)\Gamma_0^{(k)} = \Gamma_0^{(l)}.$$

Moreover, every function $\sum_{\alpha \in \mathcal{A}} h_\alpha \chi_{I_\alpha^{(k)}}$ from $\Gamma^{(k)}$ can be identified with the vector $h = (h_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$. Under this identification,

$$\Gamma_0^{(k)} = Ann\big(\lambda^{(k)}\big) := \big\{h = (h_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}^{\mathcal{A}} \colon \big\langle h, \lambda^{(k)}\big\rangle = 0\big\}$$

and the operator $S(k,l)$ is the linear automorphism of $\mathbb{R}^{\mathcal{A}}$ whose matrix in the canonical basis is $Q(k,l)^t$. Moreover, the norm on $\Gamma^{(k)}$ inherited from the supremum norm coincides with the norm of vectors.

5.2. *Correction of functions of bounded variation*

Suppose now that T is of periodic type. Let us consider the linear subspaces

$$\Gamma_{cs}^{(k)} = \left\{h \in \Gamma^{(k)} \colon \limsup_{l \to \infty} \frac{1}{l} \log\big\|S(k,l)h\big\| = \limsup_{l \to \infty} \frac{1}{l} \log\big\|\big(A^t\big)^{l-k} h\big\| \leqslant 0\right\},$$

$$\Gamma_u^{(k)} = \left\{h \in \Gamma^{(k)} \colon \limsup_{l \to \infty} \frac{1}{l} \log\big\|\big(A^t\big)^{k-l} h\big\| < 0\right\}.$$

Denote by

$$U^{(k)} : \mathrm{BV}\bigg(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)}\bigg) \to \mathrm{BV}\bigg(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)}\bigg)/\Gamma_{cs}^{(k)}$$

the natural quotient map. Let $P_0^{(k)} : \mathrm{BV}_0(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)}) \to \mathrm{BV}_0(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ be the linear (centering) operator

$$P_0^{(k)}\varphi(x) = \varphi(x) - \frac{1}{|I_\alpha^{(k)}|} \int_{I_\alpha^{(k)}} \varphi(t)\,dt \quad \text{if } x \in I_\alpha^{(k)}.$$

Proposition 5.1. *For every $\varphi \in \mathrm{BV}_0(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ the sequence*

$$\big\{U^{(k)} \circ \big(P_0^{(k)} - S(k,l)^{-1} \circ \big(S(k,l) \circ P_0^{(k)} - P_0^{(l)} \circ S(k,l)\big)\big)\varphi\big\}_{l \geqslant k} \tag{5.4}$$

is well defined and converges in the quotient norm on $\mathrm{BV}_0(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})/\Gamma_{cs}^{(k)}$ induced by $\|\cdot\|_{\mathrm{BV}}$.

Notation. Let $P^{(k)} : \mathrm{BV}_0(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)}) \to \mathrm{BV}_0(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})/\Gamma_{cs}^{(k)}$ stand for the limit operator. Note that if $\varphi \in \Gamma_0^{(k)}$ then $P_0^{(k)}\varphi = 0$, and hence $P^{(k)}\varphi = 0$.

We denote by $\mathrm{BV}^{\diamond}(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha)$ the subspace of functions $\varphi \in \mathrm{BV}(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha)$ such that $\varphi_-(x) = \varphi_+(x)$ for every $x = T^n l_\alpha$, $\alpha \in \mathcal{A}$, $\pi_0(\alpha) \neq 1$, $n \in \mathbb{Z} \setminus \{0\}$.

In general, $(S(k)\varphi)_{k\geqslant 1}$ grows exponentially like $\exp(k\theta_2)$ (see the proof of Theorem 2.4 in Appendix A). We will show that the growth can be reduced when we correct the function φ by a function h constant over the exchanged intervals. Then we will obtain a polynomial growth.

Theorem 5.2. *Suppose that $T = T_{(\pi,\lambda)}$ is of periodic type and M is the maximal size of Jordan blocks in the Jordan decomposition of its periodic matrix. Let $\varphi \in \mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(0)})$. There exist $C_1, C_2 > 0$ such that if $\hat\varphi + \Gamma_{cs}^{(0)} = P^{(0)}\varphi$, then $\hat\varphi - \varphi \in \Gamma_0^{(0)}$ and*

$$\big\|S(k)(\hat\varphi)\big\|_{\sup} \leqslant C_1 k^M \operatorname{Var}\varphi + C_2 k^{M-1}\|\hat\varphi\|_{\sup} \quad \textit{for every natural } k. \tag{5.5}$$

For every $\varphi \in \mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(0)})$ there exists a unique $h \in \Gamma_u^{(0)} \cap \Gamma_0^{(0)}$ such that $\varphi + h + \Gamma_{cs}^{(0)} = P^{(0)}\varphi$.

Notation. In what follows, $\hat\varphi$ will stand for the function φ corrected by the unique correction $h \in \Gamma_u^{(0)} \cap \Gamma_0^{(0)}$, i.e. $\hat\varphi = \varphi + h$.

Theorem 5.3 *(Denjoy–Koksma type inequality). Suppose that T is an IET of periodic type with non-degenerated spectrum and let $\varphi \in \mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(0)})$. If additionally $\kappa(\pi) = 1$ or $\varphi \in \mathrm{BV}_0^\Diamond(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(0)})$ then*

$$\big\|S(k)(\hat\varphi)\big\|_{\sup} \leqslant C_1 \operatorname{Var}\varphi + C_2\|\hat\varphi\|_{\sup} \quad \textit{for every natural } k. \tag{5.6}$$

This result is a counterpart, for IETs of periodic type, of the classical Denjoy–Koksma inequality (2.5). Indeed, if T is an exchange of two intervals I_1, I_2, that is T is a rotation on $\mathbb{T}$ by a quadratic irrational, then $\kappa(\pi) = 1$ and $\Gamma_u^{(0)} \cap \Gamma_0^{(0)} = \{0\}$, so $\hat\varphi = \varphi$. Since $S(k)(\varphi) = \varphi^{(q_{pk})}$ on $I_1^{(k)}$ and $S(k)(\varphi) = \varphi^{(q_{pk-1})}$ on $I_2^{(k)}$, the inequality (5.6) applied to rotated functions $\varphi(\cdot + x)$, $x \in \mathbb{T}$, gives the classical Denjoy–Koksma for BV functions along a subsequence of denominators.

Recall that a counterpart of the Denjoy–Koksma inequality for a.e. IET has already been proved by Marmi, Moussa and Yoccoz in [28] for BV_*^1 functions. They showed that after making a stronger correction, $\|S(k)(\hat\varphi)\|_{\sup} \leqslant C\|Q(k)\|^{-\omega}\|\varphi\|_{\mathrm{BV}}$ for some $C, \omega > 0$ whenever T is of Roth type. This was a crucial step in proving that φ is a coboundary, so it does not seem to help in proving the ergodicity of $\hat\varphi$.

For completeness the proofs of Proposition 5.1, Theorems 5.2 and 5.3 will be given in Appendix C.

If $\varphi : I \to \mathbb{R}^\ell$ is a $\mathrm{BV}^\Diamond$ function with zero mean and $\varphi = (\varphi_1, \ldots, \varphi_\ell)$, we deal with the corrected function $\hat\varphi := (\widehat{\varphi_1}, \ldots, \widehat{\varphi_\ell})$, and we have

$$\big\|S(k)(\hat\varphi)\big\|_{\sup} \leqslant C_1 \max_{1\leqslant i\leqslant \ell} \operatorname{Var}\varphi_i + C_2\|\hat\varphi\|_{\sup} \quad \text{for every natural } k.$$

5.3. Ergodicity of corrected step functions

We now consider piecewise constant zero mean cocycles $\varphi : I \to \mathbb{R}^\ell$, $\ell \geqslant 1$, which will have additional discontinuities in the interior of the exchanged intervals. Suppose that $\gamma_i \in I$, $i = 1, \ldots, s$, are discontinuities of φ different from l_α, $\alpha \in \mathcal{A}$. Denote by $\bar d_i \in \mathbb{R}^\ell$ the vector describing the jumps of coordinate functions of φ at γ_i, that is, $\bar{d}_i = \varphi_+(\gamma_i) - \varphi_-(\gamma_i) \in \mathbb{R}^\ell$. In this section we will prove the ergodicity of $\hat{\varphi}$ for almost every choice of discontinuities. Note that the corrected cocycle $\hat{\varphi}$ is also piecewise constant and it is discontinuous at γ_i with the jump vector $\bar{d}_i$ for $i = 1, \dots, s$, and hence it is still non-trivial.

Theorem 5.4. *Suppose that $T = T_{(\pi,\lambda)}$ is an IET of periodic type and it has non-degenerated spectrum. There exists a set $D \subset I^s$ of full Lebesgue measure such that if*

(i) $(\gamma_1, \dots, \gamma_s) \in D$;
(ii) *the subgroup $\mathbb{Z}(\bar{d}_1, \dots, \bar{d}_s) \subset \mathbb{R}^\ell$ generated by $\bar{d}_1, \dots, \bar{d}_s$ is dense in $\mathbb{R}^\ell$,*

then the cocycle $\hat{\varphi} : I \to \mathbb{R}^\ell$ is ergodic.

Proof. As we already mentioned we can assume that $I^{(n+1)} \subset I^{(n)}_{\alpha_1}$ for every natural n, where $\alpha_1 = (\pi_0^{(n)})^{-1}(1) = \pi_0^{-1}(1)$. Fix $\alpha \in \mathcal{A}$ and choose arbitrarily $b_0 < a_1 < b_1 < \dots < a_s < b_s < a_{s+1}$ so that $[b_0, a_{s+1}) = I_\alpha$. Let

$$F_i^{(n)} = \bigcup_{h^{(n)}_{\alpha_1} \leqslant j < h^{(n+1)}_{\alpha}} T^j\big(a_i/\rho_1^{n+1}, b_i/\rho_1^{n+1}\big), \quad \text{for } 1 \leqslant i \leqslant s,$$

$$C_i^{(n)} = \bigcup_{0 \leqslant j < h^{(n)}_{\alpha_1}} T^j\big(b_i/\rho_1^{n+1}, a_{i+1}/\rho_1^{n+1}\big), \quad \text{for } 0 \leqslant i \leqslant s$$

(ρ_1 is the Perron–Frobenius eigenvalue of the periodic matrix A). Since $[b_0/\rho_1^{n+1}, a_{s+1}/\rho_1^{n+1}) = I^{(n+1)}_\alpha$, the sets $C_i^{(n)}$, $F_i^{(n)}$ are towers for which each level is an interval. Moreover, $C_i^{(n)} \subset C_\alpha^{(n)}$ for $0 \leqslant i \leqslant s$ and

$$h_\alpha^{(n+1)} - h_{\alpha_1}^{(n)} \geqslant \sum_{\beta \in \mathcal{A}} h_\beta^{(n)} - h_{\alpha_1}^{(n)} \geqslant h_{\min}^{(n)}.$$

In view of (4.9), it follows that

$$\mu\big(C_i^{(n)}\big) = (a_{i+1} - b_i)\frac{h^{(n)}_{\alpha_1}}{\rho_1^{n+1}} \geqslant (a_{i+1} - b_i)\frac{h^{(n)}_{\min}}{\rho_1^{n+1}} \geqslant \frac{a_{i+1} - b_i}{C\nu(A)\rho_1} > 0,$$

$$\mu\big(F_i^{(n)}\big) = (b_i - a_i)\frac{h^{(n+1)}_{\alpha} - h^{(n)}_{\alpha_1}}{\rho_1^{n+1}} \geqslant (b_i - a_i)\frac{h^{(n)}_{\min}}{\rho_1^{n+1}} \geqslant \frac{b_i - a_i}{C\nu(A)\rho_1} > 0.$$

Recall that if $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic and $(\Xi_n)_{n \geqslant 1}$ is a sequence of towers for T for which

$$\liminf_{n\to\infty} \mu(\Xi_n) > 0 \quad \text{and} \quad \operatorname{height}(\Xi_n) \to \infty,$$

then (see King [22, Lemma 3.4]) $\mu(B \cap \Xi_n) - \mu(B)\mu(\Xi_n) \to 0$ for all $B \subset \mathcal{B}$. It follows that, for μ-almost every $x \in X$, the point x belongs to Ξ_n for infinitely many n. Indeed, setting

$B_l := \bigcap_{k\geqslant l} \Xi_k^c$ we have $\mu(B_l \cap \Xi_n) = 0$ for $n > l$, so $\mu(B_l) = 0$ for each $l \geqslant 1$. Therefore $\mu(\bigcup_{l\geqslant 1}\bigcap_{k\geqslant l} \Xi_k^c) = 0$.

Applying this fact to subsequences of $(F_i^{(n)})_{n\geqslant 1}$ successively for $i = 1, \dots, s$, we conclude that for a.e. $(\gamma_1, \dots, \gamma_s) \in I^s$ there exists a subsequence $(k_n)_{n\geqslant 1}$ such that

$$\gamma_i \in F_i^{(k_n)} \quad \text{for all } 1 \leqslant i \leqslant s \text{ and } n \geqslant 1.$$

Denote by $D \subset I^s$ the subset of all such $(\gamma_1, \dots, \gamma_s)$ for which γ_i does not belong to the union of orbits of l_α, $\alpha \in \mathcal{A}$, for $i = 1, \dots, s$. Therefore $\varphi \in \mathrm{BV}^\Diamond(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^\ell)$.

Suppose that for some $n \geqslant 1$ we have $\gamma_i \in F_i^{(n)}$ for all $1 \leqslant i \leqslant s$. Then the sets $T^j C_i^{(n)}$, $0 \leqslant j < h_\alpha^{(n+1)}$, $0 \leqslant i \leqslant s$, do not contain discontinuities of $\hat{\varphi}$. Thus similar arguments to those from the proof of (4.7) show that $\hat{\varphi}^{(h_\alpha^{(n+1)})}$ is constant on each $C_i^{(n)}$ and equals say $\bar{g}_i^{(n)} \in \mathbb{R}^\ell$.

Let $x \in [b_{i-1}/\rho_1^{n+1}, a_i/\rho_1^{n+1})$ and $y \in [b_i/\rho_1^{n+1}, a_{i+1}/\rho_1^{n+1})$. Since $\gamma_i \in T^{j_0}[a_i/\rho_1^{n+1}, b_i/\rho_1^{n+1})$ for some $h_{\alpha_1}^{(n)} \leqslant j_0 < h_\alpha^{(n+1)}$, it follows that $\hat{\varphi}(T^j x) = \hat{\varphi}(T^j y)$ for all $0 \leqslant j < h_\alpha^{(n+1)}$, $j \neq j_0$ and $\hat{\varphi}(T^{j_0} y) - \hat{\varphi}(T^{j_0} x) = \bar{d}_i$. Consequently,

$$\bar{g}_i^{(n)} - \bar{g}_{i-1}^{(n)} = \hat{\varphi}^{(h_\alpha^{(n+1)})}(y) - \hat{\varphi}^{(h_\alpha^{(n+1)})}(x) = \bar{d}_i.$$

It follows that

$$\hat{\varphi}^{(h_\alpha^{(n+1)})}(x) = \bar{g}_0^{(n)} + \sum_{l=1}^{i} \bar{d}_l \quad \text{for all } x \in C_i^{(n)},\ 0 \leqslant i \leqslant s.$$

Since $\varphi \in \mathrm{BV}^\Diamond(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^\ell)$, by Theorem 5.2 there exists $C > 0$ such that

$$\left\|\hat{\varphi}^{(h_\alpha^{(n+1)})}(x)\right\| = \left\|S(n+1)\hat{\varphi}(x)\right\| \leqslant C \quad \text{for all } x \in I_\alpha^{(n+1)},$$

and hence $\|\bar{g}_0^{(n)}\| \leqslant C$. Therefore for each $(\gamma_1, \dots, \gamma_s) \in D$ there exists a subsequence $(k_n)_{n\geqslant 1}$ such that

$$\hat{\varphi}^{(h_\alpha^{(k_n+1)})}(x) = \bar{g}_0^{(k_n)} + \sum_{l=1}^{i} \bar{d}_l \quad \text{for all } x \in C_i^{(k_n)},\ 0 \leqslant i \leqslant s,$$

and $\bar{g}_0^{(k_n)} \to \bar{g}_0$ in $\mathbb{R}^\ell$. Since $\liminf \mu(C_i^{(k_n)}) > 0$ for each $0 \leqslant i \leqslant s$, Corollary 2.8 implies $\bar{g}_0 + \sum_{l=1}^{i} \bar{d}_l \in E(\hat{\varphi})$ for each $0 \leqslant i \leqslant s$. Therefore $\bar{d}_l \in E(\hat{\varphi})$ for each $1 \leqslant l \leqslant s$. Since $\bar{d}_1, \dots, \bar{d}_s$ generate a dense subgroup of $\mathbb{R}^\ell$ and $E(\hat{\varphi})$ is closed, it follows that $E(\hat{\varphi}) = \mathbb{R}^\ell$. $\square$

Remark 5.5. Notice that the condition (ii) implies $s > \ell$. On the other hand, if $s > \ell$, in view of Remark 3.8, we can easily find a collection of vectors $\bar{d}_1, \dots, \bar{d}_s \in \mathbb{R}^\ell$ such that $\overline{\mathbb{Z}(\bar{d}_1, \dots, \bar{d}_s)} = \mathbb{R}^\ell$.

In order to have a more specific condition on the discontinuities γ_i, $i = 1, \dots, s$, which guarantees ergodicity, we can use a periodic type condition. Indeed, let us consider a set

$\{\gamma_1, \ldots, \gamma_s\} \subset I \setminus \{l_\alpha\colon \alpha \in \mathcal{A}\}$. The points $\gamma_1, \ldots, \gamma_s$ together with l_α, $\alpha \in \mathcal{A}$, give a new partition of I into $d+s$ intervals. Therefore T can be treated as a $d+s$-IET. Denote by (π', λ') the combinatorial data of this representation of T.

Definition 4. We say that the set $\{\gamma_1, \ldots, \gamma_s\}$ is of *periodic type with respect to* $T_{(\pi,\lambda)}$ if the IET $T_{(\pi',\lambda')}$ is of periodic type as an exchange of $d+s$ intervals.

An example of a set which is of periodic type with respect to an IET will be presented in Section 7.2.

Remark 5.6. By the definition of periodic type, (λ', π') satisfies the Keane condition. Therefore, each γ_i does not belong to the orbit of any l_α, $\alpha \in \mathcal{A}$.

In view of Theorem 23 in [31], each admissible interval $I^{(p)}$ (p is a period) for $T_{(\pi',\lambda')}$ is also admissible for $T_{(\pi,\lambda)}$. Therefore $T_{(\pi,\lambda)}$ is of periodic type as an exchange of d-intervals as well. It follows that, for every $n \geqslant 0$ and $i = 1, \ldots, s$ if $\gamma_i \in I_\alpha$, then $\gamma_i = T^j_{(\pi,\lambda)}(\gamma_i/\rho^n)$ for some $0 \leqslant j < h^{(n)}_\alpha$. Therefore similar arguments to those in the proof of Theorem 5.4 give the following result.

Theorem 5.7. *Suppose that $T = T_{(\pi,\lambda)}$ is an IET of periodic type and it has non-degenerated spectrum. Let $\varphi : I \to \mathbb{R}^\ell$ be a zero mean piecewise constant cocycle with additional discontinuity at $\gamma_i \in I \setminus \{l_\alpha\colon \alpha \in \mathcal{A}\}$ with the jump vectors $\bar{d}_i \in \mathbb{R}^\ell$ for $i = 1, \ldots, s$. If*

(i) *the set $\{\gamma_1, \ldots, \gamma_s\}$ is of periodic type with respect to $T_{(\pi,\lambda)}$;*
(ii) $\overline{\mathbb{Z}(\bar{d}_1, \ldots, \bar{d}_s)} = \mathbb{R}^\ell$,

then the cocycle $\hat{\varphi} : I \to \mathbb{R}^\ell$ is ergodic.

6. Recurrence and ergodicity of extensions of multivalued Hamiltonians

In this section we deal with a class of smooth flows on non-compact manifolds which are extensions of so-called multivalued Hamiltonian flows on compact surfaces of higher genus. Each such flow has a special representation over a skew product of an IET and a BV cocycle. This allows us to apply abstract results from previous sections to state some sufficient conditions for recurrence and ergodicity whenever the IET is of periodic type.

6.1. Special flows

In this subsection we briefly recall some basic properties of special flows. Let T be an automorphism of a σ-finite measure space $(X, \mathcal{B}, \mu)$. Let $f : X \to \mathbb{R}$ be a strictly positive function such that

$$\sum_{n \geqslant 1} f\left(T^n x\right) = +\infty \quad \text{for a.e. } x \in X. \tag{6.1}$$

By $T^f = (T^f_t)_{t \in \mathbb{R}}$ we will mean the corresponding special flow under f (see e.g. [10, Chapter 11]) acting on $(X^f, \mathcal{B}^f, \mu^f)$, where $X^f = \{(x, s) \in X \times \mathbb{R}\colon 0 \leqslant s < f(x)\}$ and $\mathcal{B}^f$ (μ^f) is the restriction of $\mathcal{B} \times \mathcal{B}(\mathbb{R})$ $(\mu \times m_{\mathbb{R}})$ to X^f. Under the action of the flow T^f each point in X^f moves vertically at unit speed, and we identify the point $(x, f(x))$ with $(Tx, 0)$. More precisely, for every $(x, s) \in X^f$ we have

$$T_t^f(x, s) = \big(T^n x, s + t - f^{(n)}(x)\big),$$

where $n \in \mathbb{Z}$ is a unique number such that $f^{(n)}(x) \leqslant s + t < f^{(n+1)}(x)$.

Remark 6.1. If T is conservative then the condition (6.1) holds automatically and the special flow T^f is conservative as well. Moreover, if T is ergodic then T^f is ergodic.

6.2. Basic properties of multivalued Hamiltonian flows

Now we will consider multivalued Hamiltonians and their associated flows, a model which has been developed by S.P. Novikov (see also [2] for the toral case). Let (M, ω) be a compact symplectic smooth surface and β be a Morse closed 1-form on M, that is $\beta = -\frac{\partial H}{\partial y}\, dx + \frac{\partial H}{\partial x}\, dy$ in local coordinates, where H is local smooth function (local Hamiltonian) such that $\det D^2 H(x, y) \neq 0$ at each critical point (x, y). Denote by $\pi : \hat{M} \to M$ the universal cover of M and by $\hat{\beta}$ the pullback of β by $\pi : \hat{M} \to M$. Since $\hat{M}$ is simply connected and $\hat{\beta}$ is also a closed form, there exists a smooth function $\hat{H} : \hat{M} \to \mathbb{R}$, called a multivalued Hamiltonian, such that $d\hat{H} = \hat{\beta}$. By assumption, $\hat{H}$ is a Morse function. Suppose additionally that all critical values of $\hat{H}$ are distinct.

Denote by $X : M \to TM$ the smooth vector field determined by

$$\beta = i_X \omega := \omega(X, \cdot).$$

Let $(\phi_t)_{t\in\mathbb{R}}$ stand for the smooth flow on M associated to the vector field X. Since $d\beta = 0$, the flow $(\phi_t)_{t\in\mathbb{R}}$ preserves the symplectic form ω, and hence it preserves the smooth measure $\nu = \nu_\omega$ determined by ω. Since β is a Morse form, the flow $(\phi_t)_{t\in\mathbb{R}}$ has finitely many fixed points (equal to zeros of β or, which is the same, equal to images of critical points of $\hat{H}$ by the map π). The set of fixed points of $(\phi_t)_{t\in\mathbb{R}}$ will by denoted by $\mathcal{F}(\beta)$. All of them are centers or non-degenerated saddles. By assumption, any two different saddles are not connected by a separatrix of the flow (called a saddle connection). Nevertheless, the flow $(\phi_t)_{t\in\mathbb{R}}$ can have saddle connections which are loops. Each such saddle connection gives a decomposition of M into two non-trivial invariant subsets.

By Theorem 14.6.3 in [21], the surface M can be represented as the finite union of disjoint $(\phi_t)_{t\in\mathbb{R}}$-invariant sets as follows

$$M = \mathcal{P} \cup \mathcal{S} \cup \bigcup_{\mathcal{T} \in \mathfrak{T}} \mathcal{T},$$

where $\mathcal{P}$ is an open set consisting of periodic orbits, $\mathcal{S}$ is a finite union of fixed points or saddle connections, and each $\mathcal{T} \in \mathfrak{T}$ is open and every positive semi-orbit in $\mathcal{T}$, that is not a separatrix incoming to a fixed point, is dense in $\mathcal{T}$. It follows that $\overline{\mathcal{T}}$ is a transitive component of $(\phi_t)_{t\in\mathbb{R}}$. Each transitive component $\overline{\mathcal{T}}$ is a surface with boundary and the boundary of $\overline{\mathcal{T}}$ is a finite union of fixed points and loop saddle connections.

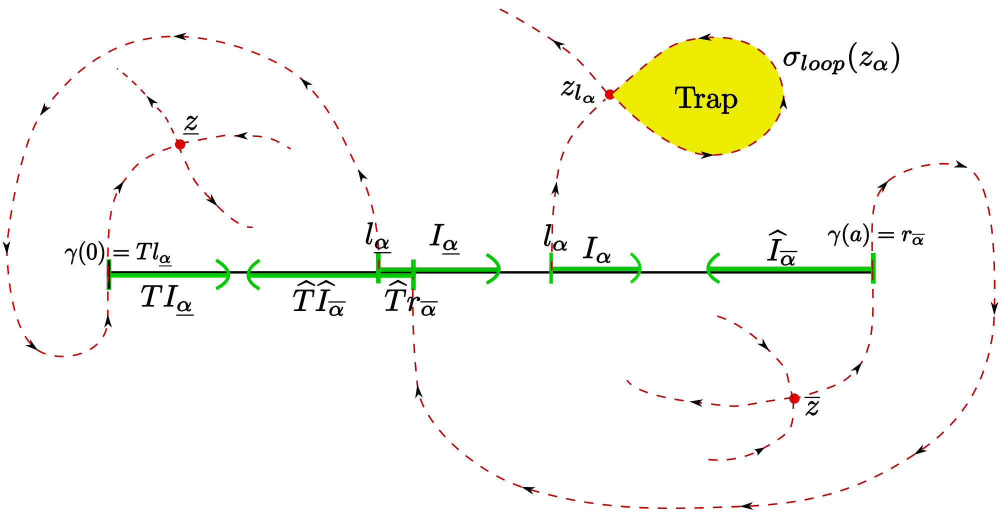

Fig. 1. Separatrices of (ϕ_t).

Remark 6.2. Let X be a smooth tangent vector field preserving a volume form ω on a surface M. A parametrization $\gamma : [a,b] \to M$ of a curve is called *induced* if

$$\int_{\gamma(s)}^{\gamma(s')} i_X\omega = s' - s \quad \text{for all } s, s' \in [a,b].$$

Let $\gamma : [a,b] \to M$ and $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to M$ be induced parameterizations of two curves. Suppose that for every $x \in [a,b]$ the positive semi-orbit of the flow starting from $\gamma(x)$ hits the curve $\tilde{\gamma}$. Denote by $T_{\gamma\tilde{\gamma}}(x) \in [\tilde{a}, \tilde{b}]$ the parameter and by $\tau_{\gamma\tilde{\gamma}}(x) > 0$ the time of the first hit. Using Stokes' theorem, it is easy to check that $T_{\gamma\tilde{\gamma}} : [a,b] \to [\tilde{a}, \tilde{b}]$ is a translation and $\tau_{\gamma\tilde{\gamma}} : [a,b] \to \mathbb{R}_+$ is a smooth function.

Fix $\mathcal{T} \in \mathfrak{T}$ and let $J \subset \mathcal{T}$ be a transversal smooth curve for $(\phi_t)_{t\in\mathbb{R}}$ such that the boundary of J consists of two points lying on an incoming and an outgoing separatrices respectively, and the segment of each separatrix between the corresponding boundary point of J and the fixed point has no intersection with J. Let $\gamma : [0,a] \to J$ stand for the induced parametrization such that the boundary points $\gamma(0)$ and $\gamma(a)$ lie on the incoming and outgoing separatrices respectively (see Fig. 1). Set $I = [0,a)$. We will identify the interval I with the curve J.

Denote by $T := T_{\gamma\gamma}$ the first-return map induced on J; T can be seen as a map $T : I \to I$. By Remark 6.2, $T : I \to I$ is an interval exchange transformation. Then $T = T_{(\pi,\lambda)}$, where $\pi \in \mathcal{S}^0_{\mathcal{A}}$ for some finite set $\mathcal{A}$ and $(\pi,\lambda) \in \mathcal{S}^0_{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}_+$ satisfies the Keane condition. Recall that l_α, $\alpha \in \mathcal{A}$, are the left end points of the exchanged intervals. Let $\mathcal{Z} = \mathcal{F}(\beta) \cap \overline{\mathcal{T}}$. Since $\overline{\mathcal{T}}$ is a transitive component, each element of $\mathcal{Z}$ is a non-degenerate saddle. Let us decompose the set $\mathcal{Z}$ of fixed points into subsets $\mathcal{Z}_0$, $\mathcal{Z}_+$ and $\mathcal{Z}_-$ of points $z \in \mathcal{Z}$ such that z has no loop connection, has a loop connection with positive orientation and has a loop connection with negative orientation respectively. For each $z \in \mathcal{Z}_+ \cup \mathcal{Z}_-$ denote by $\sigma_{loop}(z)$ the corresponding loop connection.

Denote by $\underline{z} \in \mathcal{Z}$ the fixed point such that $\gamma(0)$ belongs to its incoming separatrix $\sigma^-(\underline{z})$. Then $\gamma(0)$ is the first backward intersection with J of $\sigma^-(\underline{z})$. Set $\underline{\alpha} = \pi_1^{-1}(1) \in \mathcal{A}$. Then each point $\gamma(l_\alpha)$ with $\alpha \neq \underline{\alpha}$ corresponds to the first backward intersection with J of an incoming separatrix of a fixed point, denoted by $z_{l_\alpha} \in \mathcal{Z}$ (see Fig. 1). The point $\gamma(l_{\underline{\alpha}})$ corresponds to the second backward intersection with J of $\sigma^-(\underline{z})$ and $Tl_{\underline{\alpha}} = 0$.

Denote by $\tau : I \to \mathbb{R}_+$ the first-return time map of the flow $(\phi_t)_{t\in\mathbb{R}}$ to J. This map is well defined and smooth on the interior of each interval I_α, $\alpha \in \mathcal{A}$, and τ has a singularity of logarithmic type at each point l_α, $\alpha \in \mathcal{A}$ (see [23]) except for the right side of $l_{\underline{\alpha}}$; here the right-sided limit of τ exists. Moreover, the flow $(\phi_t)_{t\in\mathbb{R}}$ on $(\mathcal{T}, \nu|_{\mathcal{T}})$ is measure-theoretical isomorphic to the special flow T^τ. An isomorphism is established by the map $\Gamma : I^\tau \to \overline{\mathcal{T}}$, $\Gamma(x, s) = \phi_s\gamma(x)$.

6.3. Extensions of multivalued Hamiltonian flows

Following [11] we will study ergodic properties of some couplings of a multivalued Hamiltonian flow $(\phi_t)_{t\in\mathbb{R}}$ on M and the geodesic flow on $\mathbb{R}^\ell$, $\ell \geqslant 1$. More precisely, we will consider coupled differential equations on $M \times \mathbb{R}^\ell$ of the form

$$\begin{cases} \dfrac{dx}{dt} = X(x), \\ \dfrac{dy}{dt} = f(x), \end{cases}$$

for $(x, y) \in M \times \mathbb{R}^\ell$, where $f : M \to \mathbb{R}^\ell$ is smooth. Then the associated flow $(\Phi_t^f)_{t\in\mathbb{R}} = (\Phi_t)_{t\in\mathbb{R}}$ on $M \times \mathbb{R}^\ell$ is given by

$$\Phi_t(x, y) = \left(\phi_t x, y + \int_0^t f(\phi_s x)\, ds\right).$$

It follows that $(\Phi_t)_{t\in\mathbb{R}}$ is a skew product flow with the base flow $(\phi_t)_{t\in\mathbb{R}}$ on M and the cocycle $F : \mathbb{R} \times M \to \mathbb{R}^\ell$ given by

$$F(t, x) = \int_0^t f(\phi_s x)\, ds.$$

Therefore $(\Phi_t)_{t\in\mathbb{R}}$ preserves the product measure $\nu \times m_{\mathbb{R}^\ell}$. The deviation of the cocycle F was studied by Forni in [12,13] for typical $(\phi_t)_{t\in\mathbb{R}}$ with no saddle connections.

Recall that the ergodicity of $(\Phi_t^f)_{t\in\mathbb{R}}$ has been studied in [11] in the case where $(\phi_t)_{t\in\mathbb{R}}$ is a flow on $\mathbb{T}^2$ with exactly one trap and the function $f : \mathbb{T}^2 \to \mathbb{R}$ does not vanish on the saddle fixed point; the ergodicity of $(\Phi_t^f)_{t\in\mathbb{R}}$ was proven for typical $(\phi_t)_{t\in\mathbb{R}}$. In this section we will deal with higher genus surfaces. We study the recurrence and ergodic properties of the flow $(\Phi_t^f)_{t\in\mathbb{R}}$ for functions $f : M \to \mathbb{R}^\ell$ such that $f(x) = 0$ for all $x \in \mathcal{F}(\beta)$. The flow $(\Phi_t^f)_{t\in\mathbb{R}}$ will be restricted to the invariant set $\overline{\mathcal{T}} \times \mathbb{R}^\ell$ for a fixed $\mathcal{T} \in \mathfrak{T}$. Let us consider its transversal submanifold $J \times \mathbb{R}^\ell \subset \overline{\mathcal{T}} \times \mathbb{R}^\ell$. Note that every point $(\gamma(x), y) \in \gamma(\mathrm{Int}\, I_\alpha) \times \mathbb{R}^\ell$ returns to

$J \times \mathbb{R}^\ell$ and the return time is $\hat{\tau}(x, y) = \tau(x)$. Denote by $\varphi_f = \varphi : \bigcup_{\alpha\in\mathcal{A}} \operatorname{Int} I_\alpha \to \mathbb{R}^\ell$ the smooth function

$$\varphi(x) = F\big(\tau(x), \gamma(x)\big) = \int_0^{\tau(x)} f\big(\phi_s \gamma(x)\big)\, ds, \quad \text{for } x \in \bigcup_{\alpha\in\mathcal{A}} \operatorname{Int} I_\alpha.$$

Notice that

$$\int_I \varphi(x)\, dx = \int_{\mathcal{T}} f\, d\nu. \tag{6.2}$$

Let us consider the skew product $T_\varphi : (I \times \mathbb{R}^\ell, \mu \times m_{\mathbb{R}^\ell}) \to (I \times \mathbb{R}^\ell, \mu \times m_{\mathbb{R}^\ell})$, $T_\varphi(x, y) = (Tx, y + \varphi(x))$ and the special flow $(T_\varphi)^{\hat{\tau}}$ built over T_φ and under the roof function $\hat{\tau} : I \times \mathbb{R}^\ell \to \mathbb{R}_+$ given by $\hat{\tau}(x, y) = \tau(x)$.

Lemma 6.3. *The special flow $(T_\varphi)^{\hat{\tau}}$ is measure-theoretically isomorphic to the flow (Φ_t) on $(\mathcal{T} \times \mathbb{R}^\ell, \nu|_{\mathcal{T}} \times m_{\mathbb{R}^\ell})$.*

Remark 6.4. If $\int_{\mathcal{T}} f\, d\nu \neq 0$ then, by (6.2), the skew product T_φ is totally dissipative. In view of Lemma 6.3, the flow (Φ_t) on $(\mathcal{T} \times \mathbb{R}^\ell, \nu|_{\mathcal{T}} \times m_{\mathbb{R}^\ell})$ is totally dissipative as well. On the other hand, if $\ell = 1$ and (ϕ_t) on $(\mathcal{T}, \nu|_{\mathcal{T}})$ is ergodic, then $\int_{\mathcal{T}} f\, d\nu = 0$ implies the recurrence of (Φ_t) on $(\mathcal{T} \times \mathbb{R}, \nu|_{\mathcal{T}} \times m_{\mathbb{R}})$.

The following lemma will help us to find further properties of φ. Since the proof is rather straightforward and the first part follows very closely the proof of Proposition 2 in [17], we leave it to the reader.

Lemma 6.5. *Let $g : [-1, 1] \times [-1, 1] \to \mathbb{R}$ be a $\mathscr{C}^1$-function such that $g(0, 0) = 0$. Then the function $\xi : [0, 1] \to \mathbb{R}$,*

$$\xi(s) = \begin{cases} \int_s^1 g(u, \frac{s}{u})\frac{1}{u}\, du, & \text{if } s > 0, \\ \int_0^1 (g(u, 0) + g(0, u))\frac{1}{u}\, du, & \text{if } s = 0, \end{cases}$$

is absolutely continuous. If additionally g is a $\mathscr{C}^{2+\epsilon}$-function, $g'(0, 0) = 0$, and $g''(0, 0) = 0$, then ξ' is absolutely continuous.

Remark 6.6. Note that the second conclusion of the lemma becomes false if the requirement $g''(0, 0) = 0$ is omitted. Indeed, if $g(x, y) = x \cdot y$ then $\xi'(s) = -\log s - 1$, $s > 0$, is not even bounded.

For each $z \in \mathcal{Z}_+ \cup \mathcal{Z}_-$ choose an element u_z of the saddle loop $\sigma_{loop}(z)$.

Theorem 6.7. *If $f(x)=0$ for all $x\in\mathcal{F}(\beta)$, then φ is absolutely continuous on each interval I_α, $\alpha\in\mathcal{A}$, in particular $\varphi\in\mathrm{BV}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha,\mathbb{R}^\ell)$. Moreover,*

$$\int\limits_I \varphi'(x)\,dx=\sum_{z\in\mathcal{Z}_+}\int\limits_{\mathbb{R}} f(\phi_s u_z)\,ds-\sum_{z\in\mathcal{Z}_-}\int\limits_{\mathbb{R}} f(\phi_s u_z)\,ds.$$

If additionally $f'(x)=0$ and $f''(x)=0$ for all $x\in\mathcal{F}(\beta)$, then $\varphi''\in L^1(I,\mathbb{R}^\ell)$, in particular, $\varphi\in\mathrm{BV}^1(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha,\mathbb{R}^\ell)$.

Proof. First note that it suffices to consider the case $\ell=1$. Since $d\beta=0$, there exists a family of pairwise disjoint open sets $U_z\subset M$, $z\in\mathcal{Z}$, such that $z\in U_z$ and there exists a smooth function $H:\bigcup_{z\in\mathcal{Z}} U_z\to\mathbb{R}$ such that $dH=\beta$ on U_z for every $z\in\mathcal{Z}$. By the Morse Lemma, for every $z\in\mathcal{Z}$ there exist a neighborhood $(0,0)\in V_z\subset\mathbb{R}^2$ and a smooth diffeomorphism $\Upsilon_z:V_z\to U_z$ such that $\Upsilon_z(0,0)=z$ and

$$H_z(x,y):=H\circ\Upsilon_z(x,y)=x\cdot y\quad\text{for all }(x,y)\in V_z.$$

Denote by $\omega^z\in\Omega^2(V_z)$ the pullback of the form ω by $\Upsilon_z:V_z\to U_z$. Since ω^z is non-zero at each point, there exists a smooth non-zero function $p=p_z:V_z\to\mathbb{R}$ such that

$$\omega^z_{(x,y)}=p(x,y)\,dx\wedge dy.$$

Let (ϕ_t^z) stand for the pullback of the flow (ϕ_t) by $\Upsilon_z:V_z\to U_z$, i.e. the local flow on V_z given by $\phi_t^z=\Upsilon_z^{-1}\circ\phi_t\circ\Upsilon_z$. Denote by $X^z:V_z\to\mathbb{R}^2$ the vector field corresponding to (ϕ_t^z). Then $dH_z=\omega^z(X^z,\cdot)$, and hence

$$X^z(x,y)=\frac{(\frac{\partial H_z}{\partial y}(x,y),-\frac{\partial H_z}{\partial x}(x,y))}{p(x,y)}=\frac{(x,-y)}{p(x,y)}.$$

Let δ be a positive number such that $[-\delta,\delta]\times[-\delta,\delta]\subset V_z$ for every $z\in\mathcal{Z}$. Let us consider the $\mathscr{C}^\infty$-curves $\gamma_z^{\pm,0},\gamma_z^{\pm,1}:[-\delta^2,\delta^2]\to M$ given by

$$\gamma_z^{\pm,0}(s)=\Upsilon_z(\pm s/\delta,\pm\delta),\qquad\gamma_z^{\pm,1}(s)=\Upsilon_z(\pm\delta,\pm s/\delta).$$

Notice that $\gamma_z^{\pm,i}$ establishes an induced parametrization for the form $\omega(x,y)$ and the vector field X. Indeed, we have for every $s\in[-\delta^2,\delta^2]$ and $i=0,1$,

$$\int\limits_{\gamma_z^{\pm,i}(0)}^{\gamma_z^{\pm,i}(s)}\beta=\int\limits_{\gamma_z^{\pm,i}(0)}^{\gamma_z^{\pm,i}(s)}dH=H\big(\gamma_z^{\pm,i}(s)\big)-H\big(\gamma_z^{\pm,i}(0)\big)=\pm s/\delta\cdot\pm\delta=s.$$

We consider the functions $\tau_z^{\pm}$ and $\varphi_z^{\pm}$ from $[-\delta^2, 0) \cup (0, \delta^2]$ to $\mathbb{R}$, where $\tau_z^{\pm}(x)$ is the exit time of the point $\gamma_z^{\pm,0}(x)$ for the flow (ϕ_t) from the set $\Upsilon_z([-\delta, \delta] \times [-\delta, \delta])$ and

$$\varphi_z^{\pm}(x) = \int_0^{\tau_z^{\pm}(x)} f\big(\phi_s \gamma_z^{\pm,0} x\big)\, ds.$$

Note that $\tau_z^{\pm}(x)$ is the passage time from $(\pm x/\delta, \pm\delta)$ to $(\pm \operatorname{sgn}(x)\delta, \pm \operatorname{sgn}(x)x/\delta)$ for the local flow (ϕ_t^z). Let $f_z : V_z \to \mathbb{R}$ be given by $f_z = f \circ \Upsilon_z$. By assumption, f_z is a smooth function such that $f_z(0, 0) = 0$. Furthermore,

$$\varphi_z^{\pm}(x) = \int_0^{\tau_z^{\pm}(x)} f_z\big(\phi_s^z(\pm x/\delta, \pm\delta)\big)\, ds.$$

Let $(x_s, y_s) = \phi_s^z(\pm x/\delta, \pm\delta)$. Then

$$\left(\frac{d}{ds} x_s, \frac{d}{ds} y_s\right) = X^z(x_s, -y_s) = \frac{(x_s, -y_s)}{p(x_s, y_s)}, \tag{6.3}$$

and hence

$$x_s \cdot y_s = H_z(x_s, y_s) = H_z(x_0, y_0) = H_z(\pm x/\delta, \pm\delta) = x.$$

Since $x \neq 0$, it follows that $x_s \neq 0$ for all $s \in \mathbb{R}$. By using the substitution $u = x_s$, we obtain $du = \frac{d}{ds} x_s\, ds = \frac{x_s}{p(x_s, y_s)}\, ds$ and

$$\varphi_z^{\pm}(x) = \int_0^{\tau_z^{\pm}(x)} f_z(x_s, y_s)\, ds = \int_0^{\tau_z^{\pm}(x)} f_z\left(x_s, \frac{x}{x_s}\right) ds = \int_{\pm x/\delta}^{\pm \operatorname{sgn}(x)\delta} f_z\left(u, \frac{x}{u}\right) p\left(u, \frac{x}{u}\right) \frac{du}{u}.$$

By Lemma 6.5, the functions $\varphi_z^{\pm} : [-\delta^2, 0) \cup (0, \delta^2] \to \mathbb{R}$ are absolutely continuous and

$$\lim_{x \to 0^+} \varphi_z^{\pm}(x) = \int_0^{\pm\delta} f_z(u, 0) p(u, 0) \frac{du}{u} + \int_0^{\pm\delta} f_z(0, u) p(0, u) \frac{du}{u}.$$

It follows that

$$\lim_{x \to 0^+} \varphi_z^{\pm}(x) = \int_0^{+\infty} f\big(\phi_s \gamma_z^{\pm,0} 0\big)\, ds + \int_{-\infty}^0 f\big(\phi_s \gamma_z^{\pm,1} 0\big)\, ds. \tag{6.4}$$

Similar arguments to those above show that

$$\lim_{x\to 0^-} \varphi_z^{\pm}(x) = \int_0^{+\infty} f\bigl(\phi_s \gamma_z^{\pm,0} 0\bigr)\, ds + \int_{-\infty}^{0} f\bigl(\phi_s \gamma_z^{\mp,1} 0\bigr)\, ds. \tag{6.5}$$

In view of Remark 6.2, we conclude that $\varphi : I \to \mathbb{R}$ is absolutely continuous on each interval I_α, $\alpha \in \mathcal{A}$, and

$$\varphi_+(l_\alpha) = \int_0^{+\infty} f\bigl(\phi_s \gamma(l_\alpha)\bigr)\, ds + \int_{-\infty}^{0} f\bigl(\phi_s \gamma(T l_\alpha)\bigr)\, ds, \tag{6.6}$$

whenever $\alpha \neq \underline{\alpha}$ and $z_{l_\alpha} \in \mathcal{Z}_- \cup \mathcal{Z}_0$. If $\alpha \neq \underline{\alpha}$ and $z_{l_\alpha} \in \mathcal{Z}_+$, then computing $\varphi_+(l_\alpha)$ we have to take into account the loop separatrix $\sigma_{loop}(z_{l_\alpha})$, so that

$$\varphi_+(l_\alpha) = \int_0^{+\infty} f\bigl(\phi_s \gamma(l_\alpha)\bigr)\, ds + \int_{-\infty}^{0} f\bigl(\phi_s \gamma(T l_\alpha)\bigr)\, ds + \int_{-\infty}^{+\infty} f(\phi_s u_{z_{l_\alpha}})\, ds. \tag{6.7}$$

Moreover, if $f'(z_{l_\alpha}) = 0$ and $f''(z_{l_\alpha}) = 0$ then the derivative φ'' is integrable on a neighborhood of l_α. It follows that if $f'(z) = 0$ and $f''(z) = 0$ for each $z \in \mathcal{F}(\beta)$ then φ'' is integrable.

If $\alpha = \underline{\alpha}$, then, by definition, the positive semi-orbit through $\gamma(l_\alpha)$ returns to γ before approaching the fixed point $\underline{z}$. It follows that

$$\varphi_+(l_\alpha) = \int_0^{\tau_{\gamma\gamma}(l_\alpha)} f\bigl(\phi_s \gamma(l_\alpha)\bigr)\, ds. \tag{6.8}$$

Let $\bar{\alpha} = \pi_0^{-1}(d)$, i.e. $r_{\bar{\alpha}} = |I|$. Similar arguments to those used for the right-sided limits show that for every $\alpha \neq \bar{\alpha}$ we have

$$\varphi_-(r_\alpha) = \int_0^{+\infty} f\bigl(\phi_s \gamma(r_\alpha)\bigr)\, ds + \int_{-\infty}^{0} f\bigl(\phi_s \gamma(\hat{T} r_\alpha)\bigr)\, ds \quad \text{if } z_{r_\alpha} \in \mathcal{Z}_0 \cup \mathcal{Z}_+, \tag{6.9}$$

$$\varphi_-(r_\alpha) = \int_0^{+\infty} f\bigl(\phi_s \gamma(r_\alpha)\bigr)\, ds + \int_{-\infty}^{0} f\bigl(\phi_s \gamma(\hat{T} r_\alpha)\bigr)\, ds + \int_{-\infty}^{+\infty} f(\phi_s u_{z_{r_\alpha}})\, ds \quad \text{if } z_{r_\alpha} \in \mathcal{Z}_-. \tag{6.10}$$

Moreover, since $\gamma(r_{\bar{\alpha}})$ lies on an outgoing separatrix, the positive semi-orbit through $\gamma(r_{\bar{\alpha}})$ returns to the curve γ, so that

$$\varphi_-(r_{\bar{\alpha}}) = \varphi_-\bigl(|I|\bigr) = \int_0^{\tau_{\gamma\gamma}(r_{\bar{\alpha}})} f\bigl(\phi_s \gamma(r_{\bar{\alpha}})\bigr)\, ds. \tag{6.11}$$

In view of (6.6)–(6.11), we have

$$\int_I \varphi'(x)\,dx = \sum_{\alpha\in\mathcal{A}} \int_{I_\alpha} \varphi'(x)\,dx = \sum_{\alpha\in\mathcal{A}} \bigl(\varphi_-(r_\alpha) - \varphi_+(l_\alpha)\bigr)$$
$$= \sum_{z\in\mathcal{Z}_-} \int_{-\infty}^{+\infty} f(\phi_s u_z)\,ds - \sum_{z\in\mathcal{Z}_+} \int_{-\infty}^{+\infty} f(\phi_s u_z)\,ds$$
$$+ \sum_{\alpha\in\mathcal{A},\ \alpha\neq\bar\alpha} \int_{-\infty}^{0} f\bigl(\phi_s\gamma(\hat T r_\alpha)\bigr)\,ds - \sum_{\alpha\in\mathcal{A},\ \alpha\neq\underline{\alpha}} \int_{-\infty}^{0} f\bigl(\phi_s\gamma(T l_\alpha)\bigr)\,ds$$
$$+ \sum_{\alpha\in\mathcal{A},\ \alpha\neq\bar\alpha} \int_{0}^{+\infty} f\bigl(\phi_s\gamma(r_\alpha)\bigr)\,ds - \sum_{\alpha\in\mathcal{A},\ \alpha\neq\underline{\alpha}} \int_{0}^{+\infty} f\bigl(\phi_s\gamma(l_\alpha)\bigr)\,ds$$
$$+ \int_0^{\tau_{\gamma\gamma}(r_{\bar\alpha})} f\bigl(\phi_s\gamma(r_{\bar\alpha})\bigr)\,ds - \int_0^{\tau_{\gamma\gamma}(l_{\underline{\alpha}})} f\bigl(\phi_s\gamma(l_{\underline{\alpha}})\bigr)\,ds. \tag{6.12}$$

Since $\underline{\alpha} = \pi_1^{-1}(1)$ and $\bar\alpha = \pi_0^{-1}(d)$, in view of (2.1), (2.2), we have

$$\{r_\alpha\colon \alpha\in\mathcal{A},\ \alpha\neq\bar\alpha\} = \bigl\{r_\alpha\colon \alpha\in\mathcal{A},\ \pi_0(\alpha)\neq d\bigr\} = \{l_\alpha\colon \alpha\in\mathcal{A},\ \pi_0(\alpha)\neq 1\},$$
$$\{T l_\alpha\colon \alpha\in\mathcal{A},\ \alpha\neq\underline{\alpha}\} = \bigl\{T l_\alpha\colon \alpha\in\mathcal{A},\ \pi_1(\alpha)\neq 1\bigr\} = \bigl\{\hat T r_\alpha\colon \alpha\in\mathcal{A},\ \pi_1(\alpha)\neq d\bigr\}.$$

Moreover, $l_{\pi_0^{-1}(1)} = 0 = T l_{\underline{\alpha}}$ and $\hat T r_{\pi_1^{-1}(d)} = |I| = r_{\bar\alpha}$. It follows that

$$\sum_{\alpha\in\mathcal{A},\ \alpha\neq\bar\alpha} \int_{-\infty}^{0} f\bigl(\phi_s\gamma(\hat T r_\alpha)\bigr)\,ds - \sum_{\alpha\in\mathcal{A},\ \alpha\neq\underline{\alpha}} \int_{-\infty}^{0} f\bigl(\phi_s\gamma(T l_\alpha)\bigr)\,ds$$
$$= \int_{-\infty}^{0} f\bigl(\phi_s\gamma(r_{\bar\alpha})\bigr)\,ds - \int_{-\infty}^{0} f\bigl(\phi_s\gamma(\hat T r_{\bar\alpha})\bigr)\,ds,$$

$$\sum_{\alpha\in\mathcal{A},\ \alpha\neq\bar\alpha} \int_{0}^{+\infty} f\bigl(\phi_s\gamma(r_\alpha)\bigr)\,ds - \sum_{\alpha\in\mathcal{A},\ \alpha\neq\underline{\alpha}} \int_{0}^{+\infty} f\bigl(\phi_s\gamma(l_\alpha)\bigr)\,ds$$
$$= \int_{0}^{+\infty} f\bigl(\phi_s\gamma(l_{\underline{\alpha}})\bigr)\,ds - \int_{0}^{+\infty} f\bigl(\phi_s\gamma(T l_{\underline{\alpha}})\bigr)\,ds.$$

Since the negative semi-orbit of $\hat T r_{\bar\alpha}$ visits $r_{\bar\alpha}$ before approaching the fixed point $\bar z$ and the positive semi-orbit of $l_{\underline{\alpha}}$ visits $T l_{\underline{\alpha}}$ before approaching the fixed point $\underline{z}$ (see Fig. 1), we have

$$\int_{-\infty}^{0} f\big(\phi_s\gamma(\hat{T}r_{\overline{\alpha}})\big)\,ds - \int_{-\infty}^{0} f\big(\phi_s\gamma(r_{\overline{\alpha}})\big)\,ds = \int_{0}^{\tau_{\gamma\gamma}(r_{\overline{\alpha}})} f\big(\phi_s\gamma(r_{\overline{\alpha}})\big)\,ds,$$

$$\int_{0}^{+\infty} f\big(\phi_s\gamma(l_{\underline{\alpha}})\big)\,ds - \int_{0}^{+\infty} f\big(\phi_s\gamma(Tl_{\underline{\alpha}})\big)\,ds = \int_{0}^{\tau_{\gamma\gamma}(l_{\underline{\alpha}})} f\big(\phi_s\gamma(l_{\underline{\alpha}})\big)\,ds.$$

In view of (6.12), it follows that

$$\int_I \varphi'(x)\,dx = \sum_{z\in\mathcal{Z}_-}\int_{-\infty}^{+\infty} f(\phi_s u_z)\,ds - \sum_{z\in\mathcal{Z}_+}\int_{-\infty}^{+\infty} f(\phi_s u_z)\,ds. \qquad \square$$

Remark 6.8. Notice that, in view of Remark 6.6, the assumption on the vanishing of derivatives of f at fixed points is necessary to control the smoothness of φ_f.

Theorem 6.9. *Suppose that the IET T is of periodic type. Let $f: M\to\mathbb{R}^\ell$ be a smooth function such that $f(x)=0$ for all $x\in\mathcal{F}(\beta)$ and $\int_{\mathcal{T}} f\,d\nu=0$. If $\theta_2(T)/\theta_1(T)<1/\ell$ then the flow (Φ_t) on $\mathcal{T}\times\mathbb{R}^\ell$ is conservative.*

Proof. By Lemma 6.3, Theorem 6.7 and (6.2), the flow (Φ_t) on $\mathcal{T}\times\mathbb{R}^\ell$ is isomorphic to a special flow built over the skew product T_φ, where $\varphi: I\to\mathbb{R}^\ell$ is a function of bounded variation with zero mean. In view of Corollary 2.6, the skew product is conservative. Now the conservativity of (Φ_t) follows from Remark 6.1. $\square$

Let g be a Riemann metric on M and let us consider the 1-form $\vartheta^\beta\in\Omega^1(M\setminus\mathcal{F}(\beta))$ on $M\setminus\mathcal{F}(\beta)$ defined by

$$\vartheta_x^\beta Y = \frac{g_x(Y,X(x))}{g_x(X(x),X(x))}.$$

Then $\vartheta_x^\beta X(x)=1$, and hence

$$\int_{\{\phi_s x:\ s\in[a,b]\}} f\cdot\vartheta^\beta = \int_a^b f(\phi_s x)\cdot\vartheta^\beta_{\phi_s x}\big(X(\phi_s x)\big)\,ds = \int_a^b f(\phi_s x)\,ds.$$

It follows that

$$\int_{\partial\mathcal{T}} f\cdot\vartheta^\beta = \sum_{z\in\mathcal{Z}_-}\int_{-\infty}^{+\infty} f(\phi_s u_z)\,ds - \sum_{z\in\mathcal{Z}_+}\int_{-\infty}^{+\infty} f(\phi_s u_z)\,ds = \int_I \varphi'(x)\,dx.$$

Theorem 6.10. *Suppose that the IET T is of periodic type. Let $f : M \to \mathbb{R}$ be a smooth function such that $f(x) = 0$, $f'(x) = 0$ and $f''(x) = 0$ for all $x \in \mathcal{F}(\beta)$,*

$$\int\limits_{\mathcal{T}} f\,d\nu = 0 \quad \textit{and} \quad \int\limits_{\partial\mathcal{T}} f \cdot \vartheta^{\beta} \neq 0.$$

Then the corresponding flow (Φ_t) on $\mathcal{T} \times \mathbb{R}$ is ergodic.

Proof. By Lemma 6.3, Theorem 6.7 and (6.2), the flow (Φ_t) on $\mathcal{T} \times \mathbb{R}$ is isomorphic to a special flow built over the skew product T_φ, where $\varphi \in \mathrm{BV}^1(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ has zero mean and

$$s(\varphi) = \int\limits_{I} \varphi'(x)\,dx = \int\limits_{\partial\mathcal{T}} f \cdot \vartheta^{\beta} \neq 0.$$

By Lemma 3.2, the cocycle φ is cohomologous to a cocycle $\varphi_{pl} \in \mathrm{PL}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\int \varphi_{pl}(x)\,dx = 0$ and $s(\varphi_{pl}) = s(\varphi) \neq 0$. In view of Theorem 3.3, the skew product $T_{\varphi_{pl}}$ is ergodic. Consequently, the skew product T_φ and hence the flow (Φ_t) on $\mathcal{T} \times \mathbb{R}$ are ergodic, by Remark 6.1. □

Suppose that the IET T is of periodic type and $\theta_2(T)/\theta_1(T) < 1/\ell$ ($\ell \geqslant 2$). Let $f : M \to \mathbb{R}^\ell$ be a smooth function such that $f(x) = 0$, $f'(x) = 0$ and $f''(x) = 0$ for all $x \in \mathcal{F}(\beta)$,

$$\int\limits_{\mathcal{T}} f\,d\nu = 0 \quad \text{and} \quad \mathbb{R}^\ell \ni v = \int\limits_{\partial\mathcal{T}} f \cdot \vartheta^{\beta} \neq 0.$$

Let $a_2, \ldots, a_\ell$ be a basis of the subspace $\{v\}^\perp$ and let us consider the function $f_a : M \to \mathbb{R}^{\ell-1}$ be given by $f_a = (\langle a_2, f\rangle, \ldots, \langle a_\ell, f\rangle)$.

Theorem 6.11. *If the flow $(\Phi_t^{f_a})$ on $\mathcal{T} \times \mathbb{R}^{\ell-1}$ is ergodic then (Φ_t^{f}) on $\mathcal{T} \times \mathbb{R}^\ell$ is also ergodic.*

Proof. Without loss of generality we can assume that $v = (1, 0, \ldots, 0)$, $a_2 = (0, 1, 0, \ldots, 0), \ldots, a_\ell = (0, \ldots, 0, 1)$. Then $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_1 : I \to \mathbb{R}$, $\varphi_2 : I \to \mathbb{R}^\ell$ are functions with $\int_I \varphi_1'(x)\,dx \neq 0$ and $\int_I \varphi_2'(x)\,dx = 0$. Applying Proposition 3.1 we can pass to cohomologous cocycles which are piecewise linear with constant slope. Now we can apply Theorem 3.4 to prove the ergodicity of T_φ which implies the ergodicity of the flow (Φ_t) on $\mathcal{T} \times \mathbb{R}^\ell$. □

7. Examples of ergodic extensions of multivalued Hamiltonian flows

In this section we will apply Theorems 5.7, 6.10 and 6.11 to construct explicit examples of ergodic extensions of multivalued Hamiltonian flows.

We first recall a recipe for constructing a multivalued Hamiltonian flow with traps which has a special representation over a given IET $T : I \to I$ satisfying the Keane condition. This construction is based on the zippered rectangles procedure introduced by Veech and a gluing technique originated by Blokhin in [4]. Fix an arbitrary collection of distinct points $\gamma_1, \ldots, \gamma_s$ on I. The presented method will allow us to construct multivalued Hamiltonian flows $(\phi_t)_{t\in\mathbb{R}}$ with s saddle loops associated to saddle fixed points z_i, $i=1,\dots,s$, so that γ_i is the first backward intersection with I of the separatrix incoming to z_i for $i=1,\dots,s$. For such flows we will prove the existence of a smooth function $f:M\to\mathbb{R}$ for which $\varphi=\varphi_f$ is a piecewise absolutely continuous function such that $(d_1,\dots,d_s)$, with $d_i=\varphi_-(\gamma_i)-\varphi_+(\gamma_i)$ for $i=1,\dots,s$, is a given vector in $\mathbb{R}^s$ and $s(\varphi)=d_1+\dots+d_s$; see Lemma 7.3 which is a crucial point for constructing ergodic extensions. If we will take an IET T of periodic type, $\{\gamma_1,\dots,\gamma_s\}$ of periodic type with respect to T (these are explicitly constructed at the end of the section) and $(d_1,\dots,d_s)$ with $d_1+\dots+d_s\neq 0$, then Lemma 7.3 yields an $\mathbb{R}$-extension of $(\phi_t)_{t\in\mathbb{R}}$ which is ergodic, in view of Theorem 6.10.

As we have already seen, in order to construct higher rank ergodic extensions of the multivalued Hamiltonian flow $(\phi_t)_{t\in\mathbb{R}}$ we need to consider functions f for which φ_f has zero sum of jumps. Then φ_f is cohomologous to a piecewise constant function (Proposition 3.1). In view of Lemma 7.3, we can construct such $f:M\to\mathbb{R}^\ell$, moreover we can control the jump vectors $d_i\in\mathbb{R}^\ell$ of φ_f at γ_i, $i=1,\dots,s$, so that the vectors $d_1,\dots,d_s$ generate a dense subgroup of $\mathbb{R}^\ell$ (whenever ℓ is not too large). In order to apply Theorem 5.7 we have to correct φ_f by a piecewise constant function from H_π. In Lemma 7.4 we will show that each such correction can be accomplished by a function on M, i.e. for every $h\in H_\pi$ there exists a smooth function $f_h:M\to\mathbb{R}$ such that $\varphi_{f_h}=h$. It follows that after a correction by a smooth function $f:M\to\mathbb{R}^\ell$ yields an ergodic extension of $(\phi_t)_{t\in\mathbb{R}}$. We finish the section with an explicit example for which we apply the procedure described above.

7.1. Construction of multivalued Hamiltonians

Let $T=T_{(\pi,\lambda)}:I\to I$ be an arbitrary IET satisfying the Keane condition. Let us start from any translation surface (M,α) built over T by applying the zippered rectangles procedure (see [35] or [38]). Denote by $\Sigma=\{p_1,\dots,p_\kappa\}$ the set of singular points of (M,α). Let $J\subset M\setminus\Sigma$ be a curve transversal to the vertical flow and such that the first-return map to J is T. We will constantly identify J with the interval I. Denote by $S\subset M$ the union of segments of all separatrices connecting singular points with J.

We will consider so-called regular adapted coordinates on $M\setminus\Sigma$, that is coordinates ζ relatively to which $\alpha_\zeta=d\zeta$. If $p\in\Sigma$ is a singular point with multiplicity $m\geqslant 1$ then we consider singular adapted coordinates around p, that is coordinates ζ relatively to which $\alpha_\zeta=d\frac{\zeta^{m+1}}{m+1}=\zeta^m\,d\zeta$. Then all changes of regular coordinates are given by translations. If ζ' is a regular adapted coordinate and ζ is a singular adapted coordinate, then $\zeta'=\zeta^{m+1}/(m+1)+c$. Let us consider the vertical vector field Y and the associated vertical flow $(\psi_t)_{t\in\mathbb{R}}$ on (M,α), which is $\alpha_xY(x)=i$ and $\frac{d}{dt}\psi_tx=Y(\psi_tx)$ for $x\in M\setminus\Sigma$. Then for a regular adapted coordinate ζ we have $Y(\zeta)=i$ and $\psi_t\zeta=\zeta+it$. Moreover, for a singular adapted coordinate ζ we have $\zeta^mY(\zeta)=i$, and hence $Y(\zeta)=\frac{i\bar{\zeta}^m}{|\zeta|^{2m}}$.

For each $\varepsilon>0$ and $p\in\Sigma$ denote by $B_\varepsilon(p)$ the ε open ball of center p and let $g=g_\varepsilon:[0,+\infty)\to[0,1]$ be a monotonic $\mathscr{C}^\infty$-function such that $g(x)=x$ for $x\in[0,\varepsilon]$ and $g(x)=1$ for $x\geqslant 2\varepsilon$. Fix $\varepsilon>0$ small enough. In what follows, we will deal with regular adapted coordinates on $M\setminus\bigcup_{p\in\Sigma}B_{2\varepsilon}(p)$ and singular adapted coordinates on $B_{3\varepsilon}(p)$ for $p\in\Sigma$. Let us consider a tangent $\mathscr{C}^\infty$-vector field $\tilde{Y}$ on M such that in adapted coordinates ζ we have

$$\tilde{Y}(\zeta)=\begin{cases} Y(\zeta)=i, & \text{on } M\setminus\bigcup_{p\in\Sigma}B_{2\varepsilon}(p),\\ \frac{g(|\zeta|)^{2m}i\bar{\zeta}^m}{|\zeta|^{2m}}, & \text{on } B_{3\varepsilon}(p),\ p\in\Sigma.\end{cases}$$

Denote by $(\tilde{\psi}_t)_{t\in\mathbb{R}}$ the associated $\mathscr{C}^\infty$-flow on M. Then $(\tilde{\psi}_t)_{t\in\mathbb{R}}$ on $M\setminus\Sigma$ is obtained by a $\mathscr{C}^\infty$ time change in the vertical flow $(\psi_t)_{t\in\mathbb{R}}$, and $(\tilde{\psi}_t)_{t\in\mathbb{R}}$ coincides with $(\psi_t)_{t\in\mathbb{R}}$ on $M\setminus\bigcup_{p\in\Sigma}B_{2\varepsilon}(p)$.

Denote by $\tilde{\omega}$ the symplectic $\mathscr{C}^\infty$-form on M such that in adapted coordinates $\zeta=x+iy$ we have

$$\tilde{\omega}_\zeta=\begin{cases} dx\wedge dy, & \text{on } M\setminus\bigcup_{p\in\Sigma}B_{2\varepsilon}(p),\\ \frac{|\zeta|^{2m}}{g(|\zeta|)^{2m}}\,dx\wedge dy, & \text{on } B_{3\varepsilon}(p),\ p\in\Sigma.\end{cases}$$

Let us consider the $\mathscr{C}^\infty$ 1-form on M given by $\tilde{\beta}=i_{\tilde{Y}}\tilde{\omega}$. Then in adapted coordinates $\zeta=x+iy$ we have

$$\tilde{\beta}_\zeta=\begin{cases} -dx, & \text{on } M\setminus\bigcup_{p\in\Sigma}B_{2\varepsilon}(p),\\ -\Re\zeta^m\,dx+\Im\zeta^m\,dy, & \text{on } B_{3\varepsilon}(p),\ p\in\Sigma.\end{cases}$$

By Cauchy–Riemann equations, $\frac{\partial}{\partial y}\Re\zeta^m+\frac{\partial}{\partial x}\Im\zeta^m=0$, and hence $d\tilde{\beta}=0$. Therefore $(\tilde{\psi}_t)_{t\in\mathbb{R}}$ is a multivalued Hamiltonian $\mathscr{C}^\infty$-flow whose orbits on $M\setminus\Sigma$ coincide with orbits of the vertical flow. It follows that $(\tilde{\psi}_t)_{t\in\mathbb{R}}$ has a special representation over the IET $T_{(\pi,\lambda)}$. If the multiplicity of a singularity $p\in\Sigma$ is equal to $m=1$ then in singular adapted coordinates $\zeta=x+iy$ on $B_\varepsilon(p)$ we have $\tilde{\beta}=-x\,dx+y\,dy$, and hence the multivalued Hamiltonian $\hat{H}$ is equal to $\hat{H}(x,y)=(y^2-x^2)/2+const$, so p is a non-degenerate critical point of $\hat{H}$.

Let us consider the symplectic form $\nu=ce^{2x}\,dx\wedge dy$, $c\neq 0$ on the disk $D=\{(x,y)\in\mathbb{R}^2\colon (x-1/2)^2+y^2\leqslant(3/2)^2\}$ and the Hamilton differential equation

$$\frac{dx}{dt}=-y,\qquad \frac{dy}{dt}=x(x-1)+y^2.$$

Then the function $-ce^{2x}((x-1)^2+y^2)/2$ is the corresponding Hamiltonian. Denote by (h_t) the associated local Hamiltonian flow. It has two critical points: $z_0=(0,0)$ is a non-degenerate saddle and $(1,0)$ is a center. The point $(0,0)$ has a loop saddle connection which coincides with the curve $e^{2x}((x-1)^2+y^2)=1$, $x\geqslant 0$. Inside this loop connection all trajectories of (h_t) are periodic (see Fig. 2). Such domains are called traps. It is easy to show that the corresponding Hamiltonian vector field Z does not vanish on ∂D, and that it has two contact points $(2,0)$ and $(-1,0)$ and two arcs A_+ and A_- connecting them with the same length (with respect to ν). Let us cut out from $M\setminus S$ a disk $B_\delta(q)$, $\delta>0$, such that $B_{2\delta}(q)$ is disjoint from the transversal curve J and from each $B_{3\varepsilon}(p)$, $p\in\Sigma$. The vector field $\tilde{Y}$ does not vanish on $\partial B_\delta(q)$, has two contact points and two arcs $\tilde{A}_+$ and $\tilde{A}_-$ connecting them with the same length (with respect to $\tilde{\omega}$). Choose $c\neq 0$ such that all four arcs A_+, A_-, $\tilde{A}_+$ and $\tilde{A}_-$ have the same length. Note that c is unique up to sign. Therefore, by Lemma 1 in [4], there exist a $\mathscr{C}^\infty$-diffeomorphism $f:\partial D\to\partial B_\delta(q)$, a symplectic $\mathscr{C}^\infty$-form ω on $(M\setminus B_\delta(q))\cup_f D$ and a tangent $\mathscr{C}^\infty$ vector field X such that

- $\mathcal{L}_X\omega=0$;
- $\omega=\tilde{\omega}$ and $X=\tilde{Y}$ on $M\setminus B_{2\delta}(q)$;
- $\omega=\nu$ and $X=Z$ on D;
- the orbits of X on $M\setminus B_{2\delta}(q)$ are pieces of orbits of the flow $(\tilde{\psi}_t)$.

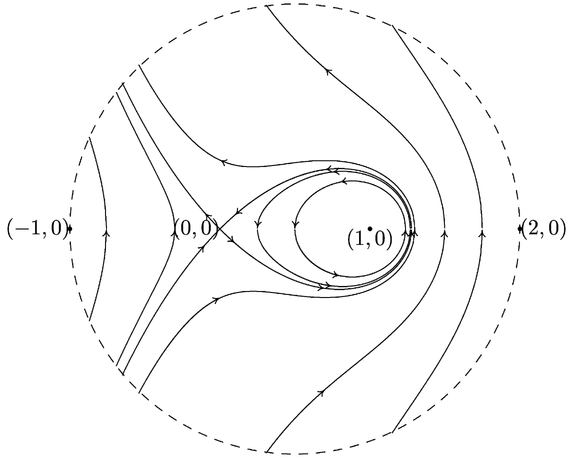

Fig. 2. The phase portrait of the Hamiltonian flow for $c > 0$.

Of course, $(M \setminus B_\delta(q)) \cup_f D$ is diffeomorphic to M, and so the vector field X and the symplectic form ω can be considered on M. Since $d(i_X\omega) = \mathcal{L}_X\omega = 0$, X is a Hamiltonian vector field with respect to ω. Denote by $(\phi_t)_{t\in\mathbb{R}}$ the Hamilton flow associated to X. Since the dynamics of $(\phi_t)_{t\in\mathbb{R}}$ and $(\psi_t)_{t\in\mathbb{R}}$ coincide on $M \setminus (\bigcup_{p\in\Sigma} B_{2\varepsilon}(p) \cup B_{2\delta}(q))$ and $J \subset M \setminus (\bigcup_{p\in\Sigma} B_{2\varepsilon}(p) \cup B_{2\delta}(q))$, the first-return map to J for $(\phi_t)_{t\in\mathbb{R}}$ is T. Denote by $\gamma \in I$ the first backward intersection with J of the separatrix incoming to z_0. Note that γ may be an arbitrary point of I different from the ends of the exchanged intervals. It suffices to choose the point $q \in M \setminus S$ and $\delta > 0$ carefully enough. Recall that the saddle point z_0 has a loop connection which will be denoted by $\sigma_{loop}(z_0)$. Then the orientation of $\sigma_{loop}(z_0)$ is positive if $c > 0$ and negative if $c < 0$.

Remark 7.1. We can repeat the procedure of producing new loop connections (positively or negatively oriented) as many times as we want. Therefore for any collection of distinct points $\{\gamma_1, \dots, \gamma_s\} \subset \bigcup_{\alpha\in\mathcal{A}} \operatorname{Int} I_\alpha$ and $\delta > 0$ small enough we can construct a multivalued Hamiltonian flow $(\phi_t)_{t\in\mathbb{R}}$ on M which has s non-degenerate saddle critical points $z_1, \dots, z_s$ such that each z_i has a loop connection $\sigma_{loop}(z_i)$ included in $B_\delta(z_i)$ for $i = 1, \dots, s$. Moreover, $(\psi_t)_{t\in\mathbb{R}}$ and $(\phi_t)_{t\in\mathbb{R}}$ coincide on $M \setminus (\bigcup_{p\in\Sigma} B_{2\varepsilon}(p) \cup \bigcup_{i=1}^{s} B_{2\delta}(z_i))$ and $\gamma_i \in I$ corresponds to the first backward intersection with J of the separatrix incoming to z_i for $i = 1, \dots, s$.

We denote by $Trap_i$ the trap corresponding to z_i, by $\epsilon(z_i) \in \{-, +\}$ the sign of the orientation of $\sigma_{loop}(z_i)$ for $i = 1, \dots, s$, and by $\mathcal{T}$ the surface M with the interior of the traps $Trap_i$, $i = 1, \dots, s$, removed.

Remark 7.2. Choose $0 < \delta' < \delta$ such that $\sigma_{loop}(z_i) \cap (M \setminus B_{\delta'}(z_i)) \neq \emptyset$ for $i = 1, \dots, s$. Let $f : M \to \mathbb{R}$ be a $\mathscr{C}^\infty$-function with $\int_{\mathcal{T}} f\omega = 0$ and such that f vanishes on each $B_{2\varepsilon}(p)$, $p \in \Sigma$ and $B_{\delta'}(z_i)$, $i = 1, \dots, s$. Then the corresponding function

$$\varphi_f : I \to \mathbb{R}, \qquad \varphi_f(x) = \varphi(x) = \int_0^{\tau(x)} f(\phi_t x)\, dt$$

($\tau : I \to \mathbb{R}_+$ is the first-return time map of the flow $(\phi_t)_{t\in\mathbb{R}}$ to J) can be extended to a $\mathscr{C}^\infty$-function on the closure of any interval of the partition $\mathcal{P}(\{l_\alpha\colon\ \alpha \in \mathcal{A}\} \cup \{\gamma_i\colon\ i = 1, \dots, s\})$. Moreover,

$$d_i(f) := \varphi_-(\gamma_i) - \varphi_+(\gamma_i) = \epsilon(z_i) \int_{-\infty}^{+\infty} f(\phi_t u_{z_i})\,dt \quad \text{and} \quad s(\varphi) = \sum_{i=1}^{s} d_i(f), \tag{7.1}$$

where u_{z_i} is an arbitrary point of $\sigma_{loop}(z_i)$ for $i = 1, \dots, s$.

Lemma 7.3. *For every $(d_1, \dots, d_s) \in \mathbb{R}^s$ there exists a $\mathscr{C}^\infty$-function $f : M \to \mathbb{R}$ which vanishes on a neighborhood of each fixed point of (ϕ_t) such that $\int_{\mathcal{T}} f\omega = 0$ and $(d_1(f), \dots, d_s(f)) = (d_1, \dots, d_s)$.*

Proof. Let us start from $f \equiv 0$. Since $\sigma_{loop}(z_i) \cap (B_{2\delta'}(z_i) \setminus B_{\delta'}(z_i)) \neq \emptyset$, we can modify f smoothly on $B_{2\delta'}(z_i) \setminus B_{\delta'}(z_i)$ such that

$$\epsilon(z_i) \int_{-\infty}^{+\infty} f(\phi_t u_{z_i})\,dt = d_i \quad \text{and} \quad \int_{(B_{2\delta'}(z_i)\setminus B_{\delta'}(z_i))\setminus Trap_i} f\omega = 0$$

for $i = 1, \dots, s$. In view of (7.1), it follows that $d_i(f) = d_i$ for $i = 1, \dots, s$. Moreover,

$$\int_{\mathcal{T}} d\omega = \sum_{i=1}^{s} \int_{(B_{2\delta'}(z_i)\setminus B_{\delta'}(z_i))\setminus Trap_i} f\omega = 0. \qquad \square$$

Lemma 7.4. *For every $h \in H_\pi$ there exists a $\mathscr{C}^\infty$-function $f : M \to \mathbb{R}$ such that $\varphi_f = \sum_{\alpha\in\mathcal{A}} h_\alpha \chi_{I_\alpha}$. If $h \in H_\pi \cap \Gamma_0$ then $\int_{\mathcal{T}} f\omega = 0$.*

Proof. Following [38], for every $\alpha \in \mathcal{A}$ denote by $[v_\alpha] \in H_1(M, \mathbb{R})$ the homology class of any closed curve v_α formed by a segment of the orbit for $(\psi_t)_{t\in\mathbb{R}}$ starting at any point $x \in \operatorname{Int} I_\alpha$ and ending at Tx together with the segment of J that joins Tx and x. Let $\Psi : H^1(M, \mathbb{R}) \to \mathbb{R}^{\mathcal{A}}$ be given by $\Psi([\varrho]) = (\int_{v_\alpha} \varrho)_{\alpha\in\mathcal{A}}$. By Lemma 2.19 in [38], the map $\Psi : H^1(M, \mathbb{R}) \to H_\pi$ establishes an isomorphism of linear spaces. Therefore for every $h \in H_\pi$ there exists a closed 1-form ϱ such that $\Psi([\varrho]) = h$ and ϱ vanishes on an open neighborhood of J. Let $f : M \to \mathbb{R}$ be given by $f(x) = \varrho_x X_x$ for $x \in M$.

For every $x \in \operatorname{Int} I_\alpha$ let v_x be the closed curve formed by the segment of orbit for $(\phi_t)_{t\in\mathbb{R}}$ starting at x and ending at Tx together with the segment of J that joins Tx and x. Then $[v_x] = [v_\alpha]$. Therefore, $h_\alpha = \int_{v_\alpha} \varrho = \int_{v_x} \varrho$. Since the form ρ vanishes on J, we have

$$\int_{v_x} \varrho = \int_0^{\tau(x)} \varrho_{\phi_t x} X(\phi_t x)\,dt = \int_0^{\tau(x)} f(\phi_t x)\,dt = \varphi_f(x).$$

Consequently, $\varphi_f(x) = h_\alpha$ for all $x \in \operatorname{Int} I_\alpha$ and $\alpha \in \mathcal{A}$. If we assume that $h \in H_\pi \cap \Gamma_0$, then

$$0 = \langle \lambda, h \rangle = \int_I \varphi_f(x)\,dx = \int_{\mathcal{T}} f\omega. \qquad \square$$

7.2. Examples

Let us consider an IET $T = T_{(\pi,\lambda)}$ and a set $\{\gamma_1, \dots, \gamma_s\} \subset I \setminus \{l_\alpha\colon \alpha \in \mathcal{A}\}$, $s \geqslant 3$. Set $\ell = s-1$. Suppose that

$$\{\gamma_1, \dots, \gamma_s\} \quad \text{is of periodic type with respect to } T \text{ and } \theta_2(T)/\theta_1(T) < 1/\ell. \tag{7.2}$$

Recall that T has to be of periodic type as well. Assume additionally that its periodic matrix has non-degenerated spectrum. An explicit example of such data for $s = 3$ is given at the end of this section.

By Remark 7.1, there exists a multivalued Hamiltonian flow $(\phi_t)_{t\in\mathbb{R}}$ with s traps (determined by saddle points z_i, $i = 1, \dots, s$) on a symplectic surface (M, ω) such that $(\phi_t)_{t\in\mathbb{R}}$ on $\mathcal{T}$ has a special representation over $T_{(\pi,\lambda)}$ and γ_i corresponds to the first backward intersection with the transversal curve of the separatrix incoming to z_i for $i = 1, \dots, s$.

By Lemma 7.3 and (7.1), there exists a $\mathscr{C}^\infty$-function $f_1 : M \to \mathbb{R}$ such that $\int_{\mathcal{T}} f_1\omega = 0$ and $s(\varphi_{f_1}) \neq 0$. In view of Theorem 6.10, the flow $(\Phi_t^{f_1})_{t\in\mathbb{R}}$ on $\mathcal{T} \times \mathbb{R}$ is ergodic.

Let $\bar{d}_1, \dots, \bar{d}_s$ be vectors in $\mathbb{R}^{\ell-1}$ such that $\overline{\mathbb{Z}(\bar{d}_1, \dots, \bar{d}_s)} = \mathbb{R}^{\ell-1}$ and $\sum_{i=1}^s \bar{d}_i = \bar{0}$. Since $s = (\ell-1)+2$, the existence of such collection follows directly from Remark 3.8. By Lemma 7.3, there exists a $\mathscr{C}^\infty$-function $f_2' : M \to \mathbb{R}^{\ell-1}$ such that f_2' vanishes on a neighborhood of each fixed point of (ϕ_t), $\int_{\mathcal{T}} f_2'\omega = \bar{0}$ and $(\varphi_{f_2'})_+(\gamma_i) - (\varphi_{f_2'})_-(\gamma_i) = \bar{d}_i$ for $i = 1, \dots, s$. Then $\varphi_{f_2'}$ has zero mean and, by (7.1), $s(\varphi_{f_2'}) = \sum_{i=1}^s \bar{d}_i = \bar{0}$.

Denote by $\bar{\varphi} : I \to \mathbb{R}^\ell$ the piecewise constant function with zero mean whose discontinuities are γ_i, $i = 1, \dots, s$, and $\bar{\varphi}_+(\gamma_i) - \bar{\varphi}_-(\gamma_i) = \bar{d}_i$ for $i = 1, \dots, s$. In view of (7.2) and Remark 5.6, $\bar{\varphi} \in \mathrm{BV}_0^\diamondsuit(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^{\ell-1})$. By Remark 7.2, $\varphi_{f_2'}$ can be extended to a $\mathscr{C}^\infty$-function on the closure of any interval of the partition $\mathcal{P}(\{l_\alpha\colon \alpha \in \mathcal{A}\} \cup \{\gamma_i\colon i = 1, \dots, s\})$. It follows that $\varphi_{f_2'} - \bar{\varphi} \in \mathrm{BV}^1(\bigsqcup_{\alpha\in\mathcal{A}}, \mathbb{R}^{\ell-1})$. Moreover, $\varphi_{f_2'} - \bar{\varphi}$ has zero mean and $s(\varphi_{f_2'} - \bar{\varphi}) = s(\varphi_{f_2'}) - s(\bar{\varphi}) = 0$. Therefore, by Proposition 3.1, $\varphi_{f_2'} - \bar{\varphi}$ is cohomologous to $\bar{h}^1 = (h_1^1, \dots, h_{\ell-1}^1)$, where $h_i^1 \in \Gamma_0$ for $i = 1, \dots, \ell-1$.

In view of Theorem 5.2 applied to the coordinate functions of $\bar{\varphi} + \bar{h}^1 \in \mathrm{BV}_0^\diamondsuit(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha, \mathbb{R}^{\ell-1})$, there exists $\bar{h}^2 = (h_1^2, \dots, h_{\ell-1}^2)$ with $h_i^2 \in \Gamma_u \cap \Gamma_0$ for $i = 1, \dots, \ell-1$ such that $\bar{\varphi} + \bar{h}^1 + \bar{h}^2 = \widehat{\bar{\varphi} + \bar{h}^1}$. Moreover, by Theorem 5.7, the cocycle $\bar{\varphi} + \bar{h}^1 + \bar{h}^2 = \widehat{\bar{\varphi} + \bar{h}^1}$ is ergodic. As $\varphi_{f_2'} + \bar{h}^2$ is cohomologous to $\bar{\varphi} + \bar{h}^1 + \bar{h}^2$, it is ergodic as well. By Lemma 7.4, there exists a $\mathscr{C}^\infty$-function $f_2'' : M \to \mathbb{R}^{\ell-1}$ with $\int_{\mathcal{T}} f_2''\omega = \bar{0}$ such that $\varphi_{f_2''} = \bar{h}^2$. Setting $f_2 = f_2' + f_2''$, we have $\int_{\mathcal{T}} f_2\omega = \bar{0}$, $\varphi_{f_2} = \varphi_{f_2'} + \bar{h}^2$, and $s(\varphi_{f_2}) = s(\varphi_{f_2'}) = \sum_{i=1}^s \bar{d}_i = \bar{0}$. It follows that the flow $(\Phi_t^{f_2})_{t\in\mathbb{R}}$ on $\mathcal{T} \times \mathbb{R}^{\ell-1}$ is ergodic. Finally applying Theorem 6.11 to $f = (f_1, f_2) : I \to \mathbb{R}^\ell$ we have the ergodicity of the flow $(\Phi_t^f)_{t\in\mathbb{R}}$ on $\mathcal{T} \times \mathbb{R}^\ell$.

Example 1. Let us consider the permutation $\left(\begin{smallmatrix} 1&2&3&4&5&6&7 \\ 6&7&4&5&3&1&2 \end{smallmatrix}\right)$ and a corresponding pair π' (note that such a permutation yields exchanges of 4 intervals with 3 “artificial” discontinuities). On the Rauzy graph $\mathcal{R}(\pi')$ let us consider the loop starting from π' and passing through the edges labeled consecutively by

$$1,0,1,1,1,1,1,1,0,1,1,0,1,1,1,0,0,1,1,1,1,0,1,0,0,0,0,1,1,1.$$

Then the resulting matrix is

$$A' := \begin{pmatrix} 9 & 8 & 20 & 20 & 15 & 5 & 5 \\ 1 & 2 & 4 & 4 & 3 & 2 & 2 \\ 2 & 2 & 6 & 5 & 4 & 1 & 1 \\ 2 & 2 & 5 & 6 & 4 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 4 & 4 & 3 & 2 & 1 \\ 1 & 1 & 3 & 3 & 2 & 1 & 2 \end{pmatrix}$$

and $(A')^2$ has positive entries. Let $\lambda' \in \mathbb{R}^7_+$ be a Perron–Frobenius eigenvector of A'. Then $T_{(\pi',\lambda')}$ is of periodic type and A' is its periodic matrix. Of course, $T_{(\pi',\lambda')}$ is an exchange of 4 intervals, more precisely, $T_{(\pi',\lambda')} = T_{(\pi_4^{sym},\lambda)}$, where $\lambda_1 = \lambda'_1 + \lambda'_2$, $\lambda_2 = \lambda'_3 + \lambda'_4$, $\lambda_3 = \lambda'_5$ and $\lambda_4 = \lambda'_6 + \lambda'_7$. As we already noticed in Section 5.3, $T_{(\pi_4^{sym},\lambda)}$ has also periodic type and the family $\gamma_1 = \lambda'_1$, $\gamma_2 = \lambda'_1 + \lambda'_2 + \lambda'_3$, $\gamma_3 = \lambda'_1 + \lambda'_2 + \lambda'_3 + \lambda'_4 + \lambda'_5 + \lambda'_6$ is of periodic type with respect to $T_{(\pi_4^{sym},\lambda)}$. Moreover,

$$A = \begin{pmatrix} 10 & 24 & 18 & 7 \\ 4 & 11 & 8 & 2 \\ 1 & 2 & 2 & 0 \\ 3 & 7 & 5 & 3 \end{pmatrix}$$

is the periodic matrix of $T_{(\pi_4^{sym},\lambda)}$, so

$$\rho_1 = \frac{13}{2} + \frac{1}{2}\sqrt{115} + \frac{1}{2}\sqrt{280 + 26\sqrt{115}}, \qquad \rho_2 = \frac{13}{2} - \frac{1}{2}\sqrt{115} + \frac{1}{2}\sqrt{280 - 26\sqrt{115}}.$$

Hence $\theta_2/\theta_1 \approx 0.164 < 1/2$, so $T_{(\pi_4^{sym},\lambda)}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ satisfy (7.2) with $s = 3$.

Remark 7.5. Similar examples can be constructed by matching the set $\{\gamma_1, \ldots, \gamma_s\}$ for a fixed IET $T = T_{\pi,\lambda}$ of periodic type. Let $p \geqslant 1$ be the period of T and let $\rho > 1$ be the Perron–Frobenius eigenvalue of the periodic matrix A of T. For every $x \in I$ let $k(x) = \inf\{k \geqslant 0: T^{-k}x \in I^{(p)}\}$. Let us consider the map $S : I \to I$, $S(x) = \rho \cdot T^{-k(x)}x$. Note that for every $\alpha \in \mathcal{A}$ the map S has at least $A_{\alpha\alpha} - 2$ fixed points in the interior of I_α. Therefore, multiplying the period of T, if necessary, for every $s \geqslant 1$ we can find s distinct fixed points $\gamma_1, \ldots, \gamma_s$ different from l_α, $\alpha \in \mathcal{A}$. In view of Theorem 23 in [31], the set $\{\gamma_1, \ldots, \gamma_s\}$ is of periodic type with respect to T.

Denote by M_2 a compact $\mathscr{C}^\infty$-surface of genus 2. We can apply the above constructions to the sequence (T_n) of IETs with $\theta_2(T_n)/\theta_1(T_n) \to 0$ constructed in Appendix B to obtain the following.

Corollary 7.6. *For every $\ell \geqslant 1$ there exist a multivalued Hamiltonian flow $(\phi_t)_{t\in\mathbb{R}}$ on M_2 and a $\mathscr{C}^\infty$-function $f : M_2 \to \mathbb{R}^\ell$ for which the flow $(\Phi^f_t)_{t\in\mathbb{R}}$ on $\mathcal{T} \times \mathbb{R}^\ell$ is ergodic.*

Acknowledgments

This research was carried out during visits of the first author to the Faculty of Mathematics and Computer Science at Nicolaus Copernicus University, and of the second author to the IRMAR at the University of Rennes 1. The authors are grateful to their hosts for their hospitality and support. They would like to thank Jacek Brzykcy for the numerical computation of the example presented in Section 7.2 and the referees for careful reading and valuable comments.

Appendix A. Deviation of cocycles: proofs

Let $T : I \to I$ be an arbitrary IET satisfying the Keane condition. For every $x \in I$ and $n \geqslant 0$ set

$$m(x, n, T) = \max\bigl\{l \geqslant 0\colon \#\bigl\{0 \leqslant k \leqslant n\colon T^k x \in I^{(l)}\bigr\} \geqslant 2\bigr\}.$$

Proposition A.1. *(See [41] or [38].) For every $x \in I$ and $n > 0$ we have*

$$\min_{\alpha\in\mathcal{A}} Q_\alpha(m) \leqslant n \leqslant d \max_{\alpha\in\mathcal{A}} Q_\alpha(m+1) = d\bigl\|Q(m+1)\bigr\|, \quad \textit{where } m = m(x, n, T).$$

Remark A.2. Assume that $T = T_{(\pi,\lambda)}$ is of periodic type and A is its periodic matrix. Then there exists $C > 0$ such that $e^{\theta_1 k}/C \leqslant \|A^k\| \leqslant Ce^{\theta_1 k}$ for every $k \geqslant 1$, where θ_1 is the largest Lyapunov exponent of A. Let $m = m(x, n, T)$. Since $\|A^n\| = \max_{\alpha\in\mathcal{A}} A^n_\alpha$, by Proposition A.1 and (4.2), it follows that

$$n \geqslant \min_{\alpha\in\mathcal{A}} Q_\alpha(m) = \min_{\alpha\in\mathcal{A}} A^m_\alpha \geqslant \frac{1}{\nu(A)} \max_{\alpha\in\mathcal{A}} A^m_\alpha = \frac{\|A^m\|}{\nu(A)} \geqslant \frac{e^{\theta_1 m}}{C\nu(A)}.$$

Thus

$$m \leqslant \frac{1}{\theta_1} \log\bigl(C\nu(A)n\bigr). \tag{A.1}$$

Proposition A.3. *(See [28].) For each bounded function $\varphi : I \to \mathbb{R}$, $x \in I$ and $n > 0$ we have*

$$\bigl|\varphi^{(n)}(x)\bigr| \leqslant 2\sum_{l=0}^{m}\bigl\|Z(l+1)\bigr\|\bigl\|S(l)\varphi\bigr\|_{\sup}, \quad \textit{where } m = m(x, n, T). \tag{A.2}$$

If additionally $\varphi \in \mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ then

$$\bigl\|S(l)\varphi\bigr\|_{\sup} \leqslant \sum_{1\leqslant j\leqslant l}\bigl\|Z(j)\bigr\|\bigl\|S(j,l)|_{\Gamma_0^{(j)}}\bigr\| \operatorname{Var}\varphi. \tag{A.3}$$

Proof of Theorem 2.4. Since λ is a positive Perron–Frobenius eigenvector of A, by Proposition 5 in [41], the restriction of A^t to the invariant space $Ann(\lambda) = \{h \in \mathbb{R}^{\mathcal{A}}\colon \langle h, \lambda\rangle = 0\}$ has the following Lyapunov exponents:

$$\theta_2 \geqslant \theta_3 \geqslant \cdots \geqslant \theta_g \geqslant 0 = \cdots = 0 \geqslant -\theta_g \geqslant \cdots \geqslant -\theta_3 \geqslant -\theta_2 > -\theta_1.$$

Thus there exists $C > 0$ such that for every $k \in \mathbb{N}$ we have

$$\left\| \left(A^t\right)^k h \right\| \leqslant C k^{M-1} \exp(k\theta_2) \|h\| \quad \text{for all } h \in Ann(\lambda).$$

Since $\Gamma_0^{(j)} = Ann(\lambda)$ and $S(j,l) = Q^t(j,l) = (A^t)^{l-j}$ on $\Gamma_0^{(j)}$, by (A.3),

$$\begin{aligned}\left\| S(l)\varphi \right\|_{\sup} &\leqslant \sum_{1 \leqslant j \leqslant l} \|A\| \left\| \left(A^t\right)^{l-j}\big|_{Ann(\lambda)} \right\| \operatorname{Var}\varphi \\ &\leqslant \sum_{0 \leqslant k < l} \|A\| C k^{M-1} \exp(k\theta_2) \operatorname{Var}\varphi \leqslant \|A\| C l^M \exp(l\theta_2) \operatorname{Var}\varphi.\end{aligned}$$

In view of (A.2), it follows that

$$\begin{aligned}\left|\varphi^{(n)}(x)\right| &\leqslant 2 \sum_{l=0}^{m} \|A\| \left\| S(l)\varphi \right\|_{\sup} \leqslant 2 \sum_{l=0}^{m} \|A\|^2 C l^M \exp(l\theta_2) \operatorname{Var}\varphi \\ &\leqslant 2\|A\|^2 C m^{M+1} \exp(m\theta_2) \operatorname{Var}\varphi,\end{aligned}$$

where $m = m(x, n, T)$. Consequently, by (A.1),

$$\left|\varphi^{(n)}(x)\right| \leqslant 2\|A\|^2 C^2 \nu(A)/\theta_1^{M+1} \log^{M+1}\left(C\nu(A)n\right) n^{\theta_2/\theta_1} \operatorname{Var}\varphi. \qquad \square$$

Appendix B. Possible values of θ_2/θ_1

In this section we will show that for each symmetric pair π_4^{sym} there are IETs of periodic type such that θ_2/θ_1 is arbitrarily small and the spectrum of the periodic matrix is non-degenerated. As it was shown in [28] for every natural n the matrix

$$M(n) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ n & n+1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ n+1 & n+2 & 2 & 2 \end{pmatrix}$$

is a resulting matrix corresponding to a loop in the Rauzy class of π_4^{sym} and starting from π_4^{sym}. Since $M(n)^2$ has positive entries, $M(n)$ is primitive and there exists an IET of periodic type for which $M(n)^2$ is its periodic matrix (see [33]). The eigenvalues $\rho_1(n) > \rho_2(n) > 1 > \rho_3(n) > \rho_4(n) > 0$ of $M(n)$ are of the form

$$\begin{aligned}\rho_1(n) &= \frac{1}{2}\left(a_n^+ + \sqrt{\left(a_n^+\right)^2 - 4}\right), &\qquad \rho_2(n) &= \frac{1}{2}\left(a_n^- + \sqrt{\left(a_n^-\right)^2 - 4}\right), \\ \rho_3(n) &= \frac{1}{2}\left(a_n^- - \sqrt{\left(a_n^-\right)^2 - 4}\right), &\qquad \rho_4(n) &= \frac{1}{2}\left(a_n^+ - \sqrt{\left(a_n^+\right)^2 - 4}\right),\end{aligned}$$

where

$$a_n^{\pm} = \frac{1}{2}\left(n + 6 \pm \sqrt{n^2 + 4}\right).$$

Since $a_n^+ \to +\infty$ and $a_n^- \to 3$ as $n \to +\infty$, it follows that

$$\frac{\theta_2(n)}{\theta_1(n)} = \frac{\log \rho_2(n)}{\log \rho_1(n)} \to 0 \quad \text{as } n \to +\infty.$$

Appendix C. Deviation of corrected functions

Proof of Proposition 5.1. First note that for every natural k the subspace $\Gamma_{cs}^{(k)} \subset \mathbb{R}^{\mathcal{A}}$ is the direct sum of invariant subspaces associated to Jordan blocks of A^t with non-positive Lyapunov exponents. It follows that there exists $C > 0$ such that

$$\left\| \left(A^t\right)^n h \right\| \leqslant C n^{M-1} \|h\| \quad \text{for all } h \in \Gamma_{cs}^{(k)} \text{ and } n \geqslant 0. \tag{C.1}$$

It is easy to show that $\Gamma_{cs}^{(k)} \subset \Gamma_0^{(k)}$. Next note that $S(k,l)\Gamma_{cs}^{(k)} = \Gamma_{cs}^{(l)}$ and the quotient linear transformation

$$S_u(k,l) : \mathrm{BV}\Big(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}\Big)/\Gamma_{cs}^{(k)} \to \mathrm{BV}\Big(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)}\Big)/\Gamma_{cs}^{(l)}$$

satisfies

$$S_u(k,l) \circ U^{(k)} = U^{(l)} \circ S(k,l). \tag{C.2}$$

Since $\Gamma_u^{(k)} \subset \mathbb{R}^{\mathcal{A}}$ is the direct sum of invariant subspaces associated to Jordan blocks of A^t with positive Lyapunov exponents, $\mathbb{R}^{\mathcal{A}} = \Gamma^{(k)} = \Gamma_{cs}^{(k)} \oplus \Gamma_u^{(k)}$ is an invariant decomposition and there exist $\theta_+ > 0$ and $C > 0$ such that

$$\left\| \left(A^t\right)^{-n} h \right\| \leqslant C \exp(-n\theta_+) \|h\| \quad \text{for all } h \in \Gamma_u^{(k)} \text{ and } n \geqslant 0.$$

Since the linear operators $A^t : \Gamma_u^{(k)} \to \Gamma_u^{(k)}$ and $A^t : \Gamma^{(k)}/\Gamma_{cs}^{(k)} \to \Gamma^{(k)}/\Gamma_{cs}^{(k)}$ are isomorphic, there exists $C' > 0$ such that

$$\left\| \left(A^t\right)^{-n} \left(h + \Gamma_{cs}^{(k)}\right) \right\| \leqslant C' \exp(-n\theta_+) \left\| h + \Gamma_{cs}^{(k)} \right\| \quad \text{for all } h + \Gamma_{cs}^{(k)} \in \Gamma^{(k)}/\Gamma_{cs}^{(k)} \text{ and } n \geqslant 0.$$

Therefore, for $0 \leqslant k \leqslant l$ the quotient map $S_u(k,l) : \Gamma^{(k)}/\Gamma_{cs}^{(k)} \to \Gamma^{(l)}/\Gamma_{cs}^{(l)}$ is invertible and

$$\left\| \left(S_u(k,l)\right)^{-1} \left(h + \Gamma_{cs}^{(l)}\right) \right\| \leqslant C' \exp\left(-(l-k)\theta_+\right) \left\| h + \Gamma_{cs}^{(l)} \right\| \tag{C.3}$$

for all $h + \Gamma_{cs}^{(l)} \in \Gamma^{(l)}/\Gamma_{cs}^{(l)}$.

Let us consider the linear operator $C^{(k)} : \mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}) \to \Gamma_0^{(k)}$ given by

$$C^{(k)}\varphi(x) = \frac{1}{|I_\alpha^{(k)}|} \int_{I_\alpha^{(k)}} \varphi(t)\,dt \quad \text{if } x \in I_\alpha^{(k)}.$$

Then $P_0^{(k)}\varphi = \varphi - C^{(k)}\varphi$ and

$$\|C^{(k)}\varphi\| \leqslant \|\varphi\|_{\sup}, \tag{C.4}$$

$$\|P_0^{(k)}\varphi\|_{\sup} \leqslant \operatorname{Var} P_0^{(k)}\varphi = \operatorname{Var}\varphi. \tag{C.5}$$

Notice that for every $\psi \in \mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(r)})$,

$$\big(S(r,r+1)\circ P_0^{(r)} - P_0^{(r+1)}\circ S(r,r+1)\big)\psi = C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\psi \in \Gamma_0^{r+1}.$$

Indeed, as $\psi = P_0^{(r)}\psi + C^{(r)}\psi$,

$$P_0^{(r+1)}\circ S(r,r+1)\psi = P_0^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\psi + P_0^{(r+1)}\circ S(r,r+1)\circ C^{(r)}\psi.$$

Moreover, $S(r,r+1)\circ C^{(r)}\psi \in \Gamma_0^{(r+1)}$, so $P_0^{(r+1)}\circ S(r,r+1)\circ C^{(r)}\psi = 0$. Hence

$$\begin{aligned}
&S(r,r+1)\circ P_0^{(r)}\psi - P_0^{(r+1)}\circ S(r,r+1)\psi\\
&\quad = S(r,r+1)\circ P_0^{(r)}\psi - P_0^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\psi\\
&\quad = C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\psi \in \Gamma_0^{(r+1)}.
\end{aligned}$$

It follows that for $\varphi \in \mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ and $0 \leqslant k \leqslant l$ we have

$$\begin{aligned}
&\big(S(k,l)\circ P_0^{(k)} - P_0^{(l)}\circ S(k,l)\big)\varphi\\
&\quad = \sum_{k\leqslant r<l}\big(S(r,l)\circ P_0^{(r)}\circ S(k,r) - S(r+1,l)\circ P_0^{(r+1)}\circ S(k,r+1)\big)\varphi\\
&\quad = \sum_{k\leqslant r<l} S(r+1,l)\circ\big(S(r,r+1)\circ P_0^{(r)} - P_0^{(r+1)}\circ S(r,r+1)\big)\circ S(k,r)\varphi\\
&\quad = \sum_{k\leqslant r<l} S(r+1,l)\circ C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\circ S(k,r)\varphi \in \Gamma_0^{(l)}.
\end{aligned}$$

Therefore the operator $U^{(k)}\circ(P_0^{(k)} - S(k,l)^{-1}\circ(S(k,l)\circ P_0^{(k)} - P_0^{(l)}\circ S(k,l)))$ is well defined and, in view of (C.2),

$$\begin{aligned}
&U^{(k)}\circ\big(P_0^{(k)} - S(k,l)^{-1}\circ\big(S(k,l)\circ P_0^{(k)} - P_0^{(l)}\circ S(k,l)\big)\big)\\
&\quad = U^{(k)}\circ P_0^{(k)} - \sum_{k\leqslant r<l} S_u(k,r+1)^{-1}\circ U^{(r+1)}\circ C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\circ S(k,r).
\end{aligned} \tag{C.6}$$

For every $\varphi \in \mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ let us consider the series

$$\sum_{r\geqslant k}\big(S_u(k,r+1)\big)^{-1}\circ U^{(r+1)}\circ C^{(r+1)}\circ S(r,r+1)\circ P_0^{(r)}\circ S(k,r)\varphi \tag{C.7}$$

in $\Gamma_0^{(k)}/\Gamma_{cs}^{(k)}$. Using (C.4), (5.2), (C.5) and (5.1) successively we obtain

$$
\begin{aligned}
&\bigl\|C^{(r+1)} \circ S(r, r+1) \circ P_0^{(r)} \circ S(k, r)\varphi\bigr\| \\
&\quad \leqslant \bigl\|S(r, r+1) \circ P_0^{(r)} \circ S(k, r)\varphi\bigr\|_{\sup} \\
&\quad \leqslant \bigl\|Z(r+1)\bigr\| \bigl\|P_0^{(r)} \circ S(k, r)\varphi\bigr\|_{\sup} \leqslant \|A\| \operatorname{Var} S(k, r)\varphi \leqslant \|A\| \operatorname{Var}\varphi.
\end{aligned}
$$

Since $\|U^{(r+1)}\| = 1$ and $U^{(r+1)} \circ C^{(r+1)} \circ S(r, r+1) \circ P_0^{(r)} \circ S(k, r)\varphi \in \Gamma_0^{(r+1)}/\Gamma_{cs}^{(r+1)}$, by (C.3), the norm of the r-th element of the series (C.7) is bounded from above by $C'\|A\| \exp(-(r-k+1)\theta_+) \operatorname{Var}\varphi$. As $K := \sum_{r \geqslant k} C'\|A\| \exp(-(r-k+1)\theta_+) < +\infty$, the series (C.7) converges in $\Gamma_0^{(k)}/\Gamma_{cs}^{(k)}$. Denoting by $\Delta P^{(k)}\varphi \in \Gamma_0^{(k)}/\Gamma_{cs}^{(k)}$ the sum of (C.7) we have

$$
\bigl\|\Delta P^{(k)}\varphi\bigr\| \leqslant K \operatorname{Var}\varphi \quad \text{for every } \varphi \in \mathrm{BV}_0\biggl(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)}\biggr) \text{ and } k \geqslant 0. \tag{C.8}
$$

In view of (C.6), it follows that the sequence (5.4) converges in $\mathrm{BV}_0(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})/\Gamma_{cs}^{(k)}$ and

$$
P^{(k)} = U^{(k)} \circ P_0^{(k)} - \Delta P^{(k)}. \qquad \square \tag{C.9}
$$

Lemma C.1. *For all $0 \leqslant k \leqslant l$ and $\varphi \in \mathrm{BV}_0(\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ we have*

$$
S_u(k, l) \circ P^{(k)}\varphi = P^{(l)} \circ S(k, l)\varphi, \tag{C.10}
$$

$$
\bigl\|P^{(k)}\varphi\bigr\|_{\sup/\Gamma_{cs}^{(k)}} \leqslant (1+K) \operatorname{Var}\varphi. \tag{C.11}
$$

Proof. First note that for $k \leqslant l \leqslant r$,

$$
\begin{aligned}
&S(k, l) \circ \bigl(P_0^{(k)} - S(k, r)^{-1} \circ \bigl(S(k, r) \circ P_0^{(k)} - P_0^{(r)} \circ S(k, r)\bigr)\bigr) \\
&\quad = \bigl(P_0^{(l)} - S(l, r)^{-1} \circ \bigl(S(l, r) \circ P_0^{(l)} - P_0^{(r)} \circ S(l, r)\bigr)\bigr) \circ S(k, l).
\end{aligned}
$$

In view of (C.2), it follows that

$$
\begin{aligned}
&S_u(k, l) \circ U^{(k)} \circ \bigl(P_0^{(k)} - S(k, r)^{-1} \circ \bigl(S(k, r) \circ P_0^{(k)} - P_0^{(r)} \circ S(k, r)\bigr)\bigr) \\
&\quad = U^{(l)} \circ \bigl(P_0^{(l)} - S(l, r)^{-1} \circ \bigl(S(l, r) \circ P_0^{(l)} - P_0^{(r)} \circ S(l, r)\bigr)\bigr) \circ S(k, l).
\end{aligned}
$$

By the definition of the operators $P^{(\cdot)}$, letting $r \to \infty$, we get $S_u(k, l) \circ P^{(k)} = P^{(l)} \circ S(k, l)$. Moreover, by (C.9), (C.5) and (C.8),

$$
\bigl\|P^{(k)}\varphi\bigr\|_{\sup/\Gamma_{cs}^{(k)}} \leqslant \bigl\|P_0^{(k)}\varphi\bigr\|_{\sup} + \bigl\|\Delta P^{(k)}\varphi\bigr\| \leqslant (1+K) \operatorname{Var}\varphi. \qquad \square
$$

Let $p : \{0, 1, \ldots, d, d+1\} \to \{0, 1, \ldots, d, d+1\}$ stand for the permutation

$$
p(j) = \begin{cases} \pi_1 \circ \pi_0^{-1}(j) & \text{if } 1 \leqslant j \leqslant d, \\ j & \text{if } j = 0, d+1. \end{cases}
$$

Following [35,36], denote by $\sigma = \sigma_\pi$ the corresponding permutation on $\{0, 1, \dots, d\}$,

$$\sigma(j) = p^{-1}\big(p(j)+1\big) - 1 \quad \text{for } 0 \leqslant j \leqslant d.$$

Then $\hat{T}_{(\pi,\lambda)} r_{\pi_0^{-1}(j)} = T_{(\pi,\lambda)} r_{\pi_0^{-1}(\sigma j)}$ for all $j \neq 0, p^{-1}(d)$. Denote by $\Sigma(\pi)$ the set of orbits for the permutation σ. Let $\Sigma_0(\pi)$ stand for the subset of orbits that do not contain zero. Then $\Sigma(\pi)$ corresponds to the set of singular points of any translation surface $S(\pi, \lambda, h)$ associated to π and hence $\#\Sigma(\pi) = \kappa(\pi) = \dim\ker \Omega_\pi + 1$. Following [36], for every $\mathcal{O} \in \Sigma(\pi)$ denote by $b(\mathcal{O}) \in \mathbb{R}^{\mathcal{A}}$ the vector given by

$$b(\mathcal{O})_\alpha = \chi_{\mathcal{O}}\big(\pi_0(\alpha)\big) - \chi_{\mathcal{O}}\big(\pi_0(\alpha) - 1\big) \quad \text{for } \alpha \in \mathcal{A}.$$

Lemma C.2. *(See [36].) For every irreducible pair π we have $\sum_{\mathcal{O}\in\Sigma(\pi)} b(\mathcal{O}) = 0$, the vectors $b(\mathcal{O})$, $\mathcal{O} \in \Sigma_0(\pi)$, are linearly independent and the linear subspace generated by them is equal to* $\ker \Omega_\pi$. *Moreover, $h \in H_\pi$ if and only if $\langle h, b(\mathcal{O})\rangle = 0$ for every $\mathcal{O} \in \Sigma(\pi)$.*

Remark C.3. Let $\Lambda^\pi : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\Sigma_0(\pi)}$ stand for the linear transformation given by $(\Lambda^\pi h)_{\mathcal{O}} = \langle h, b(\mathcal{O})\rangle$ for $\mathcal{O} \in \Sigma_0(\pi)$. By Lemma C.2, $H_\pi = \ker \Lambda^\pi$ and if $\mathbb{R}^{\mathcal{A}} = F \oplus H_\pi$ is a direct sum decomposition then $\Lambda^\pi : F \to \mathbb{R}^{\Sigma_0(\pi)}$ establishes an isomorphism of linear spaces. It follows that there exists $K_F > 0$ such that

$$\|h\| \leqslant K_F \big\|\Lambda^\pi h\big\| \quad \text{for all } h \in F.$$

Lemma C.4. *(See [36].) Suppose that $T_{(\tilde{\pi},\tilde{\lambda})} = \mathcal{R}(T_{(\pi,\lambda)})$. Then there exists a bijection $\xi : \Sigma(\pi) \to \Sigma(\tilde{\pi})$ such that $\Theta(\pi,\lambda)^{-1} b(\mathcal{O}) = b(\xi\mathcal{O})$ for $\mathcal{O} \in \Sigma(\pi)$.*

Let $T = T_{(\pi,\lambda)}$ be an IET satisfying the Keane condition. For all $\mathcal{O} \in \Sigma(\pi)$ and $\varphi \in \mathrm{BV}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ let

$$\mathcal{O}(\varphi) := \sum_{\alpha\in\mathcal{A},\ \pi_0(\alpha)\in\mathcal{O}} \varphi_-(r_\alpha) - \sum_{\alpha\in\mathcal{A},\ \pi_0(\alpha)-1\in\mathcal{O}} \varphi_+(l_\alpha).$$

Note that if $h \in \Gamma^{(0)}$ (i.e. h is a function constant on exchanged intervals), then

$$\mathcal{O}(h) = \sum_{\pi_0(\alpha)\in\mathcal{O}} h_\alpha - \sum_{\pi_0(\alpha)-1\in\mathcal{O}} h_\alpha = \sum_{\alpha\in\mathcal{A}} \big(\chi_{\mathcal{O}}\big(\pi_0(\alpha)\big) - \chi_{\mathcal{O}}\big(\pi_0(\alpha)-1\big)\big) h_\alpha = \big\langle h, b(\mathcal{O})\big\rangle.$$

Moreover,

$$\big|\mathcal{O}(\varphi)\big| \leqslant 2d\|\varphi\|_{\sup} \quad \text{for every } \varphi \in \mathrm{BV}\Big(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha\Big) \text{ and } \mathcal{O} \in \Sigma(\pi). \tag{C.12}$$

Let us consider $T_{(\tilde{\pi},\tilde{\lambda})} = \mathcal{R}(T_{(\pi,\lambda)})$ and the renormalized cocycle $\tilde{\varphi} : \tilde{I} \to \mathbb{R}$, that is

$$\tilde{\varphi}(x) = \sum_{0\leqslant i<\Theta_\beta(\pi,\lambda)} \varphi\big(T^i_{(\pi,\lambda)} x\big) \quad \text{for } x \in \tilde{I}_\beta.$$

The proof of the following lemma is straightforward and we leave it to the reader.

Lemma C.5. *If* $\varphi \in \mathrm{BV}^{\diamondsuit}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha)$ *then* $\tilde{\varphi} \in \mathrm{BV}^{\diamondsuit}(\bigsqcup_{\alpha\in\mathcal{A}} \tilde{I}_\alpha)$ *and* $(\xi\mathcal{O})(\tilde{\varphi}) = \mathcal{O}(\varphi)$ *for each* $\mathcal{O} \in \Sigma(\pi)$.

Let $T = T_{(\pi,\lambda)}$ be an IET of periodic type and let A be its periodic matrix. By Lemma C.4, there exists a bijection $\xi : \Sigma(\pi) \to \Sigma(\pi)$ such that $A^{-1}b(\mathcal{O}) = b(\xi\mathcal{O})$ for $\mathcal{O} \in \Sigma(\pi)$. Since $\xi^N = Id_{\Sigma(\pi)}$ for some $N \geqslant 1$, multiplying the period of T by N, we can assume that $\xi = Id_{\Sigma(\pi)}$. Therefore $Ab(\mathcal{O}) = b(\mathcal{O})$ for each $\mathcal{O} \in \Sigma(\pi)$, and hence $A|_{\ker \Omega_\pi} = Id$. It follows that the dimension of $\Gamma_c^{(0)} = \{h \in \mathbb{R}^{\mathcal{A}}\colon\ A^t h = h\}$ is greater than or equal to $\kappa - 1$. Denote by $\Gamma_s^{(0)} \subset \mathbb{R}^{\mathcal{A}}$ the direct sum of invariant subspaces associated to Jordan blocks of A^t with negative Lyapunov exponents. Then there exist $0 < \theta_-$ and $C \geqslant 1$ such that

$$\big\| \big(A^t\big)^n h \big\| \leqslant C \exp(-n\theta_-)\|h\| \quad \text{for all } h \in \Gamma_s^{(k)} \text{ and } n \geqslant 0. \tag{C.13}$$

Assume that T has non-degenerated spectrum, i.e. $\theta_g > 0$. Then $\dim \Gamma_s^{(0)} = \dim \Gamma_u^{(0)} = g$. Since $2g + \kappa - 1 = d$ and $\dim \Gamma_c^{(0)} = \kappa - 1$,

$$\mathbb{R}^{\mathcal{A}} = \Gamma^{(0)} = \Gamma_s^{(0)} \oplus \Gamma_c^{(0)} \oplus \Gamma_u^{(0)}$$

is an A^t-invariant decomposition. It follows that $\Gamma_s^{(0)} \oplus \Gamma_c^{(0)} = \Gamma_{cs}^{(0)} \subset \Gamma_0^{(0)}$. Therefore

$$\Gamma_0^{(0)} = \Gamma_s^{(0)} \oplus \Gamma_c^{(0)} \oplus \big(\Gamma_u^{(0)} \cap \Gamma_0^{(0)}\big).$$

Recall that $\Gamma_s^{(0)} \oplus \Gamma_u^{(0)} \subset H_\pi$. As T has non-degenerated spectrum, these subspaces have the same dimension, and so they are equal. Denote by $\Gamma_s^{(k)}$, $\Gamma_c^{(k)}$ and $\Gamma_u^{(k)}$ the subspaces of functions on $I^{(k)}$ constant on intervals $I_\alpha^{(k)}$, $\alpha \in \mathcal{A}$, corresponding to the vectors from $\Gamma_s^{(0)}$, $\Gamma_c^{(0)}$ and $\Gamma_u^{(0)}$ respectively. Then

$$\begin{gathered}\Gamma^{(k)} = \Gamma_s^{(k)} \oplus \Gamma_c^{(k)} \oplus \Gamma_u^{(k)}, \qquad H_\pi = \Gamma_s^{(k)} \oplus \Gamma_u^{(k)}, \\ \Gamma_0^{(k)} = \Gamma_s^{(k)} \oplus \Gamma_c^{(k)} \oplus \big(\Gamma_u^{(k)} \cap \Gamma_0^{(k)}\big)\end{gathered} \tag{C.14}$$

for $k \geqslant 0$ is a family of decomposition invariant with respect to the renormalization operators $S(k,l)$ for $0 \leqslant k < l$. Moreover, if additionally $\kappa = 1$ then $\Gamma_c^{(k)}$ is trivial and $\Gamma_{cs}^{(k)} = \Gamma_s^{(k)}$.

As $\xi = Id_{\Sigma(\pi)}$, by Lemma C.5, for every $\varphi \in \mathrm{BV}^{\diamondsuit}(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ and $l \geqslant k$ we have

$$S(k,l)\varphi \in \mathrm{BV}^{\diamondsuit}\Big(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)}\Big) \quad \text{and} \quad \mathcal{O}\big(S(k,l)\varphi\big) = \mathcal{O}(\varphi) \quad \text{for each } \mathcal{O} \in \Sigma(\pi). \tag{C.15}$$

Proofs of Theorems 5.2 and 5.3. Since

$$U^{(0)}\hat{\varphi} = P^{(0)}\varphi = U^{(0)} \circ P_0^{(0)}\varphi - \Delta P^{(0)}\varphi = U^{(0)}\varphi - U^{(0)} \circ C^{(0)}\varphi - \Delta P^{(0)}\varphi,$$

we have

$$\varphi - \hat{\varphi} \in U^{(0)} \circ C^{(0)}\varphi + \Delta P^{(0)}\varphi \subset \Gamma_0^{(0)}.$$

In view of (C.2) and (C.10),

$$U^{(k)} \circ S(k)\hat{\varphi} = S_u(k) \circ U^{(0)}\hat{\varphi} = S_u(k) \circ P^{(0)}\varphi = P^{(k)} \circ S(k)\varphi.$$

Therefore, by (C.11) and (5.1), we have

$$\left\| U^{(k)} \circ S(k)\hat{\varphi} \right\|_{\sup/\Gamma_{cs}^{(k)}} = \left\| P^{(k)}\big(S(k)\varphi\big) \right\|_{\sup/\Gamma_{cs}^{(k)}} \leqslant (1+K)\operatorname{Var}\big(S(k)\varphi\big) \leqslant (1+K)\operatorname{Var}\varphi.$$

It follows that for every $k \geqslant 0$ there exist $\varphi_k \in \mathrm{BV}_0(\bigsqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ and $h_k \in \Gamma_{cs}^{(k)}$ such that

$$S(k)\hat{\varphi} = \varphi_k + h_k \quad \text{and} \quad \|\varphi_k\|_{\sup} \leqslant (1+K)\operatorname{Var}\varphi. \tag{C.16}$$

As

$$\varphi_{k+1} + h_{k+1} = S(k+1)\hat{\varphi} = S(k,k+1)\big(S(k)\hat{\varphi}\big) = S(k,k+1)\varphi_k + A^t h_k, \tag{C.17}$$

setting $\Delta h_{k+1} = h_{k+1} - A^t h_k$ ($\Delta h_0 = h_0$), we have $\Delta h_{k+1} = -\varphi_{k+1} + S(k,k+1)\varphi_k$. Moreover, by (C.16),

$$\begin{aligned}\|\Delta h_{k+1}\| &= \left\|\varphi_{k+1} - S(k,k+1)\varphi_k\right\|_{\sup} \leqslant \|\varphi_{k+1}\|_{\sup} + \left\|S(k,k+1)\varphi_k\right\|_{\sup} \\ &\leqslant \big(1+\|A\|\big)(1+K)\operatorname{Var}\varphi\end{aligned}$$

and

$$\|h_0\| = \|\Delta h_0\| = \|\hat{\varphi} - \varphi_0\|_{\sup} \leqslant \|\hat{\varphi}\|_{\sup} + (1+K)\operatorname{Var}\varphi.$$

Since $h_k = \sum_{0\leqslant l\leqslant k}(A^t)^{k-l}\Delta h_l$ and $\Delta h_l \in \Gamma_{cs}^{(l)}$, by (C.1), for $k \geqslant 1$

$$\begin{aligned}\|h_k\| &\leqslant \sum_{0\leqslant l\leqslant k} \left\|\big(A^t\big)^{k-l}\Delta h_l\right\| \leqslant \sum_{0\leqslant l\leqslant k} C(k-l)^{M-1}\|\Delta h_l\| \\ &\leqslant Ck^M\big(1+\|A\|\big)(1+K)\operatorname{Var}\varphi + Ck^{M-1}\|\hat{\varphi}\|_{\sup}.\end{aligned}$$

In view of (C.16), it follows that

$$\left\|S(k)\hat{\varphi}\right\|_{\sup} \leqslant \|\varphi_k\|_{\sup} + \|h_k\| \leqslant Ck^M\big(2+\|A\|\big)(1+K)\operatorname{Var}\varphi + Ck^{M-1}\|\hat{\varphi}\|_{\sup},$$

and the inequality (5.5) is proved.

Notice that a slight change in the proof shows also (5.6) if additionally T has non-degenerated spectrum and $\kappa(\pi) = 1$. Indeed, then $\Delta h^l \in \Gamma_{cs}^{(l)} = \Gamma_s^{(l)}$ and, in view of (C.13), we have a better control over the growth of h_k, that is

$$\begin{aligned}\|h_k\| &\leqslant \sum_{0\leqslant l\leqslant k} \left\|\big(A^t\big)^{k-l}\Delta h_l\right\| \leqslant \sum_{0\leqslant l\leqslant k} C\exp\big((l-k)\theta_-\big)\|\Delta h_l\| \\ &\leqslant \frac{C(1+\|A\|)(1+K)}{1-\exp(-\theta_-)}\operatorname{Var}\varphi + C\|\hat{\varphi}\|_{\sup},\end{aligned}$$

and hence

$$\left\|S(k)\hat{\varphi}\right\|_{\sup} \leqslant \|\varphi_k\|_{\sup} + \|h_k\| \leqslant \frac{C(2+\|A\|)(1+K)}{1-\exp(-\theta_-)} \operatorname{Var}\varphi + C\|\hat{\varphi}\|_{\sup},$$

which gives (5.6) if $\kappa(\pi) = 1$. As we will see the proof of (5.6) for $\kappa(\pi) > 1$ is more involved.

Let us return to the general case of periodic type of T. Since $\hat{\varphi} - \varphi \in \Gamma_0^{(0)} = (\Gamma_u^{(0)} \cap \Gamma_0^{(0)}) \oplus \Gamma_{cs}^{(0)}$, there exist $h \in (\Gamma_u^{(0)} \cap \Gamma_0^{(0)})$ and $h' \in \Gamma_{cs}^{(0)}$ such that $\varphi + h = \hat{\varphi} + h'$. Hence

$$\varphi + h + \Gamma_{cs}^{(0)} = \hat{\varphi} + \Gamma_{cs}^{(0)} = P^{(0)}\varphi.$$

Suppose that $h_1, h_2 \in \Gamma_u^{(0)} \cap \Gamma_0^{(0)}$ are vectors such that

$$\varphi + h_1 + \Gamma_{cs}^{(0)} = \varphi + h_2 + \Gamma_{cs}^{(0)} = P^{(0)}\varphi.$$

In view of (5.5), $\|S(k)(\varphi + h_1)\|_{\sup}$ and $\|S(k)(\varphi + h_2)\|_{\sup}$ have at most polynomial growth. Therefore, $\|(A^t)^k(h_1 - h_2)\| = \|S(k)(h_1 - h_2)\|$ has at most polynomial growth as well. Since $h_1 - h_2 \in \Gamma_u^{(0)}$, it follows that $h_1 = h_2$, which completes the proof of Theorem 5.2.

In the remainder we will modify the method of the proof of (5.5) in a more subtle way in order to show (5.6) for T with non-degenerated spectrum and $\kappa(\pi) > 1$. In this case the central part of the decomposition is non-trivial and this gives a non-trivial growth for the sequence $(h_k)_{k\geqslant 0}$. To avoid this problem we will split h_k into stable and central terms and we will estimate these two terms separately.

Since $\Gamma_{cs}^{(k)} = \Gamma_c^{(k)} \oplus \Gamma_s^{(k)}$, each $h_k \in \Gamma_{cs}^{(k)}$ satisfying (C.16) can be decomposed as follows $h_k = h_k^s + h_k^c$ with $h_k^c \in \Gamma_c^{(k)}$ and $h_k^s \in \Gamma_s^{(k)} \subset H_\pi$. By Remark C.3, $\Lambda^\pi(h_k^s) = 0$. In view of (C.16) and (C.15), it follows that

$$\mathcal{O}(\hat{\varphi}) = \mathcal{O}\big(S(k)\hat{\varphi}\big) = \mathcal{O}(\varphi_k) + \mathcal{O}\big(h_k^c\big) \quad \text{for every } \mathcal{O} \in \Sigma(\pi).$$

Moreover, by (C.12) and (C.16),

$$\left|\mathcal{O}(\varphi_k)\right| \leqslant 2d\|\varphi_k\|_{\sup} \leqslant 2d(1+K)\operatorname{Var}\varphi \quad \text{and} \quad \left|\mathcal{O}(\hat{\varphi})\right| \leqslant 2d\|\hat{\varphi}\|_{\sup}$$

for every $\mathcal{O} \in \Sigma(\pi)$. Therefore

$$\left|\langle h_k^c, b(\mathcal{O})\rangle\right| = \left|\mathcal{O}\big(h_k^c\big)\right| \leqslant 2d\big((1+K)\operatorname{Var}\varphi + \|\hat{\varphi}\|_{\sup}\big) \quad \text{for every } \mathcal{O} \in \Sigma(\pi),$$

so that

$$\left\|\Lambda^\pi\big(h_k^c\big)\right\| \leqslant 2d\big((1+K)\operatorname{Var}\varphi + \|\hat{\varphi}\|_{\sup}\big). \tag{C.18}$$

By (C.14), we have $\mathbb{R}^{\mathcal{A}} = \Gamma^{(k)} = \Gamma_c^{(k)} \oplus H_\pi$, so in view of Remark C.3, there exists $K' \geqslant 1$ such that $\|h\| \leqslant K'\|\Lambda^\pi h\|$ for every $h \in \Gamma_c^{(k)}$. By (C.18), it follows that

$$\left\|h_k^c\right\| \leqslant 2dK'\big((1+K)\operatorname{Var}\varphi + \|\hat{\varphi}\|_{\sup}\big). \tag{C.19}$$

Let $\Delta h^s_{k+1} = h^s_{k+1} - A^t h^s_k$ for $k \geqslant 0$ and $\Delta h^s_0 = h^s_0$. Then from (C.17), we have

$$\Delta h^s_{k+1} = -\varphi_{k+1} + S(k,k+1)\varphi_k - h^c_{k+1} + A^t h^c_k = -\varphi_{k+1} + S(k,k+1)\varphi_k - h^c_{k+1} + h^c_k.$$

Therefore, by (C.16) and (C.19),

$$\begin{aligned}\|\Delta h^s_{k+1}\| &\leqslant \|\varphi_{k+1}\|_{\sup} + \|A\|\|\varphi_k\|_{\sup} + \|h^c_{k+1}\| + \|h^c_k\| \\ &\leqslant \big(1 + \|A\| + 4dK'\big)(1+K)\operatorname{Var}\varphi + 4dK'\|\hat{\varphi}\|_{\sup},\end{aligned}$$

$$\|\Delta h^s_0\| = \|\hat{\varphi} - \varphi_0 - h^c_0\|_{\sup} \leqslant \big(1 + 2dK'\big)\big(\|\hat{\varphi}\|_{\sup} + (1+K)\operatorname{Var}\varphi\big).$$

Since $h^s_k = \sum_{0\leqslant l\leqslant k}(A^t)^l \Delta h^s_{k-l}$ and $\Delta h^s_l \in \Gamma_s^{(l)}$, by (C.13), it follows that

$$\begin{aligned}\|h^s_k\| &\leqslant \sum_{0\leqslant l\leqslant k}\big\|\big(A^t\big)^l \Delta h^s_{k-l}\big\| \leqslant \sum_{0\leqslant l\leqslant k} C\exp(-l\theta_-)\|\Delta h^s_{k-l}\| \\ &\leqslant \frac{C(1+\|A\|+4dK')}{1-\exp(-\theta_-)}\big((1+K)\operatorname{Var}\varphi + \|\hat{\varphi}\|_{\sup}\big).\end{aligned}$$

In view of (C.16) and (C.19), it follows that

$$\|S(k)\hat{\varphi}\|_{\sup} \leqslant \|\varphi_k\|_{\sup} + \|h^c_k\| + \|h^s_k\| \leqslant \frac{C(2+\|A\|+6dK')}{1-\exp(-\theta_-)}\big((1+K)\operatorname{Var}\varphi + \|\hat{\varphi}\|_{\sup}\big),$$

which completes the proof. □

Theorem C.6. *There exist $C_3, C_4 > 0$ such that*

$$\|\hat{\varphi}^{(n)}\|_{\sup} \leqslant C_3 \log^{M+1} n \operatorname{Var}\varphi + C_4 \log^M n \|\hat{\varphi}\|_{\sup}$$

for every natural n. If additionally T has non-degenerated spectrum then

$$\|\hat{\varphi}^{(n)}\|_{\sup} \leqslant C_3 \log n \operatorname{Var}\varphi + C_4 \log n \|\hat{\varphi}\|_{\sup}.$$

Proof. By Proposition A.3 and Theorem 5.2, for every $x \in I$ we have

$$\begin{aligned}\|\hat{\varphi}^{(n)}(x)\| &\leqslant 2\|A\|\sum_{k=0}^{m}\big(C_1 k^M \operatorname{Var}\varphi + C_2 k^{M-1}\|\hat{\varphi}\|_{\sup}\big) \\ &\leqslant 2\|A\|\big(C_1 m^{M+1}\operatorname{Var}\varphi + C_2 m^M \|\hat{\varphi}\|_{\sup}\big),\end{aligned}$$

where $m = m(x,n,T)$. Now the assertion follows directly from (A.1). The rest of the proof runs as before. □

Appendix D. Example of a non-regular step cocycle

Let $T = T_{(\pi,\lambda)}$ be an IET of periodic type with periodic matrix A.

Lemma D.1. *(Cf. [28].) Suppose that $h \in \Gamma_0^{(0)}$ and $\varphi : I \to \mathbb{R}$ is the associated step cocycle. If $h \in \Gamma_s^{(0)}$ then φ is a coboundary. If $h \notin \Gamma_{cs}^{(0)}$ then φ is not a coboundary.*

Proof. If $h \in \Gamma_s^{(0)}$ then, by (C.13), $\|S(l)\varphi\|_{\sup} = \|(A^t)^l h\| \leqslant C\exp(-l\theta_-)\|h\|$. In view of Proposition A.3, it follows that

$$\left\|\varphi^{(n)}\right\|_{\sup} \leqslant 2\sum_{l=0}^{\infty}\left\|Z(l+1)\right\|\left\|S(l)\varphi\right\|_{\sup} \leqslant 2C\|A\|\|h\|\sum_{l=0}^{\infty}\exp(-l\theta_-) = \frac{2C\|A\|\|h\|}{1-\exp(-\theta_-)}$$

for every natural n. But each bounded cocycle is a coboundary (see §2.1 in [28]).

Now suppose that $h \in \Gamma_0^{(0)}$ and φ is a coboundary. Set $\varepsilon = \inf\{\mu(C_\alpha^{(n)}):\ n \geqslant 0,\ \alpha \in \mathcal{A}\}$ (see Section 4 for the definition of the tower $C_\alpha^{(n)}$). In view of (4.6), $\varepsilon > 0$. Since φ is a coboundary, there exist $M > 0$ and a sequence $(B_k)_{k\geqslant 0}$ of measurable sets with $\mu(B_k) > 1-\varepsilon$ for $n \geqslant 0$ such that $|\varphi^{(k)}(x)| \leqslant M$ for all $x \in B_k$ and $k \geqslant 0$. Recall that for every $x \in C_\alpha^{(n)}$ we have $\varphi^{(h_\alpha^{(n+1)})}(x) = ((A^t)^{n+1}h)_\alpha$. Since $C_\alpha^{(n)} \cap B_{h_\alpha^{(n+1)}} \neq \emptyset$, it follows that $|((A^t)^{n+1}h)_\alpha| \leqslant M$ for every $n \geqslant 0$ and $\alpha \in \mathcal{A}$. Thus $\|(A^t)^{n+1}h\| \leqslant M$ for every $n \geqslant 0$, and hence $h \in \Gamma_{cs}^{(0)}$. $\square$

Example 2. Let us consider an IET $T = T_{(\pi_5^{sym},\lambda)}$ of periodic type with periodic matrix

$$A = \begin{pmatrix} 18 & 28 & 31 & 38 & 18 \\ 10 & 16 & 8 & 9 & 6 \\ 13 & 20 & 36 & 46 & 18 \\ 2 & 3 & 16 & 22 & 6 \\ 39 & 61 & 63 & 77 & 37 \end{pmatrix}.$$

The existence of such IET was shown in [33]. The Perron–Frobenius eigenvalue of A is $55 + 12\sqrt{21}$ and λ is equal to $(1+\sqrt{21}, 2, 1+\sqrt{21}, 2, 7+\sqrt{21})$ up to multiplication by a positive constant. Moreover, the eigenvalues and eigenvectors of A^t are as follows:

$$\begin{aligned}
&\rho_1 = 55 + 12\sqrt{21}, && v_1 = (-1+\sqrt{21}, 1+\sqrt{21}, 3+\sqrt{21}, 5+\sqrt{21}, 4),\\
&\rho_2 = 9 + 4\sqrt{5}, && v_2 = (-2, -1-1\sqrt{5}, 2, 1+\sqrt{5}, 0),\\
&\rho_3 = 1, && v_3 = (-1, -2, 0, -1, 1),\\
&\rho_4 = 9 - 4\sqrt{5}, && v_4 = (-2, -1+1\sqrt{5}, 2, 1-\sqrt{5}, 0),\\
&\rho_5 = 55 - 12\sqrt{21}, && v_5 = (-1-\sqrt{21}, 1-\sqrt{21}, 3-\sqrt{21}, 5-\sqrt{21}, 4).
\end{aligned}$$

Note that $v_2, v_3, v_4, v_5 \in \Gamma_0^{(0)}$. Denote by $\varphi_i : I \to \mathbb{R}$ the step function corresponding to v_i for $1 < i \leqslant 5$. Since $|\rho_2| > 1 > |\rho_4|$, by Lemma D.1, φ_4 is a coboundary and φ_2 is not a coboundary.

We will show that φ_2 is a non-regular cocycle. Note that the cocycles $\varphi_2 + \varphi_4$ and $\varphi_2 - \varphi_4$ take values in $\mathbb{Z}$ and $\sqrt{5}\mathbb{Z}$ respectively. Since φ_4 is a coboundary, it follows that $E(\varphi_2) \subset \mathbb{Z}$

and $E(\varphi_2) \subset \sqrt{5}\mathbb{Z}$, and hence $E(\varphi_2) = \{0\}$. Since φ_2 is not a coboundary, $\overline{E}(\varphi_2) = \{0, \infty\}$, and hence it is non-regular.

References

[1] J. Aaronson, An Introduction to Infinite Ergodic Theory, Math. Surveys Monogr., vol. 50, AMS, Providence, RI, 1997.

[2] V.I. Arnold, Topological and ergodic properties of closed 1-forms with incommensurable periods, Funktsional. Anal. i Prilozhen. 25 (1991) 1–12 (in Russian), translation in: Funct. Anal. Appl. 25 (1991) 81–90.

[3] L. Baggett, K. Merrill, Smooth cocycles for an irrational rotation, Israel J. Math. 79 (1992) 281–288.

[4] A.A. Blokhin, Smooth ergodic flows on surfaces, Trans. Moscow Math. Soc. 27 (1972) 117–134.

[5] M. Boshernitzan, A condition for minimal interval exchange maps to be uniquely ergodic, Duke Math. J. 52 (1985) 723–752.

[6] N. Chevallier, J.-P. Conze, Examples of recurrent or transient stationary walks in $\mathbb{R}^d$ over a rotation of $\mathbb{T}^2$, in: Contemp. Math., vol. 485, 2009, pp. 71–84.

[7] J.-P. Conze, Ergodicité d'une transformation cylindrique, Bull. Soc. Math. France 108 (1980) 441–456.

[8] J.-P. Conze, Recurrence, ergodicity and invariant measures for cocycles over a rotation, in: Contemp. Math., vol. 485, 2009, pp. 45–70.

[9] J.-P. Conze, E. Gutkin, On recurrence and ergodicity for geodesic flows on noncompact periodic polygonal surfaces, preprint, 2010, arXiv:1008.0136.

[10] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai, Ergodic Theory, Springer-Verlag, New York, 1982.

[11] B. Fayad, M. Lemańczyk, On the ergodicity of cylindrical transformations given by the logarithm, Mosc. Math. J. 6 (2006) 657–672.

[12] G. Forni, Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus, Ann. of Math. (2) 146 (1997) 295–344.

[13] G. Forni, Deviation of ergodic averages for area-preserving flows on surfaces of higher genus, Ann. of Math. (2) 155 (2002) 1–103.

[14] K. Frączek, On ergodicity of some cylinder flows, Fund. Math. 163 (2) (2000) 117–130.

[15] K. Frączek, M. Lemańczyk, On symmetric logarithm and some old examples in smooth ergodic theory, Fund. Math. 180 (2003) 241–255.

[16] K. Frączek, M. Lemańczyk, On disjointness properties of some smooth flows, Fund. Math. 185 (2005) 117–142.

[17] K. Frączek, M. Lemańczyk, Smooth singular flows in dimension 2 with the minimal self-joining property, Monatsh. Math. 156 (2009) 11–45.

[18] K. Frączek, M. Lemańczyk, E. Lesigne, Mild mixing property for special flows under piecewise constant functions, Discrete Contin. Dyn. Syst. 19 (2007) 691–710.

[19] G. Greschonig, Recurrence in unipotent groups and ergodic nonabelian group extensions, Israel J. Math. 147 (2005) 245–267.

[20] P. Hubert, B. Weiss, Ergodicity for infinite periodic translation surfaces, preprint, 2010, available from: http://www.math.bgu.ac.il/~barakw/papers/.

[21] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, with a supplementary chapter by Katok and Leonardo Mendoza, Encyclopedia Math. Appl., vol. 54, Cambridge University Press, Cambridge, 1995.

[22] J. King, Joining-rank and the structure of finite rank mixing transformations, J. Anal. Math. 51 (1988) 182–227.

[23] A.V. Kočergin, On the absence of mixing in special flows over the rotation of a circle and in flows on a two-dimensional torus, Dokl. Akad. Nauk SSSR 205 (1972) 949–952.

[24] L. Kuipers, H. Niederreiter, Uniform Distribution of Sequences, John Wiley & Sons, New York, 1974.

[25] J. Kułaga, On self-similarity problem for smooth flows on orientable surfaces, preprint, 2010, arXiv:1011.6166.

[26] M. Lemańczyk, F. Parreau, D. Volný, Ergodic properties of real cocycles and pseudo-homogeneous Banach spaces, Trans. Amer. Math. Soc. 348 (1996) 4919–4938.

[27] M. Lemańczyk, M. Mentzen, H. Nakada, Semisimple extensions of irrational rotations, Studia Math. 156 (2003) 31–57.

[28] S. Marmi, P. Moussa, J.-C. Yoccoz, The cohomological equation for Roth-type interval exchange maps, J. Amer. Math. Soc. 18 (2005) 823–872.

[29] I. Oren, Ergodicity of cylinder flows arising from irregularities of distribution, Israel J. Math. 44 (1983) 127–138.

[30] D. Pask, Skew products over irrational rotation, Israel J. Math. 69 (1990) 65–74.

[31] G. Rauzy, Échanges d'intervalles et transformations induites, Acta Arith. 34 (1979) 315–328.
[32] K. Schmidt, Cocycle of Ergodic Transformation Groups, Lecture Notes in Math., vol. 1, Macmillan Co., India, 1977.
[33] Ya.G. Sinai, C. Ulcigrai, Weak mixing in interval exchange transformations of periodic type, Lett. Math. Phys. 74 (2005) 111–133.
[34] W.A. Veech, Projective Swiss cheeses and uniquely ergodic interval exchange transformations, in: Ergodic Theory and Dynamical Systems, I, Birkhäuser, 1981, pp. 113–193.
[35] W.A. Veech, Gauss measures for transformations on the space of interval exchange maps, Ann. of Math. (2) 115 (1982) 201–242.
[36] W.A. Veech, The metric theory of interval exchange transformations, I. Generic spectral properties, Amer. J. Math. 106 (1984) 1331–1358.
[37] M. Viana, Ergodic theory of interval exchange maps, Rev. Mat. Complut. 19 (2006) 7–100.
[38] M. Viana, Dynamics of interval exchange transformations and Teichmüller flows, lecture notes available from: http://w3.impa.br/~viana/out/ietf.pdf.
[39] J.-C. Yoccoz, Continued fraction algorithms for interval exchange maps: an introduction, in: Frontiers in Number Theory, Physics, and Geometry. I, Springer-Verlag, Berlin, 2006, pp. 401–435.
[40] A. Zorich, Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents, Ann. Inst. Fourier (Grenoble) 46 (1996) 325–370.
[41] A. Zorich, Deviation for interval exchange transformations, Ergodic Theory Dynam. Systems 17 (1997) 1477–1499.