On the Fundamental Solution of Delay-Differential Equations in Banach Spaces

SHIN-ICHI NAOKAGIRI

Department of Applied Mathematics, Faculty of Engineering, Kobe University, Kobe, 657, Japan

Received April 16, 1980; revised November 13, 1980

1. INTRODUCTION

In this paper we consider the linear delay-differential equation involving \( m \) delay terms

\[
\frac{dx(t)}{dt} = Ax(t) + \sum_{r=1}^{m} A_r x(t-rt),
\]

where \( \tau \) is a positive constant, \( x(t) \) belongs to a Banach space \( X \), and \( A, A_1, \ldots, A_m \) are linear, not necessarily bounded operators on \( X \). It is assumed that \( A \) generates a strongly continuous semi-group \( T(t) \) and \( A_r, r = 1, \ldots, m \), are relatively bounded compared with \( A \) (see Section 2). For the case \( A_r = 0 \) (the null operator on \( X \)), \( r = 1, \ldots, m \), Eq. (1.1) is reduced to

\[
\frac{dx(t)}{dt} = Ax(t).
\]

The study of (1.2) is rather classic and a great number of research papers and monographs exist (see [5, 9, 10, 15, 16] and their references). In these works, various types of existence, uniqueness, differentiability and continuous dependence theorems are established on the basis of the construction and the representation of the semi-group \( T(t) \) relating to (1.2).

Our purpose here is to give two representations of the fundamental solution of (1.1) in terms of \( T(t) \) and \( A_r \) and establish a variation of constants formula for (1.1). Such expressions are useful to obtain the fundamental theorems for (1.1) and some system theoretical results [11, 12].

We briefly explain the content of this paper. In Section 2, the notations and notions used throughout the paper and the system description are given. A definition of the mild solution is also given in Section 2 and the treatment is based on the perturbation theory (i.e., Eq. (1.1) is considered as a pertur...
bation of (1, 2)). In Section 3, two new forms of the fundamental solution \( G(t) \) are established. The main results are contained in Section 4. Two concrete representations of the mild solution in terms of \( G(t) \) and in terms of \( T(t) \) are obtained by using the Hausdorff–Young inequality and Fubini's theorem concerning Bochner integrals. Finally, some examples and applications are given in Section 5.

2. SYSTEM DESCRIPTION AND MILD SOLUTION

First we give the notations and terminology used throughout this paper. Let \( R \) be the set of real numbers and let \( R^+ \) be the set of non-negative numbers. Let \( X \) be a Banach space. We denote by \( B(X) \) the Banach space of all bounded linear operators from \( X \) into itself. Given \( a < b \), we denote by \( L_p(a, b; X) \) and \( C^r(a, b; X) \) \( (r = 0, 1, 2,...) \) the Banach space of all equivalence classes of strongly measurable functions from \( [a, b] \) into \( X \) which are \( p \)-Bochner integrable \( (1 < p < \infty) \) or essentially bounded \( (p = \infty) \) on \( [a, b] \) and the Banach space of all \( r \)-times (strongly) continuously differentiable functions from \( [a, b] \) into \( X \), respectively. Put \( C^0(a, b; X) = C(a, b; X) \). When \( X = R \), \( L_p(a, b; X) \) will be denoted by \( L_p[a, b] \). \( L_p^{\text{loc}}(R^+; X) \) and \( C(R^+; X) \) will denote the Fréchet spaces \( \bigcap_{t>0} L_p(0, t; X) \) and \( \bigcap_{t>0} C(0, t; X) \), respectively.

Let us consider now the differential system with \( m \) delay terms

\[
\frac{dx(t)}{dt} = Ax(t) + \sum_{r=1}^{m} A_r x(t - r\tau) + f(t), \quad t > 0, 
\]

\( S: \)

\[
x(0) = x_0, \quad x(s) = g(s), \quad s \in [-m\tau, 0),
\]

where \( \tau > 0 \) is a constant, \( x(t), f(t), g(t) \in X \) and the operators \( A \) and \( A_r \) \( (r = 1,\ldots, m) \), possibly unbounded, are assumed to satisfy the following assumptions \( H_0 \) and \( H_1 \), respectively.

- \( H_0 \). \( A \) generates a strongly continuous semi-group \( \{T(t): t \geq 0\} \) on \( X \).
- \( H_1 \). \( A_r, r = 1,\ldots, m \), are closed linear operators with dense domains \( D(A_r) \) in \( X \).

In the system \( S \), \( x_0 \in X \) is called an initial value, \( f(\cdot) \) and \( g(\cdot) \) are called a forcing function and an initial function, respectively.

First of all we shall give a definition of the mild solution of the system \( S \). To do so we need the next integrability condition for \( A_r \).

- \( H_\$ \). For each \( r \) there exists a function \( M_r(\cdot) \in L_{\infty}[0, \tau] \) such that \( \| T(t) A_r x \| \leq M_r(t) \| x \| \) for a.a. \( t \in [0, \tau] \) and all \( x \in D(A_r) \).
Let assumption $H_q^q$, $q \in [1, \infty]$, be satisfied. Then for any $x \in X$, there exists only one element $y(t, x)$ in $X$ for a.a. $t \in [0, \tau]$ as limits of $T(t)A_x x_n$ such that $x_n \in D(A_x)$ and $x_n \to x$ in $X$. The operator $(T(t)A_x)$ defined by $(T(t)A_x)x = y(t, x)$ for $x$ in $X$ is well defined and bounded for a.a. $t \in [0, \tau]$. This means that for each $r$ and for a.a. $t \in [0, \tau]$, $T(t)A_x$ can be extended to the bounded operator $(T(t)A_x)$ on $X$ and the extended operator $(T(t)A_x)$ satisfies the inequality $\| (T(t)A_x) \| \leq M_x(t)$ for a.a. $t$ in $[0, \tau]$.

Let $x_0, f(\cdot)$ and $g(\cdot)$ be given with

$$x_0 \in X,$$

$$f(\cdot) \subseteq L^p_{\text{loc}}(R^+; X),$$

$$g(\cdot) \subseteq L^p(-\tau, 0; X),$$

and let assumption $H_q^{q'}$ with $p'^{-1} + q'^{-1} = 1$ be satisfied, where $p$, $p' \in [1, \infty]$. Then the function

$$x_1(t; x_0, f, g) = T(t)x_0 + \int_0^t (T(t-s)A_x) f(s) \, ds$$

$$+ \sum_{r=1}^m \int_0^t (T(t-s)A_x) g(s - r \tau) \, ds$$

is well defined, the integrals being Bochner integrals in $X$, and is strongly continuous on $[0, \tau]$. We give a short proof of this. The first term of (2.6) is clearly strongly continuous by assumption $H_0$. The integrand of the second term of (2.6) is strongly measurable and Bochner integrable by $H_0$ and (2.4) and hence again by $H_0$ the second term is strongly continuous. By assumption $H_q^{q'}$, $(T(t-s)A_x)$ is bounded and its norm is bounded by $M_x(t-s)$, $M_x(\cdot) \subseteq L^q_{\text{loc}}[0, \tau]$, for a.a. $s \in [0, t]$ and hence by (2.5) the function $(T(t-s)A_x) g(s - r \tau)$ is strongly measurable (see Hille and Phillips [9, Chap. 3]) and Bochner integrable on $[0, \tau]$ (note that $g(s - r \tau) \in D(A_x)$ is not assumed). Therefore, all integrands of the third term are Bochner integrable and hence the third term is strongly continuous. Thus, $x_1(\cdot; x_0, f, g) \subseteq C(0, \tau; X)$. In general, for any natural number $k$ we define $x_{k+1}(t; x_0, f, g)$ inductively by

$$x_{k+1}(t; x_0, f, g) = T(t)x_k(t; x_0, f, g) + \int_0^t (T(t-s)A_x) f(k \tau + s) \, ds$$

$$+ \sum_{r=1}^m \int_0^t (T(t-s)A_x) \begin{cases} x_{k-r+1}(s; x_0, f, g), & \text{if } r \leq k, \\ g(s - (r-k) \tau) & \text{otherwise} \end{cases} ds.$$
SHIN-ICHI NAKAGIRI

Since \( x_1 \in C(0, \tau; X) \subset L_p^0(0, \tau; X) \), it follows as above that \( x_2 \) is well defined and \( x_2 \in C(0, \tau; X) \). Continuing this process, we find that \( x_k \in C(0, \tau; X) \) for all \( k = 1, 2, \ldots \). Define \( x(t; x_0, f, g) \) by \( x(0; x_0, f, g) = x_0 \) and \( x(t; x_0, f, g) = x_k(t - (k - 1) \tau; x_0, f, g) \) if \( t \in ((k - 1) \tau, k\tau] \). We shall say that the function \( x(\cdot; x_0, f, g) \in C(R_+; X) \) is the mild solution of \( S \). Since there will be no confusion about the operator \( (T(t) A_t) \), we denote this operator simply by \( T(t) A_t \).

Under the above assumptions which guarantee the unique existence of the mild solution \( x(t; x_0, f, g) \), there arises the solution mapping \( \mathcal{S} : X \times L_p^{loc}(R^+; X) \times L_p(-\infty, 0; X) \to C(R^+; X) \) defined by \( \mathcal{S}(x_0, f, g) = x(\cdot; x_0, f, g) \). The domain of \( \mathcal{S} \) is a Fréchet space endowed with the product topology of \( X, L_p^{loc}(R^+; X) \) and \( L_p(-\infty, 0; X) \). Clearly \( C(R^+; X) \) is a Fréchet space. By standard but complicated arguments concerning the integral representation of mild solutions given in Section 4, we can verify that \( \mathcal{S} \) is linear and continuous.

**Remark 2.1.** Consider the case where \( A \) generates an analytic semigroup \( \Gamma(t) \), i.e., there exist \( 0 < \omega < \pi \) and \( M \geq 1 \) such that the resolvent set \( \rho(-A) \) contains \( \{ \lambda : \arg \lambda > \omega \} \), \( \| \lambda R(\lambda; -A) \| \leq M \) if \( \lambda < 0 \) and for every \( \epsilon > 0 \) there exists an \( M_\epsilon \geq 1 \) such that \( \| \lambda \Gamma(\lambda; -A) \| \leq M_\epsilon \) if \( \| \lambda \| > \omega + \epsilon \). In this case the fractional power \( (-A)^\alpha \), \( \alpha \geq 0 \), can be defined and

\[
\lim_{t \to 0^+} \| t^\alpha (T(t)(-A)^\alpha) \| < \infty
\]

holds (see Tanabe [15, p. 69]). Let \( A_r, r = 1, \ldots, m \), be linear combinations of fractional powers \( \sum_{i=1}^k c_i(-A)^{\alpha_i} \), where \( c_i \) are constants and \( 0 \leq \alpha_i < 1 \) \((i = 1, \ldots, k)\). Then assumptions \( H_1 \) and \( H_2^f \) with any \( q \) such that \( q < \min \{ 1/\alpha_i : r = 1, \ldots, m, i = 1, \ldots, k_r \} \) are satisfied.

**Remark 2.2.** Let every \( A_r \) be bounded and commute with \( A \), and let \( x_0 \in D(A), f(\cdot) \in C^1(R^+; X), g(\cdot) \in C^1(-\infty, 0; X) \) be satisfied. Then the mild solution \( x(t) = x(t; x_0, f, g) \) defined in this section is the unique strong solution of \( S \), i.e., \( x(\cdot) \in C^1(R^+; X), x(t) \in D(A) \) for \( t \geq 0 \) and \( x(t) \) satisfies Eq. (2.1). With respect to strong solutions and the differentiability of solutions for unbounded \( A_r \), we refer to Nakagiri [12].

**Remark 2.3.** For the nonautonomous case where \( A \) and \( A_t \) depend on time \( t \) and \( A \) generates a strongly differentiable mild evolution operator \( U(t, s), 0 \leq s \leq t \) (see [15, 16] for such constructions), we can extend the concept of a mild solution under assumptions similar to \( H_1 \) and \( H_2^f \).
3. Construction and Representation of the Fundamental Solution

In this section we construct the fundamental solution of the system $S$ and give its explicit representations in terms of $T(t)$ and $A_r$.

Let $f = 0$, $g = 0$ and let assumptions $H_0, H_1$ and $H_2$ (which is weaker than $H_2^q$ for all $q \in (1, \infty]$) be satisfied. Then, as in Section 2, we can construct the mild solution $x(t; x_0) = x(t; x_0, 0, 0) \in X$ for any $x_0 \in X$. The mapping $\mathbb{F}: \mathbb{R}^+ \times X \to X$ defined by $\mathbb{F}(t, x_0) = x(t; x_0)$ gives rise to generate a one parameter family of bounded operators $\{G(t): t \geq 0\}$, where $G(t)$ is defined by $G(t) x = \mathbb{F}(t, x)$ for $x \in X$ and satisfies the following properties:

(i) $G(t) = T(t)$ for all $t \in [0, \tau]$ and $G(t) \in B(X)$ for all $t \geq 0$.

(ii) For each $x_0 \in X$, $G(t) x_0$ is continuous on $\mathbb{R}^+$.

These are easy to verify.

Analogously to the finite dimensional case, we shall call $G(t)$ the fundamental solution of $S$. This terminology will be justified by Theorem 4.2 in the next section. Note that $G(t)$ does not satisfy the semi-group property $G(t + s) = G(t) G(s)$ in general (see Theorem 3.2).

If the number of delay terms $m$ equals 1, the following formal expression of $G(t)$ is obtained from (2.7) by the method of steps (El'sgol'ts [7]).

$$G(0) = I,$$ the identity operator in $B(X)$

and

$$G(t) = T(t) + \sum_{i=1}^{k} \int_{t}^{t} T(t - s_i) A_1 \, ds_i \cdots \int_{1}^{t} T(s_2 - s_1) A_1 T(s_1 - \tau) \, ds_1$$

if $t \in (k\tau, (k + 1) \tau]$. Here and later on it is understood that $\sum_{i=1}^{0} = 0$. The meaning of the integrals is uncertain in the above formula. In what follows, we shall give a definite meaning of integrals appeared in the expression of $G(t)$ for general $m$ delay terms.

Compared with the special case $m = 1$, it is quite complicated to write down the explicit form of $G(t)$. To overcome this complexity, we utilize the method of multisystems introduced and developed by Asner and Halanay [1], Popov [14] and Zmood [17] which is used to solve the pointwise degeneracy problem in Euclidean $n$ space.

Let $x_0 \in X$ and

$$G_k(t) = G(t + (k - 1) \tau) \quad \text{for} \quad t \in [0, \tau] \quad \text{and} \quad k = 1, 2, \ldots \quad (3.1)$$
Consider the following $k$-system

$$\frac{d}{dt} \begin{bmatrix} G_1(t) x_0 \\ G_2(t) x_0 \\ \vdots \\ G_m(t) x_0 \\ G_k(t) x_0 \end{bmatrix} = \begin{bmatrix} A \\ A_1 \\ \vdots \\ A_m \\ 0 \end{bmatrix} \begin{bmatrix} A \\ A_1 \\ \vdots \\ A_m \end{bmatrix} \begin{bmatrix} A \\ A_1 \\ \vdots \\ A_m \end{bmatrix} \begin{bmatrix} O \\ A_1 \\ \vdots \\ A_m \end{bmatrix} \begin{bmatrix} A \\ A_1 \end{bmatrix} x_0 = G_1(t) x_0 \begin{bmatrix} G_1(t) x_0 \\ G_2(t) x_0 \\ \vdots \\ G_m(t) x_0 \\ G_k(t) x_0 \end{bmatrix}, \quad (3.2)$$

where $O$ is the null operator on $X$. Since $G_k(t) x_0$ is the function translated by $(k - 1)\tau$ of the mild solution $x(t; x_0)$ of $S$, (3.2) is equivalent to step-by-step integration of (1.1). From (3.2) and the definition of the mild solution, we obtain the relation for the operators $G_i(t)$, i.e., $G_i(t) x_0$, $i = 1, \ldots, k$, are given inductively by

$$G_i(t) x_0 = T(t) G_i(0) x_0 + \sum_{r=1}^{\min(l-1, m)} \int_0^t T(t-s) A_r G_{l-r}(s) x_0 \, ds, \quad (3.3)$$

where $G_i(0) x_0 = G_{l-i}(\tau) x_0$. From (3.3) follows the next recursive formula for $G_k(t)$.

$$G_k(t) = T(t)$$

and

$$G_k(t) = T(t) G_{k-1}(\tau) + \sum_{i=1}^{\min(k-1, m)} \int_0^t T(t-s) A_i G_{k-i}(s) ds \quad \text{for } k \geq 2. \quad (3.4)$$

The expression of formula (3.4) is formal. We now give the definite meaning of (3.4). We first consider $G_2(t)$. The operator $G_2(t)$ is given formally by

$$G_2(t) = T(t + \tau) + \int_0^t T(t-s) A_1 T(s) ds.$$
DELAY-DIFFERENTIAL EQUATIONS IN BANACH SPACES

\[ T(t-s)A_1T(s)x_n \to T(t-s)A_1T(s)x \text{ in } X \text{ for a.a. } s \in [0, t] \text{ because of the boundedness of } T(s) \text{ and } T(t-s)A_1. \] This shows that \( (\mathcal{F}x)(s) = h(s) \) for a.a. \( s \in [0, t] \), and hence \( \mathcal{F} \) is closed and therefore bounded. It is in this sense that the integral appearing in the formula of \( G_2(t) \) should be interpreted. Furthermore, \( G_2(\cdot) x \in C(0, \tau; X) \) for each \( x \in X \). We can prove by induction that the same is true for \( G_k(t), k \geq 3 \). Clearly the norm of \( G_k(t) \) is bounded on \([0, \tau]\) which is also assured by property (ii) of \( G(t) \).

Formula (3.4) seems rather simple, but this is only an induction formula and for our further investigation we have to obtain a concrete expression of \( G_k(t), k \geq 2 \), in terms of \( T(t) \) and \( A_r \) only. So we use the following technique.

Let \( x_i = G_i(0)x_0, 1 \leq i \leq k \). Then by (3.3), \( G_i(t)x_0 \) can be written as

\[ G_i(t)x_0 = T_i(t)x_0 + T_{i-1}(t)x_2 + \cdots + T_2(t)x_{i-1} + T_1(t)x_i. \] (3.5)

Here the operators \( T_1(t), \ldots, T_k(t), t \geq 0 \), are defined inductively by

\[ T_1(t) = T(t) \]

and

\[ T_k(t) = \sum_{l=1}^{\min(k-1, m)} \int_0^t T(t-s)A_1T_{k-l}(s)ds \quad \text{for } k = 2, 3, \ldots \] (3.6)

The interpretation of the integrals in (3.6) is same as given in (3.4). Formula (3.6) is also inductive but simpler than (3.4), and from this formula we can derive the explicit form of \( T_k(t) \). Before giving the form, let us define the index sets \( A(j, k) \) for all \( j = 1, 2, \ldots \) and \( k = 1, 2, \ldots \) by

\[ A(j, k) = \{ (i_1, \ldots, i_j) : 1 \leq i_1, \ldots, i_j \leq m \text{ and } i_1 + \cdots + i_j = k \}. \]

Note that \( A(j, k) = \emptyset \) for \( j > k \).

We now obtain from (3.6) the following integral expression of \( T_k(t) \) for \( k \geq 2 \).

\[ T_k(t) = \sum_{j=1}^{k-1} \sum_{\Lambda(j, k-1)} \int_0^t T(t-s)A_{i_1} \cdots \times \int_0^{s_1} T(s_1-s)A_{i_j} T(s)ds \] ds_1 \cdots ds_{j-1}. \] (3.7)

For example,

\[ T_2(t) = \int_0^t T(t-s)A_1T(s)ds, \]

\[ T_3(t) = \int_0^t T(t-s)A_2T(s)ds + \int_0^t T(t-s_1)A_1 \int_0^{s_1} T(s_1-s)A_1T(s)ds \] ds_1. \]
The meaning of the iterated integrals appearing in (3.7) is similar to those given above. We note that \( T_k(t) \) is strongly continuous on \( \mathbb{R}^+ \) for each \( k = 1, 2, \ldots \).

Now, it is possible to give an expression of \( G(t) \) in terms of \( T_k(t) \). From (3.3), (3.5) and (3.6) it follows that

\[
G(t) = \sum_{i=1}^{k} T_i(t - (i - 1) \tau), \quad t \in [(k - 1) \tau, k\tau],
\]

which will be used in the next section.

Next we consider another representation of \( G(t) \). We denote the \( k \times k \) diagonal matrix of operators appearing in (3.2) by \( \mathcal{A}_k \) and put \( Z_k(t)x_0 = \langle G_1(t)x_0, \ldots, G_k(t)x_0 \rangle \), where the superscript \( t \) means the transpose operator. Then (3.2) is written simply by

\[
\frac{d}{dt} Z_k(t)x_0 = \mathcal{A}_k Z_k(t)x_0.
\]

In the finite dimensional case, the operator \( \mathcal{A}_k \) in (3.9) defines a bounded operator and hence the exponential function \( e^{t \mathcal{A}_k} \) can be defined in the usual manner. In this case, the exponential function \( e^{t \mathcal{A}_k} \) is used to obtain another expression of \( G(t) \) in the literature [1, 14, 17]. We modify the method to obtain such expression of \( G(t) \) in our infinite dimensional case. Put

\[
\mathcal{A}_k(t) = \begin{pmatrix}
T(t) & T(t) & O \\
T_2(t) & T(t) & T(t) \\
T_3(t) & T_2(t) & T(t) \\
& & \ddots \\
T_k(t) & T_{k-1}(t) & \cdots & T_2(t) & T(t)
\end{pmatrix}.
\]

Then \( Z_k(t)x_0 \) is given exactly by

\[
Z_k(t)x_0 = \mathcal{A}_k(t) Z_k(0)x_0,
\]

where

\[
Z_1(0) = I \quad \text{and} \quad Z_k(0) = \left( \mathcal{A}_{k-1}(\tau) Z_{k-1}(0) \right) \quad \text{for} \quad k \geq 2.
\]

Therefore, we obtain the another explicit representation of \( G(t) \).

\[
G(t) = G_k(t - (k - 1) \tau) = I_k \mathcal{A}_k(t - (k - 1) \tau) Z_k(0), \quad t \in [(k - 1) \tau, k\tau],
\]
where $I_k = (O, ..., O, I)$. Form (3.12) is suitable to differentiate $G(t)$ and this form is used to obtain the main theorems in [1, 12, 14, 17]. Notice that the matrix of operators $S_k(t)$ corresponds to $e^{tA}$; however, the form of $S_k(t)$ in terms of $T_l(t)$, $l = 1, ..., k$, is not given in [1, 14, 17].

Summing up the above arguments, we have the following theorem.

**Theorem 3.1.** Let the assumptions $H_0$, $H_1$, and $H_1'$ be satisfied. Then the set of one parameter family of strongly continuous operators

$$\{T_k(t): k = 1, 2, ...\}$$

can be constructed and is given by (3.7), and the fundamental solution $G(t)$, $t \geq 0$, is given by

$$G(t) = \sum_{l=1}^{k} T_l(t - (i - 1) \tau)$$

or by

$$G(t) = I_k S_k(t - (k - 1) \tau) Z_k(0),$$

where $t \in [(k - 1) \tau, k\tau]$ and $I_k$, $S_k$, $Z_k(0)$ are defined in (3.10), (3.11) and (3.12).

As far as we know, the expression (3.7), (3.8) of $G(t)$ is not found in the literature in spite of its simplicity even in the case of finite dimensional space.

We shall now give the special form of $G(t)$ in terms of $T(t)$ and $A$, under the additional assumption $H_3$.

$H_3$. For each $r A_r$ commutes with $T(t)$ for all $t \geq 0$, i.e., for any $x \in D(A_r)$ and $t \geq 0$, $T(t)x \in D(A_r)$ and $A_r T(t)x = T(t) A_r x$.

By assumption $H_3$ and (3.7), $T_k(t)$ is given formally by

$$T_k(t) = \sum_{j=1}^{k-1} \sum_{\Lambda(j, k-1)} \frac{t^j}{j!} T(t) A_{i_1} \cdots A_{i_j}$$

if $k \geq 2$. (3.13)

It is verified that the representation (3.13) is exactly true on $D_k = \bigcap_{j=1}^{k-1} \bigcap_{\Lambda(j, k-1)} D(A_{i_1} \cdots A_{i_j})$ ($D_k$ may be empty). Therefore, $G(t)$, $t \in [(k - 1) \tau, k\tau]$, is represented formally by

$$G(t) = T(t) + \sum_{l=2}^{k} \sum_{j=1}^{l-1} \sum_{\Lambda(j, l-1)} \frac{1}{j!} (t - (i - 1) \tau)^j \times T(t - (i - 1) \tau) A_{i_1} \cdots A_{i_j}$$

(3.14)
and this representation is also true on $\bigcap_{i=2}^{k} D_i$. Note that in the case given in Remark 2.1, $\bigcap_{i=2}^{k} D_i$ is dense in $X$.

By using (3.14), we can easily prove the next theorem.

**Theorem 3.2.** Let the assumptions in Theorem 3.1 and assumption $H_3$ be satisfied. If $T(t)$ is analytic or $D(AA_r)$, $r = 1, \ldots, m$, are all dense in $X$, then the fundamental solution $G(t)$ satisfies the semi-group property $G(t + s) = G(t)G(s)$ for all $t, s \in R^+$ if and only if $A_r = 0$ for all $r = 1, \ldots, m$.

**Proof.** The “if” part is always true. We shall show the “only if” part. Let $G(t)$ satisfy the semi-group property on the interval $[0, 2\pi]$. Then by (3.14), $tT(t)A_1x = 0$ on $[0, \pi]$ for each $x \in D(A_1)$. If $T(t)$ is analytic, we can differentiate this equality and obtain $A_1x = 0$ by putting $t = 0$. Since $D(A_1)$ is dense in $X$, we see $A_1 = 0$. For each $x \in D(AA_1)$, $T(t)A_1x$ is differentiable on $[0, \pi]$. Then if $D(AA_1)$ is dense, we have $A_1 = O$ by the same reason as above. Similarly we can verify $A_2 = \cdots = A_m = 0$ if $T(t)$ is analytic or $D(AA_r)$, $r = 2, \ldots, m$, are all dense.

**Remark 3.1.** Consider the case where the semi-group $T(t)$ is strongly measurable in $L^1_{\text{loc}}(R^+; B(X))$ (such a case is met when $X$ is separable and reflexive). Then the integral representations (3.4), (3.6) and (3.7) are well defined as Bochner integrals in $B(X)$ under assumption $H_1$. For this and related concepts like weak measurability, we refer to [9, 15].

**Remark 3.2.** If all operators $A_1, \ldots, A_m$ are bounded, then the matrix of operators $A_k$ is the infinitesimal generator of the semi-group $\{G_k(t): t \geq 0\}$ on the $k$-direct product Banach space $X^k$.

**4. Representation of the Mild Solution**

In this section we shall give two different types of concrete representation of the mild solution without induction. The first one is expressed by the operators $T_k(t)$ and the second by the fundamental solution $G(t)$. The second one is well known as a variation of constants formula and is extended for more general delay systems both in finite dimensional space (Bellman and Cooke [2], Oğuztöreli [13]) and infinite dimensional space (Delfour and Mitter [3, 4] in which all operators appearing in the system are bounded).

First we obtain the following expression of the mild solution in terms of $x_0$, $f(\cdot)$, $g(\cdot)$ and the operators $T_i(t)$ and $(T_i(t)A_r)$, $i = 1, 2, \ldots$, $r = 1, \ldots, m$. The definition of $(T_i(t)A_r)$, which is a bounded extension of $T_i(t)A_r$, will be given in the proof of Theorem 4.1.
**Theorem 4.1.** Let $x_0 \in X$, $f(\cdot) \in L^\text{loc}_p(R^+; X)$, $g(\cdot) \in L^p_{\text{loc}}(-mt; 0; X)$ and let assumptions $H_0$, $H_1$ and $H_2'$ with $1/p' + 1/q' = 1$ be satisfied. Then the mild solution $x(t; x_0, f, g)$ is given by

$$x(t; x_0, f, g) = \sum_{i=1}^{k} T_i(t - (i - 1) \tau) x_0$$

$$+ \sum_{i=1}^{k} \int_0^{t-(i-1)\tau} T_i(t - (i - 1) \tau - s) f(s) \, ds$$

$$+ \sum_{i=1}^{k} \sum_{r=1}^{m} \int_{-rt}^{t-(i-1)\tau} (T_i(t - (i - 1 + r) \tau - s) A_r) \hat{g}(s) \, ds, \quad (4.1)$$

where $t \in [(k-1)\tau, k\tau]$ and

$$\hat{g}(s) = g(s), \quad s \in [-mt, 0)$$

$$= 0, \quad s \in [0, \infty).$$

**Proof.** Since condition $H_2'$, $q' \in [1, \infty]$ is stronger than $H_1$, the operators $T_i(t)$ can be constructed. First we shall show that the right-hand side of (4.1) is meaningful. It is evident that the first term is well defined. In the second term the integrals are well defined as Bochner integrals because of the strong continuity of $T_i(t)$ and condition (2.4). To prove that the integrals in the third term are well defined as Bochner integrals, we have to show the following property $C_1$.

$C_1$. For any $t \geq 0$, $i = 1, 2, \ldots$ and $r = 1, \ldots, m$, there exists $M_{t,i,r}(\cdot)$ in $L^q_{\text{loc}}[0, t]$ such that

$$\|(T_i(s) A_r)\| \leq M_{t,i,r}(s) \quad \text{for a.a. } s \in [0, t]. \quad (4.2)$$

The definition of $(T_i(s) A_r)$ in $C_1$ for $i \geq 2$, $r = 1, \ldots, m$ will be given later (note that $T_i(s) = T(s)$). It will be proved that these operators can be defined for a.a. $s \in [0, t]$ if $H_2'$ is satisfied.

Let $t \geq 0$ be fixed. Then by $H_2'$, $(4.2)_{i,r,t}$ holds for $i = 1$, $r = 1, \ldots, m$ and all $t \in [0, \tau]$. If $t \geq \tau$, then by the semi-group property of $T(t)$ we have

$$\|(T_i(s) A_r)\| \leq M_r(s) \quad \text{for a.a. } s \in [0, \tau)$$

$$\leq KM_r(s - \tau) \quad \text{for a.a. } s \in [\tau, 2\tau)$$

$$\vdots$$

$$\leq KM_r(s - (k - 1) \tau) \quad \text{for a.a. } s \in [(k - 1) \tau, t), \quad (4.3)$$

where $K$ is a bound of $\|T(s)\|$ on $[0, t]$ and $M_r(s)$ is the function defined in $H_2'$. The function defined by the right-hand side of (4.3) is clearly an element
of $L_q[0, t]$ and this shows that $(4.2)_{1, r, t}$ is valid for $i = 1, r = 1, \ldots, m$ and all $t \geq 0$. It is easy to see that $M_{t, 1, r}(-)$ in $(4.2)_{1, r, t}$ belongs to $L_1[0, t] \cap L_q[0, t]$ for all $r$ and $t$.

We now consider the integral

$$
\int_0^s (T(s - \bar{s}) A_1)(T(\bar{s}) A_r) x \, d\bar{s} \quad \text{for } x \in X. \quad (4.4)
$$

The integrand in $(4.4)$ is strongly measurable on $[0, s]$ by assumption $H'_1$. If this integrand is Bochner integrable, then the integral representation $(4.4)$ is meaningful and we can define the operator $(T_2(s) A_r)$ by such integrals. Therefore, it is obvious that $(T_2(s) A_r)$ can be defined in this sense if the convolution

$$
M_{t, 1, 1} * M_{t, 1, r}(s) = \int_0^s M_{t, 1, 1}(s - \bar{s}) M_{t, 1, r}(\bar{s}) \, d\bar{s}
$$

is finite because of the inequality

$$
\int_0^s \| (T(s - \bar{s}) A_1)(T(\bar{s}) A_r) x \| \, d\bar{s} \leq \left( \int_0^s \| (T(s - \bar{s}) A_1) \| \cdot \| (T(\bar{s}) A_r) \| \, d\bar{s} \right) \| x \| \leq M_{t, 1, 1} * M_{t, 1, 1}(s) \| x \|.
$$

It follows from the Hausdorff–Young inequality that

$$
M_{t, 1, 1} * M_{t, 1, r}(s) < \infty \quad \text{for a.a. } s \in [0, t]
$$

and

$$
\| M_{t, 1, 1} * M_{t, 1, r}(-) \|_{L_q[0, t]} \leq \| M_{t, 1, 1}(-) \|_{L_1[0, t]} \cdot \| M_{t, 1, r}(-) \|_{L_q[0, t]}.
$$

Here we use assumption $H'_2$. So it is verified that $(T_2(s) A_r)$ is defined as a bounded operator on $X$ for a.a. $s \in [0, t]$ and satisfies the inequality

$$
\| (T_2(s) A_r) \| \leq M_{t, 2, r}(s) \quad \text{for a.a. } s \in [0, t], \quad (4.5)
$$

where $M_{t, 2, r}(-) = M_{t, 1, 1} * M_{t, 1, r}(-) \in L_q[0, t]$. Hence $(4.2)_{2, r, t}$ is shown for all $r$ and $t$. If $x \in D(A_r)$, then by the definition of $T_2(s)$ in $(3.7)$, we have

$$
T_2(s) A_r x = \left( \int_0^s (T(s - \bar{s}) A_1)(T(\bar{s}) A_r) \, d\bar{s} \right) A_r x \quad - \int_0^s (T(s - \bar{s}) A_1)(T(\bar{s}) A_r x \, d\bar{s}.
$$
This means that for a.a. $s \in [0, t]$, $(T_i(s) A_r)$ gives a bounded extension of $T_i(s) A_r$.

By induction with respect to $i$ using formula (3.6), we see that $(4.2)_{i,r,t}$ is true for all $i, r$ and $t$. Here the meaning of $(T_i(s) A_r)$, $i \geq 3$, is understood in an analogous way as above. This proves the property $C_1$, and hence the right-hand side of expression (4.1) is shown to be meaningful. In what follows we simply denote the operator $(T_i(s) A_r)$ by $T_i(s) A_r$, as in Section 2.

Next we shall prove equality (4.1) by mathematical induction on $k$. It is clear from (2.6) that (4.1) is true for $k = 1$. Let $x_k(t; x_0), x_k(t; f)$ and $x_k(t; g)$ be the function translated by $(k - 1) \tau$ of the first, second and third term of the right-hand side of (4.1), respectively.

Put

$$y_{k+1}(t; x_0) = T(t) x_{k+1}(\tau; x_0)$$

$$+ \sum_{r=1}^{m} \int_{0}^{t} T(t - s) A_r \begin{cases} x_{k-r+1}(s; x_0), & 1 \leq r \leq k \\ 0, & \text{otherwise} \end{cases} ds.$$  

Then

$$y_{k+1}(t; x_0) = \sum_{i=1}^{k} T(t) T_i((k + 1 - i) \tau) x_0 + \sum_{r=1}^{m} \int_{0}^{t} T(t - s) A_r \begin{cases} \sum_{i=1}^{k-r} T_i(s + (k - r - i) \tau) x_0, & 1 \leq r \leq k \\ 0, & \text{otherwise} \end{cases} ds$$

$$= T(t + k\tau) x_0 + \sum_{i=2}^{k} T(t) T_i((k + 1 - i) \tau) x_0$$

$$+ \sum_{i=2}^{k+1} \left( \sum_{(j_1, j_2) \in A(2, i)} \int_{0}^{t} T(t - s) A_{j_1} T_{j_2}(s + (k + 1 - i) \tau) x_0 ds \right).$$  

(4.6)

where $(j_1, j_2) \in A(2, i)$. The integral terms of (4.6) are calculated as follows.

$$\int_{0}^{t} T(t - s) A_{j_1} T_{j_2}(s + (k + 1 - i) \tau) x_0 ds$$

$$= \int_{(k+1-i)\tau}^{t+(k+1-i)\tau} T(t + (k + 1 - i) \tau - s) A_{j_1} T_{j_2}(s) x_0 ds$$

$$= \int_{0}^{t+(k+1-i)\tau} T(t + (k + 1 - i) \tau - s) A_{j_1} T_{j_2}(s) x_0 ds - T(t) \int_{0}^{(k+1-i)\tau} T((k + 1 - i) \tau - s) A_{j_1} T_{j_2}(s) x_0 ds.$$  

(4.7)
Here we use a change of integral variables and the semi-group property of $T(t)$. By (4.7) and the formula (3.6), the last term of (4.6) is given by

$$\sum_{l=2}^{k+1} (T_l(t + (k + 1 - i) \tau) x_0 - T(t) T_l((k + 1 - i) \tau) x_0) \quad (4.8)$$

Hence

$$y_{k+1}(t; x_0) = \sum_{l=1}^{k+1} T_l(t + (k + 1 - i) \tau) x_0 = x_{k+1}(t; x_0). \quad (4.9)$$

Put

$$y_{k+1}(t; f) = T(t) x_k(\tau; f) + \int_0^t T(t - s) f(s + k \tau) ds + \sum_{r=1}^m \int_0^t T(t - s) A_r \begin{cases} x_{k-r+1}(s; f), & 1 \leq r \leq k \\ 0 & \text{otherwise} \end{cases} ds;$$

then by similar computations as above it follows that

$$y_{k+1}(t; f) = \int_0^{k \tau} T(t) T_1(k \tau - s) f(s) ds + \int_{k \tau}^{t+k \tau} T(t + k \tau - s) f(s) ds + \sum_{l=2}^{k+1} \left( \sum_{(2,i)} \int_0^t T(t - s) A_i, \right. \times \int_0^{s+(k+1-i) \tau} T_{j_2}(s + (k + 1 - i) \tau - s_1) f(s_1) ds_1 ds_1 \left. \right)$$

$$= I_1 + I_2 + \sum_{l=2}^{k+1} I_{3,l} + \sum_{l=2}^{k+1} \left( \sum_{(2,i)} I_{4,l,j_1,j_2} \right). \quad (4.10)$$

The sum of the first two terms of (4.10) is given by

$$I_1 + I_2 = \int_0^{t+(k+1-1) \tau} T_1(t + (k + 1 - 1) \tau - s) f(s) ds. \quad (4.11)$$

We devide the integral $I_{4,l,j_1,j_2}$ into two parts as

$$I_{4,l,j_1,j_2} = \int_0^t \left( \int_0^{(k+1-1) \tau} + \int_{(k+1-1) \tau}^{s+(k+1-1) \tau} \right) T(t - s) A_{j_1}, \times T_{j_2}(s + (k + 1 - i) \tau - s_1) f(s_1) ds_1 ds$$

$$= J_{1,l,j_1,j_2} + J_{2,l,j_1,j_2}.$$
Note that $J^i_{\Lambda (2,1)} = 0$ if $i = k + 1$. By Fubini's theorem concerning Bochner integrals and a change of integral variables $s \rightarrow s + (k + 1 - i) \tau - s_1$, it follows that

$$J^i_{1,1,1} = \int_0^{(k+1-i)\tau} \int_0^t T(t-s)A_{j_1} \times T_{j_2}(s + (k + 1 - i) \tau - s_1) \, ds \, f(s_1) \, ds_1$$

$$= \int_0^{(k+1-i)\tau} \left( \int_0^{t+(k+1-i)\tau-s_1} T(t+(k + 1 - i) \tau - s_1-s)A_{j_1} \times T_{j_2}(s) \, ds \right) f(s_1) \, ds_1$$

$$- \int_0^{(k+1-i)\tau} T(t) \left( \int_0^{(k+1-i)\tau-s_1} T((k + 1 - i) \tau - s_1-s)A_{j_1} \times T_{j_2}(s) \, ds \right) f(s_1) \, ds_1, \quad (4.12)$$

and hence by (3.6)

$$\sum_{\Lambda (2,1)} J^i_{1,1,1} = \int_0^{(k+1-i)\tau} T(t+(k + 1 - i) \tau - s) f(s) \, ds$$

$$- \int_0^{(k+1-i)\tau} T(t) T((k + 1 - i) \tau - s) f(s) \, ds. \quad (4.13)$$

On the other hand, again by a change of integral variables and Fubini's theorem, it follows that

$$J^i_{2,1,1} = \int_0^t \left( \int_0^{t-s_1} T(t-s_1-s)A_{j_1} T_{j_2}(s) \, ds \right) \times f(s_1 + (k + 1 - i) \tau) \, ds_1,$$

and hence again by (3.6)

$$\sum_{\Lambda (2,1)} J^i_{2,1,1} = \int_0^{(k+1-i)\tau} T(t+(k + 1 - i) \tau - s) f(s) \, ds. \quad (4.14)$$

Therefore, by (4.10), (4.11), (4.13) and (4.14) we have that
\[ y_{k+1}(t; f) = I_1 + I_2 + \sum_{i=2}^{k} I_{3,i} + \sum_{i=2}^{k+1} \left( \sum_{\lambda(2,i)} I_{4,i,j_1,j_2} \right) \]
\[ = \sum_{i=1}^{k+1} \int_{0}^{i(1-i)} T_i(t + (k + 1 - i) \tau - s) f(s) \, ds \]
\[ = x_{k+1}(t; f). \quad (4.15) \]

We can verify by similar calculations that
\[ y_{k+1}(t; g) = T(t) x_k(t; g) \]
\[ + \sum_{r=1}^{m} \int_{0}^{t} T(t - s) A_r \left\{ \begin{array}{ll}
  x_{k-r+1}(s; g), & 1 \leq r < k \\
  g(s - (r - k) \tau), & \text{otherwise}
\end{array} \right\} \, ds \]
\[ = x_{k+1}(t; g). \quad (4.16) \]

From (4.9), (4.15) and (4.16) we see that (4.1) is also true on the interval \([k \tau, (k + 1) \tau]\), and this completes the proof by mathematical induction.

We shall now give another representation of the mild solution in terms of the fundamental solution \(G(t)\) which is well known as a variation of constants formula and has a simpler form than that given in Theorem 4.1. Before giving the form, we note some property which is needed in the integral representation of the mild solution.

Under assumption \(H_T'\), the following fact \(C_2\) holds.

\(C_2\). For any \(t \geq 0\) and \(r = 1, \ldots, m\), there exists \(M_{t,r}(\cdot) \in L_q([-\infty, 0])\) such that
\[ \| (G(t - r \tau - s) A_r) \| \leq M_{t,r}(s) \quad \text{for a.a. } s \in [-\infty, 0], \]
where \((G(s) A_r) = 0\) if \(s < 0\).

The sense of \((G(s) A_r)\) is that given as \(\sum_{i=1}^{j} T_i(s - (i - 1) \tau) A_r\) if \(s \in [(j - 1) \tau, j \tau]\). To assure \(C_2\), we first consider the case \(t - r \tau < 0\). In this case, by (3.8) we have
\[ \| (G(t - r \tau - s) A_r) \| \leq 0 \quad \text{for } s \in (t - r \tau, 0] \]
\[ \leq \sum_{i=1}^{j} M_{t,i,r}(t - (r + i - 1) \tau - s) \quad \text{for a.a. } s \in I_j \quad (j = 1, \ldots, j_t), \quad (4.17) \]
where \(I_j = (t - (r + j) \tau, t - (r + j - 1) \tau] \cap [-\infty, 0]\), \(j_t\) is the smallest natural number such that \((j + 1) \tau \geq t\) and \(M_{t,i,r}(\cdot)\) is the function given in \(C_1\). Set \(M_{t,r}(\cdot)\) the function defined by the right-hand side of (4.17). Then \(M_{t,r}(\cdot) \in L_q([-\infty, 0])\), and hence \(C_2\) is assured. In the case \(t - r \tau \geq 0\), the
inequality in $C_2$ is also verified by choosing some appropriate function $M_{t,r}(\cdot)$ in $L_q([-rt,0])$.

**Theorem 4.2.** Under the same assumptions in Theorem 4.1, the mild solution $x(t;x_0,f,g)$ is given by

$$
x(t;x_0,f,g) = G(t)x_0 + \int_0^t G(t-s)f(s)\,ds + \sum_{r=1}^{m} \int_{-rt}^{0} (G(t-rt-s)A_r)g(s)\,ds,
$$

(4.18)

where $(G(s)A_r) = O$ if $s < 0$.

**Proof.** The right-hand side of (4.18) is meaningful by fact $C_2$. We shall show equality (4.18). Let $t \in [(k-1)\tau, k\tau]$ be fixed. Then it follows from (3.8) that

$$
\int_0^t G(t-s)f(s)\,ds = \sum_{i=1}^{k-1} \int_{-i\tau}^{t-(i-1)\tau} G(t-s)f(s)\,ds + \int_{0}^{t-(k-1)\tau} G(t-s)f(s)\,ds
$$

$$
= \sum_{i=1}^{k-1} \int_{-i\tau}^{t-(i-1)\tau} T_i(t-(j-1)\tau-s)f(s)\,ds + \int_{0}^{t-(k-1)\tau} T_i(t-(i-1)\tau-s)f(s)\,ds
$$

$$
= \sum_{i=1}^{k} \left( \sum_{j=1}^{k-i} \int_{-i\tau}^{t-(i+j-2)\tau} T_i(t-(i-1)\tau-s)f(s)\,ds + \int_{0}^{t-(k-1)\tau} T_i(t-(i-1)\tau-s)f(s)\,ds \right)
$$

$$
= \sum_{i=1}^{k} \int_{0}^{t-(i-1)\tau} T_i(t-(i-1)\tau-s)f(s)\,ds.
$$

By analogous calculations we can verify that the last term of (4.18) equals to the last term of (4.1). Hence (4.18) is proved by Theorem 4.1.

5. **Examples and Applications**

Throughout this section, we suppose that $X$ is a complex separable Hilbert space with inner product $(\cdot,\cdot)$ and $A$ is self-adjoint on $X$. Then the semi-
group \( T(t) \) generated by \( A \) is also self-adjoint. Therefore, from Nagy's theorem it follows that there exists a resolution of unity \( \{ E(\cdot) \} \) on some half infinite interval \((-\infty, c]\) corresponding to \( T(t) \) such that \( T(t) \) is given by

\[
T(t) = \int_{-\infty}^{c} e^{\lambda t} dE(\lambda)
\]

if there exists \( t > 0 \) such that \( T(t)x = 0 \) implies \( x = 0 \) (cf. [6, 9]).

We consider the following delay-differential equation

\[
\frac{dx(t)}{dt} = Ax(t) + A_1x(t-\tau),
\]

where \( A \) and \( A_1 \) satisfy assumptions \( H_2^1 \) and \( H_3 \).

**Example 1.** Let \( A \) have the compact resolvent \( R(\lambda_0; A) \) for some \( \lambda_0 \in \rho(A) \). Then there exists a sequence of real distinct eigenvalues \( \lambda_n \) and corresponding eigenfunctions \( \psi_{nj} \in D(A) \) of \( A, n = 1, 2, \ldots, j = 1, \ldots, m_n \), such that

(i) there exists a real constant \( c \) such that

\[ c \geq \lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n > \cdots \quad \text{and} \quad \lim_{n \to \infty} \lambda_n = -\infty; \]

(ii) the system \( \{ \psi_{nj} : j = 1, \ldots, m_n, \quad n = 1, 2, \ldots \} \) is a complete orthonormal system in \( X \); and

(iii) the spectrum \( \sigma(A) = \{ \lambda_n : n = 1, 2, \ldots \} \) and for any \( \lambda_n \in \sigma(A) \), \( \psi_{nj} \) satisfies \( A\psi_{nj} = \lambda_n\psi_{nj}, j = 1, \ldots, m_n \).

Therefore, by (i)–(iii) and (5.1), we have that the semi-group \( T(t) \) is analytic and is represented by

\[
T(t)x = \sum_{n=1}^{\infty} e^{\lambda_n t} \sum_{j=1}^{m_n} (x, \psi_{nj}) \psi_{nj}, \quad t \geq 0 \quad \text{for each} \quad x \in X. \quad (5.3)
\]

Since \( A_1 \) satisfies \( H_3 \), the fundamental solution \( G(t) \) of (5.2) is given by

\[
G(t)x = \sum_{n=1}^{\infty} \sum_{i=1}^{k} \frac{(t-(i-1)\tau)^{i-1}}{(i-1)!} e^{\lambda_n(i-(i-1)\tau)} \sum_{j=1}^{m_n} (A_1^{i-1}x, \psi_{nj}) \psi_{nj}, \quad t \in [(k-1)\tau, k\tau],
\]

for \( x \in D(A_1^{k-1}) \). This formula immediately follows from (3.14) and (5.3).

Let us consider the following particular case. Let \( X = L_2[0,1] \) and let \( A = (\partial/\partial \xi)^2, \quad \xi \in [0,1], \) be the Laplace operator with domain \( D(A) = \)
\( x \in L_2[0, 1] \): \((\partial/\partial \xi)^2 x \in L_2[0, 1]\) and \(x(0) = x(1) = 0\). Here \((\partial/\partial \xi)^2\) is understood in the sense of distributions. It is well known that \(A\) is self-adjoint and \(-A\) is of type \((\pi/2, 1)\) (see Remark 2.1). Let \(A_1 = (-A)^{1/2}\). Then the fundamental solution \(G(t)\) in (5.4) is represented by

\[
G(t) x = \sum_{n=1}^{\infty} \sum_{i=1}^{k} \frac{2n\pi (t - (i - 1)\tau)}{(i - 1)!} \exp(-n^2\tau^2 (t - (i - 1)\tau)(x, \sin n\pi \xi)) \sin n\pi \xi, \quad t \in [(k - 1)\tau, k\tau],
\]

for \(x \in D((-A)^{(k-1)/2})\).

**Example 2.** Let \(A_1 = I\). Then the fundamental solution \(G(t)\) of (5.2) is given by

\[
G(t) x = \sum_{i=1}^{k} \frac{(t - (i - 1)\tau)^{i-1}}{(i - 1)!} \int_{-\infty}^{c} e^{\lambda(t - (i - 1)\tau)} d\nu(\lambda),
\]

\(t \in [(k - 1)\tau, k\tau]\). (5.5)

We consider the following case. Let \(X = L_2[\mathbb{R}^n]\) and let \(A = \sum_{j=1}^{n} (\partial/\partial \xi_j)^2\), \(\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n\), be the Laplace operator with domain \(D(A) = \{x \in L_2[\mathbb{R}^n]: \sum_{j=1}^{n} (\partial/\partial \xi_j)^2 x \in L_2[\mathbb{R}^n]\}\). Then \(A\) is self-adjoint on \(X\). In this case, \(G(t), t \in [(k - 1)\tau, k\tau]\), in (5.5) is represented by

\[
(G(t) x)(\xi) = \sum_{i=1}^{k} \frac{(t - (i - 1)\tau)^{i-1}}{(i - 1)!} \prod_{i=1}^{\infty} \int_{-\infty}^{\infty} \frac{(t - (i - 1)\tau - n^2)}{(2\sqrt{\pi})^n} \exp\left(-\frac{\sum_{j=1}^{n} (\xi_j - \eta_j)^2}{4(t - (i - 1)\tau)}\right) x(\eta_1, ..., \eta_n) \, d\eta_1 \cdots d\eta_n
\]

for a.e. \(\xi \in \mathbb{R}^n\),

for any \(x(\cdot) \in L_2[\mathbb{R}^n]\) satisfying \(|x(\xi)| \leq M e^{\|\xi\|} \), \(\xi \in \mathbb{R}^n\) (see Friedman [8] and [6]).

**References**