A Note on Matrix Inversion

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ABSTRACT

We generalize a recent result of Thompson on inverses of block matrices over principal ideal domains to isomorphisms of direct sums of modules over an arbitrary ring.

Recently, Thompson [4] showed that if

$$\alpha=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an invertible $2n\times 2n$ matrix over a principal ideal domain, expressed in $n\times n$ block form as shown, and if

$$\alpha^{-1}=\begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

then $\mathscr{S}(A)=\mathscr{S}(W)$, $\mathscr{S}(D)=\mathscr{S}(X)$, $\mathscr{S}(B)=\mathscr{S}(Y)$, and $\mathscr{S}(C)=\mathscr{S}(Z)$, where $\mathscr{S}$ denotes Smith canonical form. Here, we give a generalization to isomorphisms of direct sums of modules over arbitrary rings. Indeed, the result is even valid in arbitrary abelian categories, as the interested reader can easily verify by using the appropriate embedding theorem [3, p. 101].

Recall that the Smith canonical form of a matrix over a principal ideal domain is the diagonal form under matrix equivalence, which can be so arranged that each nonzero diagonal entry divides the next; see [2, p. 26]. Further, if matrices are viewed as homomorphisms of free modules in the usual way, then it is well known that two matrices have the same Smith form (up to units) if and only if they have isomorphic cokernels. For a history of

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this result and its extensions, see [1]. Our generalization of Thompson's result can now be stated as:

THEOREM. *Let M, N, U, and V be left modules over a ring R. Let $\alpha: M \oplus N \to U \oplus V$ be an isomorphism, given in matrix form as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, so that $A: M \to U$, $B: N \to U$, $C: M \to V$, and $D: N \to V$. Let α^{-1} have matrix form $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$. Then A and W have isomorphic cokernels and isomorphic kernels. The analogous statements hold for D and X, for B and Y, and for Z and C.*

Proof. Since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

are mutually inverse, we obtain

$$AX + BZ = 1_U, \tag{1}$$

$$CY + DW = 1_V, \tag{2}$$

$$AY + BW = 0, \tag{3}$$

$$CX + DZ = 0, \tag{4}$$

$$XA + YC = 1_M, \tag{5}$$

$$ZB + WD = 1_N, \tag{6}$$

$$XB + YD = 0, \tag{7}$$

$$ZA + WC = 0. \tag{8}$$

Define $\beta: N/W(V) \to U/A(M)$ by $\beta(n + W(V)) = B(n) + A(M)$. By (3), $BW(V) = AY(V)$, whence β is well defined. If $\beta(n + W(V) = 0$, then $B(n) = A(m)$ for some $m \in M$. By (6), $n = ZB(n) + WD(n) = ZA(m) + WD(n)$. But $ZA(m) = -WC(m)$, by (8). Hence $n \in W(V)$, so $n + W(V) = 0$ in $N/W(V)$. Thus, β is a monomorphism. That β is an epimorphism is equivalent to the equation $U = B(N) + A(M)$, which follows immediately from (1). Thus, β is an isomorphism from coker W to coker A.

For $m \in \ker W$, $AY(m) = -BW(m) = 0$, by (3). Hence, Y restricts to a homomorphism $\psi: \ker W \to \ker A$. If $m \in \ker \psi$, then $m \in (\ker W) \cap (\ker Y)$. By (2), $m = CY(m) + DW(m) = 0$, whence ψ is a monomorphism. For $m \in \ker A$, we have $m = XA(m) + YC(m) = YC(m)$, by (5). Also, $WC(m) = -ZA(m)$ by (8). Hence, $C(m) \in \ker W$, and $m = \psi(C(m))$. Hence, ψ is an epimorphism. Thus, ψ gives an isomorphism from $\ker W$ to $\ker A$. The other isomorphisms asserted in the theorem are established analogously.

REFERENCES

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