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Counting dimer coverings on self-similar Schreier graphs[☆]

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Dedicated to Toni Machì

ABSTRACT

We study partition functions for the dimer model on families of finite graphs converging to infinite self-similar graphs and forming approximation sequences to certain well-known fractals. The graphs that we consider are provided by actions of finitely generated groups by automorphisms on rooted trees, and thus their edges are naturally labeled by the generators of the group. It is thus natural to consider weight functions on these graphs taking different values according to the labeling. We study in detail the well-known example of the Hanoi Towers group $H^{(3)}$, closely related to the Sierpiński gasket.

1. Introduction

The dimer model is widely studied in different areas of mathematics ranging from combinatorics to probability theory to algebraic geometry. It originated in statistical mechanics where it was introduced in the purpose of investigating absorption of diatomic molecules on surfaces. In particular, one wants to find the number of ways in which diatomic molecules, called dimers, can cover a doubly periodic lattice, so that each dimer covers two adjacent lattice points and no lattice point remains uncovered. First exact results on the dimer model in a finite rectangle of $\mathbb{Z}^2$ were obtained by Kasteleyn [12,13] and independently Temperley and Fisher [17] in the 60s. A much more recent breakthrough is the solution of the dimer model on arbitrary planar bipartite periodic graphs by Kenyon, Okounkov and Sheffield [15]. We refer to the lecture notes by Kenyon [14] for an introduction into the dimer model.

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Let $Y = (V, E)$ be a finite graph with the vertex set V and the edge set E. A *dimer* is a graph consisting of two vertices connected by a non-oriented edge. A *dimer covering* D of Y is an arrangement of dimers on Y such that each vertex of V is the endpoint of exactly one dimer. In other words, dimer coverings correspond exactly to *perfect matchings* in Y. Let $\mathcal{D}$ denote the set of dimer coverings of Y, and let $w : E \longrightarrow \mathbb{R}_+$ be a weight function defined on the edge set of Y. The physical meaning of the weight function can be, for example, the interaction energy between the atoms in a diatomic molecule. We associate with each dimer covering $D \in \mathcal{D}$ its weight

$$W(D) := \prod_{e \in D} w(e),$$

i.e., the product of the weights of the edges belonging to D. To each weight function w on Y corresponds the Boltzmann measure on $\mathcal{D}$, $\mu = \mu(Y, w)$ defined as

$$\mu(D) = \frac{W(D)}{\Phi(w)}.$$

The normalizing constant that ensures that this is a probability measure is one of the central objects in the theory, it is called the *partition function*:

$$\Phi(w) := \sum_{D \in \mathcal{D}} W(D).$$

If the weight function is constant equal to 1, the partition function just counts all the dimer coverings on Y.

For a growing sequence of finite graphs, $\{Y_n\}_n$, one can ask whether the limit

$$\lim_{n \to \infty} \frac{\log(\Phi_n(w_n))}{|V(Y_n)|}$$

exists, where w_n is the weight function on Y_n, and $\Phi_n(w_n)$ is the associated partition function. If yes, it is then called the *thermodynamic limit*. For $w_n \equiv 1$, it specializes to the *entropy* of the absorption of diatomic molecules per site.

Let us recall the method developed by Kasteleyn [12] to compute the partition function of the dimer model on finite planar graphs. It consists in, given a finite graph $Y = (V, E)$, constructing an anti-symmetric matrix such that the absolute value of its Pfaffian is the partition function of the dimer model on Y. Recall that the Pfaffian $Pf(M)$ of an anti-symmetric matrix $M = (m_{ij})_{i,j=1,\ldots,N}$, with N even, is

$$Pf(M) := \sum_{\pi \in Sym(N)} \operatorname{sgn}(\pi) m_{p_1 p_2} \cdots m_{p_{N-1} p_N},$$

where the sum runs over all permutations $\pi = \begin{pmatrix} 1 & 2 & \cdots & N \\ p_1 & p_2 & \cdots & p_N \end{pmatrix}$ such that $p_1 < p_2, p_3 < p_4, \ldots, p_{N-1} < p_N$ and $p_1 < p_3 < \cdots < p_{N-1}$. One has $(Pf(M))^2 = \det(M)$.

Given an orientation on Y and a weight function w on E, consider the oriented adjacency matrix $A = (a_{ij})_{i,j=1,\ldots,|V|}$ of (Y, w) with this orientation. It is of course anti-symmetric.

Definition 1.1. A good orientation on Y is an orientation of the edges of Y such that the number of clockwise oriented edges around each face of Y is odd.

Theorem 1.2 (*[12]*).

1. *Let $Y = (V, E)$ be a planar graph with a good orientation, let w be a weight function on E. If A is the associated oriented adjacency matrix, then*

$$\Phi(w) = |Pf(A)|.$$

2. *If Y is planar, a good orientation on Y always exists.*

In this paper we apply the Kasteleyn method to study dimers partition functions on families of finite graphs that form approximating sequences for some well-known fractals, and on the other hand converge in local convergence to interesting self-similar graphs. The graphs that we consider are Schreier graphs of certain finitely generated groups and thus come naturally endowed with a labeling of the edges of the graph by the generators of the group. It is therefore natural to think about the edges with different labels as being of different type, and to consider weight functions on them that take different values according to the type of the edge.

We now turn to Schreier graphs of self-similar groups and recall some basic facts and definitions. Let T be the infinite regular rooted tree of degree q, i.e., the rooted tree where each vertex has q offsprings. Every vertex of the n-th level of the tree can be regarded as an element of the set X^n of words of length n over the alphabet $X = \{0, 1, \ldots, q-1\}$; the set X^ω of infinite words over X can be identified with the set ∂T of infinite geodesic rays in T emanating from the root. Now let $G < \mathrm{Aut}(T)$ be a group acting on T by automorphisms, generated by a finite symmetric set $S \subset G$. Throughout the paper we will assume that the action of G is transitive on each level of the tree (note that any action by automorphisms is level-preserving).

Definition 1.3. The n-th *Schreier graph* Σ_n of the action of G on T, with respect to the generating set S, is a (labeled, oriented) graph with $V(\Sigma_n) = X^n$, and edges (u, v) between vertices u and v such that u is moved to v by the action of some generator $s \in S$. The edge (u, v) is then labeled by s.

For an infinite ray $\xi \in \partial T$, the *orbital Schreier graph* Σ_ξ has vertex set $G \cdot \xi$ and the edge set determined by the action of generators on this orbit, as above.

The graphs Σ_n are Schreier graphs of stabilizers of vertices of the n-th level of the tree and the graphs Σ_ξ are Schreier graphs of stabilizers of infinite rays. It is not difficult to see that the orbital Schreier graphs are infinite and that the finite Schreier graphs $\{\Sigma_n\}_{n=1}^{\infty}$ form a sequence of graph coverings. Finite Schreier graphs converge to infinite Schreier graphs in the space of rooted (labeled) graphs with local convergence (rooted Gromov–Hausdorff convergence [11, Chapter 3]). More precisely, for an infinite ray $\xi \in X^\omega$ denote by ξ_n the n-th prefix of the word ξ. Then the sequence of rooted graphs $\{(\Sigma_n, \xi_n)\}$ converges to the infinite rooted graph (Σ_ξ, ξ) in the space $\mathcal{X}$ of (rooted isomorphism classes of) rooted graphs endowed with the following metric: the distance between two rooted graphs (Y_1, v_1) and (Y_2, v_2) is

$$Dist((Y_1, v_1), (Y_2, v_2)) := \inf\left\{\frac{1}{r+1}; B_{Y_1}(v_1, r) \text{ is isomorphic to } B_{Y_2}(v_2, r)\right\},$$

where $B_Y(v, r)$ is the ball of radius r in Y centered in v.

Definition 1.4. A finitely generated group $G < \mathrm{Aut}(T)$ is *self-similar* if, for all $g \in G, x \in X$, there exist $h \in G, y \in X$ such that

$$g(xw) = yh(w),$$

for all finite words w over the alphabet X.

Self-similarity implies that G can be embedded into the wreath product $Sym(q) \wr G$, so that any automorphism $g \in G$ can be represented as

$$g = \tau(g_0, \ldots, g_{q-1}), \tag{1}$$

where $\tau \in Sym(q)$ describes the action of g on the first level of T and $g_i \in G, i = 0, \ldots, q-1$ is the restriction of g on the full subtree of T rooted at the vertex i of the first level of T (observe that any such subtree is isomorphic to T). Hence, if $x \in X$ and w is a finite word over X, we have

$$g(xw) = \tau(x)g_x(w).$$

See [16] and references therein for more information about this interesting class of groups, also known as *automata groups*.

In many cases, self-similarity of a group action allows to formulate a number of rules that allow to construct inductively the sequence of Schreier graphs $\{\Sigma_n\}_{n\geq 1}$ [2,16] and thus to describe inductively the action of the group on the n-th level of the tree. More precisely, the action of $g \in G$ on the n-th level can be represented by a permutation matrix of size q, whose entries are matrices of size q^{n-1}. If g is as in (1), the nonzero entries of the matrix are at position $(i, \tau(i))$ and correspond to the action of the restriction g_i of g on the subtree of depth $n-1$ rooted at i, for each $i = 0, \ldots, q-1$.

In the next sections we will systematically use this description. Our idea is to define recursively an oriented adjacency matrix associated with the action of the generators on the n-th level, with some prescribed signs. The rows and columns of this matrix are indexed by the words of length n over the alphabet $\{0, 1, \ldots, q-1\}$, in their lexicographic order. The signs can be interpreted as corresponding to a good orientation of the graph Σ_n, in the sense of Kasteleyn. This allows to compute the partition function and the number of dimer coverings by studying the Pfaffian of this matrix.

In this paper we compute the partition function of the dimer model on the following examples of planar Schreier graphs associated with self-similar actions:

- the first Grigorchuk group of intermediate growth (see [8] for a detailed account and further references);
- the Basilica group that can be described as the iterated monodromy group of the complex polynomial $z^2 - 1$ (see [16] for connections of self-similar groups to complex dynamics);
- the Hanoi Towers group $H^{(3)}$ whose action on the ternary tree models the famous Hanoi Towers game on three pegs, see [9], and whose Schreier graphs are closely related to the Sierpiński gasket. Let us mention that counting dimers on the Schreier graphs of the Hanoi towers group $H^{(3)}$ is related to the computation of the partition function for the Ising model on the Sierpiński triangle, via Fisher's correspondence [7], see Section 4.5 in our paper [6], devoted to the Ising model on the self-similar Schreier graphs.

Finally we also compute the partition function of the dimer model on the (finite approximations of) the Sierpiński triangle. These graphs cannot be labeled so as to become Schreier graphs of a self-similar group, but they are very similar to the Schreier graphs of the group $H^{(3)}$. They have a few natural labeling of the edges in three different types, of which we describe three, and provide computations for two of those.

1.1. Plan of the paper

The rest of the paper is structured as follows. In Section 2 we study the dimer model on the Schreier graphs associated with the action of the Grigorchuk group and of the Basilica group on the rooted binary tree. Even if the model on these graphs can be easily computed directly, we prefer to apply the general Kasteleyn theory: the partition function at each finite level is described, the thermodynamic limit and the entropy are explicitly computed. In Section 3 the dimer model is studied on the Schreier graphs of the Hanoi Towers group $H^{(3)}$. First, we follow a combinatorial approach using recursion and the property of self-similarity of these graphs (see Section 3.2). A recursive description of the partition function is given in Theorem 3.1. The thermodynamic limit is not explicitly computed, although its existence is proven in two particular cases (see Propositions 3.4 and 3.5). Then, the problem is studied by using Kasteleyn method (Section 3.3): the Pfaffian of the oriented adjacency matrix is recursively investigated via the Schur complement. The description of the partition function that we give in Theorem 3.7 uses iterations of a rational map. In Section 4, the dimer model is studied on finite approximations of the well-known Sierpiński gasket: these are self-similar graphs closely related to the Schreier graphs of the group $H^{(3)}$. Two different weight functions on the edges of these graphs are considered and for both of them the partition function, the thermodynamic limit and the entropy are computed. In Section 5 we perform, for the Schreier graph of $H^{(3)}$ and the Sierpiński gasket, a statistical analysis of the random variable defined as the number of occurrences of a fixed label in a random dimer covering.

2. The partition function of the dimer model on the Schreier graphs of the Grigorchuk group and of the Basilica group

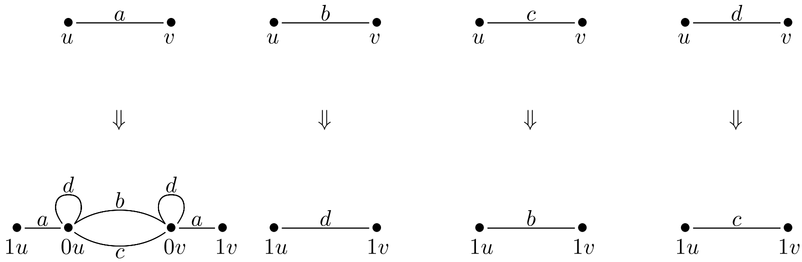
In this section we study the dimer model on two examples of Schreier graphs: the Schreier graphs of the Grigorchuk group and of the Basilica group. Even if in these cases the problem can be easily solved combinatorially, we prefer to apply here the Kasteleyn theory because we will follow the same strategy in the next sections to solve the problem on more complicated graphs.

2.1. The Grigorchuk group

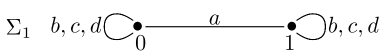
Let us start with the Grigorchuk group: this is the self-similar group acting on the rooted binary tree generated by the automorphisms:

$$a = \epsilon(id, id), \qquad b = e(a, c), \qquad c = e(a, d), \qquad d = e(id, b),$$

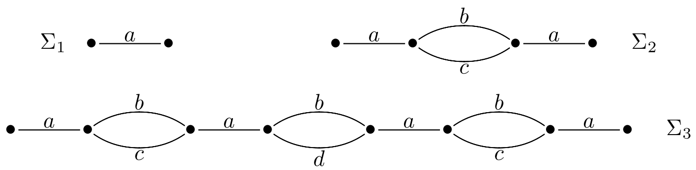
where e and ϵ are, respectively, the trivial and the non-trivial permutations in $Sym(2)$ (observe that all the generators are involutions). The following substitutional rules describe how to construct recursively the graph Σ_{n+1} from Σ_n, starting from the Schreier graph of the first level Σ_1 [1,8]. More precisely, the construction consists in replacing the labeled subgraphs of Σ_n on the top of the picture by new labeled graphs (on the bottom).



starting from



In the study of the dimer model on these graphs, we consider the graphs without loops. We keep the notation Σ_n for these graphs. The following pictures give examples for $n = 1, 2, 3$.



In general, the Schreier graph Σ_n, without loops, has a linear shape and it has 2^{n-1} simple edges, all labeled by a, and $2^{n-1} - 1$ cycles of length 2 whose edges are labeled by b, c, d.

What we need in order to apply the Kasteleyn theory is an adjacency matrix giving a good orientation to Σ_n. We start by providing the (unoriented weighted) adjacency matrix Δ_n of Σ_n, which refers to the graph with loops, that one can easily get by using the self-similar definition of the generators of the group. It is defined by putting

$$a_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad b_1 = c_1 = d_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and, for every $n \geq 2$,

$$a_n = \begin{pmatrix} 0 & I_{n-1} \\ I_{n-1} & 0 \end{pmatrix}, \qquad b_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & c_{n-1} \end{pmatrix},$$

$$c_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & d_{n-1} \end{pmatrix}, \qquad d_n = \begin{pmatrix} I_{n-1} & 0 \\ 0 & b_{n-1} \end{pmatrix},$$

where a_n, b_n, c_n, d_n and I_n are matrices of size 2^n. If we put $A_n = aa_n$, $B_n = bb_n$, $C_n = cc_n$ and $D_n = dd_n$, then Δ_n is given by

$$\Delta_n = A_n + B_n + C_n + D_n = \begin{pmatrix} ba_{n-1} + ca_{n-1} + dI_{n-1} & aI_{n-1} \\ aI_{n-1} & bc_{n-1} + cd_{n-1} + db_{n-1} \end{pmatrix}.$$

We want now to modify Δ_n in order to get an oriented adjacency matrix Δ'_n for Σ_n, corresponding to a good orientation in the sense of Kasteleyn. To do this, it is necessary to delete the nonzero diagonal entries in Δ_n (this is equivalent to delete loops in the graph) and to construct an anti-symmetric matrix whose entries coincide, up to the sign, with the corresponding entries of Δ_n. Finally, we have to verify that each cycle of Σ_n, with the orientation induced by Δ'_n, has an odd number of edges clockwise oriented. So let us define the matrices

$$a'_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad b'_1 = c'_1 = d'_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, for every $n \geq 2$, we put

$$a'_n = \begin{pmatrix} 0 & I_{n-1} \\ -I_{n-1} & 0 \end{pmatrix}, \qquad b'_n = \begin{pmatrix} a'_{n-1} & 0 \\ 0 & c'_{n-1} \end{pmatrix},$$
$$c'_n = \begin{pmatrix} a'_{n-1} & 0 \\ 0 & d'_{n-1} \end{pmatrix}, \qquad d'_n = \begin{pmatrix} I_{n-1} & 0 \\ 0 & b'_{n-1} \end{pmatrix}.$$

For each n, we put $A'_n = aa'_n$, $B'_n = bb'_n$, $C'_n = cc'_n$ and $D'_n = dd'_n$. Moreover, set

$$J_1 = \begin{pmatrix} b+c+d & 0 \\ 0 & b+c+d \end{pmatrix} \quad \text{and} \quad J_n = \begin{pmatrix} dI_{n-1} & 0 \\ 0 & \overline{J_{n-1}} \end{pmatrix} \quad \text{for } n \geq 2,$$

where, for every $n \geq 1$, the matrix $\overline{J_n}$ is obtained from J_n with the following substitutions:

$$b \mapsto d \qquad c \mapsto b \qquad d \mapsto c.$$

Define

$$\Delta'_1 = A'_1 + B'_1 + C'_1 + D'_1 - J_1 = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

and, for each $n \geq 2$,

$$\Delta'_n = A'_n + B'_n + C'_n + D'_n - J_n = \begin{pmatrix} ba'_{n-1} + ca'_{n-1} & aI_{n-1} \\ -aI_{n-1} & bc'_{n-1} + cd'_{n-1} + db'_{n-1} - \overline{J_{n-1}} \end{pmatrix}.$$

Note that the matrix J_n is introduced to erase the nonzero diagonal entries of Δ_n, corresponding to loops.

Proposition 2.1. *The matrix Δ'_n induces a good orientation on the Schreier graph Σ_n of the Grigorchuk group.*

Proof. It is easy to show by induction that Δ'_n is anti-symmetric and that each entry of Δ'_n coincides, up to the sign, with the corresponding entry of the adjacency matrix Δ_n of Σ_n, where loops are deleted. Finally, we know that all cycles in the Schreier graph have length 2 and this ensures that each cycle has a good orientation in the sense of Kasteleyn. □

Theorem 2.2. *The partition function of the dimer model on the Schreier graph Σ_n of the Grigorchuk group is*

$$\Phi_n(a, b, c, d) = a^{2^{n-1}}.$$

Proof. It is easy to check, by using the self-similar definition of the generators of the group, that

$$a(1^{n-1}0) = 01^{n-2}0 \qquad b(1^{n-1}0) = c(1^{n-1}0) = d(1^{n-1}0) = 1^{n-1}0$$

and

$$a(1^n) = 01^{n-1} \qquad b(1^n) = c(1^n) = d(1^n) = 1^n.$$

This implies that the vertices $1^{n-1}0$ and 1^n are the (only) two vertices of degree 1 of Σ_n, for each n. This allows us to easily compute $\det(\Delta'_n)$ by an iterated application of the Laplace expansion. We begin from the element a at the entry $(2^{n-1}, 2^n)$, which is the only nonzero element of the column 2^n. So we can "burn" the row 2^{n-1} and the column 2^n. Similarly, row 2^n and column 2^{n-1} can be deleted and a second factor a appears in $\det(\Delta'_n)$. With these deletions, we have "deleted" in the graph all edges going to and coming from the vertex 01^{n-1} (corresponding to the row and column 2^{n-1}). So the vertex 001^{n-2} (which is adjacent to 01^{n-1} in Σ_n) has now degree 1 and on the lines corresponding to it there is just a letter a (or $-a$) corresponding to the edge joining it to 101^{n-2}. Hence, the Laplace expansion can be applied again with respect to this element, and so on. Observe that each simple edge labeled a of Σ_n contributes a^2 to $\det(\Delta'_n)$. The assertion follows since the number of simple edges is 2^{n-1}. □

Corollary 2.3. *The thermodynamic limit is $\frac{1}{2}\log a$. In particular, the entropy of absorption of diatomic molecules per site is zero.*

Proof. A direct computation gives

$$\lim_{n\to+\infty} \frac{\log(\Phi_n(a,b,c,d))}{|V(\Sigma_n)|} = \lim_{n\to+\infty} \frac{\log(\Phi_n(a,b,c,d))}{2^n} = \frac{1}{2}\log a.$$

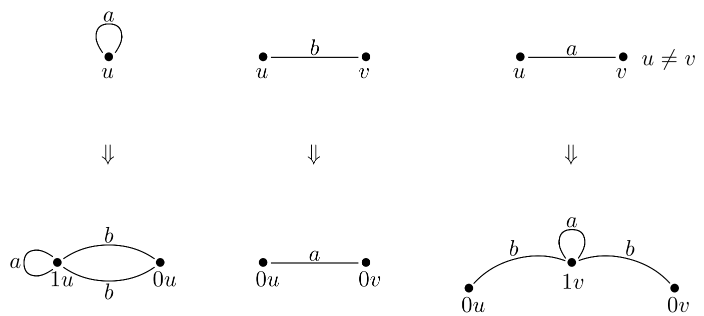
By putting $a = 1$, we get the entropy. □

2.2. The Basilica group

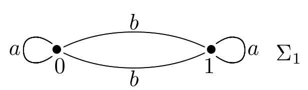
The Basilica group [10] is the self-similar group generated by the automorphisms:

$$a = e(b, id), \qquad b = \epsilon(a, id).$$

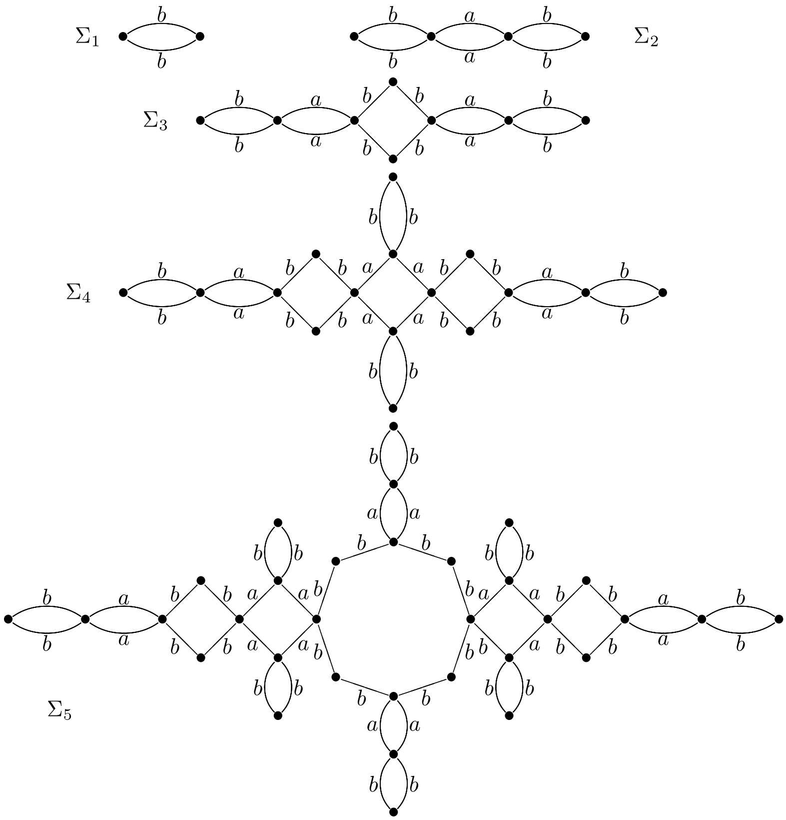
It acts level-transitively on the binary tree, and the following substitutional rules [5] allow to construct inductively Σ_{n+1} from Σ_n,



starting with the Schreier graph Σ_1 on the first level.



We consider here the dimer model on the Schreier graphs of the Basilica group without loops, as in the following pictures, for $n = 1, \ldots, 5$.



It follows from the substitutional rules described above that each Σ_n is a cactus, (i.e., a separable graph whose blocks are either cycles or single edges), and that the maximal length of a cycle in Σ_n is $\lceil \frac{n}{2} \rceil$. We further compute the number of cycles in Σ_n, that will be needed later. Denote by a_j^i the number of cycles of length j labeled by a in Σ_i and, similarly, denote by b_j^i the number of cycles of length j labeled by b in Σ_i.

Proposition 2.4. *For any $n \geq 4$ consider the Schreier graph Σ_n of the Basilica group. For each $k \geq 1$, the number of cycles of length 2^k labeled by a is*

$$a_{2^k}^n = \begin{cases} 2^{n-2k-1} & 1 \leq k \leq \dfrac{n-1}{2} - 1 \\ 2 & k = \dfrac{n-1}{2} \end{cases}, \quad \textit{for } n \textit{ odd},$$

$$a_{2^k}^n = \begin{cases} 2^{n-2k-1} & 1 \leq k \leq \dfrac{n}{2} - 1 \\ 1 & k = \dfrac{n}{2} \end{cases}, \quad \textit{for } n \textit{ even}$$

and the number of cycles of length 2^k labeled by b is

$$b_{2^k}^n = \begin{cases} 2^{n-2k} & 1 \leq k \leq \dfrac{n-1}{2} - 1 \\ 2 & k = \dfrac{n-1}{2} \\ 1 & k = \dfrac{n+1}{2} \end{cases}, \quad \textit{for } n \textit{ odd},$$

$$b^n_{2^k} = \begin{cases} 2^{n-2k} & 1 \leq k \leq \dfrac{n}{2} - 1 \\ 2 & k = \dfrac{n}{2} \end{cases}, \quad \textit{for } n \textit{ even}.$$

Proof. The recursive formulas for the generators imply that, for each $n \geq 3$, one has

$$a^n_2 = b^{n-1}_2 \quad \text{and} \quad b^n_2 = a^{n-1}_1 = 2^{n-2}$$

and in general $a^n_{2^k} = a^{n-2(k-1)}_2$ and $b^n_{2^k} = b^{n-2(k-1)}_2$. In particular, for each $n \geq 4$, the number of 2-cycles labeled by a is 2^{n-3} and the number of 2-cycles labeled by b is 2^{n-2}. More generally, the number of cycles of length 2^k is given by

$$a^n_{2^k} = 2^{n-2k-1}, \qquad b^n_{2^k} = 2^{n-2k},$$

where the last equality is true if $n - 2k + 2 \geq 4$, i.e., for $k \leq \frac{n}{2} - 1$. Finally, for n odd, there is only one cycle of length $2^{\frac{n+1}{2}}$ labeled by b and there are four cycles of length $2^{\frac{n-1}{2}}$, two of them labeled by a and two labeled by b; for n even, there are three cycles of length $2^{\frac{n}{2}}$, two of them labeled by b and one labeled by a. $\square$

Corollary 2.5. *For each $n \geq 4$, the number of cycles labeled by a in Σ_n is*

$$\begin{cases} \dfrac{2^{n-1}+2}{3} & n \textit{ odd}, \\ \dfrac{2^{n-1}+1}{3} & n \textit{ even} \end{cases}$$

and the number of cycles labeled by b in Σ_n is

$$\begin{cases} \dfrac{2^{n}+1}{3} & n \textit{ odd}, \\ \dfrac{2^{n}+2}{3} & n \textit{ even}. \end{cases}$$

The total number of cycles of length ≥ 2 is $2^{n-1}+1$ and the total number of edges, without loops, is $3 \cdot 2^{n-1}$.

Also in this case we construct an adjacency matrix Δ'_n associated with a good orientation of Σ_n, in the sense of Kasteleyn. We first present the (unoriented weighted) adjacency matrix Δ_n of the Schreier graph of the Basilica group. Define the matrices

$$a_1 = a_1^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad b_1 = b_1^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, for every $n \geq 2$, we put

$$a_n = \begin{pmatrix} b_{n-1} & 0 \\ 0 & I_{n-1} \end{pmatrix}, \qquad a_n^{-1} = \begin{pmatrix} b_{n-1}^{-1} & 0 \\ 0 & I_{n-1} \end{pmatrix},$$
$$b_n = \begin{pmatrix} 0 & a_{n-1} \\ I_{n-1} & 0 \end{pmatrix}, \qquad b_n^{-1} = \begin{pmatrix} 0 & I_{n-1} \\ a_{n-1}^{-1} & 0 \end{pmatrix}.$$

If we put $A_n = aa_n$, $A_n^{-1} = aa_n^{-1}$, $B_n = bb_n$ and $B_n^{-1} = bb_n^{-1}$, then Δ_n is given by

$$\Delta_n = A_n + A_n^{-1} + B_n + B_n^{-1} = \begin{pmatrix} a(b_{n-1} + b_{n-1}^{-1}) & b(a_{n-1} + I_{n-1}) \\ b(a_{n-1}^{-1} + I_{n-1}) & 2aI_{n-1} \end{pmatrix}.$$

We modify now Δ_n in order to get the oriented adjacency matrix Δ'_n. To do this, we need to delete the nonzero diagonal entries and to construct an anti-symmetric matrix whose entries are equal, up to the sign, to the corresponding entries of Δ_n. Finally, we have to check that each elementary cycle of Σ_n, with the orientation induced by Δ'_n, has an odd number of edges clockwise oriented. We define the matrices

$$a'_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a'^{-1}_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad b'_1 = b'^{-1}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, for every $n \geq 2$, we put

$$a'_n = \begin{pmatrix} b'_{n-1} & 0 \\ 0 & I_{n-1} \end{pmatrix}, \qquad a'^{-1}_n = \begin{pmatrix} b'^{-1}_{n-1} & 0 \\ 0 & -I_{n-1} \end{pmatrix},$$
$$b'_n = \begin{pmatrix} 0 & a'_{n-1} \\ -I_{n-1} & 0 \end{pmatrix}, \qquad b'^{-1}_n = \begin{pmatrix} 0 & I_{n-1} \\ a'^{-1}_{n-1} & 0 \end{pmatrix}.$$

(Observe that here the exponent -1 is just a notation and it does not correspond to the inverse in algebraic sense.) Put $A'_n = aa'_n$, $A'^{-1}_n = aa'^{-1}_n$, $B'_n = bb'_n$ and $B'^{-1}_n = bb'^{-1}_n$. Then

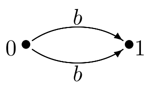
$$\Delta'_1 = A'_1 + A'^{-1}_1 + B'_1 + B'^{-1}_1 = \begin{pmatrix} 0 & 2b \\ -2b & 0 \end{pmatrix}$$

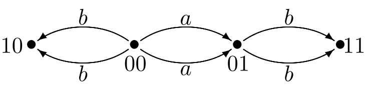
and, for each $n \geq 2$,

$$\Delta'_n = A'_n + A'^{-1}_n + B'_n + B'^{-1}_n = \begin{pmatrix} a(b'_{n-1} + b'^{-1}_{n-1}) & b(a'_{n-1} + I_{n-1}) \\ b(a'^{-1}_{n-1} - I_{n-1}) & 0 \end{pmatrix}.$$

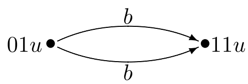
Proposition 2.6. *Δ'_n induces a good orientation on the Schreier graph Σ_n of the Basilica group.*

Proof. It is easy to show by induction that Δ'_n is anti-symmetric and that each entry of Δ'_n coincides, up to the sign, with the corresponding entry of the adjacency matrix Δ_n of Σ_n, where loops are deleted. We also prove the assertion about the orientation by induction. For $n = 1, 2$ we have $\Delta'_1 = \begin{pmatrix} 0 & 2b \\ -2b & 0 \end{pmatrix}$ and $\Delta'_2 = \begin{pmatrix} 0 & 2a & 2b & 0 \\ -2a & 0 & 0 & 2b \\ -2b & 0 & 0 & 0 \\ 0 & -2b & 0 & 0 \end{pmatrix}$, which correspond to





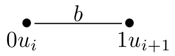
Now look at the second block $b(a'_{n-1} + I_{n-1}) = b \begin{pmatrix} b'_{n-2} + I_{n-2} & 0 \\ 0 & 2I_{n-2} \end{pmatrix}$ of Δ'_n. The matrix $2bI_{n-2}$ corresponds to the 2-cycles



which come from the a-loops of Σ_{n-1} centered at $1u$ and so they are 2^{n-2}. The block $b(b'_{n-2} + I_{n-2})$ corresponds to the b-cycles of length $2^k \geq 4$. These cycles come from the a-cycles of level $n-1$ but they have double length. In particular, bb'_{n-2} corresponds to the b-cycles at level $n-2$ (well oriented by induction), that correspond to the a-cycles at level $n-1$ with the same good orientation given by the substitutional rule

$$u \xrightarrow{b} v \qquad \Longrightarrow \qquad 0u \xrightarrow{a} 0v$$

Such a cycle labeled a with vertices $u_1, u_2, \ldots, u_{2^{k-1}}$ (of length $2^{k-1} \geq 2$) at level $n-1$ gives rise to a b-cycle of length 2^k in Σ_n following the third substitutional rule. In this new cycle, by induction, there is an odd number of clockwise oriented edges of type



All the other edges have the same orientation (given by the matrix bI_{n-2}). Since these edges are in even number, this implies that the b-cycle is well oriented. A similar argument can be developed for edges labeled by a and this completes the proof. □

Theorem 2.7. *The partition function of the dimer model on the Schreier graph Σ_n of the Basilica group is*

$$\Phi_n(a,b) = \begin{cases} 2^{\frac{2^n+1}{3}} b^{2^{n-1}} & n \text{ odd}, \\ 2^{\frac{2^n+2}{3}} b^{2^{n-1}} & n \text{ even}. \end{cases}$$

Proof. For small n the assertion can be directly verified. Suppose now $n \geq 5$. Observe that we have $\det(\Delta'_n) = b^{2^n}(\det(a'_{n-1} + I_{n-1}))^2$, since the matrices $a'_{n-1} + I_{n-1}$ and $a'^{-1}_{n-1} - I_{n-1}$ have the same determinant. Let us prove by induction on n that, for every $n \geq 5$, $(\det(a'_{n-1} + I_{n-1}))^2 = 2^{2^{n-1}} \cdot 2^{2l'}$, where l' is the number of cycles labeled by b in Σ_n having length greater than 2. One can verify by direct computation that $\det(\Delta'_5) = 2^{22}$ and $\det(\Delta'_6) = 2^{44}$. Now

$$\det(\Delta'_n) = (\det(a'_{n-1} + I_{n-1}))^2 = \begin{vmatrix} b'_{n-2} + I_{n-2} & 0 \\ 0 & 2I_{n-2} \end{vmatrix}^2 = 2^{2^{n-1}} \cdot (\det(b'_{n-2} + I_{n-2}))^2$$

$$= 2^{2^{n-1}} \begin{vmatrix} I_{n-3} & a'_{n-3} \\ -I_{n-3} & I_{n-3} \end{vmatrix}^2 = 2^{2^{n-1}}(\det(a'_{n-3} + I_{n-3}))^2 = 2^{2^{n-1}} \cdot 2^{2^{n-3}} \cdot 2^{2l''},$$

where the last equality follows by induction and l'' is the number of b-cycles in Σ_{n-2} having length greater than 2. Now observe that l'' is also equal to the number of a-cycles of length greater than 2 in Σ_{n-1} but also to the number of b-cycles of length greater than 4 in Σ_n. We already proved that $b^n_4 = b^{n-2}_2 = 2^{n-4}$ and so $2^{2^{n-3}} = 2^{2b^n_4}$. Similarly $2^{2^{n-1}} = 2^{2b^n_2}$. Then one gets the assertion by using computations made in Proposition 2.4. □

Corollary 2.8. *The thermodynamic limit is $\frac{1}{3}\log 2 + \frac{1}{2}\log b$. In particular, the entropy of absorption of diatomic molecules per site is $\frac{1}{3}\log 2$.*

3. The dimer model on the Schreier graphs of the Hanoi Towers group $H^{(3)}$

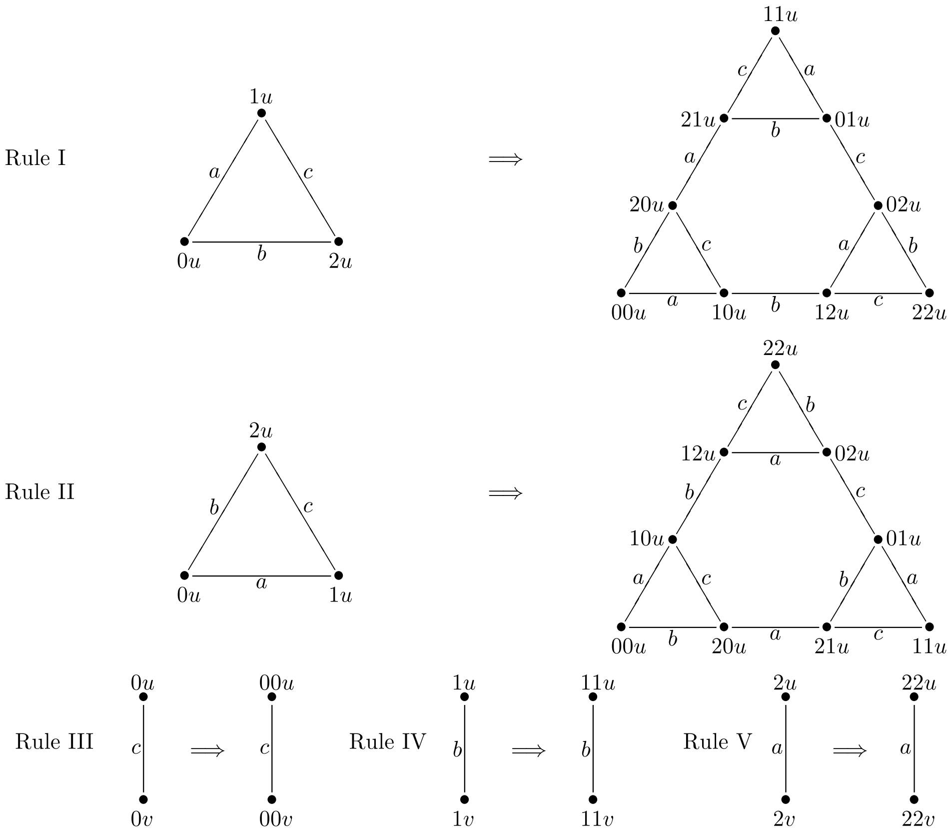
We present here a more sophisticated example of dimers computation on Schreier graphs — the Schreier graphs of the action of the Hanoi Towers group $H^{(3)}$ on the rooted ternary tree.

3.1. The Schreier graphs

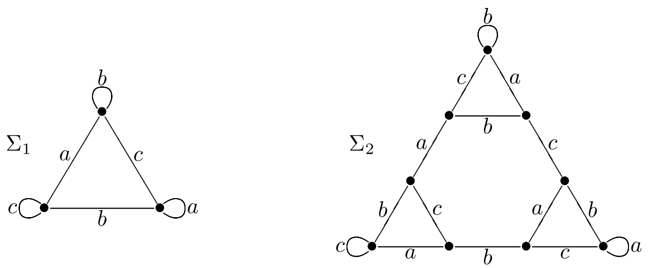
The group $H^{(3)}$ is generated by the automorphisms of the ternary rooted tree having the following self-similar form [9]:

$$a = (01)(id, id, a) \qquad b = (02)(id, b, id) \qquad c = (12)(c, id, id),$$

where (01), (02) and (12) are transpositions in $Sym(3)$. Observe that a, b, c are involutions. The associated Schreier graphs are self-similar in the sense of [19], that is, each Σ_{n+1} contains three copies of Σ_n glued together by three edges. These graphs can be recursively constructed via the following substitutional rules [9]:



The starting point is the Schreier graph Σ_1 of the first level.

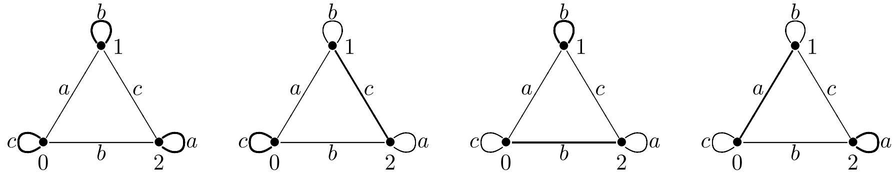


In fact, the substitutional rules determine not only the graphs Σ_n but the graphs together with a particular embedding in the plane. Throughout the paper we will always consider the graphs embedded in the plane, as drawn on the Figures, up to translations. Observe that, for each $n \geq 1$, the graph Σ_n has three loops, at the vertices 0^n, 1^n and 2^n, labeled by c, b and a, respectively. Moreover, these are the only loops in Σ_n. Since the number of vertices of the Schreier graph Σ_n is 3^n (and so an odd number), we let a dimer covering of Σ_n cover either zero or two outmost vertices: we will consider covered by a loop the vertices not covered by any dimer. For this reason we do not erase the loops in this example.

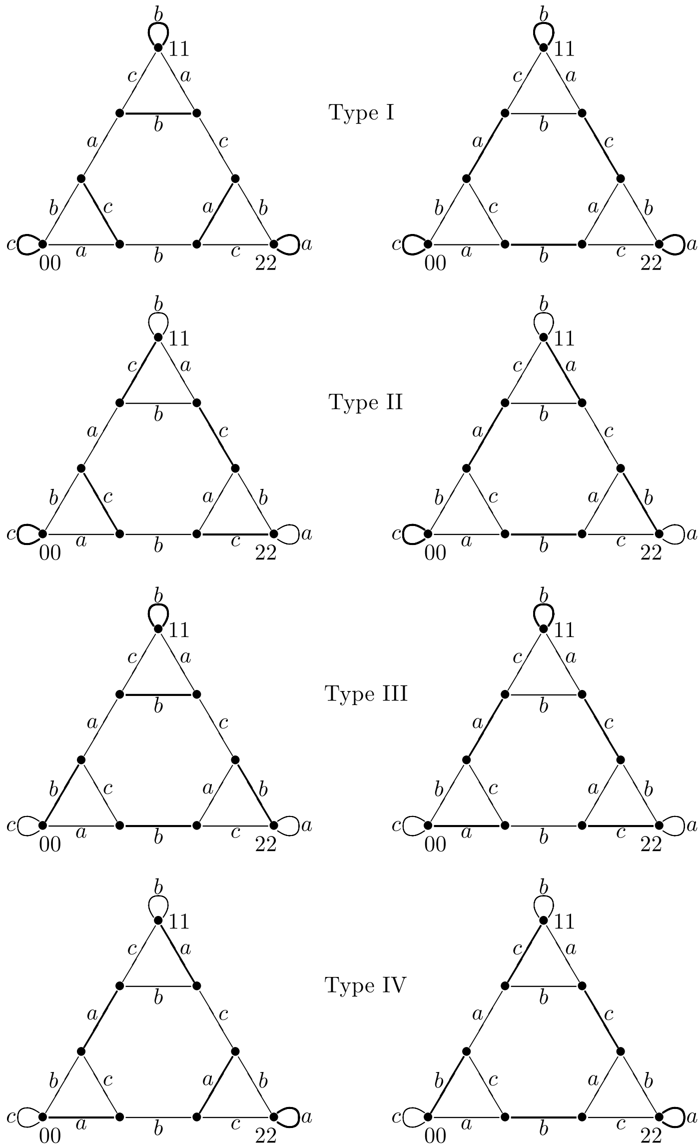
The two subsections below correspond to the calculation of the dimers on Hanoi Schreier graphs by two different methods: combinatorial (Section 3.2), using the self-similar structure of the graph; and via the Kasteleyn theory (Section 3.3), using self-similarity of the group $H^{(3)}$ in the construction of the oriented adjacency matrix.

3.2. A combinatorial approach

There are four possible dimer configurations on Σ_1:



At level 2, we have eight possible dimer configurations:



More generally, we will say that a dimer covering is of type I if it contains all the three loops, of type II if it only contains the leftmost loop (at vertex 0^n), of type III if it only contains the upmost loop (at vertex 1^n) and of type IV if it only contains the rightmost loop (at vertex 2^n).

For Σ_n, $n \geq 1$, let us denote by $\Phi_n^i(a, b, c)$ the partition function of the dimer coverings of type i, for $i = \mathrm{I, II, III, IV}$, so that $\Phi_n = \Phi_n^{\mathrm{I}} + \Phi_n^{\mathrm{II}} + \Phi_n^{\mathrm{III}} + \Phi_n^{\mathrm{IV}}$. In what follows we omit the variables a, b, c in the partition functions.

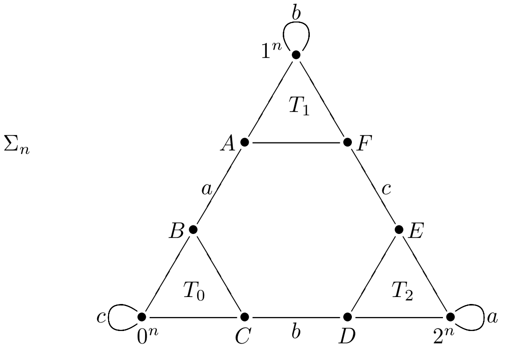
Theorem 3.1. *For each $n \geq 1$, the functions $\{\Phi_n^i\}$, $i = \mathrm{I, II, III, IV}$, satisfy the system of equations*

$$\begin{cases} \Phi_{n+1}^{\mathrm{I}} = \left(\Phi_n^{\mathrm{I}}\right)^3 \cdot \dfrac{1}{abc} + \Phi_n^{\mathrm{II}}\Phi_n^{\mathrm{III}}\Phi_n^{\mathrm{IV}} \\ \Phi_{n+1}^{\mathrm{II}} = \left(\Phi_n^{\mathrm{II}}\right)^3 \cdot \dfrac{1}{c} + \Phi_n^{\mathrm{I}}\Phi_n^{\mathrm{III}}\Phi_n^{\mathrm{IV}} \cdot \dfrac{1}{ab} \\ \Phi_{n+1}^{\mathrm{III}} = \left(\Phi_n^{\mathrm{III}}\right)^3 \cdot \dfrac{1}{b} + \Phi_n^{\mathrm{I}}\Phi_n^{\mathrm{II}}\Phi_n^{\mathrm{IV}} \cdot \dfrac{1}{ac} \\ \Phi_{n+1}^{\mathrm{IV}} = \left(\Phi_n^{\mathrm{IV}}\right)^3 \cdot \dfrac{1}{a} + \Phi_n^{\mathrm{I}}\Phi_n^{\mathrm{II}}\Phi_n^{\mathrm{III}} \cdot \dfrac{1}{bc}, \end{cases} \tag{2}$$

with the initial conditions

$$\begin{cases} \Phi_1^{\mathrm{I}} = abc \\ \Phi_1^{\mathrm{II}} = c^2 \\ \Phi_1^{\mathrm{III}} = b^2 \\ \Phi_1^{\mathrm{IV}} = a^2. \end{cases}$$

Proof. We prove the assertion by induction on n. The initial conditions can be easily verified. The induction step follows from the substitutional rules. More precisely, in Σ_n we have a copy T_0 of Σ_{n-1} reflected with respect to the bisector of the angle with vertex 0^{n-1}; a copy T_1 of Σ_{n-1} reflected with respect to the bisector of the angle with vertex 1^{n-1}; a copy T_2 of Σ_{n-1} reflected with respect to the bisector of the angle with vertex 2^{n-1}.

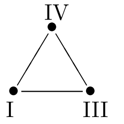


Using this information, let us analyze the dimer coverings of Σ_n as constructed from dimer coverings of T_i, $i = 0, 1, 2$, that can be in turn interpreted as dimer coverings of Σ_{n-1}.

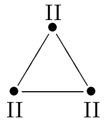
First suppose that the dimer covering of Σ_n contains only one loop, and without loss of generality assume it is at 0^n. There are then two possible cases.

- The copy of Σ_{n-1} corresponding to T_0 was covered using three loops (type I). Then the covering of Σ_n must cover the edges connecting T_0 to T_1 and T_2, (labeled a and b respectively). So in the copy of Σ_{n-1} corresponding to T_1 we had a dimer covering with only a loop in A (type IV, by reflection) and in T_2 a covering with only a loop in D (type III, by reflection). These coverings cover vertices E

and F and so the edge labeled c joining E and F does not belong to the cover of Σ_n. We describe this situation as:

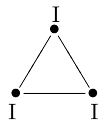


- The copy of Σ_{n-1} corresponding to T_0 was covered with only one loop in 0^{n-1} (type II), so that the vertices B and C are covered. This implies that the edges joining A, B and C, D, labeled a and b respectively are not covered in Σ_n. So in the copy of Σ_{n-1} corresponding to T_1 there was no loop at 1^{n-1} (or A), which implies that in it only the loop at F was covered (type II, by reflection). Similarly the covering of the copy of Σ_{n-1} corresponding to T_2 could only contain a loop in E (type II, by reflection). Consequently, in Σ_n, the edge joining E and F (labeled c) must belong to the covering. Schematically this situation can be described as follows.

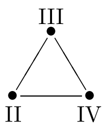


Now suppose that the covering of Σ_n contains loops at 0^n, 1^n, 2^n. There are again two possible cases.

- We have on T_0 a dimer covering with three loops (type I), so that the dimer covering of Σ_n must use the edges joining the copy T_0 to the copies T_1 and T_2. So in the copy of Σ_{n-1} corresponding to T_1 (and similarly for T_2) we necessarily have a covering with three loops (type I).



- We have on T_0 a dimer covering with only one loop in 0^{n-1} (type II), so that the vertices B and C are covered. This implies that the edges joining A, B and C, D are not covered in Σ_n. So in T_1 we cannot have a loop at A, which implies that the corresponding copy of Σ_{n-1} was covered with only a loop in 1^{n-1} (type III). Similarly T_2 was covered with only a loop in 2^{n-1} (type IV).



The claim follows. $\square$

Corollary 3.2. *For each $n \geq 1$, the number of dimer coverings of type I, II, III, IV of the Schreier graph Σ_n is the same and it is equal to $2^{\frac{3^{n-1}-1}{2}}$. Hence, the number of dimer coverings of the Schreier graph Σ_n is equal to $2^{\frac{3^{n-1}+3}{2}}$. The entropy of absorption of diatomic molecules per site is $\frac{1}{6}\log 2$.*

Proof. By construction in the proof of Theorem 3.1 we see that the number of configurations of type I of Σ_n is given by $2h^3$, where h is (by inductive hypothesis) the common value of the number of configurations of type I, II, III, IV of Σ_{n-1}, equal to $2^{\frac{3^{n-2}-1}{2}}$. So the number of configurations of type I in Σ_n is equal to

$$2 \cdot 2^{\frac{3(3^{n-2}-1)}{2}} = 2^{\frac{3^{n-1}-1}{2}}.$$

Clearly the same count holds for the coverings of type II, III and IV of Σ_n, and this completes the proof. $\square$

Remark 3.3. Analogues of Theorem 3.1 can be deduced for the dimers partition function on Schreier graphs of any self-similar (automata) group with bounded activity (generated by a bounded automaton). Indeed, Nekrashevych in [16] introduces an inductive procedure (that he calls "inflation") that produces a sequence of graphs that differ from the Schreier graphs only in a bounded number of edges (see also [3], where this construction is used to study growth of infinite Schreier graphs). This inductive procedure allows to describe the partition function for the dimer model on them by writing a system of recursive equations as in (2).

Unfortunately, we were not able to find an explicit solution of the system (2). We will come back to these equations in Section 5 where we study some statistics for the dimer coverings on Σ_n. Meanwhile, in the next Section 3.3, we will attempt to compute the partition function in a different way, using the Kasteleyn theory.

In the rest of this subsection we present the solution of the system (2) in the particular case, when all the weights on the edges are put to be equal, and deduce the thermodynamic limit.

Proposition 3.4. *The partition function for $a = b = c$ is*

$$\Phi_n(a, a, a) = 2^{\frac{3^{n-1}-1}{2}} \cdot a^{\frac{3^n+1}{2}} (a + 3).$$

In this case, the thermodynamic limit is $\frac{1}{6}\log 2 + \frac{1}{2}\log a$.

Proof. By putting $a = b = c$, the system (2) reduces to

$$\begin{cases} \Phi^{\mathrm{I}}_{n+1} = \left(\Phi^{\mathrm{I}}_n\right)^3 \cdot \dfrac{1}{a^3} + (\Phi^{\mathrm{II}}_n)^3 \\ \Phi^{\mathrm{II}}_{n+1} = \left(\Phi^{\mathrm{II}}_n\right)^3 \cdot \dfrac{1}{a} + \Phi^{\mathrm{I}}_n(\Phi^{\mathrm{II}}_n)^2 \cdot \dfrac{1}{a^2} \end{cases} \tag{3}$$

with initial conditions $\Phi^{\mathrm{I}}_1 = a^3$ and $\Phi^{\mathrm{II}}_1 = a^2$, since $\Phi^{\mathrm{II}}_n = \Phi^{\mathrm{III}}_n = \Phi^{\mathrm{IV}}_n$. One can prove by induction that $\frac{\Phi^{\mathrm{I}}_n}{\Phi^{\mathrm{II}}_n} = a$, so that the first equation in (3) becomes

$$\Phi^{\mathrm{I}}_{n+1} = \frac{2(\Phi^{\mathrm{I}}_n)^3}{a^3},$$

giving

$$\begin{cases} \Phi^{\mathrm{I}}_n = 2^{\frac{3^{n-1}-1}{2}} \cdot a^{\frac{3^n+3}{2}} \\ \Phi^{\mathrm{II}}_n = 2^{\frac{3^{n-1}-1}{2}} \cdot a^{\frac{3^n+1}{2}}. \end{cases}$$

The partition function is then obtained as

$$\Phi_n = \Phi^{\mathrm{I}}_n + 3\Phi^{\mathrm{II}}_n = 2^{\frac{3^{n-1}-1}{2}} \cdot a^{\frac{3^n+1}{2}} (a + 3).$$

The thermodynamic limit is

$$\lim_{n\to\infty} \frac{\log(\Phi_n)}{3^n} = \frac{1}{6}\log 2 + \frac{1}{2}\log a.$$

Finally, by putting $a = 1$, we get the entropy of absorption of diatomic molecules as in Corollary 3.2. □

Next, we deduce the existence of the thermodynamic limit of the function $\Phi_n(1, 1, c)$, for $c \geq 1$. Observe that $\Phi_n(1, 1, c) = \Phi^{\mathrm{I}}_n(1, 1, c) + \Phi^{\mathrm{II}}_n(1, 1, c) + 2\Phi^{\mathrm{III}}_n(1, 1, c)$, since $\Phi^{\mathrm{III}}_n(1, 1, c) = \Phi^{\mathrm{IV}}_n(1, 1, c)$. A similar argument holds for the functions $\Phi_n(a, 1, 1)$ and $\Phi_n(1, b, 1)$.

Proposition 3.5. *For every $c \geq 1$, the thermodynamic limit* $\lim_{n\to\infty} \frac{\log(\Phi_n(1,1,c))}{3^n}$ *exists.*

Proof. It is clear that, for every $c \geq 1$, the sequence $\varepsilon_n := \frac{\log(\Phi_n(1,1,c))}{3^n}$ is positive. We claim that ε_n is decreasing. We have:

$$\begin{aligned}
\frac{\varepsilon_{n+1}}{\varepsilon_n} &= \frac{\log(\Phi_{n+1}(1,1,c))}{3\cdot\log(\Phi_n(1,1,c))} \\
&= \frac{3\cdot\log(\Phi_n(1,1,c)) + \log\left(\frac{\Phi_{n+1}(1,1,c)}{\Phi_n(1,1,c)^3}\right)}{3\cdot\log(\Phi_n(1,1,c))} \\
&= 1 + \frac{\log\left(\frac{\Phi_{n+1}(1,1,c)}{\Phi_n(1,1,c)^3}\right)}{3\cdot\log(\Phi_n(1,1,c))}.
\end{aligned}$$

Since $\log(\Phi_n(1,1,c)) > 0$ for every $c \geq 1$, it suffices to prove that $\frac{\Phi_{n+1}(1,1,c)}{\Phi_n(1,1,c)^3}$ is less or equal to 1 for every $c \geq 1$.

$$\begin{aligned}
\frac{\Phi_{n+1}(1,1,c)}{\Phi_n(1,1,c)^3} &= \frac{\Phi^{\mathrm{I}}_{n+1}(1,1,c) + \Phi^{\mathrm{II}}_{n+1}(1,1,c) + 2\Phi^{\mathrm{III}}_{n+1}(1,1,c)}{(\Phi^{\mathrm{I}}_n(1,1,c) + \Phi^{\mathrm{II}}_n(1,1,c) + 2\Phi^{\mathrm{III}}_n(1,1,c))^3} \\
&= \frac{\left(\frac{(\Phi^{\mathrm{I}}_n)^3}{c} + \Phi^{\mathrm{II}}_n(\Phi^{\mathrm{III}}_n)^2 + \frac{(\Phi^{\mathrm{II}}_n)^3}{c} + \Phi^{\mathrm{I}}_n(\Phi^{\mathrm{III}}_n)^2 + 2(\Phi^{\mathrm{III}}_n)^3 + 2\frac{\Phi^{\mathrm{I}}_n\Phi^{\mathrm{II}}_n\Phi^{\mathrm{III}}_n}{c}\right)}{\left(\Phi^{\mathrm{I}}_n + \Phi^{\mathrm{II}}_n + 2\Phi^{\mathrm{III}}_n\right)^3} \\
&\leq \frac{\left((\Phi^{\mathrm{I}}_n)^3 + \Phi^{\mathrm{II}}_n(\Phi^{\mathrm{III}}_n)^2 + (\Phi^{\mathrm{II}}_n)^3 + \Phi^{\mathrm{I}}_n(\Phi^{\mathrm{III}}_n)^2 + 2(\Phi^{\mathrm{III}}_n)^3 + 2\Phi^{\mathrm{I}}_n\Phi^{\mathrm{II}}_n\Phi^{\mathrm{III}}_n\right)}{\left(\Phi^{\mathrm{I}}_n + \Phi^{\mathrm{II}}_n + 2\Phi^{\mathrm{III}}_n\right)^3} \\
&\leq 1.
\end{aligned}$$

This implies the existence of $\lim_{n\to\infty} \frac{\log(\Phi_n(1,1,c))}{3^n}$. $\square$

3.3. Partition function by the Kasteleyn method

In order to define a matrix inducing a good orientation, in the sense of Kasteleyn, on the Schreier graphs of $H^{(3)}$, we introduce the matrices

$$a_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad c_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then, for every n even, we put

$$a_n = \begin{pmatrix} 0 & -I_{n-1} & 0 \\ I_{n-1} & 0 & 0 \\ 0 & 0 & a_{n-1} \end{pmatrix}, \qquad b_n = \begin{pmatrix} 0 & 0 & I_{n-1} \\ 0 & b_{n-1} & 0 \\ -I_{n-1} & 0 & 0 \end{pmatrix},$$
$$c_n = \begin{pmatrix} c_{n-1} & 0 & 0 \\ 0 & 0 & -I_{n-1} \\ 0 & I_{n-1} & 0 \end{pmatrix},$$

and for every $n > 1$ odd, we put

$$a_n = \begin{pmatrix} 0 & I_{n-1} & 0 \\ -I_{n-1} & 0 & 0 \\ 0 & 0 & a_{n-1} \end{pmatrix}, \qquad b_n = \begin{pmatrix} 0 & 0 & -I_{n-1} \\ 0 & b_{n-1} & 0 \\ I_{n-1} & 0 & 0 \end{pmatrix},$$
$$c_n = \begin{pmatrix} c_{n-1} & 0 & 0 \\ 0 & 0 & I_{n-1} \\ 0 & -I_{n-1} & 0 \end{pmatrix},$$

where a_n, b_n, c_n and I_n are square matrices of size 3^n. Now we put $A_n = aa_n$, $B_n = bb_n$ and $C_n = cc_n$ for each $n \geq 1$ and define $\Delta_n = A_n + B_n + C_n$, so that

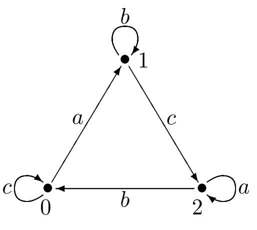
$$\Delta_1 = \begin{pmatrix} c & a & -b \\ -a & b & c \\ b & -c & a \end{pmatrix}$$

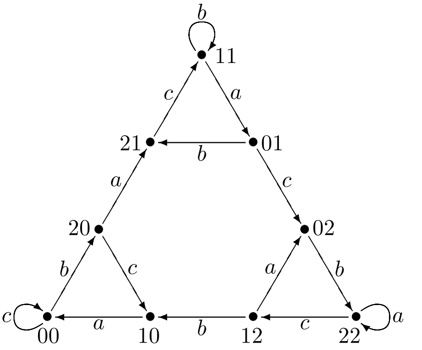
and, for each $n > 1$,

$$\Delta_n = \begin{pmatrix} C_{n-1} & -aI_{n-1} & bI_{n-1} \\ aI_{n-1} & B_{n-1} & -cI_{n-1} \\ -bI_{n-1} & cI_{n-1} & A_{n-1} \end{pmatrix} \quad \text{for } n \text{ even},$$

$$\Delta_n = \begin{pmatrix} C_{n-1} & aI_{n-1} & -bI_{n-1} \\ -aI_{n-1} & B_{n-1} & cI_{n-1} \\ bI_{n-1} & -cI_{n-1} & A_{n-1} \end{pmatrix} \quad \text{for } n \text{ odd}.$$

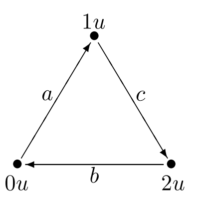
We want to prove that, for each $n \geq 1$, the oriented adjacency matrix Δ_n induces a good orientation on Σ_n. Then we will apply Kasteleyn theory to get the partition function of the dimer model on Σ_n. One can easily verify that also in this case the entries of the matrix Δ_n coincide, in absolute value, with the entries of the (unoriented weighted) adjacency matrix of the Schreier graphs of the group. The problem related to loops and their orientation will be discussed later. The figures below describe the orientation induced on Σ_1 and Σ_2 by the matrices Δ_1 and Δ_2, respectively.



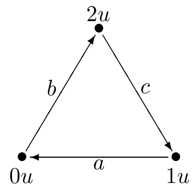


Proposition 3.6. *For each $n \geq 1$, the matrix Δ_n induces a good orientation on Σ_n.*

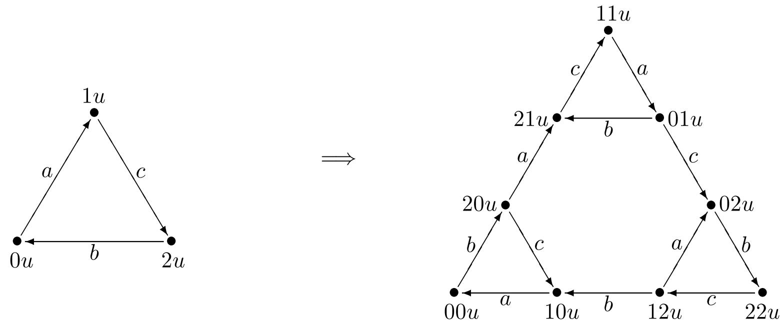
Proof. Observe that in Σ_1 the sequence of labels a, b, c appears in anticlockwise order. Following the substitutional rules, we deduce that for every n odd we can read in each elementary triangle the sequence a, b, c in anticlockwise order. On the other hand, for n even, the occurrences of a, b, c in each elementary triangle of Σ_n follow a clockwise order. We prove our claim by induction on n. For $n = 1$, the matrix Δ_1 induces on Σ_1 the orientation shown in the picture above, so that the assertion is true for $n = 1$. Now observe that, for every n odd, the blocks $\pm aI_{n-1}$, $\pm bI_{n-1}$, $\pm cI_{n-1}$ in Δ_n ensure that each elementary triangle in Σ_n has the same orientation given by



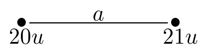
For n even, the sequence a, b, c is clockwise and the blocks $\pm aI_{n-1}$, $\pm bI_{n-1}$, $\pm cI_{n-1}$ in Δ_n ensure that the orientation induced on the edges is clockwise as the following picture shows:



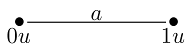
So we conclude that for every n all the elementary triangles of Σ_n are clockwise oriented. Now construct the graph Σ_{n+1} from Σ_n and suppose n odd (the same proof works in the case n even). Rule I gives



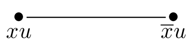
In order to understand why the edges not belonging to an elementary triangle have this orientation, we observe that the edge



has the same orientation as the edge



since the entry $(20u, 21u)$ of the matrix Δ_{n+1} is the same as the entry $(0u, 1u)$ of the matrix Δ_n. Similarly for the other two edges joining vertices $01u, 02u$ and $10u, 12u$. This implies that each elementary hexagon has a good orientation. Now note that in Σ_{n+1} we have 3^{n-i} elementary cycles of length $3 \cdot 2^i$, for each $i = 0, 1, \ldots, n$. We already know that in Σ_{n+1} all the elementary triangles and hexagons have a good orientation. Observe that each cycle in Σ_n having length $k = 3 \cdot 2^m$, with $m \geq 1$, gives rise in Σ_{n+1} to a cycle of length $2k$. In this new cycle of Σ_{n+1}, k edges join vertices starting with the same letter and keep the same orientation as in Σ_n (so they are well oriented by induction); the remaining k edges belong to elementary triangles and have the form



where $x \neq \bar{x}$ and $x, \bar{x} \in \{0, 1, 2\}$, $u \in \Sigma_n$. Since the last k edges belong to elementary triangles, they are oriented in the same direction and, since k is even, they give a good orientation to the cycle. The same argument works for each elementary cycle and so the proof is completed. □

The matrix Δ_n cannot be directly used to find the partition function because it is not anti-symmetric (there are three nonzero entries in the diagonal corresponding to loops) and it is of odd size. Let $\Gamma_{n,c}$ be the matrix obtained from Δ_n by deleting the row and the column indexed by 0^n and where the entries $(1^n, 1^n)$ and $(2^n, 2^n)$ are replaced by 0, so that the partition function of dimer coverings of type II is given by $c\sqrt{\det(\Gamma_{n,c})}$. Similarly, we define $\Gamma_{n,b}$, $\Gamma_{n,a}$ for dimer coverings of type III, IV, respectively. Now let Λ_n be the matrix obtained from Δ_n by deleting the three rows and the three columns indexed by 0^n, 1^n and 2^n, so that the partition function of the dimer coverings of type I is $abc\sqrt{\det \Lambda_n}$. This gives

$$\Phi_n(a, b, c) = c\sqrt{\det(\Gamma_{n,c})} + b\sqrt{\det(\Gamma_{n,b})} + a\sqrt{\det(\Gamma_{n,a})} + abc\sqrt{\det \Lambda_n}. \tag{4}$$

In order to compute $\det(\Gamma_{n,c})$ (the case of $\Gamma_{n,b}$, $\Gamma_{n,a}$ and Λ_n is analogous), we put

$$a'_1 = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad b'_1 = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, \qquad c'_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix}.$$

Then for every $n > 1$ odd we put

$$a'_n = \begin{pmatrix} 0 & aI_{n-1} & 0 \\ -aI_{n-1} & 0 & 0 \\ 0 & 0 & a'_{n-1} \end{pmatrix}, \qquad b'_n = \begin{pmatrix} 0 & 0 & -bI_{n-1} \\ 0 & b'_{n-1} & 0 \\ bI_{n-1} & 0 & 0 \end{pmatrix},$$

$$c'_n = \begin{pmatrix} c'_{n-1} & 0 & 0 \\ 0 & 0 & cI_{n-1} \\ 0 & -cI_{n-1} & 0 \end{pmatrix},$$

and for every n even we put

$$a'_n = \begin{pmatrix} 0 & -aI_{n-1} & 0 \\ aI_{n-1} & 0 & 0 \\ 0 & 0 & a'_{n-1} \end{pmatrix}, \qquad b'_n = \begin{pmatrix} 0 & 0 & bI_{n-1} \\ 0 & b'_{n-1} & 0 \\ -bI_{n-1} & 0 & 0 \end{pmatrix},$$

$$c'_n = \begin{pmatrix} c'_{n-1} & 0 & 0 \\ 0 & 0 & -cI_{n-1} \\ 0 & cI_{n-1} & 0 \end{pmatrix}.$$

Finally, set

$$\overline{A}_1 = \overline{B}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \overline{C}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix}.$$

Then for every $n > 1$ odd we put

$$\overline{A}_n = \begin{pmatrix} 0 & aI^0_{n-1} & 0 \\ -aI^0_{n-1} & 0 & 0 \\ 0 & 0 & a'_{n-1} \end{pmatrix}, \qquad \overline{B}_n = \begin{pmatrix} 0 & 0 & -bI^0_{n-1} \\ 0 & b'_{n-1} & 0 \\ bI^0_{n-1} & 0 & 0 \end{pmatrix}, \qquad \overline{C}_n = c'_n,$$

and for every n even we put

$$\overline{A}_n = \begin{pmatrix} 0 & -aI^0_{n-1} & 0 \\ aI^0_{n-1} & 0 & 0 \\ 0 & 0 & a'_{n-1} \end{pmatrix}, \qquad \overline{B}_n = \begin{pmatrix} 0 & 0 & bI^0_{n-1} \\ 0 & b'_{n-1} & 0 \\ -bI^0_{n-1} & 0 & 0 \end{pmatrix}, \qquad \overline{C}_n = c'_n,$$

with

$$I^0_n = I_n - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, let $\overline{\Delta}_n = \overline{A}_n + \overline{B}_n + \overline{C}_n$ for each $n \geq 1$, so that

$$\overline{\Delta}_n = \begin{pmatrix} c'_{n-1} & aI^0_{n-1} & -bI^0_{n-1} \\ -aI^0_{n-1} & b'_{n-1} & cI_{n-1} \\ bI^0_{n-1} & -cI_{n-1} & a'_{n-1} \end{pmatrix} \quad \text{for } n \text{ odd},$$

$$\overline{\Delta}_n = \begin{pmatrix} c'_{n-1} & -aI^0_{n-1} & bI^0_{n-1} \\ aI^0_{n-1} & b'_{n-1} & -cI_{n-1} \\ -bI^0_{n-1} & cI_{n-1} & a'_{n-1} \end{pmatrix} \quad \text{for } n \text{ even}.$$

The introduction of the matrices I_n^0 guarantees that $\det(\overline{\Delta}_n) = \det(\Gamma_{n,c})$, since we have performed all the necessary cancellations in Δ_n. Geometrically this corresponds to erasing the loops rooted at the vertices 1^n and 2^n and the edges connecting the vertex 0^n to the rest of Σ_n.

Next, we define a rational function $F : \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ as follows

$$F(x_1, x_2, x_3, x_4, x_5, x_6) = \left(x_1, x_2, x_3, \frac{x_1 x_4^3 + x_2 x_3 x_5 x_6}{x_1 x_2 x_3 + x_4 x_5 x_6}, \frac{x_2 x_5^3 + x_1 x_3 x_4 x_6}{x_1 x_2 x_3 + x_4 x_5 x_6}, \frac{x_3 x_6^3 + x_1 x_2 x_4 x_5}{x_1 x_2 x_3 + x_4 x_5 x_6} \right).$$

Denote $F^{(k)}(\underline{x})$ the k-th iteration of the function F, and F_i the i-th projection of F so that

$$F(\underline{x}) = (F_1(\underline{x}), \ldots, F_6(\underline{x})).$$

Set

$$F^{(k)}(a, b, c, a, b, c) = (a, b, c, a^{(k)}, b^{(k)}, c^{(k)}).$$

Theorem 3.7. *For each $n \geq 3$, the partition function $\Phi_n(a, b, c)$ of the dimer model on the Schreier graph Σ_n of the Hanoi Tower group $H^{(3)}$ is*

$$\begin{aligned}\Phi_n(a, b, c) = {} & \prod_{k=0}^{n-3} \left(abc + a^{(k)} b^{(k)} c^{(k)} \right)^{3^{n-k-2}} \Big(abc \left(a^{(n-2)} b^{(n-2)} + a^{(n-2)} c^{(n-2)} + b^{(n-2)} c^{(n-2)} \right. \\ & \left. + \ a^{(n-2)} b^{(n-2)} c^{(n-2)} + abc \right) + a^2 (a^{(n-2)})^3 + b^2 (b^{(n-2)})^3 + c^2 (c^{(n-2)})^3 \Big).\end{aligned}$$

Proof. We explicitly analyze the case of dimer coverings of type II. It follows from the discussion above that $\Phi_n^{\mathrm{II}}(a, b, c) = c\sqrt{\det(\overline{\Delta}_n)}$. More precisely, the factor c corresponds to the label of the loop at 0^n, and the factor $\sqrt{\det(\overline{\Delta}_n)}$ is the absolute value of the Pfaffian of the oriented adjacency matrix of the graph obtained from Σ_n by deleting the edges connecting the vertex 0^n to the rest of the graph and the loops rooted at 1^n and 2^n. The cases corresponding to coverings of type III and IV are analogous. If we expand twice the matrix $\overline{\Delta}_n$ using the recursion formula and perform the permutations (17) and (58) for both rows and columns, we get the matrix (for n odd)

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \left(\begin{array}{cccccc|ccc} 0 & 0 & 0 & -cI_{n-2} & -aI_{n-2} & 0 & bI_{n-2}^0 & 0 & 0 \\ 0 & 0 & -cI_{n-2} & 0 & -bI_{n-2} & 0 & 0 & aI_{n-2} & 0 \\ 0 & cI_{n-2} & 0 & 0 & 0 & aI_{n-2} & 0 & 0 & -bI_{n-2} \\ cI_{n-2} & 0 & 0 & 0 & 0 & bI_{n-2} & -aI_{n-2}^0 & 0 & 0 \\ aI_{n-2} & bI_{n-2} & 0 & 0 & 0 & 0 & 0 & -cI_{n-2} & 0 \\ 0 & 0 & -aI_{n-2} & -bI_{n-2} & 0 & 0 & 0 & 0 & cI_{n-2} \\ \hline -bI_{n-2}^0 & 0 & 0 & aI_{n-2}^0 & 0 & 0 & c'_{n-2} & 0 & 0 \\ 0 & -aI_{n-2} & 0 & 0 & cI_{n-2} & 0 & 0 & b'_{n-2} & 0 \\ 0 & 0 & bI_{n-2} & 0 & 0 & -cI_{n-2} & 0 & 0 & a'_{n-2} \end{array} \right).$$

Note that each entry is a square matrix of size 3^{n-2}. Hence, the Schur complement formula gives

$$\begin{aligned}\det(\overline{\Delta}_n) &= \det(M_{11}) \cdot \det(M_{22} - M_{21} M_{11}^{-1} M_{12}) \\ &= (2abc)^{2 \cdot 3^{n-2}} \begin{vmatrix} c'_{n-2} & -\dfrac{a^4 + b^2 c^2}{2abc} I_{n-2}^0 & \dfrac{b^4 + a^2 c^2}{2abc} I_{n-2}^0 \\ \dfrac{a^4 + b^2 c^2}{2abc} I_{n-2}^0 & b'_{n-2} & -\dfrac{c^4 + a^2 b^2}{2abc} \\ -\dfrac{b^4 + a^2 c^2}{2abc} I_{n-2}^0 & \dfrac{c^4 + a^2 b^2}{2abc} & a'_{n-2} \end{vmatrix}. \end{aligned} \tag{5}$$

The matrix obtained in (5) has the same shape as $\overline{\Delta}_{n-1}$, so we can use recursion by defining

$$\overline{\Delta}_k(x_1, \ldots, x_6) = \begin{pmatrix} c'_{k-1} & -F_4(x_1, \ldots, x_6) I_{k-1}^0 & F_5(x_1, \ldots, x_6) I_{k-1}^0 \\ F_4(x_1, \ldots, x_6) I_{k-1}^0 & b'_{k-1} & -F_6(x_1, \ldots, x_6) \\ -F_5(x_1, \ldots, x_6) I_{k-1}^0 & F_6(x_1, \ldots, x_6) & a'_{k-1} \end{pmatrix}.$$

Hence, (5) becomes

$$\begin{aligned}\det(\overline{\Delta}_n) &= (2abc)^{2\cdot 3^{n-2}} \cdot \det(\overline{\Delta}_{n-1}(a,b,c,a,b,c))\\ &= (2abc)^{2\cdot 3^{n-2}}(abc + a^{(1)}b^{(1)}c^{(1)})^{2\cdot 3^{n-3}} \cdot \det\left(\overline{\Delta}_{n-2}(F^{(1)}(a,b,c,a,b,c))\right)\\ &= \prod_{k=0}^{n-3}\left(abc + a^{(k)}b^{(k)}c^{(k)}\right)^{2\cdot 3^{n-k-2}} \cdot \det\left(\overline{\Delta}_2(F^{(n-3)}(a,b,c,a,b,c))\right)\\ &= \prod_{k=0}^{n-3}\left(abc + a^{(k)}b^{(k)}c^{(k)}\right)^{2\cdot 3^{n-k-2}} \cdot \left(ab(a^{(n-2)}b^{(n-2)}) + c(c^{(n-2)})^3\right)^2.\end{aligned}$$

A similar recurrence holds for coverings of type III and IV. For dimer coverings of type I, by using the Schur complement again, we get

$$\det(\Lambda_n) = \prod_{k=0}^{n-3}\left(abc + a^{(k)}b^{(k)}c^{(k)}\right)^{2\cdot 3^{n-k-2}} \cdot \left(abc + a^{(n-2)}b^{(n-2)}c^{(n-2)}\right)^2.$$

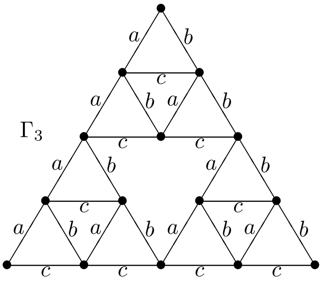
Then we use (4) and the proof is completed. □

Remark 3.8. The proof above, with $a = b = c = 1$, gives the number of dimers coverings of Σ_n that we had already computed to be $2^{\frac{3^{n-1}+3}{2}}$ in Corollary 3.2.

4. The dimer model on the Sierpiński gasket

In this section we study the dimer model on a sequence of graphs $\{\Gamma_n\}_{n\geq 1}$ forming finite approximations to the well-known Sierpiński gasket. The local limit of these graphs is an infinite graph known as the infinite Sierpiński triangle. The graphs Γ_n are not Schreier graphs of a self-similar group. However, they are self-similar in the sense of [19], and their structure is very similar to that of the Schreier graphs Σ_n of the group $H^{(3)}$, studied in the previous section. More precisely, one can obtain Γ_n from Σ_n by contracting the edges joining two different elementary triangles.

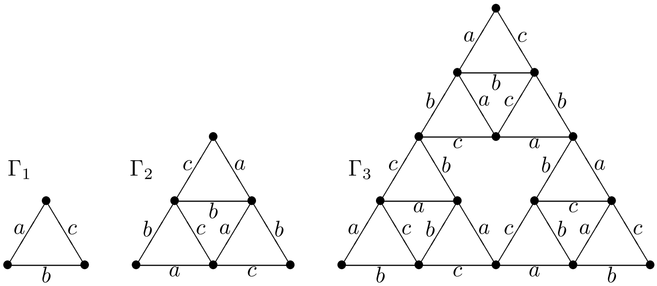
One can think of a few natural ways to label the edges of these graphs with weights of three types. The one that springs first into mind is the "directional" weight, where the edges are labeled a, b, c according to their direction in the graph drawn on the plane, see the picture of Γ_3 with the directional labeling below. Note that the labeled graph Γ_n is obtained from the labeled graph Γ_{n-1} by taking three translated copies of the latter (and identifying three pairs of corners, see the picture).



The dimer model on Γ_n with this labeling was previously studied in [18]: the authors wrote down a recursion between levels n and $n+1$, obtaining a system of equations involving the partition functions, but did not arrive at an explicit solution. Unfortunately, we were not able to compute the generating function of the dimer covers corresponding to this "directional" weight function either. Below we describe two other natural labelings of Γ_n for which we were able to compute the partition functions: we refer to them as the "Schreier" labeling and the "rotation-invariant" labeling.

4.1. The "Schreier" labeling

In the "Schreier" labeling, at a given corner of labeled Γ_n we have a copy of labeled Γ_{n-1} reflected with respect to the bisector of the corresponding angle, see the picture below.



It turns out that this labeling of the graph Γ_n can be alternatively described by considering the labeled Schreier graph Σ_n of the Hanoi Towers group and then contracting the edges connecting copies of Σ_{n-1} in Σ_n, hence the name "Schreier" labeling.

For every $n \geq 1$, the number of vertices of Γ_n is $|V(\Gamma_n)| = \frac{3}{2}(3^{n-1} + 1)$. This implies that, for n odd, $|V(\Gamma_n)|$ is odd and so we allow dimer coverings touching either two or none of the corners. If n is even, $|V(\Gamma_n)|$ is even, and so we allow dimer coverings touching either one or three corners. We say that a dimer covering of Γ_n is:

- of type f, if it covers no corner;
- of type g^{ab} (respectively g^{ac}, g^{bc}), if it covers the corner of Γ_n where two edges a and b (respectively a and c, b and c) meet, but does not cover any other corner;
- of type h^{ab} (respectively h^{ac}, h^{bc}), if it does not cover the corner of Γ_n where two edges a and b (respectively a and c, b and c) meet, but it covers the remaining two corners;
- of type t, if it covers all three corners.

Observe that for n odd we can only have configurations of type f and h, and for n even we can only have configurations of type g and t.

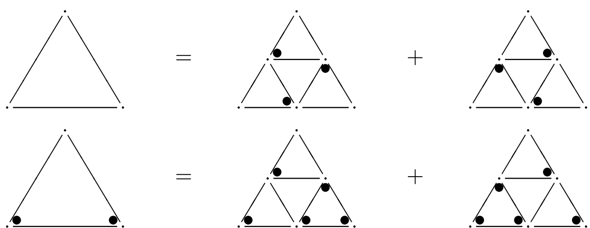
From now on, we will denote by f_n, g_n^{ab}, g_n^{ac}, g_n^{bc}, h_n^{ab}, h_n^{ac}, h_n^{bc}, t_n the summand in the partition function $\Phi_n(a, b, c)$ counting the coverings of the corresponding type. For instance, for $n = 1$, the only nonzero terms are $f_1 = 1$, $h_1^{ab} = c$, $h_1^{ac} = b$, $h_1^{bc} = a$, so that we get

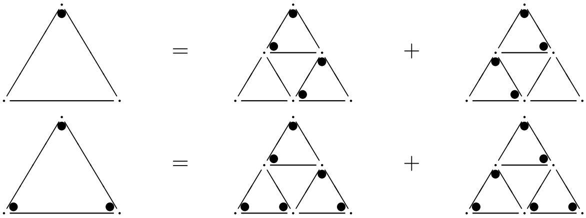
$$\Phi_1(a, b, c) = 1 + a + b + c.$$

For $n = 2$, the only nonzero terms are $t_2 = 2abc$, $g_2^{ab} = 2ab$, $g_2^{ac} = 2ac$, $g_2^{bc} = 2bc$, so that we get

$$\Phi_2(a, b, c) = 2(abc + ab + ac + bc).$$

In the following pictures, the dark bullet next to a vertex means that this vertex is covered by a dimer in that configuration. Since the graph Γ_{n+1} consists of three copies of Γ_n, a dimer covering of Γ_{n+1} can be constructed from three coverings of Γ_n. For example, a configuration of type f for Γ_{n+1} can be obtained using three configurations of Γ_n of type g^{ab}, g^{ac}, g^{bc}, as the first of the following pictures shows.





By using these recursions and arguments similar to those in the proof of Theorem 3.1, one can show that for n odd (respectively, for n even), the number of coverings of types f, h^{ab}, h^{ac}, h^{bc} (respectively of types t, g^{ab}, g^{ac}, g^{bc}) is the same, and so is equal to the quarter of the total number of dimer coverings of Γ_n.

Theorem 4.1. *For each n, the partition function of the dimer model on Γ_n is:*

$$\begin{cases} \Phi_n(a,b,c) = 2(4abc)^{\frac{3^{n-1}-3}{4}}(abc+ab+ac+bc) & \text{for } n \text{ even} \\ \Phi_n(a,b,c) = (4abc)^{\frac{3^{n-1}-1}{4}}(1+a+b+c) & \text{for } n \text{ odd.} \end{cases}$$

Hence, the number of dimer coverings of Γ_n is $2^{\frac{3^{n-1}+3}{2}}$.

Proof. The recursion shows that, for n odd, one has:

$$\begin{cases} f_n = 2g_{n-1}^{ab}g_{n-1}^{ac}g_{n-1}^{bc} \\ h_n^{ab} = 2t_{n-1}g_{n-1}^{ac}g_{n-1}^{bc} \\ h_n^{ac} = 2t_{n-1}g_{n-1}^{ab}g_{n-1}^{bc} \\ h_n^{bc} = 2t_{n-1}g_{n-1}^{ab}g_{n-1}^{ac}. \end{cases} \tag{6}$$

Similarly, for n even, one has:

$$\begin{cases} t_n = 2h_{n-1}^{ab}h_{n-1}^{ac}h_{n-1}^{bc} \\ g_n^{ab} = 2f_{n-1}h_{n-1}^{ac}h_{n-1}^{bc} \\ g_n^{ac} = 2f_{n-1}h_{n-1}^{ab}h_{n-1}^{bc} \\ g_n^{bc} = 2f_{n-1}h_{n-1}^{ab}h_{n-1}^{ac}. \end{cases} \tag{7}$$

The solutions of systems (6) and (7), with initial conditions $f_1 = 1$, $h_1^{ab} = c$, $h_1^{ac} = b$, $h_1^{bc} = a$, can be computed by induction on n. We find:

$$\begin{cases} t_n = 2abc(4abc)^{\frac{3^{n-1}-3}{4}} \\ g_n^{ab} = 2ab(4abc)^{\frac{3^{n-1}-3}{4}} \\ g_n^{ac} = 2ac(4abc)^{\frac{3^{n-1}-3}{4}} \\ g_n^{bc} = 2bc(4abc)^{\frac{3^{n-1}-3}{4}} \end{cases} \quad n \text{ even}, \qquad \begin{cases} f_n = (4abc)^{\frac{3^{n-1}-1}{4}} \\ h_n^{ab} = c(4abc)^{\frac{3^{n-1}-1}{4}} \\ h_n^{ac} = b(4abc)^{\frac{3^{n-1}-1}{4}} \\ h_n^{bc} = a(4abc)^{\frac{3^{n-1}-1}{4}} \end{cases} \quad n \text{ odd}.$$

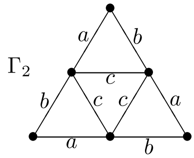
The assertion follows from the fact that $\Phi_n(a,b,c) = f_n + h_n^{ab} + h_n^{ac} + h_n^{bc}$ for n odd and $\Phi_n(a,b,c) = t_n + g_n^{ab} + g_n^{ac} + g_n^{bc}$ for n even. The number of dimer coverings of Γ_n is obtained as $\Phi_n(1,1,1)$. □

Corollary 4.2. *The thermodynamic limit is $\frac{1}{6}\log(4abc)$. In particular, the entropy of absorption of diatomic molecules per site is $\frac{1}{3}\log 2$.*

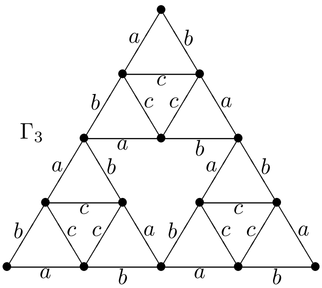
The number of dimer coverings and the value of the entropy have already appeared in [4], where the dimers on Γ_n with the weight function constant 1 were considered.

Note also that the number of dimer coverings found for Sierpiński graphs Γ_n coincides with the number of dimer coverings for the Schreier graphs Σ_n of the group $H^{(3)}$ (see Section 3).

4.2. "Rotation-invariant" labeling



This labeling is, for $n \geq 2$, invariant under rotation by $2\pi/3$. For $n \geq 3$, the copy of Γ_{n-1} at the left (respectively upper, right) corner of Γ_n is rotated by 0 (respectively $2\pi/3$, $4\pi/3$).



We distinguish here the following types of dimers coverings: we say that a dimer covering is of type g (respectively of type h, of type f, of type t) if exactly one (respectively exactly two, none, all) of the three corners of Γ_n is (are) covered. Observe that by symmetry of the labeling we do not need to define $g^{ab}, g^{ac}, g^{bc}, h^{ab}, h^{ac}, h^{bc}$. It is easy to check that in this model for n odd we can only have configurations of type f and h, for n even we can only have configurations of type g and t. By using recursion, one also checks that for n even, the number of coverings of type t is one third of the number of coverings of type g, and that for n odd, the number of coverings of type f is one third of the number of dimer coverings of type h.

Next, we compute the partition function $\Phi_n(a, b, c)$ associated with the "rotation-invariant" labeling. We will denote by f_n, g_n, h_n, t_n the summands in $\Phi_n(a, b, c)$ corresponding to coverings of types f, g, h, t. For instance, for $n = 2$, we have $g_2 = 3c(a + b)$, $t_2 = a^3 + b^3$, so that

$$\Phi_2(a, b, c) = a^3 + b^3 + 3c(a + b).$$

Theorem 4.3. *The partition function of Γ_n, for each $n \geq 2$, is given by:*

$$\begin{cases} \Phi_n(a,b,c) = 2^{\frac{3^{n-2}-1}{2}}(a^3+b^3)^{\frac{3^{n-2}-1}{4}}(ac+bc)^{\frac{3^{n-1}-3}{4}}(a^3+b^3+3c(a+b)) & \text{for } n \text{ even} \\ \Phi_n(a,b,c) = 2^{\frac{3^{n-2}-1}{2}}(a^3+b^3)^{\frac{3^{n-2}-3}{4}}(ac+bc)^{\frac{3^{n-1}-1}{4}}(3(a^3+b^3)+c(a+b)) & \text{for } n \text{ odd}. \end{cases}$$

Proof. Similarly to how we computed the partition function for the "Schreier" labeling, we get, for $n \geq 3$ odd:

$$\begin{cases} f_n = 2\left(\dfrac{g_{n-1}}{3}\right)^3 \\ h_n = 6t_{n-1}\left(\dfrac{g_{n-1}}{3}\right)^2 \end{cases} \tag{8}$$

and for n even:

$$\begin{cases} t_n = 2\left(\dfrac{h_{n-1}}{3}\right)^3 \\ g_n = 6f_{n-1}\left(\dfrac{h_{n-1}}{3}\right)^2. \end{cases} \tag{9}$$

The solutions of systems (8) and (9), with the initial conditions $g_2 = 3c(a+b)$ and $t_2 = a^3 + b^3$, can be computed by induction on n: one gets, for n even,

$$\begin{cases} t_n = 2^{\frac{3^{n-2}-1}{2}} (a^3+b^3)^{\frac{3^{n-2}+3}{4}} (ac+bc)^{\frac{3^{n-1}-3}{4}} \\ g_n = 3 \cdot 2^{\frac{3^{n-2}-1}{2}} (a^3+b^3)^{\frac{3^{n-2}-1}{4}} (ac+bc)^{\frac{3^{n-1}+1}{4}} \end{cases}$$

and for $n \geq 3$ odd

$$\begin{cases} f_n = 2^{\frac{3^{n-2}-1}{2}} (a^3+b^3)^{\frac{3^{n-2}-3}{4}} (ac+bc)^{\frac{3^{n-1}+3}{4}} \\ h_n = 3 \cdot 2^{\frac{3^{n-2}-1}{2}} (a^3+b^3)^{\frac{3^{n-2}+1}{4}} (ac+bc)^{\frac{3^{n-1}-1}{4}}. \end{cases}$$

The assertion follows from the fact that $\Phi_n(a, b, c) = f_n + h_n$ for n odd and $\Phi_n(a, b, c) = t_n + g_n$ for n even. □

Corollary 4.4. *The thermodynamic limit is* $\frac{1}{9}\log 2 + \frac{1}{18}\log(a^3+b^3) + \frac{1}{6}\log c(a+b)$.

By putting $a = b = c = 1$, one finds the same value of entropy as in the "Schreier" labeling, as expected.

5. Statistics

In this section we study the statistics of occurrences of edges with a given label in a random dimer covering, for the Schreier graphs of $H^{(3)}$ and for the Sierpiński triangles. We compute the mean and the variance and, in some cases, we are able to find the asymptotic behavior of the moment generating function of the associated normalized random variable.

5.1. Schreier graphs of the Hanoi Towers group

Denote by c_n (by symmetry, a_n and b_n can be studied in the same way) the random variable that counts the number of occurrences of edges labeled c in a random dimer covering on Σ_n. In order to study it, introduce the function $\Phi_n^i(c) = \Phi_n^i(1, 1, c)$, for $i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, and observe that $\Phi_n^{\mathrm{III}}(c) = \Phi_n^{\mathrm{IV}}(c)$. Moreover, denote by $\mu_{n,i}$ and $\sigma_{n,i}^2$ the mean and the variance of c_n in a random dimer covering of type i, respectively. Note that we have $\mu_{n,\mathrm{III}} = \mu_{n,\mathrm{IV}}$ and $\sigma_{n,\mathrm{III}}^2 = \sigma_{n,\mathrm{IV}}^2$.

Theorem 5.1. *For each $n \geq 1$,*

$$\mu_{n,\mathrm{I}} = \frac{3^{n-1}+1}{2}, \qquad \mu_{n,\mathrm{II}} = \frac{3^{n-1}+3}{2}, \qquad \mu_{n,\mathrm{III}} = \frac{3^{n-1}-1}{2},$$
$$\sigma_{n,\mathrm{I}}^2 = \frac{3^n - 6n + 3}{4}, \qquad \sigma_{n,\mathrm{II}}^2 = \frac{3^n + 10n - 13}{4}, \qquad \sigma_{n,\mathrm{III}}^2 = \frac{3^n - 2n - 1}{4}.$$

Proof. For $a = b = 1$, the system (2) reduces to

$$\begin{cases} \Phi_{n+1}^{\mathrm{I}} = \dfrac{\left(\Phi_n^{\mathrm{I}}\right)^3}{c} + \Phi_n^{\mathrm{II}}\left(\Phi_n^{\mathrm{III}}\right)^2 \\ \Phi_{n+1}^{\mathrm{II}} = \dfrac{\left(\Phi_n^{\mathrm{II}}\right)^3}{c} + \Phi_n^{\mathrm{I}}\left(\Phi_n^{\mathrm{III}}\right)^2 \\ \Phi_{n+1}^{\mathrm{III}} = \left(\Phi_n^{\mathrm{III}}\right)^3 + \dfrac{\Phi_n^{\mathrm{I}}\Phi_n^{\mathrm{II}}\Phi_n^{\mathrm{III}}}{c} \end{cases}$$

with initial conditions $\Phi_1^{\mathrm{I}}(c) = c$, $\Phi_1^{\mathrm{II}}(c) = c^2$ and $\Phi_1^{\mathrm{III}}(c) = 1$. Now put, for every $n \geq 1$, $q_n = \Phi_n^{\mathrm{II}}/\Phi_n^{\mathrm{I}}$ and $r_n = \Phi_n^{\mathrm{III}}/\Phi_n^{\mathrm{I}}$. Observe that both q_n and r_n are functions of the only variable c. In particular, for each n, one has $q_n(1) = r_n(1) = 1$, since the number of dimer covering is the same for each type of configuration. By computing the quotient $\Phi^{\mathrm{II}}_{n+1}/\Phi^{\mathrm{I}}_{n+1}$ and dividing each term by $(\Phi^{\mathrm{I}}_n)^3$, one gets

$$q_{n+1} = \frac{\frac{q_n^3}{c} + r_n^2}{\frac{1}{c} + q_n r_n^2}. \tag{10}$$

Similarly, one has

$$r_{n+1} = \frac{r_n^3 + \frac{q_n r_n}{c}}{\frac{1}{c} + q_n r_n^2}. \tag{11}$$

Using (10) and (11), one can show by induction that $q'_n(1) = 1$ and $r'_n(1) = -1$, for every $n \geq 1$. Moreover, $q''_n(1) = 4(n-1)$ and $r''_n(1) = n+1$.

From the first equation of the system, one gets $\Phi^{\mathrm{I}}_n(c) = \frac{(\Phi^{\mathrm{I}}_{n-1})^3}{c}\left(1 + q_{n-1}(c) r^2_{n-1}(c)\right)$. By applying the logarithm and using recursion, we have:

$$\log(\Phi^{\mathrm{I}}_n(c)) = 3^{n-1}\log(\Phi^{\mathrm{I}}_1(c)) - \sum_{k=0}^{n-2} 3^k \log c + \sum_{k=1}^{n-1} 3^{n-1-k}\log(1 + c q_k(c) r_k^2(c)).$$

Taking the derivative and putting $c = 1$, one gets

$$\mu_{n,\mathrm{I}} = \left.\frac{\Phi^{\mathrm{I}'}_n(c)}{\Phi^{\mathrm{I}}_n(c)}\right|_{c=1} = \frac{3^{n-1}+1}{2},$$

what is one third of the total number of edges involved in such a covering, as it was to be expected because of the symmetry of the graph and of the labeling. Taking once more derivative, one gets

$$\left.\frac{\Phi^{\mathrm{I}''}_n(c)\Phi^{\mathrm{I}}_n(c) - (\Phi^{\mathrm{I}'}_n(c))^2}{(\Phi^{\mathrm{I}}_n(c))^2}\right|_{c=1} = \frac{3^{n-1} - 6n + 1}{4}.$$

Hence,

$$\sigma^2_{n,\mathrm{I}} = \frac{\Phi^{\mathrm{I}'}_n(1)}{\Phi^{\mathrm{I}}_n(1)} + \frac{\Phi^{\mathrm{I}''}_n(1)\Phi^{\mathrm{I}}_n(1)}{(\Phi^{\mathrm{I}}_n(1))^2} - \frac{(\Phi^{\mathrm{I}'}_n(1))^2}{(\Phi^{\mathrm{I}}_n(1))^2} = \frac{3^n - 6n + 3}{4}.$$

In a similar way one can find $\mu_{n,\mathrm{II}}, \sigma^2_{n,\mathrm{II}}, \mu_{n,\mathrm{III}}, \sigma^2_{n,\mathrm{III}}$. □

Observe that one has $\mu_{n,\mathrm{II}} > \mu_{n,\mathrm{I}} > \mu_{n,\mathrm{III}}$: this corresponds to the fact that the distribution of labels a, b, c is uniform in a configuration of type I, but not in the other ones. In fact, a configuration of type II has a loop labeled c, but a configuration of type III (resp. IV) has a loop labeled b (resp. a): so the label c is "dominant" in type II, whereas the label b (resp. a) is "dominant" in type III (resp. IV).

5.2. Sierpiński triangles

Theorem 5.2. *For Sierpiński triangles with the "Schreier" labeling, for each $n \geq 1$, the random variable c_n has*

$$\mu_n = \frac{3^{n-1}}{4}, \qquad \sigma^2_n = \frac{3}{16}.$$

Moreover, the associated probability density function is

$$f(x) = \begin{cases} \frac{3}{4}\delta\left(x + \frac{1}{\sqrt{3}}\right) + \frac{1}{4}\delta\left(x - \sqrt{3}\right) & n \text{ odd} \\ \frac{3}{4}\delta\left(x - \frac{1}{\sqrt{3}}\right) + \frac{1}{4}\delta\left(x + \sqrt{3}\right) & n \text{ even}, \end{cases}$$

where δ denotes the Dirac function.

Proof. Putting $a = b = 1$, one gets

$$\begin{cases} \Phi_n(c) = (4c)^{\frac{3^{n-1}-1}{4}}(c+3) & \text{for } n \text{ odd} \\ \Phi_n(c) = 2(4c)^{\frac{3^{n-1}-3}{4}}(3c+1) & \text{for } n \text{ even.} \end{cases}$$

The mean and the variance of c_n can be computed as in the previous theorem, by using logarithmic derivatives. Now let $C_n = \frac{c_n - \mu_n}{\sigma_n}$ be the normalized random variable, then the moment generating function of C_n is given by

$$\mathbb{E}(e^{sC_n}) = e^{-\mu_n s/\sigma_n}\mathbb{E}(e^{sx_n/\sigma_n}) = e^{-\mu_n s/\sigma_n}\frac{\Phi_n(e^{s/\sigma_n})}{\Phi_n(1)}.$$

We get

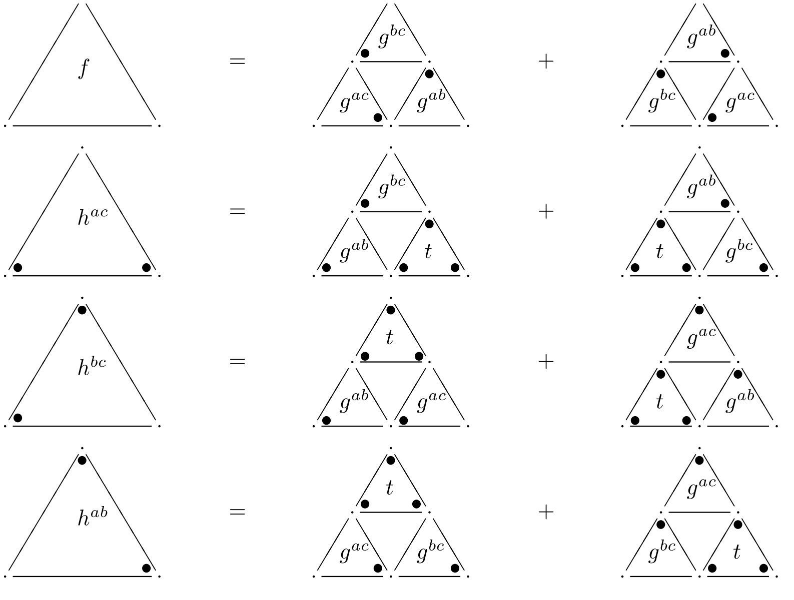
$$\mathbb{E}(e^{sC_n}) = \begin{cases} \dfrac{e^{\sqrt{3}s} + 3e^{-\frac{s}{\sqrt{3}}}}{4} & n \text{ odd} \\ \dfrac{e^{-\sqrt{3}s} + 3e^{\frac{s}{\sqrt{3}}}}{4} & n \text{ even.} \end{cases}$$

and the claim follows. □

Observe that the moment generating functions that we have found only depend on the parity of n. The following theorem gives an interpretation of the probability density functions given in Theorem 5.2.

Theorem 5.3. *For n odd, the normalized random variable C_n is equal to $\sqrt{3}$ in each covering of type h^{ab} and to $-1/\sqrt{3}$ in each covering of type f, h^{ac}, h^{bc}. For n even, the normalized random variable C_n is equal to $-\sqrt{3}$ in each covering of type g^{ab} and to $1/\sqrt{3}$ in each covering of type t, g^{ac}, g^{bc}.*

Proof. The assertion can be proved by induction. For $n = 1, 2$ a direct computation shows that the assertion is true. We give here only the proof for $n > 2$ odd. The following pictures show how to get a labeled dimer covering for Γ_n, n odd, starting from three dimer coverings of Γ_{n-1}. One can easily check that these recursions hold, by using the definition of the labeling of Γ_n.



If we look at the first three pictures, we see that the variable C_n in a dimer covering of type f, h^{ac}, h^{bc} is given, by induction, by the sum of two contributions $1/\sqrt{3}$ and one contribution $-\sqrt{3}$, which gives $-1/\sqrt{3}$. The fourth picture shows that the variable C_n in a dimer covering of type h^{ab} is given, by induction, by the sum of three contributions $1/\sqrt{3}$, which gives $\sqrt{3}$. A similar proof can be given for n even. The statement follows. $\square$

Theorem 5.4. *For Sierpiński graphs with the "rotation-invariant" labeling, for each $n \geq 2$, the random variables a_n and b_n have*

$$\mu_n = \frac{3^{n-1}}{4} \qquad \sigma_n^2 = \frac{4 \cdot 3^{n-1} + 3}{4}$$

and they are asymptotically normal. The random variable c_n has

$$\mu_n = \frac{3^{n-1}}{4} \qquad \sigma_n^2 = \frac{3}{16}$$

and the associated probability density function is

$$f(x) = \begin{cases} \frac{3}{4}\delta\left(x - \frac{1}{\sqrt{3}}\right) + \frac{1}{4}\delta\left(x + \sqrt{3}\right) & \text{for } n \text{ even} \\ \frac{3}{4}\delta\left(x + \frac{1}{\sqrt{3}}\right) + \frac{1}{4}\delta\left(x - \sqrt{3}\right) & \text{for } n \text{ odd.} \end{cases}$$

Proof. By putting $b = c = 1$ in the partition functions given in Theorem 4.3, one gets

$$\begin{cases} \Phi_n(a) = 2^{\frac{3^{n-2}-1}{2}}(a^3+1)^{\frac{3^{n-2}-1}{4}}(a+1)^{\frac{3^{n-1}-3}{4}}(a^3+3a+4) & \text{for } n \text{ even} \\ \Phi_n(a) = 2^{\frac{3^{n-2}-1}{2}}(a^3+1)^{\frac{3^{n-2}-3}{4}}(a+1)^{\frac{3^{n-1}-1}{4}}(3a^3+a+4) & \text{for } n \text{ odd.} \end{cases}$$

Similarly, one can find

$$\begin{cases} \Phi_n(c) = 2^{\frac{3^{n-1}-1}{2}}c^{\frac{3^{n-1}-3}{4}}(3c+1) & \text{for } n \text{ even} \\ \Phi_n(c) = 2^{\frac{3^{n-1}-1}{2}}c^{\frac{3^{n-1}-1}{4}}(c+3) & \text{for } n \text{ odd.} \end{cases}$$

Then one proceeds as in the previously studied cases. $\square$

A similar interpretation as in the case of the "Schreier" labeling can be given.

Theorem 5.5. *For n even, the normalized random variable C_n is equal to $-\sqrt{3}$ in each covering of type t and to $1/\sqrt{3}$ in each covering of type g. For n odd, the normalized random variable C_n is equal to $\sqrt{3}$ in each covering of type f and to $-1/\sqrt{3}$ in each covering of type h.*

Remark 5.6. In [18] the authors study the statistical properties of the dimer model on Γ_n endowed with the "directional" labeling: for n even (which is the only case allowing a perfect matching), they get the following expressions for the mean and the variance of the number of labels c:

$$\mu_n = \frac{3^{n-1}+1}{4} \qquad \sigma_n^2 = \frac{3^{n-1}-3}{4}.$$

Moreover, they show that the associated normalized random variable tends weakly to the normal distribution.

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