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# Mean value from representation of rational number as sum of two Egyptian fractions

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ABSTRACT

For given positive integers $n$ and $a$, let $R(n;a)$ denote the number of positive integer solutions $(x,y)$ of the Diophantine equation

$$\frac{a}{n}=\frac{1}{x}+\frac{1}{y}.$$

Write

$$S(N;a)=\sum_{\substack{n\leqslant N\\(n,a)=1}} R(n;a).$$

Recently Jingjing Huang and R.C. Vaughan proved that for $4\leqslant N$ and $a\leqslant 2N$, there is an asymptotic formula

$$S(N;a)=\frac{3}{\pi^2 a}\prod_{p\mid a}\frac{p-1}{p+1}\cdot N\big(\log^2 N+c_1(a)\log N+c_0(a)\big)+\Delta(N;a).$$

In this paper, we shall get a more explicit expression with better error term for $c_0(a)$.



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## 1. Introduction

Representation of rational number as sum of unit fractions, or Egyptian fractions, is an interesting topic in number theory. For its history and related problems, one can see R.K. Guy's book [1].

Recently Jingjing Huang and R.C. Vaughan [2] studied the representation of rational number as sum of two Egyptian fractions. They established two mean value theorems, one of which is

**Proposition 1.** *For given positive integers n and a, let $R(n;a)$ denote the number of positive integer solutions $(x,y)$ of the Diophantine equation*

$$\frac{a}{n}=\frac{1}{x}+\frac{1}{y}.$$

*Write*

$$S(N;a)=\sum_{\substack{n\leqslant N\\(n,a)=1}}R(n;a). \tag{1}$$

*Then for $4\leqslant N$ and $a\leqslant 2N$, there is an asymptotic formula*

$$S(N;a)=\frac{3}{\pi^2 a}\prod_{p|a}\frac{p-1}{p+1}\cdot N\big(\log^2 N+c_1(a)\log N+c_0(a)\big)+\Delta(N;a),$$

*where*

$$c_1(a)=6\gamma-4\frac{\zeta'(2)}{\zeta(2)}-2+\sum_{p|a}\frac{6p+2}{p^2-1}\cdot\log p \tag{2}$$

*and*

$$c_0(a)=-2\log^2 a-4\log a\sum_{p|a}\frac{\log p}{p-1}+O\left(\frac{a}{\varphi(a)}\cdot\log a\right), \tag{3}$$

*and*

$$\Delta(N;a)\ll N^{\frac{1}{2}}\log^5 N\cdot\frac{a}{\varphi(a)}\prod_{p|a}\left(1-\frac{1}{p^{\frac{1}{2}}}\right)^{-1}. \tag{4}$$

*Here p denotes prime number, $\gamma$ is the Euler constant and $\varphi(a)$ is the Euler totient function.*

In this paper, we shall apply results in [3] and [4] to get a more explicit expression with better error term for $c_0(a)$ in (3). We shall prove

**Theorem.** *Let $S(N;a)$ be defined in (1). Then for $4\leqslant N$ and $3\leqslant a\leqslant 2N$, we have*

$$S(N;a)=\frac{3}{\pi^2 a}\prod_{p|a}\frac{p-1}{p+1}\cdot N\big(\log^2 N+c_1(a)\log N+c_0(a)\big)+\Delta(N;a),$$

*where $c_1(a)$ and $\Delta(N;a)$ are the same as in (2) and (4), while*

$$
\begin{aligned}
c_0(a) = & -2\log^2 a - 4\log a \sum_{p|a} \frac{\log p}{p-1} - 2\left(\sum_{p|a} \frac{\log p}{p-1}\right)^2 \\
& + \left(\sum_{p|a} \frac{3p+1}{p^2-1} \cdot \log p\right)^2 - \sum_{p|a} \frac{3p^3+2p^2+3p}{(p^2-1)^2} \cdot \log^2 p \\
& + \left(6\gamma - 4\frac{\zeta'(2)}{\zeta(2)} - 2\right) \sum_{p|a} \frac{3p+1}{p^2-1} \cdot \log p \\
& + \frac{2}{\zeta^2(2)}\big(2\zeta'(2) + \zeta(2)\big)^2 - \frac{4}{\zeta(2)}\big(\zeta''(2) + \zeta'(2)\big) \\
& - 6\gamma\left(2\frac{\zeta'(2)}{\zeta(2)} + 1\right) + 8\gamma^2 - 2\gamma_1 + 2\zeta(2) + O\left(\frac{1}{\varphi(a)}\right). \qquad (5)
\end{aligned}
$$

*Here $p$ denotes prime number, $\varphi(a)$ is the Euler totient function, $\gamma$ is the Euler constant and*

$$
\gamma_1 = \lim_{H\to\infty}\left(\sum_{h=1}^{H} \frac{\log h}{h} - \frac{1}{2}\log^2 H\right). \qquad (6)
$$

## 2. Preliminaries

**Proposition 2.** *For the given integer $a \geqslant 3$, let $\chi$ be a Dirichlet character* mod $a$ *and $\chi_0$ denote the principal character. Then we have*

$$
\begin{aligned}
\sum_{\substack{\chi \,(\mathrm{mod}\ a) \\ \chi\neq\chi_0}} \big|L(1,\chi)\big|^2 = & \ \zeta(2)\varphi(a)\prod_{p|a}\left(1-\frac{1}{p^2}\right) - \frac{\varphi^2(a)}{a^2}\left(\log a + \sum_{p|a}\frac{\log p}{p-1}\right)^2 \\
& + \frac{\varphi^2(a)}{a^2}\big(\gamma^2 + 2\gamma_1 - 2\zeta(2)\big) + O\left(\frac{\varphi(a)}{a^2}\right),
\end{aligned}
$$

*where $p$ denotes prime number, $\varphi(a)$ is the Euler totient function, $\gamma$ is the Euler constant and $\gamma_1$ is defined in* (6)*.*

This is Theorem 1 in [3].

**Proposition 3.** *For the given integer $a \geqslant 3$, we have*

$$
\sum_{\substack{\chi \,(\mathrm{mod}\ a) \\ \chi(-1)=-1}} \big|L(1,\chi)\big|^2 = \frac{\pi^2}{12}\cdot\frac{\varphi^2(a)}{a^2}\left(a\prod_{p|a}\left(1+\frac{1}{p}\right) - 3\right).
$$

One can see Theorem A in p. 440 of [4].

**Lemma 1.** *In the neighborhood of $s=1$, there is the Laurent expansion*

$$
\zeta^3(s) = \frac{1}{(s-1)^3} + \frac{3\gamma}{(s-1)^2} + \frac{3\gamma^2 - 3\gamma_1}{s-1} + \cdots,
$$

*where $\gamma$ is the Euler constant and $\gamma_1$ is defined in* (6)*.*

**Proof.** We know

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots.$$

Hence,

$$\begin{aligned}
\zeta^3(s) &= \left(\frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots\right)^2 \left(\frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots\right) \\
&= \left(\frac{1}{(s-1)^2} + \gamma^2 + \frac{2\gamma}{s-1} - 2\gamma_1 + \cdots\right)\left(\frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots\right) \\
&= \left(\frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + \left(\gamma^2 - 2\gamma_1\right) + \cdots\right)\left(\frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots\right) \\
&= \frac{1}{(s-1)^3} + \frac{\gamma}{(s-1)^2} - \frac{\gamma_1}{s-1} + \frac{2\gamma}{(s-1)^2} + \frac{2\gamma^2}{s-1} + \frac{\gamma^2 - 2\gamma_1}{s-1} + \cdots \\
&= \frac{1}{(s-1)^3} + \frac{3\gamma}{(s-1)^2} + \frac{3\gamma^2 - 3\gamma_1}{s-1} + \cdots. \qquad \square
\end{aligned}$$

**Lemma 2.** *If* $a \geqslant 3$, *then*

$$\sum_{p|a} \frac{\log p}{p-1} \ll \log\log a.$$

**Proof.** It is enough to prove for sufficiently large $a$. When $x \geqslant \log a$, the function $\frac{\log x}{x}$ decreases monotonously. Thus

$$\begin{aligned}
\sum_{p|a} \frac{\log p}{p-1} &\ll \sum_{p|a} \frac{\log p}{p} \\
&= \sum_{\substack{p|a \\ p \leqslant \log a}} \frac{\log p}{p} + \sum_{\substack{p|a \\ \log a < p}} \frac{\log p}{p} \\
&\leqslant \sum_{p \leqslant \log a} \frac{\log p}{p} + \frac{\log\log a}{\log a} \sum_{p|a} 1 \\
&\ll \log\log a + \frac{\log\log a}{\log a} \cdot \log a \\
&\ll \log\log a. \qquad \square
\end{aligned}$$

**Lemma 3.** *If* $a \geqslant 3$, *then*

$$\sum_{p|a} \frac{3p^3 + 2p^2 + 3p}{(p^2-1)^2} \cdot \log^2 p \ll \log\log^2 a.$$

**Proof.** Assume that $a$ is sufficiently large. When $x \geqslant \log a$, the function $\frac{\log^2 x}{x}$ decreases monotonously. Thus

$$\begin{aligned}\sum_{p|a} \frac{3p^3+2p^2+3p}{(p^2-1)^2} \cdot \log^2 p &\ll \sum_{p|a} \frac{\log^2 p}{p} \\ &= \sum_{\substack{p|a \\ p\leqslant \log a}} \frac{\log^2 p}{p} + \sum_{\substack{p|a \\ \log a<p}} \frac{\log^2 p}{p} \\ &\leqslant \log\log a \sum_{p\leqslant \log a} \frac{\log p}{p} + \frac{\log\log^2 a}{\log a} \sum_{p|a} 1 \\ &\ll \log\log^2 a. \qquad \square\end{aligned}$$

## 3. The proof of Theorem

According to the discussion in [2], we have

$$S(N;a) = \frac{1}{\varphi(a)} \sum_{\chi \,(\mathrm{mod}\ a)} \chi(-1)\mathrm{Res}_{s=1}\left(f_\chi(s)\frac{N^s}{s}\right) + \Delta(N;a), \tag{7}$$

where

$$f_\chi(s) = \frac{L(s,\chi_0)}{L(2s,\chi_0)} \cdot L(s,\chi)L(s,\bar{\chi}) \tag{8}$$

and

$$\Delta(N;a) \ll N^{\frac{1}{2}} \log^5 N \cdot \frac{a}{\varphi(a)} \prod_{p|a}\left(1-\frac{1}{p^{\frac{1}{2}}}\right)^{-1}.$$

Write

$$S_1(N;a) = \frac{1}{\varphi(a)} \sum_{\substack{\chi \,(\mathrm{mod}\ a) \\ \chi\neq\chi_0}} \chi(-1)\mathrm{Res}_{s=1}\left(f_\chi(s)\frac{N^s}{s}\right) \tag{9}$$

and

$$S_2(N;a) = \frac{1}{\varphi(a)}\mathrm{Res}_{s=1}\left(f_{\chi_0}(s)\frac{N^s}{s}\right). \tag{10}$$

The discussion in [2] shows that when $\chi \neq \chi_0$,

$$\mathrm{Res}_{s=1}\left(f_\chi(s)\frac{N^s}{s}\right) = \mathrm{Res}_{s=1}\left(\frac{L(s,\chi_0)L(s,\chi)L(s,\bar{\chi})N^s}{L(2s,\chi_0)s}\right) = \frac{6N}{\pi^2}\prod_{p|a}\frac{p}{p+1} \cdot \left|L(1,\chi)\right|^2.$$

Hence,

$$S_1(N;a)=\frac{6N}{\pi^2}\cdot\frac{1}{\varphi(a)}\prod_{p|a}\frac{p}{p+1}\cdot\sum_{\substack{\chi\,(\mathrm{mod}\ a)\\ \chi\neq\chi_0}}\chi(-1)\left|L(1,\chi)\right|^2.$$

Since $\chi(-1)=1$ or $-1$, by Propositions 2 and 3, we get

$$\begin{aligned}
&\sum_{\substack{\chi\,(\mathrm{mod}\ a)\\ \chi\neq\chi_0}}\chi(-1)\left|L(1,\chi)\right|^2\\
&=\sum_{\substack{\chi\,(\mathrm{mod}\ a)\\ \chi\neq\chi_0\\ \chi(-1)=1}}\left|L(1,\chi)\right|^2-\sum_{\substack{\chi\,(\mathrm{mod}\ a)\\ \chi\neq\chi_0\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2\\
&=\sum_{\substack{\chi\,(\mathrm{mod}\ a)\\ \chi\neq\chi_0}}\left|L(1,\chi)\right|^2-2\sum_{\substack{\chi\,(\mathrm{mod}\ a)\\ \chi\neq\chi_0\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2\\
&=\sum_{\substack{\chi\,(\mathrm{mod}\ a)\\ \chi\neq\chi_0}}\left|L(1,\chi)\right|^2-2\sum_{\substack{\chi\,(\mathrm{mod}\ a)\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2\\
&=\zeta(2)\varphi(a)\prod_{p|a}\left(1-\frac{1}{p^2}\right)-\frac{\varphi^2(a)}{a^2}\left(\log a+\sum_{p|a}\frac{\log p}{p-1}\right)^2\\
&\quad+\frac{\varphi^2(a)}{a^2}\left(\gamma^2+2\gamma_1-2\zeta(2)\right)+O\left(\frac{\varphi(a)}{a^2}\right)\\
&\quad-\zeta(2)\cdot\frac{\varphi^2(a)}{a^2}\left(a\prod_{p|a}\left(1+\frac{1}{p}\right)-3\right)\\
&=\zeta(2)\varphi(a)\prod_{p|a}\left(1-\frac{1}{p^2}\right)-\frac{\varphi^2(a)}{a^2}\left(\log a+\sum_{p|a}\frac{\log p}{p-1}\right)^2\\
&\quad+\frac{\varphi^2(a)}{a^2}\left(\gamma^2+2\gamma_1-2\zeta(2)\right)+O\left(\frac{\varphi(a)}{a^2}\right)\\
&\quad-\zeta(2)\varphi(a)\prod_{p|a}\left(1-\frac{1}{p^2}\right)+3\zeta(2)\cdot\frac{\varphi^2(a)}{a^2}\\
&=-\frac{\varphi^2(a)}{a^2}\left(\log a+\sum_{p|a}\frac{\log p}{p-1}\right)^2+\frac{\varphi^2(a)}{a^2}\left(\gamma^2+2\gamma_1+\zeta(2)\right)+O\left(\frac{\varphi(a)}{a^2}\right).
\end{aligned}$$

Hence,

$$S_1(N;a)=\frac{6N}{\pi^2}\cdot\frac{\varphi(a)}{a^2}\prod_{p|a}\frac{p}{p+1}\cdot\left(-\left(\log a+\sum_{p|a}\frac{\log p}{p-1}\right)^2+\left(\gamma^2+2\gamma_1+\zeta(2)\right)+O\left(\frac{1}{\varphi(a)}\right)\right)$$

$$= \frac{3}{\pi^2 a} \prod_{p|a} \frac{p-1}{p+1} \cdot N\Biggl(-2\log^2 a - 4\log a \sum_{p|a} \frac{\log p}{p-1} - 2\biggl(\sum_{p|a} \frac{\log p}{p-1}\biggr)^2 + 2\gamma^2 + 4\gamma_1 + 2\zeta(2) + O\biggl(\frac{1}{\varphi(a)}\biggr)\Biggr).$$

Now we proceed to calculate

$$\mathrm{Res}_{s=1}\biggl(f_{\chi_0}(s)\frac{N^s}{s}\biggr).$$

The discussion in [2] yields

$$f_{\chi_0}(s) = \frac{L^3(s,\chi_0)}{L(2s,\chi_0)} = \frac{\zeta^3(s)}{\zeta(2s)} \prod_{p|a} \frac{(p^s-1)^2}{p^s(p^s+1)} = \frac{\zeta^3(s)}{\zeta(2s)} \cdot G(s), \tag{11}$$

where

$$G(s) = \prod_{p|a} \frac{(p^s-1)^2}{p^s(p^s+1)}. \tag{12}$$

By Lemma 1,

$$\begin{aligned}
\mathrm{Res}_{s=1}\biggl(f_{\chi_0}(s)\frac{N^s}{s}\biggr) &= \mathrm{Res}_{s=1}\zeta^3(s) \cdot \frac{N^s}{\zeta(2s)s} \cdot G(s) \\
&= \mathrm{Res}_{s=1}\frac{1}{(s-1)^3} \cdot \frac{N^s}{\zeta(2s)s} \cdot G(s) \\
&\quad + 3\gamma\,\mathrm{Res}_{s=1}\frac{1}{(s-1)^2} \cdot \frac{N^s}{\zeta(2s)s} \cdot G(s) \\
&\quad + \bigl(3\gamma^2 - 3\gamma_1\bigr)\mathrm{Res}_{s=1}\frac{1}{s-1} \cdot \frac{N^s}{\zeta(2s)s} \cdot G(s).
\end{aligned}$$

We shall calculate these residues respectively.

1. It is easy to see

$$\bigl(3\gamma^2 - 3\gamma_1\bigr)\mathrm{Res}_{s=1}\frac{1}{s-1} \cdot \frac{N^s}{\zeta(2s)s} \cdot G(s) = N \cdot \frac{G(1)}{\zeta(2)} \cdot \bigl(3\gamma^2 - 3\gamma_1\bigr).$$

2. We have

$$\begin{aligned}
&3\gamma\,\mathrm{Res}_{s=1}\frac{1}{(s-1)^2} \cdot \frac{N^s}{\zeta(2s)s} \cdot G(s) \\
&\quad = 3\gamma\biggl(\frac{N^s}{\zeta(2s)s} \cdot G(s)\biggr)'\bigg|_{s=1} \\
&\quad = 3\gamma\biggl(\biggl(\frac{N^s}{\zeta(2s)s}\biggr)' \cdot G(s) + \frac{N^s}{\zeta(2s)s} \cdot G'(s)\biggr)\bigg|_{s=1}
\end{aligned}$$

$$= 3\gamma\left(\frac{N^s \log N \cdot \zeta(2s)s - N^s(\zeta'(2s)\cdot 2s + \zeta(2s))}{(\zeta(2s)s)^2}\cdot G(s) + \frac{N^s}{\zeta(2s)s}\cdot G'(s)\right)\Bigg|_{s=1}$$
$$= 3\gamma G(1)\left(\frac{N\log N\cdot\zeta(2) - N(2\zeta'(2)+\zeta(2))}{\zeta^2(2)} + \frac{N}{\zeta(2)}\cdot\frac{G'(1)}{G(1)}\right)$$
$$= N\cdot\frac{G(1)}{\zeta(2)}\cdot 3\gamma\left(\frac{\zeta(2)\log N - (2\zeta'(2)+\zeta(2))}{\zeta(2)} + \frac{G'(1)}{G(1)}\right)$$
$$= N\cdot\frac{G(1)}{\zeta(2)}\cdot\left(3\gamma\log N - 3\gamma\left(2\frac{\zeta'(2)}{\zeta(2)}+1\right) + 3\gamma\cdot\frac{G'(1)}{G(1)}\right).$$

The derivative of $\log G(s)$ is

$$\frac{G'(s)}{G(s)} = \sum_{p|a}\bigl(2\log\bigl(p^s-1\bigr) - s\log p - \log\bigl(p^s+1\bigr)\bigr)'$$
$$= \sum_{p|a}\left(\frac{2}{p^s-1}\cdot p^s\log p - \log p - \frac{1}{p^s+1}\cdot p^s\log p\right)$$
$$= \sum_{p|a}\left(\frac{2}{p^s-1}+\frac{1}{p^s+1}\right)\log p. \tag{13}$$

Hence,

$$\frac{G'(1)}{G(1)} = \sum_{p|a}\left(\frac{2}{p-1}+\frac{1}{p+1}\right)\log p = \sum_{p|a}\frac{3p+1}{p^2-1}\cdot\log p, \tag{14}$$

which is a formula in p. 1652 of [2].

3. We have

$$\operatorname{Res}_{s=1}\frac{1}{(s-1)^3}\cdot\frac{N^s}{\zeta(2s)s}\cdot G(s)$$
$$= \frac{1}{2!}\left(\frac{N^s}{\zeta(2s)s}\cdot G(s)\right)''\Bigg|_{s=1}$$
$$= \frac{1}{2}\left(\left(\frac{N^s}{\zeta(2s)s}\right)''\cdot G(s) + 2\left(\frac{N^s}{\zeta(2s)s}\right)'\cdot G'(s) + \frac{N^s}{\zeta(2s)s}\cdot G''(s)\right)\Bigg|_{s=1}$$
$$= \frac{1}{2}\cdot\left(\frac{N^s}{\zeta(2s)s}\right)''\cdot G(s)\Bigg|_{s=1} + \left(\frac{N^s}{\zeta(2s)s}\right)'\cdot G'(s)\Bigg|_{s=1} + \frac{1}{2}\cdot\frac{N^s}{\zeta(2s)s}\cdot G''(s)\Bigg|_{s=1}.$$

We shall calculate these expressions respectively.

a) Since

$$\left(\frac{N^s}{\zeta(2s)s}\right)'' = \left(\frac{N^s\log N\cdot\zeta(2s)s - N^s(\zeta'(2s)\cdot 2s+\zeta(2s))}{(\zeta(2s)s)^2}\right)'$$
$$= \frac{(N^s\log N\cdot\zeta(2s)s - N^s(\zeta'(2s)\cdot 2s+\zeta(2s)))'(\zeta(2s)s)^2}{(\zeta(2s)s)^4}$$

$$
\begin{aligned}
&- \frac{(N^s \log N \cdot \zeta(2s)s - N^s(\zeta'(2s) \cdot 2s + \zeta(2s)))2\zeta(2s)s(\zeta(2s)s)'}{(\zeta(2s)s)^4} \\
&= \frac{(N^s \log N \cdot \zeta(2s)s)'}{(\zeta(2s)s)^2} - \frac{(N^s(\zeta'(2s) \cdot 2s + \zeta(2s)))'}{(\zeta(2s)s)^2} \\
&- \frac{N^s \log N \cdot \zeta(2s)s \cdot 2(\zeta'(2s) \cdot 2s + \zeta(2s))}{(\zeta(2s)s)^3} \\
&+ \frac{N^s(\zeta'(2s) \cdot 2s + \zeta(2s)) \cdot 2(\zeta'(2s) \cdot 2s + \zeta(2s))}{(\zeta(2s)s)^3} \\
&= \frac{(N^s \log^2 N \cdot \zeta(2s)s + N^s \log N \cdot \zeta'(2s)2s + N^s \log N \cdot \zeta(2s))}{(\zeta(2s)s)^2} \\
&- \frac{N^s \log N(\zeta'(2s) \cdot 2s + \zeta(2s)) + N^s(\zeta''(2s) \cdot 4s + 2\zeta'(2s) + 2\zeta'(2s))}{(\zeta(2s)s)^2} \\
&- \frac{N^s \log N \cdot 2(\zeta'(2s) \cdot 2s + \zeta(2s))}{(\zeta(2s)s)^2} + \frac{N^s \cdot 2(2s\zeta'(2s) + \zeta(2s))^2}{(\zeta(2s)s)^3},
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2} \cdot \left(\frac{N^s}{\zeta(2s)s}\right)'' \cdot G(s)\bigg|_{s=1} &= N \cdot \frac{G(1)}{2} \cdot \left(\frac{\zeta(2)\log^2 N + (2\zeta'(2) + \zeta(2))\log N}{\zeta^2(2)}\right. \\
&\quad - \frac{(2\zeta'(2) + \zeta(2))\log N + 4\zeta''(2) + 4\zeta'(2)}{\zeta^2(2)} \\
&\quad \left. - \frac{2(2\zeta'(2) + \zeta(2))\log N}{\zeta^2(2)} + \frac{2(2\zeta'(2) + \zeta(2))^2}{\zeta^3(2)}\right) \\
&= N \cdot \frac{G(1)}{\zeta(2)} \cdot \left(\frac{1}{2}\log^2 N - \frac{1}{\zeta(2)}\big(2\zeta'(2) + \zeta(2)\big)\log N\right. \\
&\quad \left. + \frac{1}{\zeta^2(2)}\big(2\zeta'(2) + \zeta(2)\big)^2 - \frac{2}{\zeta(2)}\big(\zeta''(2) + \zeta'(2)\big)\right).
\end{aligned}
$$

b) We have

$$
\begin{aligned}
\left(\frac{N^s}{\zeta(2s)s}\right)' \cdot G'(s)\bigg|_{s=1} &= \frac{N^s \log N \cdot \zeta(2s)s - N^s(\zeta'(2s) \cdot 2s + \zeta(2s))}{(\zeta(2s)s)^2} \cdot G'(s)\bigg|_{s=1} \\
&= \frac{N \log N \cdot \zeta(2) - N(2\zeta'(2) + \zeta(2))}{\zeta^2(2)} \cdot G'(1) \\
&= N \cdot \frac{G(1)}{\zeta(2)} \cdot \frac{\zeta(2)\log N - (2\zeta'(2) + \zeta(2))}{\zeta(2)} \cdot \frac{G'(1)}{G(1)} \\
&= N \cdot \frac{G(1)}{\zeta(2)} \cdot \left(\log N - 2\frac{\zeta'(2)}{\zeta(2)} - 1\right) \cdot \frac{G'(1)}{G(1)}.
\end{aligned}
$$

c) Differentiating the equality in (13)

$$
G'(s) = G(s) \sum_{p|a} \left(\frac{2}{p^s - 1} + \frac{1}{p^s + 1}\right) \log p,
$$

we get

$$G''(s)=G'(s)\sum_{p|a}\left(\frac{2}{p^s-1}+\frac{1}{p^s+1}\right)\log p+G(s)\sum_{p|a}\left(-\frac{2p^s\log p}{(p^s-1)^2}-\frac{p^s\log p}{(p^s+1)^2}\right)\log p$$
$$=G(s)\left(\frac{G'(s)}{G(s)}\sum_{p|a}\left(\frac{2}{p^s-1}+\frac{1}{p^s+1}\right)\log p-\sum_{p|a}\left(\frac{2}{(p^s-1)^2}+\frac{1}{(p^s+1)^2}\right)p^s\log^2 p\right).$$

Therefore

$$G''(1)=G(1)\left(\frac{G'(1)}{G(1)}\sum_{p|a}\left(\frac{2}{p-1}+\frac{1}{p+1}\right)\log p-\sum_{p|a}\left(\frac{2}{(p-1)^2}+\frac{1}{(p+1)^2}\right)p\log^2 p\right)$$
$$=G(1)\left(\left(\frac{G'(1)}{G(1)}\right)^2-\sum_{p|a}\frac{3p^3+2p^2+3p}{(p^2-1)^2}\cdot\log^2 p\right),$$

which is a formula in p. 1652 of [2].

We have

$$\frac{1}{2}\cdot\frac{N^s}{\zeta(2s)s}\cdot G''(s)\bigg|_{s=1}=\frac{N}{2\zeta(2)}\cdot G''(1)$$
$$=N\cdot\frac{G(1)}{\zeta(2)}\cdot\left(\frac{1}{2}\left(\frac{G'(1)}{G(1)}\right)^2-\frac{1}{2}\sum_{p|a}\frac{3p^3+2p^2+3p}{(p^2-1)^2}\cdot\log^2 p\right).$$

Combining results in cases a), b) and c), we get

$$\mathrm{Res}_{s=1}\frac{1}{(s-1)^3}\cdot\frac{N^s}{\zeta(2s)s}\cdot G(s)=N\cdot\frac{G(1)}{\zeta(2)}\cdot\left(\left\{\frac{1}{2}\log^2 N-\frac{1}{\zeta(2)}\big(2\zeta'(2)+\zeta(2)\big)\log N\right.\right.$$
$$+\left.\frac{1}{\zeta^2(2)}\big(2\zeta'(2)+\zeta(2)\big)^2-\frac{2}{\zeta(2)}\big(\zeta''(2)+\zeta'(2)\big)\right\}$$
$$+\left\{\frac{G'(1)}{G(1)}\cdot\log N-\left(2\frac{\zeta'(2)}{\zeta(2)}+1\right)\cdot\frac{G'(1)}{G(1)}\right\}$$
$$\left.+\left\{\frac{1}{2}\left(\frac{G'(1)}{G(1)}\right)^2-\frac{1}{2}\sum_{p|a}\frac{3p^3+2p^2+3p}{(p^2-1)^2}\cdot\log^2 p\right\}\right)$$
$$=N\cdot\frac{G(1)}{\zeta(2)}\cdot\left(\frac{1}{2}\log^2 N-\frac{1}{\zeta(2)}\big(2\zeta'(2)+\zeta(2)\big)\log N\right.$$
$$+\frac{G'(1)}{G(1)}\cdot\log N+\frac{1}{2}\left(\frac{G'(1)}{G(1)}\right)^2$$
$$-\frac{1}{2}\sum_{p|a}\frac{3p^3+2p^2+3p}{(p^2-1)^2}\cdot\log^2 p-\left(2\frac{\zeta'(2)}{\zeta(2)}+1\right)\cdot\frac{G'(1)}{G(1)}$$
$$\left.+\frac{1}{\zeta^2(2)}\big(2\zeta'(2)+\zeta(2)\big)^2-\frac{2}{\zeta(2)}\big(\zeta''(2)+\zeta'(2)\big)\right).$$

Now we can get from cases 1, 2 and 3 that

$$
\begin{aligned}
\operatorname{Res}_{s=1}\left(f_{\chi_0}(s)\frac{N^s}{s}\right) &= N\cdot\frac{G(1)}{\zeta(2)}\cdot\left(\frac{1}{2}\log^2 N-\frac{1}{\zeta(2)}\big(2\zeta'(2)+\zeta(2)\big)\log N\right.\\
&\quad+\frac{G'(1)}{G(1)}\cdot\log N+\frac{1}{2}\left(\frac{G'(1)}{G(1)}\right)^2\\
&\quad-\frac{1}{2}\sum_{p|a}\frac{3p^3+2p^2+3p}{(p^2-1)^2}\cdot\log^2 p-\left(2\frac{\zeta'(2)}{\zeta(2)}+1\right)\cdot\frac{G'(1)}{G(1)}\\
&\quad\left.+\frac{1}{\zeta^2(2)}\big(2\zeta'(2)+\zeta(2)\big)^2-\frac{2}{\zeta(2)}\big(\zeta''(2)+\zeta'(2)\big)\right)\\
&\quad+N\cdot\frac{G(1)}{\zeta(2)}\cdot\left(3\gamma\log N-3\gamma\left(2\frac{\zeta'(2)}{\zeta(2)}+1\right)+3\gamma\cdot\frac{G'(1)}{G(1)}\right)\\
&\quad+N\cdot\frac{G(1)}{\zeta(2)}\cdot\big(3\gamma^2-3\gamma_1\big)\\
&= N\cdot\frac{G(1)}{\zeta(2)}\cdot\left(\frac{1}{2}\log^2 N+3\gamma\log N\right.\\
&\quad-\frac{1}{\zeta(2)}\big(2\zeta'(2)+\zeta(2)\big)\log N+\frac{G'(1)}{G(1)}\cdot\log N\\
&\quad+\frac{1}{2}\left(\frac{G'(1)}{G(1)}\right)^2-\frac{1}{2}\sum_{p|a}\frac{3p^3+2p^2+3p}{(p^2-1)^2}\cdot\log^2 p\\
&\quad+\left(3\gamma-2\frac{\zeta'(2)}{\zeta(2)}-1\right)\cdot\frac{G'(1)}{G(1)}+\frac{1}{\zeta^2(2)}\big(2\zeta'(2)+\zeta(2)\big)^2\\
&\quad\left.-\frac{2}{\zeta(2)}\big(\zeta''(2)+\zeta'(2)\big)-3\gamma\left(2\frac{\zeta'(2)}{\zeta(2)}+1\right)+3\gamma^2-3\gamma_1\right).
\end{aligned}
$$

Since

$$
\frac{1}{\varphi(a)}\cdot\frac{G(1)}{\zeta(2)}=\frac{1}{a}\prod_{p|a}\frac{p}{p-1}\cdot\frac{6}{\pi^2}\prod_{p|a}\frac{(p-1)^2}{p(p+1)}=\frac{6}{\pi^2 a}\prod_{p|a}\frac{p-1}{p+1},
$$

$$
\begin{aligned}
S_2(N;a) &= \frac{1}{\varphi(a)}\operatorname{Res}_{s=1}\left(f_{\chi_0}(s)\frac{N^s}{s}\right)\\
&= \frac{3}{\pi^2 a}\prod_{p|a}\frac{p-1}{p+1}\cdot N\left(\log^2 N+\left(6\gamma-4\frac{\zeta'(2)}{\zeta(2)}-2\right)\log N\right.\\
&\quad+\left(\sum_{p|a}\frac{6p+2}{p^2-1}\cdot\log p\right)\log N+\left(\sum_{p|a}\frac{3p+1}{p^2-1}\cdot\log p\right)^2\\
&\quad-\sum_{p|a}\frac{3p^3+2p^2+3p}{(p^2-1)^2}\cdot\log^2 p+\left(6\gamma-4\frac{\zeta'(2)}{\zeta(2)}-2\right)\sum_{p|a}\frac{3p+1}{p^2-1}\cdot\log p\\
&\quad+\frac{2}{\zeta^2(2)}\big(2\zeta'(2)+\zeta(2)\big)^2-\frac{4}{\zeta(2)}\big(\zeta''(2)+\zeta'(2)\big)\\
&\quad\left.-6\gamma\left(2\frac{\zeta'(2)}{\zeta(2)}+1\right)+6\gamma^2-6\gamma_1\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
S_1(N;a)+S_2(N;a) &= \frac{3}{\pi^2 a}\prod_{p|a}\frac{p-1}{p+1}\cdot N\biggl(-2\log^2 a-4\log a\sum_{p|a}\frac{\log p}{p-1}\\
&\quad -2\biggl(\sum_{p|a}\frac{\log p}{p-1}\biggr)^2+2\gamma^2+4\gamma_1+2\zeta(2)+O\biggl(\frac{1}{\varphi(a)}\biggr)\biggr)\\
&\quad +\frac{3}{\pi^2 a}\prod_{p|a}\frac{p-1}{p+1}\cdot N\biggl(\log^2 N+\biggl(6\gamma-4\frac{\zeta'(2)}{\zeta(2)}-2\biggr)\log N\\
&\quad +\biggl(\sum_{p|a}\frac{6p+2}{p^2-1}\cdot\log p\biggr)\log N+\biggl(\sum_{p|a}\frac{3p+1}{p^2-1}\cdot\log p\biggr)^2\\
&\quad -\sum_{p|a}\frac{3p^3+2p^2+3p}{(p^2-1)^2}\cdot\log^2 p+\biggl(6\gamma-4\frac{\zeta'(2)}{\zeta(2)}-2\biggr)\sum_{p|a}\frac{3p+1}{p^2-1}\cdot\log p\\
&\quad +\frac{2}{\zeta^2(2)}\bigl(2\zeta'(2)+\zeta(2)\bigr)^2-\frac{4}{\zeta(2)}\bigl(\zeta''(2)+\zeta'(2)\bigr)\\
&\quad -6\gamma\biggl(2\frac{\zeta'(2)}{\zeta(2)}+1\biggr)+6\gamma^2-6\gamma_1\biggr)\\
&= \frac{3}{\pi^2 a}\prod_{p|a}\frac{p-1}{p+1}\cdot N\biggl(\log^2 N+\biggl(6\gamma-4\frac{\zeta'(2)}{\zeta(2)}-2\biggr)\log N\\
&\quad +\biggl(\sum_{p|a}\frac{6p+2}{p^2-1}\cdot\log p\biggr)\log N-2\log^2 a-4\log a\sum_{p|a}\frac{\log p}{p-1}\\
&\quad -2\biggl(\sum_{p|a}\frac{\log p}{p-1}\biggr)^2+\biggl(\sum_{p|a}\frac{3p+1}{p^2-1}\cdot\log p\biggr)^2\\
&\quad -\sum_{p|a}\frac{3p^3+2p^2+3p}{(p^2-1)^2}\cdot\log^2 p+\biggl(6\gamma-4\frac{\zeta'(2)}{\zeta(2)}-2\biggr)\sum_{p|a}\frac{3p+1}{p^2-1}\cdot\log p\\
&\quad +\frac{2}{\zeta^2(2)}\bigl(2\zeta'(2)+\zeta(2)\bigr)^2-\frac{4}{\zeta(2)}\bigl(\zeta''(2)+\zeta'(2)\bigr)\\
&\quad -6\gamma\biggl(2\frac{\zeta'(2)}{\zeta(2)}+1\biggr)+8\gamma^2-2\gamma_1+2\zeta(2)+O\biggl(\frac{1}{\varphi(a)}\biggr)\biggr)\\
&= \frac{3}{\pi^2 a}\prod_{p|a}\frac{p-1}{p+1}\cdot N\bigl(\log^2 N+c_1(a)\log N+c_0(a)\bigr),
\end{aligned}
$$

where $c_1(a)$ and $c_0(a)$ are defined in (2) and (5). By Lemmas 2 and 3, we can see that terms of $c_0(a)$ in (5) are arranged in decreasing order of $a$.

So far the proof of Theorem is finished.

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