Tableaux and Matrix Correspondences

KIEM-PHONG VO

Bell Laboratories, Murray Hill, New Jersey 07974

AND

ROGER WHITNEY*

Math Department, University of California, San Diego, LaJolla, California 92093

Communicated by the Managing Editors

Received November 12, 1981

The Robinson-Schensted correspondence, a bijection between nonnegative matrices and pair of tableaux, is investigated. The representation, in the tableau, of the dihedral symmetries of the matrix are derived in a simple manner using a shape-preserving anti-isomorphism of the platic monoid. The Robinson-Schensted correspondence is shown to be equivalent to the Hillman-Grassl bijection between reverse plane partitions and tabloids. A generalized insertion method for the Robinson-Schensted correspondence is obtained.

I. INTRODUCTION

A partition \( \lambda \) of an integer \( n \) (denoted \( \lambda \vdash n \)) is a sequence of positive integers \( (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k) \) whose sum is \( n \). The Ferrers diagram of \( \lambda \) is a collection of cells arranged with \( \lambda_1 \) cells in the first (bottom) row, \( \lambda_2 \) cells in the second row, etc. The rows are left justified. A tabloid with shape \( \lambda \) is a filling of the Ferrers diagram of \( \lambda \) with nonnegative integers, one integer per cell. We shall make the convention that rows and columns of a tabloid are read from left to right and bottom to top, respectively. A plane partition is a tabloid with weakly decreasing rows and columns. A reverse plane partition is a tabloid with weakly increasing rows and columns. A tableau is a reverse plane partition with positive entries and strictly increasing columns. Finally a standard (Young) tableau is a tableau with shape \( \lambda \vdash n \) whose entries comprise the set \( \{1, 2, \ldots, n\} \).

The studies of tableaux and plane partitions were initiated by Young [17]

* Current address: Department of Information and Computer Science, University of California, Irvine, California 92717.
and MacMahon around the beginning of the century and still are pursued vigorously at the present. At the heart of the subject are various correspondences relating collections of these objects to other combinatorial objects or just among themselves. Earliest discovered was the correspondence between permutations and pairs of standard tableaux of the same shapes found by Robinson [10]. This correspondence was rediscovered in quite a different form by Schensted [11] in his work characterizing maximal increasing subsequences of a given sequence of integers. Finally, the correspondence was augmented to its most general form by Knuth [6] to relate nonnegative integral matrices and pairs of tableaux. Schutzenberger and Lascoux have done a number of fundamental studies and elevated the entire subject to its present state as a major mathematical tool [9, 14]. Burge [2] used this correspondence and some other variations of it to prove combinatorially a few of Schur's identities. Another entirely different correspondence between reverse plane partitions and tabloids was discovered by Hillman and Grassl [4]. This correspondence was used to give combinatorial proofs of generating function identities for reverse plane partitions due to Stanley [16].

In this work, we first define a natural anti-isomorphism on the plactic monoid [9] which shall be shown to be the well-known Schutzenberger operator (or the evacuation operator). The representations of the dihedral symmetries will be derived in a much simpler way than the method used by Gansner.

A simple observation on the Hillman–Grassl correspondence gives two amazing results. In this correspondence, Hillman and Grassl constructed zigzag paths in a reverse plane partition (RPP) starting at the upper left corner of the RPP. These paths are used to define a tabloid. We note that we get a similar correspondence starting the zigzag paths at the lower right corner of the RPP. We then have:

1. The two tabloids produced are the same. That is, the Hillman–Grassl correspondence can be produced in two different ways. In particular, this removes the seemingly artificial emphases on columns or rows in constructing the zigzag paths.

2. Let $M$ be a square tabloid (a matrix), then at each step the process of row inserting $M$ into the pair of tableaux $(P, Q)$ is identical to the new way of producing the Hillman–Grassl map $M \rightarrow \text{RPP}$; using the Frobenius map to translate RPPs into tableaux. Thus, the Hillman–Grassl correspondence is just row insertion is disguise.
Using these facts, we find a number of interesting facts about row and columns insertions. A generalized insertion method is obtained. This insertion can start anywhere in a biword and read contiguous elements either to the right of to the left in any order. This is in contrast with row insertion where elements are read only from left to right and column insertion where elements are read only from right to left.

II. THE KNUTH AND BURGE CORRESPONDENCES

Let $A$ be the ordered alphabet $\{1, 2, \ldots, n\}$. A word is a finite sequence of letters from $A$. The set of all words is denoted $A^*$. A-biletter is a column $\binom{a}{b}$ with $a, b$ from $A$. We shall consider two linear orders on the set of biletters.

A biletter $\binom{a}{b}$ is said to lexicographically precede another biletter $\binom{c}{d}$, denoted $\binom{a}{b} <_K \binom{c}{d}$ if either $a < c$ or $a = c$ and $b < d$. On the other hand, the locally reverse lexicographic order is defined by $\binom{a}{b} <_B \binom{c}{d}$ if $a < c$ or $a = c$ and $b > d$.

**Example 1.** Let $a = \binom{1}{2}$, $b = \binom{2}{1}$, $c = \binom{1}{1}$. Then we have

$$a <_K b, \quad a <_K c, \quad a <_B b, \quad \text{and} \quad c <_B a.$$ 

Let $B$ be the collection of all finite multisets of biletters. A multiset of biletters can be represented as a sequence of biletters ordered by either of the above linear orders. We shall generally omit inner parentheses in a such representation. The top and bottom rows of a sequence of biletters can be considered as words in $A^*$. The top row is always weakly increasing. The lower word consists of blocks with the same top letters. Within each blocks, the letters will be weakly increasing or decreasing depending on whether the biletters were ordered in $<_K$ or $<_B$.

Collection $B$ can also be identified with the set of $n \times n$ nonnegative integral matrices. If $M = (m_{ij})$ is such a matrix, the corresponding element of $B$ is the multiset of biletters in which the biletter $\binom{j}{j}$ appears exactly $m_{ij}$ times. When the biletters are ordered by $<_K$, we have the Knuth correspondence biword. When they are ordered by $<_B$, we have the Burge correspondence biword. The terminologies used here are due to Gansner.

**Example 2.**

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{Knuth biword:} \quad \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 1 & 1 & 2 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \quad \text{Burge biword:} \quad \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 1 & 2 & 1 & 1 & 3 & 1 \end{bmatrix}.$$
Let $T$ be a tableau with entries in $A$. Let $a$ be a letter in $A$. We can insert $a$ into $T$ in two different ways, row and column insertions. The process can be best described with Algol-like recursive procedures. In these procedures $\emptyset$ denotes the empty tableau. First the row insertion algorithm:

```plaintext
procedure R_INSERT(a, T)
begin
  if $T = \emptyset$ then $T := a$;
  else begin
    Let $b_1 \leq \cdots \leq b_k$ be the bottom row of $T$,
    and $T'$ be the rest of $T$;
    if $a \geq b_k$ then append $a$ to the bottom row;
    else begin
      Let $i$ be so that $b_{i-1} < a \leq b_i$;
      $t := b_i$;
      $b_i := a$;
      R_INSERT($t$, $T'$);
    end
  end
end
```

The elements $b_i$'s in the algorithm are said to be bumped. It is clear that the shape of the new tableau covers the shape of $T$ (in the partition lattice).

**Example 3.**

```
\begin{array}{ccc}
  & 4 & 4 \\
2 & 4 & \rightarrow 2 & 4 & \leftarrow 3 & \rightarrow 2 & 3 & \rightarrow 2 & 3 \\
1 & 2 & 3 & 3 & \leftarrow 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3
\end{array}
```

Now, the column insertion algorithm:

```plaintext
procedure C_INSERT(a, T)
begin
  if $T = \emptyset$ then $T := a$;
  else begin
    Let $b_1 < \cdots < b_k$ be the left most column of $T$,
    and $T'$ be the rest of $T$;
    if $a > b_k$ then add $a$ to the top of the left
    most column;
    else begin
```

```
Let \( i \) be so that \( b_{i-1} < a \leq b_i \);
\[
\begin{align*}
&\quad t := b_i; \\
&\quad b_i := a; \\
&\quad C \_ INSERT(t, T');
\end{align*}
\]

Note that here we are using the convention of reading columns of tableaux from bottom to top.

**Example 4.**

\[
\begin{array}{cccccc}
4 & 2 & 4 & 4 & 4 & 4 \\
3 & 4 & 3 & 2 & 3 & 3 \\
1 & 2 & 4 & 1 & 2 & 4 \\
\end{array}
\]

Inserting an element into a tableau creates a new corner in the tableau. Given the coordinates of a corner, we can reverse the process to delete an element from the tableau. We shall give the procedure \( R \_ DELETE \), the inverse of \( R \_ INSERT \). The corresponding algorithm \( C \_ DELETE \) shall be left as an exercise. The arguments of \( R \_ DELETE \) will be the coordinates \((p, q)\) of a corner. A tableau \( T = (t_{pq}) \), the value of \( T_{pq} \), denoted by \( v \), and an initial flag \( \text{TRUE} \). The algorithm returns the modified tableau and the deleted entry.

procedure \( R \_ DELETE(p, q, b, T, \text{flag}) \)
begin
if \( \text{flag} = \text{TRUE} \) then delete the corner cell \((p, q)\) from \( T \);
if \( p = 1 \) the return \( v \); \( (p \) is the row number) 
else begin
\quad Let \( i := p - 1 \), and \( j \) be s.t. \( t_{ij} < v \leq t_{i,j+1} \);
\quad \( t := t_{ij} \);
\quad \( t_{ij} := v \);
\quad return \( R \_ DELETED(i, j, t, T, \text{FALSE}) \);
end
end

If \( a \) was just row inserted into \( T \) to create the corner \((p, q)\), an immediate call to \( R \_ DELETE \) will return \( a \), and restore \( T \) to its previous state.
**Example 5.** The sequence of operations to delete the element 4:

\[
\begin{array}{ccc}
4 & 2 & 3 \\
1 & 2 & 2
\end{array}
\rightarrow
\begin{array}{ccc}
2 & 4 & 3 \\
1 & 2 & 2
\end{array}
\rightarrow
\begin{array}{ccc}
2 & 4 & 3 \\
1 & 2 & 3
\end{array}
\rightarrow
\begin{array}{ccc}
2 & 4 & 3 \\
1 & 2 & 3
\end{array}
\]

Henceforth, we shall use the notations \( \rightarrow^R \) and \( \rightarrow^C \) for row and column insertions, respectively. If \( w = a_1 a_2 \cdots a_k \) is a word in \( A^* \), we obtain the row insertion tableau \( I_R(w) \) and the column insertion \( I_C(w) \) of \( w \) as follows:

\[
I_R(w) = \left( a_k \rightarrow^R \left( \ldots \left( a_2 \rightarrow^R \left( a_1 \rightarrow^R \emptyset \right) \right) \ldots \right) \right),
\]

\[
I_C(w) = \left( a_k \rightarrow^C \left( \ldots \left( a_2 \rightarrow^C \left( a_1 \rightarrow^C \emptyset \right) \right) \ldots \right) \right),
\]

where \( \emptyset \) stands for the empty tableau.

Certain different words can give rise to the same tableau after being row (or column) inserted. We can completely characterize them.

**Definition.** Let \( u, v \in A^* \), and let \( w \) be a three-letter word of either of the following forms:

(a) \( w = bac \), where \( a < b \leq c \)

(b) \( w = acb \), where \( a \leq b < c \).

Let \( w' = bca \) or \( cab \) depending on whether \( w \) was of type (a) or (b), respectively. We say that \( uwv \) and \( uw'u \) are obtained from one another by a Knuth exchange. In general, if \( v, w \) are words in \( A^* \) and can be obtained from one another by a finite sequence of Knuth exchanges, we say \( v \) and \( w \) are Knuth equivalent. In this case, we write \( v \sim_K w \).

Similarly, by changing all \( \leq \) to \( < \) and \( < \) to \( \leq \) in the above, we define the Burge equivalence of words. We shall write \( v \sim_B w \) for two Burge equivalent words \( v, w \).

The following theorem is due to Knuth [6], and Burge [2]:

**Theorem 1.** Let \( u, v \) be words in \( A \).

(a) \( u \sim_K v \) if and only if \( I_R(u) = I_R(v) \),

(b) \( u \sim_B v \) if and only if \( I_C(u) = I_C(v) \).

If \( w = a_1 a_2 \cdots a_k \) is a word, let \( Rw = a_k a_{k-1} \cdots a_1 \) be the reverse word of \( w \). Using Theorem IId, one can prove the following theorem due to Schensted [11] relating row and column insertions:

**Theorem 2.** Let \( w \) be a word in \( A^* \), and \( Rw \) its reverse word. Then:
Given a biword \((q^i_p) = (q_1, q_2, \ldots, q_k)\) in either lexicographic or locally reverse lexicographic order, we can obtain a unique pair of tableaux of the same shape \((P, Q)\). The algorithm ENCODE that produces the pair of tableaux is called by ENCODE\((q^i_p), \emptyset, \emptyset, \text{INSERTFUNC}\). INSERTFUNC is \(R\_\text{INSERT}\) if the biword was ordered by \(<_K\), and \(C\_\text{INSERT}\) if it was ordered by \(<_B\).

\[
\begin{align*}
\text{procedure ENCODE}((q^i_p), P, Q, \text{INSERTING}) \quad & \quad \text{begin} \\
P := Q := \emptyset; \quad & \quad \text{for } i = 1 \text{ until } k \text{ do} \\
& \quad \text{begin} \\
& \quad \quad \text{INSERTFUNC}(p_i, P); \\
& \quad \quad \text{Put } q_i \text{ in the corresponding new cell in } Q; \\
& \quad \text{end}
\end{align*}
\]

The following theorem is due to Robinson, Schensted, Knuth, and Burge. For a proof, see [1].

\textbf{THEOREM 3.} Both ENCODE maps (using \(R\_\text{INSERT}\) or \(C\_\text{INSERT}\)) defined above are bijections between the set of biwords \(B\) (or the set of \(n \times n\) matrices) and the set of pairs of tableaux having the same shape and entries no larger than \(n\).

We shall use the notations \(\leftrightarrow^K\) and \(\leftrightarrow^B\) to denote the Knuth and the Burge correspondence, respectively.

\textbf{EXAMPLE 6.}

\[
\begin{align*}
(1 & 1 2 2 2 3 3) \quad \leftrightarrow^K_3 \quad 3 & 3 \\
(1 & 3 3 1 1 2 1 3) \quad \leftrightarrow^K_2 3 & 2 \quad 2 \\
& \quad 1 1 1 1 3, 1 1 1 2 3 \\
(1 & 1 1 2 2 2 3 3) \quad \leftrightarrow^n_3 & 3 \\
(3 & 3 1 2 1 1 3 1) \quad \leftrightarrow^n_2 & 2 \\
& \quad 1 1 1 1 3, 1 1 1 2 3
\end{align*}
\]

For a complete account, we shall give the procedure DECODE, the inverse of ENCODE. Given a pair of tableaux \((P, Q)\) of the same shape, and a delete function \(R\_\text{DELETE}\) or \(C\_\text{DELETE}\), DECODE\((\emptyset, P, Q, \text{DELETEFUNC}\) returns a biword \((q^i_p)\) in \(<_K\) or \(<_B\) order, respectively.
procedure DECODE((E), P, Q, DELETEFUNC)
begin
    p := q := ∅;
    while Q ≠ ∅ do
        begin
            Let (i, j) be indices so that q_{ij} is maximal in Q and is also lowest and furthest to the right;
            Prepend q_{ij} to q, and delete it from Q;
            i = DELETEFUNC(i, j, p_{ij}, P, TRUE);
            Prepend t to p;
        end
    end
See [6] for a proof that this is the inverse of ENCODE.

EXAMPLE 7. Using R - DELETE:

\[
\begin{array}{cccc}
4 & 4 \\
3 & 3 & \rightarrow & 4 & 3 & (4) \\
1 & 2 & 1 & 3 & 13 & 13 & (2)
\end{array}
\]

\[
\rightarrow 4 & 3 & (3 & 4) & \rightarrow & 4 & , & (3 & 3 & 4) & \rightarrow & (1 & 3 & 3 & 4) \\
1 & 1 & (3 & 2) & & (1 & 3 & 2) & & (4 & 1 & 3 & 2)
\]

III. THE EVACUATION OPERATOR AND THE DIHEDRAL SYMMETRIES

We recall that A is the alphabet \{1, 2, ..., n\}. Let T be a tableau with entries from A. We define the tableau words of T as:

DEFINITION 1.

(a) The Knuth tableau word \(w_K(T)\) is obtained by reading \(T\) row by row from the top down, and left to right.

(b) The Burge tableau word \(w_B(T)\) is simply the reverse of \(w_K(T)\), i.e., \(w_B(T) = R(w_K(T))\).

EXAMPLE 1.

\[
\begin{array}{cccc}
4 & 4 & \rightarrow & 4 & 3 & 1 & 1 & 1 \\
T = 2 & 3 & 3 & & & & & \\
1 & 1 & 1 & \leftarrow & 1 & 1 & 3 & 3 & 2 & 4
\end{array}
\]

LEMMA 1. Let \(T\) be a tableau. Then \(I_K(w_K(T)) = I_C(w_B(T)) = T\).
Proof. The fact that \( I_R(w_k(T)) = T \) is left as an exercise. By the definition of \( w_B \) and Theorem II.2, we have \( I_C(w_B(T)) = I_C(Rw_k(T)) = I_R(w_k(T)) \).

1. The Evacuation Operator

Definition 2. Let \( w = a_1a_2 \cdots a_k \in A^* \). We define the evacuation operator \( E: A^* \rightarrow A^* \) as

\[
Ew = (n + 1 - a_k)(n + 1 - a_{k-1}) \cdots (n + 1 - a_1).
\]

The following result can be directly verified using the elementary Knuth or Burge exchanges:

Lemma 2. Let \( u, v \) be words in \( A^* \). Then:

(a) \( a \approx_K v \) if and only if \( Eu \approx_K Ev \).

(b) \( u \approx_B v \) if and only if \( Eu \approx_B Ev \).

If \( T \) is a tableau, by Theorem II.2, \( I_R(Ew_k(T)) = I_C(Ew_B(T)) \). So we can define

Definition 3. Let \( T \) be a tableau. Define

\[
ET = I_R(Ew_k(T)) = I_C(Ew_B(T)).
\]

Theorem 3. Let \( T \) be a tableau. Then, \( E^2T = T \).

Proof. First, we have \( w_k(ET) = w_k(I_R(Ew_k(T))) \approx_K Ew_k(T) \) by the definition of \( E \) and Theorem II.1. Now using Lemma III.2, we have

\[
E^2T = I_R(Ew_k(ET)) = I_R E^2w_k(T) = T.
\]

The last equality follows from Lemma III.1.

Example 2. Let \( A = \{1, 2, 3, 4\} \),

\[
T = \begin{array}{ccc}
2 & 3 & 3 \\
1 & 1 & 1
\end{array}
\quad \text{and} \quad w_k(T) = \begin{array}{cccccc}
4 & 2 & 3 & 3 & 1 & 1 \\
1 & 1 & 1
\end{array}
\]

So

\[
Ew_k(T) = \begin{array}{ccccccc}
4 & 4 & 4 & 2 & 2 & 3 & 1 \\
1 & 2 & 3
\end{array}
\quad \text{and} \quad ET = \begin{array}{ccccccc}
4 & 4 & 4 & 2 & 2 & 3 & 1 \\
1 & 2 & 3
\end{array}
\]
The fact that $T$ and $ET$ have the same shape in the above example is not a mere coincidence as we shall see later.

2. The Dihedral Symmetries

Let $M$ be an $n \times n$ nonnegative matrix, Let $\tau$ denote reflexion around the main diagonal (transposition), and $\rho$ denote rotation by $90^\circ$ counter-clockwise. Let $(\frac{q}{p})$ and $(\frac{q'}{p'})$ be the Knuth biwords of $M$ and $\tau M$, respectively. Note since $\tau$ interchanges the rows and columns of $M$ the elements of $p$ and $q$ are rearranged to form $q'$ and $p'$, respectively. We have

**Lemma 4.**

In the Knuth correspondence:

(a) $M \leftrightarrow (\frac{q}{p})$,
(b) $\tau M \leftrightarrow (\frac{q'}{p'})$ (Ref. around the main diagonal),
(c) $\rho^2 M \leftrightarrow (\frac{E q}{E p})$ (Rot. by $180^\circ$),
(d) $\tau \rho^2 M \leftrightarrow (\frac{E q'}{E p'})$ (Ref. around the off-diagonal, i.e., the diagonal running from lower left corner to upper right corner.)

In the Burge correspondence:

(e) $\rho M \leftrightarrow (\frac{E q'}{R p'})$ (Rot. by $90^\circ$),
(f) $\rho \tau M \leftrightarrow (\frac{E q}{R p})$ (Ref. about the horizontal midline),
(g) $\tau \rho M \leftrightarrow (\frac{E q'}{R p'})$ (Ref. about the vertical midline),
(h) $\rho^3 M \leftrightarrow (\frac{R q'}{E p'})$ (Rot. by $270^\circ$).

**Proof.** Follow directly from considerations of matrix entries. See also Example 3 below.

We now have

**Theorem 5.** Let $M$ be any nonnegative integral $n \times n$ matrix. Then the following are equivalent. ($P$ and $Q$ are tableau of the same shape.)

In the Knuth correspondence:

(a) $M \leftrightarrow (P, Q)$,
(b) $\tau M \leftrightarrow (Q, P)$ (Ref. around the main diagonal),
(c) $\rho^2 M \leftrightarrow (E P, E Q)$ (Rot. by $180^\circ$),
(d) $\tau \rho^2 M \leftrightarrow (E Q, E P)$ (Ref. around the off-diagonal).
In the Burge correspondence:

(e) \( \rho M \leftrightarrow (Q, EP) \) (Rot. by 90°),

(f) \( \rho \tau M \leftrightarrow (P, EQ) \) (Ref. around the horizontal midline),

(g) \( \tau \rho M \leftrightarrow (EP, Q) \) (Ref. around the vertical midline),

(h) \( \rho^3 M \leftrightarrow (EQ, P) \) (Rot. by 270°).

Proof. We will show that (b), (c), (d) are equivalent to (a).

(a) \( \Leftrightarrow \) (b) is due to Schensted, a proof can be found in [11]. Now, from Lemmas III.2 and III.4, we have that \( \rho^2 M \leftrightarrow^K (EP, T_1) \) and \( \tau \rho^2 M \leftrightarrow^K (EQ, T_2) \). Using the fact that \( \tau^2 \) is the identity, we have \( \tau^2 \rho^2 M = \rho^2 M \). So \( (T_2, EQ) = (EP, T_1) \). Hence, \( T_1 = EP \), and \( T_2 = EP \), showing (c) and (d) are equivalent to (a).

Now, appealing to Theorem II.2 and Lemma III.2, in the Burge correspondence, we have \( \rho M \leftrightarrow^B (Q, T_1) \), \( \rho \tau M \leftrightarrow^B (P, T_2) \), \( \tau \rho M \leftrightarrow^B (EP, T_3) \), and \( \rho^3 M \leftrightarrow^B (EQ, T_4) \), where the \( T_i \)'s tableaux remain to be determined. Schensted's result cited above also hold for the Burge correspondence. Using this, fact, and the fact that \( \tau \rho \tau = \rho^3 \), we see that the following diagrams commute:

\[
\begin{align*}
\tau \rho M & = \rho M \\
(T_3, EP) & = (Q, T_1) \\
\tau \rho M & = \rho^3 M \\
(T_2, P) & = (EQ, T_4)
\end{align*}
\]

Thus \( T_3 = Q \), \( T_2 = EQ \), \( T_1 = EP \), and \( T_4 = P \), proving (e), (f), (g), (h), are equivalent to (a).

Example 3.

In the Knuth correspondence:

\[
\begin{array}{cccccccc}
1 & 0 & 2 & & & & & \\
M = 2 & 1 & 0 & \leftrightarrow & \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \end{pmatrix} & \leftrightarrow & 3 & 3 \\
1 & 0 & 1 & & & & & \\
1 & 2 & 1 & \leftrightarrow & \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 3 & 3 \end{pmatrix} & \leftrightarrow & 3 & 3 \\
\tau M = 0 & 1 & 0 & \leftrightarrow & \begin{pmatrix} 1 & 2 & 2 & 3 & 2 & 1 & 1 & 3 \end{pmatrix} & \leftrightarrow & 2 & 2 \\
2 & 0 & 1 & & & & & \\
\rho^2 M = 0 & 1 & 2 & \leftrightarrow & \begin{pmatrix} 1 & 3 & 2 & 3 & 3 & 1 & 1 & 3 \end{pmatrix} & \leftrightarrow & 2 & 3 \\
2 & 0 & 1 & & & & & \\
\end{array}
\]
In the Burge correspondence:

\[
\begin{align*}
&\tau \rho^3 M = 0 \quad 1 \quad 0 \leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ 1 & 3 & 2 & 1 & 2 & 2 & 3 \end{pmatrix} \leftrightarrow 2 \quad 3 \\
&1 \quad 2 \quad 1 \\
&1 \quad 2 \quad 1 \quad 2 \quad 2 \quad 3, \quad 1 \quad 1 \quad 1 \quad 3 \quad 3
\end{align*}
\]

Of course, we also have the analog for the Burge correspondence of \( M \):

**Theorem 6.** Let \( M \) be an \( n \times n \) nonnegative integral matrix. Then the following are equivalent.

In the Burge correspondence:

(a) \( M \leftrightarrow (R, S) \),
(b) \( \tau M \leftrightarrow (S, R) \) (Ref. about the main diagonal),
(c) \( \rho^2 M \leftrightarrow (ER, ES) \) (Rot. by \( 90^\circ \)),
(d) \( \tau \rho^2 M \leftrightarrow (ES, ER) \) (Ref. about the off diagonal).

In the Knuth correspondence:

(e) \( \rho M \leftrightarrow (S, ER) \) (Rot. by \( 90^\circ \)),
(f) \( \rho \tau M \leftrightarrow (R, ES) \) (Ref. about the horizontal midline),
(g) \( \tau \rho M \leftrightarrow (ER, S) \) (Ref. about the vertical midline),
(h) \( \tau \rho^3 M \leftrightarrow (ES, R) \) (Rot. by \( 270^\circ \)).

**Proof.** Similar so that of Theorem 5.

Theorems 5 and 6 are due to Gansner [3]. The idea of using the actions of the dihedral symmetries to shorten the proofs is due to Baclawski [1].
3. Applications

So far, we have not mentioned explicitly that the evacuation operator depends on a specific number $n$ as in Definition III.1. From now on, we shall write $E_n$ to emphasize this fact.

**Corollary 7.** Let $T$ be a tableau. Then $T$ and $E_nT$ have the same shape.

**Proof.** $(T, T)$ corresponds to a symmetric matrix $M$ as in (b) of Theorem 5. By (e) of the same theorem, $(T, ET)$ corresponds to $\rho M$ (in the Burge correspondence). Thus, $T$ and $ET$ have the same shape by Theorem II.3.

In the remainder of this section, we shall let the alphabet $A$ be any totally ordered set, finite or infinite. It is well known that $A^*$ is in 1–1 correspondence with the set of pairs of tableaux of the same shape so that the first tableau has entries in $A$, and the second tableau is standard. This can be seen most easily as follows. Let $w = a_1a_2 \cdots a_k \in A^*$, and $q = 1, 2, \ldots, k$. Now consider the biword $(\,^q\!\!w\,)$ and use ENCODE with $R_\text{INSERT}$ to get the pair of tableaux $(T(w), Q)$. This is usually known as row insertion. Alternatively, one can consider the biword $(\,^q\!\!w_{\text{R}}\,)$ and use $C_\text{INSERT}$ with ENCODE to get $(T(w), Q')$. This is column insertion. The relationship of $Q$ and $Q'$ is remarkable.

**Theorem 8.** Let $w = a_1 \cdots a_k$, $q = 1, 2, \ldots, k$, $(T(w), Q)$ be the row insertion of $(\,^q\!\!w\,)$, and $(T(w, Q'))$ be the column insertion of $(\,^q\!\!w_{\text{R}}\,)$, then $Q' = E_kQ$.

**Proof.** Let $D$ be the distinguishing operator that assigns to each letter $a_i$ a number $d_i$ from $\{1, 2, \ldots, k\}$ so that $d_i < d_j$ whenever $a_i < a_j$ or $a_i = a_j$ and $i < j$. Now, in the biwords, replace $w$ by $d_1d_2 \cdots d_k$. It is well known that $D$ commutes with row (column) insertion (for example, see [1]). Thus, we can consider $(\,^q\!\!w_{\text{D}}\,)$ as the Knuth biword of a $k \times k$ permutation matrix $M$, and $(\,^q\!\!w_{\text{R}}\,)$ as the Burge biword of $\rho M$. The result now follows easily from Theorem 5.

Let $w \in A^*$, we shall write $|w|_a$ for the number of times the letter $a$ appears in $w$. A word $w$ is called a lattice permutation if for any initial subword (prefix) $u$ of $w$, i.e., $w = uv$ for some $v$, we have $|u|_a \geq |u|_b$ whenever $a \leq b$. A word $w$ is Yamanouchi if its reversal $Rw$ is a lattice permutation. A tableau $T$ is Yamanouchi if its Knuth tableau word is Yamanouchi. It is easy to see that in this case, the tableau must consist of rows having the same letters. Now if $w$ is a lattice permutation with letters consisting of $\{a_1 < a_2 < \cdots < a_k\}$, we define the position tableau $Po$ of $w$ as the tableau having in the $i$th row the positions of $a_i$ in $w$. 
Example 4.

\[ w = 1 \quad 2 \quad 1 \quad 1 \quad 3 \quad 2 \quad 1 \quad 3 \quad 2 \leftarrow P = 2 \quad 6 \quad 9 \]
\[ 1 \quad 3 \quad 4 \quad 7 \]

Lemma 9. Let Po be a position tableau of the lattice permutation w. Then Po is a standard tableau. In fact, Po is the right tableau obtained by column insertion of the lattice permutation w.

Proof. In fact, without any more effort, we can show that \( I_C(w) \) is a Yamanouchi tableau. Note that the word \( w' \) obtained by removing the last letter of w is also a lattice permutation. So we can assume by induction that \( I_C(w') \) is Yamanouchi and Po' is the position tableau of \( w' \) is also the right tableau. A moment reflection shows that if the last letter of w is \( a_i \), when column inserted, it must go into the i-th row. The lattice permutation condition implies that all bumped elements are from the same row, and stay in the same row. Thus the position of \( a_i \) is recorded correctly in Po' to form Po.

Corollary 10. Let \( y = y_1y_2 \cdots y_k \) be a Yamanouchi word. Let P' be the position tableau of Ry. Let \( (I_R(y), Q) \) be the pair of tableaux obtained by row inserting y. Then \( I_R(y) \) is Yamanouchi and \( Q = E_k(P_o) \).

Proof. The fact that \( I_R(y) \) is Yamanouchi follows from the above proof and Theorem II.2. The fact that \( Q = E_k(P_o) \) is a result from Lemma 9 and Theorem III.8.

We now study the fixed of evacuation. Again, let \( A = \{1, 2, \ldots, n\} \). A word \( w = a_1a_2 \cdots a_k \in A^* \) is said to be invariant if \( E_n w = w \). That is, \( a_1 \cdots a_k = (n+1-a_k)(n+1-a_{k-1}) \cdots (n+1-a_1) \). As a result of Lemma III.2:

Lemma 11. Let T be a tableau. Then, \( E_n T = T \) if and only if \( w_K(T)(w_B(T)) \) is Knuth (Burge) equivalent to some invariant word. In this case, we say that T is invariant.

Corollary 12. \( E_n T = T \) if and only if the Knuth (Burge) biword corresponding to \((T, T)\) has an invariant lower word.

Proof. Let \( \left( \begin{array}{c} q \\ p \end{array} \right) \) and M be the Knuth biword, and the \( n \times n \) matrix corresponding to \((T, T)\). If \( E_n T = T \), then \((T, T) = (E_n T, E_n T)\). So M is fixed under reflexion through the off-diagonal, and rotation by 180°. Lemma III.4 gives directly that \( E_n p = p \). The converse follows directly from Lemma 11.
Specialization of the Knuth correspondence to permutation matrices gives us the familiar result that every permutation corresponds uniquely to a pair of standard tableaux of the same shape. Theorem III.5 shows that every standard tableau corresponds to an involution. The above result shows that every invariant standard tableau corresponds to an involution \( \sigma \) with an extra symmetry, namely, \( \sigma(i) = j \) iff \( \sigma(n + 1 - i) = n + 1 - j \). Let \( S_n \) be the set of permutations of \( n \) elements. We have

**Corollary 13.** There is a bijection between the sets

\[
I_n = \{ \sigma \in S_n : \sigma(i) = j \text{ iff } \sigma(j) = i \text{ and } \sigma(n + 1 - i) = n + 1 - j \},
\]

\[
J_n = \{ T : \text{invariant standard tableau with } n \text{ entries} \}.
\]

Let \( j_n = |J_n| \). The proof of the following facts will be left as an exercise.

**Corollary 14.**

(a) \( j_{2n} = j_{2n+1} \) for \( n \geq 0 \).

It can be seen directly that \( j_0 = j_1 = 1 \), and \( j_2 = 2 \). For \( n \geq 2 \), we have

(b) \( j_{2n} = 2j_{2n-2} + (2n - 2)j_{2n-4} \),

(c) \( j_{2n} = 2^n + 2^{n-1} \binom{n}{2} + \sum_{i=2}^{\lfloor n/2 \rfloor} 2^{n-i-1} \binom{n}{i} (2i - 1)!/(i - 1)! \).

**IV. The Hillman-Grassl Correspondence**

Let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k) \) be a partition of some integer \( n \). The set of cells of \( \lambda \) is defined by

\[
C(\lambda) = \{(i, j) : 1 \leq i \leq k \text{ and } 1 \leq j \leq \lambda_{k-i+1} \}.
\]

Note that the cells are now defined from top down, and left to right (as the convention for reading entries of a matrix), not as the convention set forth in the Introduction. It is important to note that the rows of tabloids and reverse plane partitions are numbered from the top down, and the rows of tableau from the bottom up.

**Example 1.** Let \( \lambda = (3, 3, 2) \):

\[
\begin{array}{ccc}
1 & * & * \\
2 & * & * & * \\
3 & * & * & * \\
\end{array}
\]

\[
C(\lambda) = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}.
\]
Let $T(\lambda)$ be the set of tabloids with shape $\lambda$. If $T \in T(\lambda)$, we let $T_{ij}$ be the entry in cell $(i, j)$. Let $R(\lambda)$ be the set of reverse plane partitions with shape $\lambda$. We denote by $\emptyset$ the tabloid and reverse plane partitions of any shape that is identically zero. The Hillman–Grassl correspondence is a bijection between $R(\lambda)$ and $T(\lambda)$. At the foundation of the correspondence are two path finding algorithms that define paths in a RPP (reverse plane partition) such that after subtracting 1 from each element of a path, we again have a RPP. We shall now describe both algorithms.

**procedure DOWNPATH**($R$, $P$) 
begin 
$P := \emptyset$; 
$i := 1$; 
Let $j$ be the least index so that $R_{1j} \neq 0$; 
while $(i, j) \in C(\lambda)$ do 
begin 
Add $(i, j)$ to $P$; 
if $R_{i+1,j} = R_{ij}$ then $i := j + 1$; 
end 
end 

**procedure UPPATH**($R$, $P$) 
begin 
$P := \emptyset$; 
Let $(i, j)$ be the lexicographically largest cell of $R$ so that $R_{ij} \neq 0$; 
while $(i, j) \in C(\lambda)$ do 
begin 
Add $(i, j)$ to $P$; 
if $R_{i,j-1} = R_{ij}$ then $j := j - 1$; else $i := i - 1$; 
end 
end 

**Example 2.** Let 

$$ R = \begin{pmatrix} 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 2 \end{pmatrix} $$

Let $PD(R)$ and $PU(R)$ be the down path and the up path, respectively. Then, we have 

$$ PD(R) = \{(1, 1), (1, 2), (2, 2), (2, 3)\} $$

and 

$$ PU(R) = \{(3, 3), (3, 2), (2, 2), (1, 2)\}. $$
The Hillman–Grassl correspondence is given below.

\begin{verbatim}
procedure Hillman–Grassl(R, T)
begin
    T := ∅;
    while R ≠ ∅ do
        begin
            PATHFUNC(R, P);
            Subtract 1 from R_{ij} for each (i, j) in P;
            m := max\{i: (i, j) ∈ P\};
            n := min\{j: (i, j) ∈ P\};
            T_{mn} := T_{mn} + 1;
        end
end
\end{verbatim}

Here, PATHFUNC could be either DOWNPATH or UPPATH. Let \( HG_D : R(\lambda) \rightarrow T(\lambda) \) and \( HG_U : R(\lambda) \rightarrow T(\lambda) \) be the functions defined by using DOWNPATH and UPPATH, respectively. The fact that \( HG_D \) is a bijection was proved in [4]. Later we shall see that \( HG_D = HG_U \).

**Example 3.** Let \( R \) be as in Example 2. The following is the sequence of constructions of \( T \) by \( HG_D \):

\begin{align*}
R & \quad T & \quad R & \quad T & \quad R & \quad T \\
2 & 3 & 0 & 0 & \overset{1}{1} & 2 & 0 & 0 & 2 & 0 & 0 \\
1 & 3 & 3 & 0 & 0 & 0 & \rightarrow & 1 & 2 & 2 & 1 & 0 & 0 & \rightarrow & 0 & 2 & 2 & 1 & 0 & 0 \\
1 & 2 & 2 & , & 0 & 0 & 0 & \overset{1}{1} & 2 & 2 & , & 0 & 0 & 0 & 1 & 1 & , & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\rightarrow & 0 & 1 & 1 & 1 & 0 & \rightarrow & 0 & 0 & 1 & 1 & 0 & \rightarrow & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0, & 1 & 1 & 0
\end{align*}

Now, the sequence of constructions of \( T \) by \( HG_U \):

\begin{align*}
R & \quad T & \quad R & \quad T & \quad R & \quad R \\
2 & 3 & 0 & 0 & 2 & 2 & 0 & 0 & \overset{1}{1} & 2 & 0 & 0 \\
1 & 3 & 2 & 0 & 0 & 0 & \rightarrow & 1 & 2 & 3 & 0 & 0 & 0 & \rightarrow & 0 & 2 & 3 & 0 & 0 & 0 \\
1 & 2 & 2 & , & 0 & 0 & 0 & 1 & 1 & 1, & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\rightarrow & 0 & 2 & 2 & 0 & 0 & 1 & \rightarrow & 0 & 1 & 1 & 0 & 1 & 1 & \rightarrow & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0, & 1 & 1 & 0 & 0 & 0 & 0, & 1 & 1 & 0 & 0 & 0, & 1 & 1 & 0
\end{align*}
Before showing that $HG_P = HG_U$, we need a few technical lemmas. Let $R \in R(\lambda)$ and $T \in T(\lambda)$. Let $DP(R, T)$ be the pair of tabloids $(R', T')$ obtained by applying one pass of the "while" loop in the HILLMAN-GRASSL procedure using DOWNPATH. Similarly, we define $UP(R, T)$. We have the following interesting fact:

**Lemma 1.** $UP(DP(R, T)) = DP(UP(R, T))$.

*Proof.* Let $PD$ and $PU$ be the down path and the up path in $R$. We observe the following facts:

(a) $PD$ starts in a column to the left of the column that $PU$ ends in.

(b) $PU$ starts in a row below the row that $PD$ ends in.

These are true since $PD$ starts in the leftmost nonzero column and $PU$ starts in the bottommost nonzero row. Now we claim:

(c) The intersection of $PD$ and $PU$ is either just one cell or a connected path segment in $R$.

To see this, let $a$, $b$ be in the intersection of $PD$ and $PU$.

Consider first the situation in Fig. 1(i). From the construction of $PU$, we must have $a < b$. On the other hand, from the construction of $PD$, we have $b < a$. This means that the path segment between $a$ and $b$ of $PU$ must consist of all $a$'s. This, in turn, implies that $PD$ must be the same as $PU$ in this segment. So this situation cannot happen. Similarly, we dispose of the situation in Fig. 20. This shows (c).

Now let $D$ be the down path in $R_U$, where $(R_U, T_U) = UP(R, T)$, and $U$ be the up path in $R_D$, where $(R_D, T_D) = DP(R, T)$. Note that from start to end, $D$ and $PD$ do not differ until they reach the intersection of $PD$ and $PU$. By (c), we have two cases:

(i) $PD$ and $PU$ intersect at 1 cell (Fig. 2(a)): Since $PD$ moves to the right, we have $a > b$. Then, $a - 1 > b - 1$, so $D$ will move to the right at this point even after applying $UP$. Clearly, $D$ and $PD$ agree after this point. So $D = PD$. Similarly, we show that $U = PU$.

![Figure 1](attachment:image.png)
Suppose $D \neq PD$. Let $D' = D - PD$. It is clear that $D'$ must start on the intersection of $PD$ and $PU$. So, the starting point of $D'$ must be $a - 1$. A moments consideration of Fig. 2b shows that all elements of $D'$ must be $a - 1$. Further, the end point of $D'$ must be on the intersection of $PD$ and $PU$. Otherwise, at the intersection of $D'$ and $PU$, the value will be $a - 1$. This implies that in the construction of $PU$, we would have used $D'$, a contradiction. So $D = PD$. Now let $U' = U - PU$. Using the same type of arguments we have that $U = PU$. It is easy to see now that the elements of $R$ affected by $UP \ast DP$ are the same as those affected by $DP \ast UP$. This proves the lemma.

**Lemma 2.** Let $n$ be the least integer such that $DP^n(R, \phi) = (\emptyset, T_n)$, where $\emptyset$ is the zero tabloid. Then,

$$DP^n(R, T) = UP(DP^{n-1}(R, T)).$$

**Proof.** Let $(R_{n-1}, T_{n-1}) = DP^{n-1}(R, \emptyset)$. Then $R_{n-1}$ must be

$$\begin{align*} 000...0111...1 \\
0.........01 \\
0.........01 \\
\vdots \\
0.........11...1 \\
0.........01 \\
\end{align*}$$

For this type of reverse plane partition, clearly, $DP = UP$. 

\[Figure 2\]
We can now prove

**Theorem 3.** $HG_u(R) = HG_d(R)$ for all $R \in R(\lambda)$.

**Proof.** We have $HG_d(R) = T_n$ (with $n$ as in Lemma 2). But

$$DP^n(R, \emptyset) = UP(DP^{n-1}(R, \emptyset))$$

by Lemma 2

$$= DP^{n-1}(UP(R, \emptyset))$$

by Lemma 1.

Now apply induction, we have that $DP^n(R, \emptyset) = UP^n(R, \emptyset)$.

Thus, we have two different ways to produce the Hillman-Grassl correspondence. In Section V we shall use $HG_u$ and its inverse $HG_u^{-1}: T(\lambda) \rightarrow R(\lambda)$. First, we need to define the inverse of UPPATH. From a suitable starting position $(i, j)$ in a RPP, INV = UPPATH finds a path so that adding $a$ to the entries of the path, we again obtain a RPP.

procedure INV = UPPATH($i, j, e, R, P$)
begin
$P := \emptyset$;
while $(i, j) \in C(\lambda)$ and $i \leq e$ do
begin
Add$(i, j)$ to $P$;
if $R_{i,j+1} = R_{ij}$ then $j := j + 1$; else $i := i + 1$;
end
end

The corresponding function INV = DOWNPATH for DOWNPATH will be left as an exercise. The inverse of $HG_u$ is defined as:

procedure INV = HILLMAN = GRASSL($T, R$)
begin
$R := \emptyset$;
while $T \neq \emptyset$ do
begin
Let $e$ be the least integer so that row $e$
of $T \neq 0$;
Let $j$ be the least integer so that $T_{ej} \neq 0$;
Let $i$ be the least integer so that $(i, j) \in C(\lambda)$;
INV = UPPATH($i, j, e, R, P$);
for each $(m, n)$ in $P$ do $R_{mn} := R_{mn} + 1$;
$T_{ej} := T_{ej} - 1$;
end
end
The proof that this procedure defines $HG_U^{-1}$ as well as the proof of the elementary properties in Lemma 4 of the Hillman–Grassl correspondence will be left as exercises.

From now on, we shall use $HG$ and $HG^{-1}$ to denote $HG_U$ and its inverse, respectively. The off-diagonal is the diagonal running from the lower to the upper right corner of the matrix.

**Lemma 4.** (a) Let $R \in R(\lambda)$, and let $\tau p^2 R$ be the transpose of $R$ around the off-diagonal. Then, $HG(R) = T$ iff $HG(\tau p^2 R) = \tau p^2 T$, where $\tau p^2 T$ is defined in the same way as $\tau p^2 R$.

(b) Let $R_C(\lambda) = \{ R \in R(\lambda): R$ is column strict with positive entries$\}$ and $T_C(\lambda) = \{ T \in T(\lambda): T_{ij} \geq 1$ for $(i, j)$ above or on the off-diagonal of $T$$. Then $HG: R_C(\lambda) \rightarrow T_C(\lambda)$ is a bijection.

(c) A similar result holds for two strict RPPs and tabloids with nonzero entry on or below the off diagonal. Further, $HG$ is a bijection between on the set of RPPs that are both row and column strict, and the set of tabloids with nonzero entries.

V. THE EQUIVALENCE OF HILLMAN–GRASSL AND ROW INSERTION

In this section we shall use the shape $\lambda = (n, n, ..., n) \rightarrow n^2$. Given a $n \times n$ nonnegative integral matrix $M$, we think of $M$ as a tabloid in $T(\lambda)$. Thus, applying the inverse Hillman–Grassl correspondence to $M$, we obtain a reverse plane partition of shape $\lambda$, $HG^{-1}(M)$. From Section II, using row insertion, $M$ also corresponds to a pair of tableaux $(P, Q)$ of the same shape, with entries no larger than $n$. Now, there is a correspondence between such pairs of tableaux and reverse plane partitions due to Frobenius. The natural question to ask is whether there is any relations between the reverse plane partitions. The first is to describe the Frobenius correspondence.

Let $\lambda = (n, n, ..., n) \rightarrow n^2$. So the Ferrers diagram of $\lambda$ is an $n \times n$ square. For $k = 1, 2, ..., 2n - 1$, let $D_k = \{ (i, j) \in C(\lambda): i + j = k + 1 \}$ be the $k$th diagonal of $\lambda$. The elements of each $D_k$ are ordered lexicographically.

**Example 1.** Let

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & * & * \\
\lambda = (3, 3, 3) = 2 & * & * \\
3 & * & *
\end{array}
\]
Then,

\[ D_1 = \{(1, 1)\}, \]
\[ D_2 = \{(1, 2), (2, 1)\}, \]
\[ D_3 = \{(1, 3), (2, 2), (3, 1)\}, \]
\[ D_4 = \{(2, 3), (3, 2)\}, \]
\[ D_5 = \{(3, 3)\}. \]

Now let \((P, Q)\) be a pair of tableaux, both of shape \(\lambda\), with entries from \(\{1, 2, \ldots, n\}\). For \(1 \leq k \leq n\), let \(\mu_k\) and \(\mu'_k\) be the shapes of \(P\) and \(Q\) restricted to entries from \(\{1, 2, \ldots, k\}\), respectively. The Frobenius function \(F: (P, Q) \rightarrow R\) is defined as follows. For \(1 \leq k \leq n\), fill the first cell of \(D_k\) with the last element of \(\mu_k\), the second cell of \(D_k\) with the next to last element of \(\mu_k\), etc., any unfilled cell is filled with 0. At the same time, \(D_{n-k}\) is filled with elements of \(\mu'_k\) in the same way.

**Example 2.** Let

\[
(P, Q) = \begin{array}{cccc}
3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 \\
1 & 1 & 2 & 2 \\
1 & 1 & 1 & 3 \\
\end{array}
\]

Then, \(\mu_1 = (2)\), \(\mu_2 = (2, 4)\), \(\mu_3 = (2, 3, 4)\), and \(\mu'_1 = (3)\), \(\mu'_2 = (3, 3)\), \(\mu'_3 = (2, 3, 4)\). So

\[
F(P, Q) = \begin{array}{ccc}
2 & 4 & 4 \\
2 & 3 & 3 \\
\end{array}
\]

It is not too hard to see that \(F\), so defined, is a bijection.

From now on, unless otherwise specified, \(HG\) and \(HG^{-1}\) will denote \(HG_U\) and \(HG_U^{-1}\), respectively. Let \(M\) be an \(n \times n\) matrix. Let \(R_0\) be the zero tabloid with shape \(\lambda\). Let \(UP^{-1}(M, R_0)\) be the pair of tabloids obtained after one pass in the 'while' loop of INV _HILLMAN _GRASSL. We write \((M_1, R_1) = UP^{-1}(M, R_0)\). Similarly, let \((M_k, R_k) = UP^{-k}(M, R_0)\) for \(k \geq 1\). Let \(r\) be the least integer so that \(M_r = 0\). Clearly, \(R_r = HG^{-1}(M)\). It is also clear that \(r\) is the sum of the entries of \(M\). Let \((q_p)\) be the Knuth biword of \(M\). We have the following remarkable fact:

**Lemma 1.** For \(1 \leq k \leq r\), the upper left half of \(R_k\) corresponds to the tableau \(P_k = I_{R}(p_1, p_2, \ldots, p_k)\) under the Frobenius correspondence.
Note that this says that the Hillman–Grassl correspondence and row insertion are identical on the $P$-tableau. The information is just encoded differently. The Frobenius correspondence translates the information from one code to the other.

**Proof.** We induct on $k$. The case $k = 1$ is trivial. Now, assume the assertion for $k - 1$. Note that $(q_k, p_k)$ is the cell of the nonzero entry of $M_{k-1}$ found in the definition of the inverse of the Hillman–Grassl map. Also, the entries of the diagonal $D_i$ of $R_{k-1}$ define the shape of the tableau $P_{k-1}$ restricted to $\{1, 2, \ldots, i\}$ ($i \leq n$). Consider the intersection of the path of $UP^{-1}(M_{k-1}, R_{k-1})$ and the first two rows of $R_{k-1}$. A typical situation is depicted in Fig. 3.

Adding 1 to each element of $R_{k-1}$ along this path can be seen as inserting $p_k$ into the first row of $P_{k-1}$, and bumping $p_k + j + 1$ in the $P_{k-1}$ tableau. When $p_k + j + 1$ is inserted in the second row, we add 1, in $R_{k-1}$, to the entry just below in column $p_k + j + 1$. Now we repeat the process on the second row, then on the third row, etc. This is clearly the same as forming the inverse up path. Thus, the lemma is proved.

Let $M$ be an $n \times n$ matrix. Let $(P, Q)$ be the Knuth correspondence tableaux of $M$, denoted $M \leftrightarrow^K (P, Q)$. As a corollary to Lemma 1, we have

**Corollary 2.** Let $M \leftrightarrow^K (P, Q)$. Then $F^{-1}(HG^{-1}(M)) = (P, S)$, where $S$ is a tableau with the same shape as $P$.

Recall that $\tau p^2 M(\tau p^2 R)$ is the reflection of $M(R)$ about the off-diagonal. We have

1. $\tau p^2 M \leftrightarrow^K (E_n Q, E_n P)$,
2. $HG^{-1}(\tau p^2 M) = \tau p^2 R$ from Lemma IV.4,
3. $F^{-1}(R) = (A, B)$ iff $F^{-1}(\tau p^2 R) = (B, A)$ by the definition of the Frobenius map.

Putting all this together, we get

$$F^{-1}(HG^{-1}(\tau p^2 M)) - F^{-1}(\tau p^2 M) = (S, S').$$

![Figure 3](image-url)
Now apply Corollary 2, we get $S = E_nQ$. This proves

**Theorem 3.** Let $M \leftrightarrow^K (P, Q)$. Then, $F^{-1}(HG^{-1}(M)) = (P, EQ)$.

Although Theorem 3 shows that we get $EQ$ for the right tableau when we apply $HG^{-1}$ and $F^{-1}$ to $M$, the information for the $Q$ tableau is encoded in the reverse up paths. In fact the information is encoded in the paths in exactly the same way that it is encoded in the $Q$ tableau during row insertion. This will complete the demonstration that row insertion and Hillman–Grassl are the same process.

We remind the reader of the general step in producing the $Q$ tableau. We have already inserted $(P_1, \ldots, P_{k-1}, q_{k-1})$ to get $(P_{k-1}, Q_{k-1})$. Now insert $p_k$ into the tableau $P_{k-1}$ to get $P_k$. In the process of bumping, one row of $P_{k-1}$ is increased in length by 1. At the end of the corresponding row of $Q_{k-1}$ we place $q_k$ to produce $Q_k$. Under the Frobenius map $F$ the row of $P_{k-1}$ that increases in length is the row that the reverse up path intersects the off-diagonal of $R$. Also $q_k$ is the row that the reverse up path ends on. This gives

**Theorem 4.** Let $M, (P, Q), R$ be as above. Let $RP_i (i = 1, 2, ..., r)$ be the reverse up paths used during the construction of $HG^{-1}(M)$. Let $r_i$ be the row that $RP_i$ intersects the off-diagonal of $R$, and let $q_i$ be the row of $R$ that $RP_i$ ends in. Then, by putting $q_i$ at the end of row $r_i$ ($i = 1, 2, ..., r$) we construct the $Q$ tableau in exactly the same steps as we construct it during row insertion.

**Example 3.**

\[
\begin{array}{c}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{array}
\]

Let $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, its Knuth biword is $w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 2 & 3 & 1 & 2 \end{pmatrix}$.

We shall apply $HG^{-1}$ to $M$. At each step the uppaths are underlined, and the inverse Frobenius map is applied. Also the $r$'s and $q$'s in Theorem 4 are used to construct the $Q$ tableau. Finally both $P$ and $Q$ are constructed by row insertion. The bumping paths are also underlined in $P$. The interested reader will find it instructive to relate the uppaths in $R$ and the bumping paths in $P$. 

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VI. Applications

1. The Schutzenberger Operator

Let $A = \{1, 2, \ldots, n\}$ be as in Section II. Let $T$ a tableau with entries in $A$. The Schutzenberger operator $S_n$ maps $T$ to another tableau $S_nT$ with the same shape. In this section, we shall show that $S_n$ is the same as $E_n$. A basic move in $S_n$ makes the least element in the current tableau largest then bubbles it up. This will be made precise below. To facilitate description of the algorithms, all elements outside of a tableau are assumed to have infinite value. If $a$ is an element in the tableau, $u(a)$, $r(a)$ will denote the element directly above or to the right of $a$, respectively.
procedure BUBBLE_UP(T)
begin
  Let $a$ be the element in the lower left corner of $T$;
  $a := n + 1 - a$;
  while $u(a) \neq \infty$ or $r(a) \neq \infty$ do
  begin
    if $u(a) > r(a)$ then exchange $a$ and $r(a)$;
    else exchange $a$ and $u(a)$;
  end
end

Now, the Schützenberger operator:

procedure SCHUTZENBERGER(T)
begin
  while $T \neq \emptyset$ do
  begin
    BUBBLE_UP(T);
    $T := T -$ the corner cell that the bubbled-up
    element ends in;
  end
end

EXAMPLE 1. Let

$$
\begin{align*}
3 \\
T &= 2 \ 3 \\
1 &\ 1 \ 2
\end{align*}
$$

then $S_3 T$ is formed as follows:

$$
\begin{array}{ccccccc}
3 & 3 & 3 & 3 & 3 & 3 \\
2 & 3 & \rightarrow & 2 & 3 & \rightarrow & 2 & 3 & \rightarrow & 2 & 3 & \rightarrow & 2 & 3 & \rightarrow \\
1 & 1 & 2 & 3 & 1 & 2 & 1 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 3 \\
3 & 3 & \rightarrow & 3 & 3 & \rightarrow & 3 & 3 & \rightarrow & 3 & 3 & \rightarrow & 3 & 2 & \rightarrow \\
2 & 2 & 3 & 2 & 2 & 3 & 2 & 2 & 3 & 2 & 2 & 3 & 2 & 3 & 3 \\
3 & 3 & \rightarrow & 3 & 3 & \rightarrow & 3 & 3 & \rightarrow & 3 & 3 & \rightarrow & 3 & 3 & \rightarrow \\
2 & 2 & \rightarrow & 2 & 2 & \rightarrow & 2 & 2 & \rightarrow & 2 & 2 & \rightarrow & 2 & 2 & \rightarrow \\
2 & 3 & 3 & 3 & 3 & 3 & 1 & 3 & 3 & 3 & 3 & 1 & 3 & 3 & 1 & 3 & 3 & 1 & 3 & 3
\end{array}
$$

LEMMA 1. Let $M \leftrightarrow^k (P, Q)$, $R = H G^{-1}(M)$, and $P U_1, P U_2, \ldots, P U_r$, he
the uppaths in the order they are used by $HG_U$ in constructing $M$ from $R$. Restricted below the off-diagonal of $R$, these paths correspond exactly to applying the Schützenberger operator $S_n$ to $EQ$.

Proof. The proof is similar in spirit to the proof of Lemma V.1.

As a result of Theorem V.4 and Lemma 1 we have

**Theorem 2.** The Schützenberger operator $S_n$ is the same as the evacuation operator $E_n$.

Proof. In Lemma 1, to construct $S_nE_nQ$, we bubble up $q_i$ to the end of row $r_i$. By Theorem B.4, this is just $Q$. So $S_nE_nQ = Q$. So $S_n$ is the inverse of $E_n$. $E_n$ is an involution as seen in Theorem III.3, so $S_n = E_n$.

As a corollary, we have proved:

**Corollary 3.** $S_n$ is an involution.

We can now state the analogues of Theorem V.4 and Lemma 1

**Theorem 4.** Let $M \leftrightarrow^K (P, Q), HG^{-1}(M) = R$. Let $PD_1, PD_2, \ldots, PD_r$ be the downpaths in the order they are used in the map $HG: R \rightarrow M$. Restricted above the off-diagonal of $R$, these paths correspond to applying $S_n$ to $P$.

**Theorem 5.** With the same hypotheses as above, the paths $PD_i$'s restricted below the off-diagonal of $R$ correspond to row inserting the tableau $EQ$.

Proof of Theorems 4, 5.

A down path in $M$ is an uppath in $\tau p^2M$. Thus, applying Lemmas V.1 and VI.1 and Theorem V.4 to $\tau p^2M$ and using the fact that $\tau p^2M \leftrightarrow^K (EQ, EP)$, we obtain the results.

**Example 2.** Let $M$ be the same matrix as in Example V.3. We shall apply $HG_D^{-1}$ to $M$. At each step, the down paths are underlined and the inverse Frobenius map is applied to get $(P, EQ)$. 
Note that in this example, the tableau $P_i$ can be obtained by applying BUBBLE_UP to $P_{i+1}$ and removing the bubbled element (see Example 1). The number removed from $P_{i+1}$ is the column index of the entry in $M$ reduced at step $i$. Since BUBBLE_UP is reversible, we can describe the map $F^{-1}HG_D^{-1}$ on the tableau level. We shall call this downpath insertion. We shall describe BUBBLE_DOWN, the inverse of BUBBLE_UP, and the procedure for downpath inserting, DP_ENCODE. The arguments of BUBBLE_DOWN will be a tableau and any of its corner element. Following the convention used in BUBBLE_UP, we let $d(a)$, $l(a)$ be the elements directly below or to the left of the element $a$.

procedure BUBBLE_DOWN(T, a)
begin
    while $d(a) \neq 0$ or $l(a) \neq 0$ do
begin
if \( d(a) < 1(a) \) then exchange \( a \) and \( 1(a) \);
else exchange \( a \) and \( d(a) \);
end

In BUBBLE_DOWN we use the convention that all entries not in the shape are 0.

```
procedure DP_ENCODE(M)
begin
    \( S := \emptyset \);
    \( T := \emptyset \);
    while \( M \neq \emptyset \) do
        begin
            Let \( c \) be the largest nonzero column of \( M \);
            Let \( r \) be the largest index so that \( M_{rc} \neq 0 \);
            \( M_{rc} := M_{rc} - 1 \);
            R_INSERT(\( n - 1 - r \), \( T \));
            Put \( c \) in the corresponding new cell in \( S \);
            BUBBLE_DOWN(\( S, c \));
        end
    end
```

Using Theorems 4 and 5, we have

**Corollary 6.** Let \( M \leftarrow^k (P, Q) \). Let the downpath-inserted tableaux of \( M \) be \( (S, T) \). Then,

1. \( (S, T) = (P, EQ) \).
2. \( DP\_ ENCODE \) and \( F^{-1}HG^{-1}_D \) are the same process.

2. The Generalized Insertion Process

We are now in a position to define a generalized insertion, which gives a bijection between biwords in lexicographic order and pairs of tableaux of the same shape. Special cases of this are row and column insertions. The generalized insertion starts anywhere in the word and reads to the left and the right. Let \( k \) be the length of the biword \( w \). Let \( s \) be in \( \{ 1, 2, ..., k \} \), and DIR be any string with \( s - 1 \) L's and \( k - s \) R's. Then, \( s \) will be the location at which the insertion starts and DIR will tell us which direction to read next (R for right, L for left). The insertion is defined by the procedure \( G\_ ENCODE \). This procedure takes \( w, s, \) DIR and outputs a pair of tableaux \( (S, T) \), denoted \( w \leftarrow^G (S, T) \).
procedure \( G_{-} \) ENCODE(\( i \)), \( s \), DIR
begin
\( S := p_{s} \);
\( T := q_{s} \);
\( h := r := 1 \);
for \( i = 1 \) to \( k - 1 \) do
begin
if \( \text{DIR}_{i} = R \) then
begin
\( R_{-} \) INSERT(\( p_{s+r} \), S);
Put \( q_{s+r} \) in the corresponding new cell in \( T \);
\( r := r + 1 \);
end
else
begin
\( C_{-} \) INSERT(\( p_{s-h} \), S);
Put \( q_{s-h} \) in the corresponding new cell in \( T \);
\( \text{BUBBLE-DOWN}(T, q_{s-h}) \);
\( h := h + 1 \);
end
end
end

**Example 3.** Let \( w = (1 1 2 2 3 3) \), \( s = 4 \), and \( \text{DIR} = \text{LRLLR} \). Let \( \rightarrow^{R} \), \( \rightarrow^{C} \), and \( \rightarrow^{B} \) be \( R_{-} \) INSERT, \( C_{-} \) INSERT, and \( \text{BUBBLE-DOWN} \):

\[
\begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{pmatrix}
\rightarrow^{R} \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{pmatrix}
\rightarrow^{C} \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{pmatrix}
\rightarrow^{B} \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{pmatrix}
\rightarrow^{R} \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{pmatrix}
\rightarrow^{B} \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{pmatrix}
\rightarrow^{B} \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{pmatrix}
\rightarrow^{B} \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{pmatrix}
\rightarrow^{B} \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{pmatrix}
\rightarrow^{B} \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{pmatrix}
The main result on the generalized insertion is

**Theorem 7.** Let \( w \) be a biword lexicographic order. Let \( s \) and \( \text{DIR} \) be as in the definition of the generalized insertion. Let \( w \leftrightarrow^K (P, Q) \). Then, independent of \( s \) and \( \text{DIR} \), \( w \leftrightarrow^G (P, Q) \).

**Proof.** When the length of \( w \) is 1, the assertion is trivial. So, assume the assertion for \( k - 1 \), and show it for \( k \). Let \( w = (a_1; \ldots; a_k) \), and \( b_1 = (a_1) \), \( b_k = (a_k) \). Either \( b_1 \) or \( b_k \) will be the last pair inserted. When \( b_k \) is the last pair inserted, \( G_{- \text{ENCODE}} \) is the same as row insertion. So the result follows in this case. Now let \( w' = w - b_1 \) and \( (P', Q') \) the pair of Knuth correspondence tableaux. When inserting \( b_1 \), we have to column insert \( p_1 \) into \( P' \). Theorem II.2 gives that the result is \( P \). Next, we have to put \( q_1 \) at the corresponding new cell in \( Q' \) then bubble it down. We must show that the result is \( Q \). Let \( M \) and \( M' \) be the matrices corresponding to \( w \) and \( w' \), respectively. \( M \) and \( M' \) only differ by 1 in the entry \((q_1, p_1)\). By Theorem V.3 we have \( F^{-1}(HG_D^{-1}(M')) = (P', EQ') \) and \( F^{-1}(HG_D^{-1}(M)) = (P, EQ) \). Applying Theorem III.5, we get \( F^{-1}(HG_D^{-1}(M'^{t})) = (Q', EP') \) and \( F^{-1}(HG_D^{-1}(M'^{t})) = (Q, EP) \). Here \( M'^{t} \) denotes the ordinary transpose of \( M \). \( M'^{t} \) and \( M'^{t} \) differ by 1 in the \((p_1, q_1)\) entry. In downpath insertion, this is the last entry of \( M'^{t} \) to be inserted. Thus, right before downpath inserting this entry we must have \((Q', EP')\). This involves adding \( q_1 \) to the new cell in \( Q \) corresponding to the new cell of \( EP \) and bubbling \( q_1 \) down. Since the evacuation operator preserved shape, the new cell of \( EP \) is the new cell we add in \( P' \) to get \( P \). Thus, the result is proved.

**VIII. Remarks and Conclusions**

Let \( A \) be a linearly ordered alphabet. Let \( v, w \in A^* \). The product of \( v \cdot w \) of \( v, w \) is obtained by concatenating \( v \) and \( w \). Under this product \( A^* \) is a monoid with an identity, the empty word \( \varnothing \). The set of equivalent classes of words under the Knuth equivalence relation is usually called the plactic monoid. An excellent survey of the subject can be found in [9]. It can be seen (without much difficulty) that the evacuation operator is the unique
anti-isomorphism on the plactic monoid that preserves shapes of tableaux. We have shown a simple way to characterize the dihedral symmetries of a square matrix using this natural definition of the evacuation as an anti-isomorphism.

After translating row insertion and the Knuth correspondence from the tableau level to the reverse plane partition level, one naturally wonders what the translations of column insertion and the Burge correspondence are. It is not too hard to translate column insertion to the RPP level. Gansner [3] was the first one to relate the Hillman–Grassl bijection to tableaux by showing that this map is the same as the Burge correspondence on \( \rho \tau M \) for any matrix \( M \).

REFERENCES

1. K. Baclawski, Lecture Notes, Spring Quarter, University of California, San Diego, 1980.