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On Principal Solutions of Nonlinear Differential Equations

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1. In this paper we are concerned with the oscillation and asymptotic behavior of principal solutions of nonlinear differential equations of the form

$$y^{(n)} + f(x, y, y', \dots, y^{(n-1)}) = 0, \qquad 0 \leqslant x < \infty, \qquad f \in C[\mathbf{R}_+ x \mathbf{R}^n, \mathbf{R}]. \quad (1.1)$$

A principal solution $y(x) \equiv y(x, \alpha)$ of (1.1) at the point $\alpha \geqslant 0$ is defined by the conditions

$$y(\alpha) = y'(\alpha) = \cdots = y^{(n-2)}(\alpha) = 0, \qquad y^{(n-1)}(\alpha) = 1. \tag{1.2}$$

We assume that principal solutions $y(x, \alpha)$ of (1.1) are defined for all $\alpha \geqslant 0$ and exist for all $t \geqslant 0$, but uniqueness is not required. The differential equation (1.1) is said to be oscillatory if for every $\alpha \geqslant 0$ the principal solution $y(x, \alpha)$ has at least one zero in (α, ∞). Otherwise (1.1) is said to be non-oscillatory.

Our results generalize well-known oscillation theorems of Leighton and Nehari [2, pp. 371–375]. Similar results were also obtained in Refs. [1] and [3], but only for special cases of Eq. (1.1).

2. We first prove the following lemma.

LEMMA 2.1. *If $y(x)$ is a solution of* (1.1) *such that*

$$f(x, y(x), y'(x), \dots, y^{(n-1)}(x)) \geqslant 0, \; x \geqslant \alpha \geqslant 0, \tag{2.1}$$

then

$$\lim_{x\to\infty} y^{(n-1)}(x) = \lim_{x\to\infty} (k-1)!(x-\alpha)^{1-k} y^{(n-k)}(x), \qquad k = 1, 2, \dots, n, \tag{2.2}$$

and both limits are finite.

Proof. Integrating (1.1) k times over $[\alpha, x]$ we obtain

$$y^{(n-k)}(x) - \sum_{j=0}^{k-1} \frac{y^{(n-k+j)}(\alpha)}{j!} (x-\alpha)^j + \frac{1}{(k-1)!} \int_\alpha^x (x-t)^{k-1} f(t, y(t), \dots, y^{(n-1)}(t))\, dt = 0. \quad (2.3)$$

Since $f(t, y(t), \dots, y^{(n-1)}(t)) \geqslant 0$, (1.1) and (2.3) yield the inequality

$$y^{(n-k)}(x) - \sum_{j=0}^{k-1} \frac{y^{(n-k+j)}(x)}{j!} (x-\alpha)^j + \frac{1}{(k-1)!} (x-\alpha)^{k-1} \mid y^{(n-1)}(\alpha) - y^{(n-1)}(x) | \geqslant 0,$$

and from this it follows that

$$(k-1)! y^{(n-k)}(x)(x-\alpha)^{1-k} + O\left(\frac{1}{x-\alpha}\right) \geqslant y^{(n-1)}(x). \quad (2.4)$$

From (1.1) and (2.1) it follows that

$$y^{(n)}(x) \leqslant 0, \qquad x \in [\alpha, \infty);$$

therefore the function $y^{(n-1)}(x)$ is nonincreasing in $x \in [\alpha, \infty)$ and so the $\lim_{x\to\infty} y^{(n-1)}(x)$ exists and is a finite number.

Now from (2.4) we obtain

$$\liminf_{x\to\infty} (k-1)! y^{(n-k)}(x)(x-\alpha)^{1-k} \geqslant \lim_{x\to\infty} y^{(n-1)}(x). \quad (2.5)$$

Again from (2.3) and any point $x_1 \epsilon (\alpha, x)$ we get the inequality

$$y^{(n-k)}(x) - \sum_{j=0}^{k-1} \frac{y^{(n-k+j)}(\alpha)}{j!} (x-\alpha)^j + \frac{1}{(k-1)!} \int_\alpha^{x_1} (x-x_1)^{k-1} f(t, y(t), \dots, y^{(n-1)}(t))\, dt \leqslant 0.$$

From this and (1.1) we obtain

$$(k-1)!\, y^{(n-k)}(x)(x-\alpha)^{1-k} - y^{(n-1)}(\alpha) + O\left(\frac{1}{x-\alpha}\right) + \left(\frac{x-x_1}{x-\alpha}\right)^{k-1} [y^{(n-1)}(\alpha) - y^{(n-1)}(x_1)] \leqslant 0,$$

and so

$$\limsup_{x\to\infty}(k-1)!\, y^{(n-k)}(x)(x-\alpha)^{1-k} \leqslant y^{(n-1)}(x_1). \tag{2.6}$$

Since (2.6) holds for all $x_1 \epsilon (\alpha, x)$ and since $\lim_{x_1\to\infty} y^{(n-1)}(x_1)$ exists, we conclude by letting $x_1 \to \infty$ that

$$\limsup_{x\to\infty}(k-1)!\, y^{(n-k)}(x)(x-\alpha)^{1-k} \leqslant \lim_{x\to\infty} y^{(n-1)}(x). \tag{2.7}$$

From (2.5) and (2.7) the desired result (2.2) follows. Both limits are finite since as we observed previously $y^{(n-1)}(x)$ is a nonincreasing function of $x \in [0, \infty)$.

THEOREM 2.2. *Let the following conditions be satisfied*:

(i) $f(x, x_1, x_2, \dots, x_n)$ *is nondecreasing in* $x_i \geqslant 0$, $i = 1, 2, \dots, n$, *and* $f(x, 0, 0, \dots, 0) \geqslant 0$ *for* $x \geqslant 0$.

(ii) $$\int_0^\infty f(t, t^{n-1}, t^{n-2}, \dots, t, 1) < \infty.$$

Then (1.1) *is nonoscillatory. Moreover, for* α *sufficiently large the principle solution* $y(x) \equiv y(x, \alpha)$ *of* (1.1), *together with all its derivatives up to order* $n-1$, *are positive for* $x > \alpha$ *and*

$$\lim_{x\to\infty} x^{1-k} y^{(n-k)}(x) = \frac{1}{(k-1)!}, \qquad k = 1, 2, \dots, n. \tag{2.8}$$

Proof. Let $\epsilon > 0$ be given, $\epsilon < 1$. Choose $\alpha_0 \geqslant 0$ large enough to assure that

$$\int_{\alpha_0}^\infty f(t, t^{n-1}, t^{n-2}, \dots, t, 1) \leqslant \epsilon. \tag{2.9}$$

This is possible because of (ii).

We shall prove that the conclusion of Theorem 2.2 is true for any $\alpha \geqslant \alpha_0$.

For simplicity we set $y(x) \equiv y(x, \alpha)$. From (2.3) and (1.2) we easily get

$$(x-\alpha)^{k-1} = (k-1)!\, y^{(n-k)}(x) + \int_\alpha^x (x-t)^{k-1} f(t, y(t), \dots, y^{(n-1)}(t))\, dt, \qquad k = 1, 2, \dots, n. \tag{2.10}$$

We now claim that $y^{(i)}(x) > 0$, $x \in (\alpha, \infty)$, $i = 0, 1,..., n-1$. If the claim were false there should exist a point $\beta > \alpha$ such that for $i = 0, 1,..., n-1$, $y^{(i)}(x) \geqslant 0$ in (α, β) while $y^{(j)}(\beta) = 0$ for some j.

By (i),

$$f(t, y(t), y'(t),..., y^{(n-1)}(t)) \geqslant 0, \qquad x \in [\alpha, \beta],$$

and by (2.10) we get

$$(x-\alpha)^{k-1} \geqslant (k-1)!\, y^{(n-k)}(x), \qquad \alpha \leqslant x \leqslant \beta, \qquad k = 1, 2,..., n. \tag{2.11}$$

Applying (2.10) for $k = n - j$ at the point $x = \beta$ and using (i), (2.11), and (2.9) we obtain the following contradiction:

$$\begin{aligned}(\beta-\alpha)^{n-j-1} &= (k-1)! y^{(j)}(\beta) + \int_\alpha^\beta (\beta - t)^{n-j-1} f(t, y(t),..., y^{(n-1)}(t)\, dt \\ &\leqslant (\beta-\alpha)^{n-j-1} \int_\alpha^\beta f\left(t, \frac{(t-\alpha)^{n-1}}{(n-1)!}, \frac{(t-\alpha)^{n-2}}{(n-2)!},..., (t-\alpha), 1\right) dt \\ &\leqslant (\beta-\alpha)^{n-j-1} \int_\alpha^\beta f(t, t^{n-1}, t^{n-2},..., t, 1)\, dt \\ &\leqslant (\beta-\alpha)^{n-j-1} \int_\alpha^\infty f(t, t^{n-1}, t^{n-2},..., t, 1)\, dt \leqslant \epsilon(\beta-\alpha)^{n-j-1}.\end{aligned}$$

Therefore the claim is correct. (2.11) holds for all $x \geqslant \alpha$ and

$$f(t, y(t),..., y^{(n-1)}(t)) \geqslant 0, \qquad t \geqslant \alpha.$$

Now (2.10) for $x \geqslant \alpha$ gives

$$\begin{aligned}(x-\alpha)^{k-1} &\leqslant (k-1)! y^{(n-k)}(x) + (x-\alpha)^{k-1} \int_\alpha^x f(t, y(t),..., y^{(n-1)}(t))\, dt \\ &\leqslant (k-1)! y^{(n-k)}(x) + (x-\alpha)^{k-1} \int_\alpha^x f(t, t^{n-1},..., t, 1)\, dt \\ &\leqslant (k-1)! y^{(n-k)}(x) + (x-\alpha)^{k-1} \int_{\alpha_0}^\infty f(t, t^{n-1},..., t, 1)\, dt \\ &\leqslant (k-1)! y^{(n-k)}(x) + \epsilon(x-\alpha)^{k-1},\end{aligned}$$

and so from this and (2.11), which is true for any β, we get

$$(1-\epsilon) \leqslant (k-1)!(x-\alpha)^{1-k} y^{(n-k)}(x) \leqslant 1, \qquad x \geqslant \alpha. \tag{2.12}$$

Then (2.8) follows from (2.12) and the fact that ϵ is arbitrary.

Under certain conditions on $f(t, x_1, x_2, \ldots, x_n)$, Theorem 2.2 has a converse, i.e., conditions (i) and (2.8) imply (ii). This is the case, for example, if $n = 4$ and $f(t, x_1, \ldots, x_4) \equiv p(t)\, x_1$, as was shown in Ref. [2].

A more general condition on $f(t, x_1, \ldots, x_n)$ to obtain the previous converse is to assume that for all A_k, $k = 1, \ldots, n-1$, there exists an M fixed such that

$$\int_0^\infty f(t, A_{n-1}t^{n-1}, \ldots, A_1 t, 1)\, dt \geqslant M \int_0^\infty f(t, t^{n-1}, \ldots, t, 1)\, dt \qquad (2.13)$$

In fact, one can prove something more, viz.,

THEOREM 2.3. *Assume that condition* (i) *of Theorem* 2.2 *is true. Assume that condition* (2.8) *holds for some solution* (*not necessarily principal*) *of* (1.1). *Then*

$$\int_0^\infty f(t, A_{n-1}t^{n-1}, \ldots, A_1 t, 1)\, dt < \infty, \qquad (2.14)$$

where

$$A_k = \frac{1}{k!}, \qquad k = 1, 2, \ldots, n-1.$$

Proof. For $\epsilon > 0$, ϵ sufficiently small, and from (2.8), it follows that for $x \geqslant x_0$, x_0 sufficiently large,

$$y^{(n-k)}(x) > \left[\frac{1}{(k-1)!} - \epsilon\right] x^{k-1} > 0, \qquad x \geqslant x_0. \qquad (2.15)$$

Integrating (1.1) over $[x_0, x]$ we get

$$y^{(n-1)}(x_0) - y^{(n-1)}(x) = \int_{x_0}^{x} f(t, y(t), y'(t), \ldots, y^{(n-1)}(t))\, dt. \qquad (2.16)$$

Define

$$\bar{A}_k = \left[\frac{1}{(k-1)!} - \epsilon\right], \qquad k = 1, 2, \ldots, n-1.$$

We have

$$A_k = \lim_{\epsilon \to 0} \bar{A}_k, \qquad k = 1, 2, \ldots, n-1,$$

and $\bar{A}_k$ decreases in ϵ, $k = 1, 2, \ldots, n-1$.
Using (2.15) and (i) it follows from (2.16) that

$$y^{(n-1)}(x_0) - y^{(n-1)}(x) \geqslant \int_{x_0}^{x} f(t, \bar{A}_{n-1}t^{n-1}, \ldots, \bar{A}_1 t, 1)\, dt. \qquad (2.17)$$

Taking limits in (2.17) as $x \to \infty$ and as $\epsilon \to 0$, and using the monotone convergence theorem in the right side of (2.17), we get

$$y^{(n-1)}(x_0) - 1 \geqslant \int_{x_0}^{\infty} f(t, A_{n-1}t^{n-1}, \ldots, A_1 t, 1)\, dt. \tag{2.18}$$

The desired result follows from (2.18) by observing that the left side of (2.18) is a finite number.

THEOREM 2.4. *Let the following conditions be satisfied:*

(i) $f(x, x_1, x_2, \ldots, x_n)$ *is nondecreasing in* $x_i \geqslant 0$, $i = 1, 2, \ldots, n$, *and* $f(x, x_1, x_2, \ldots, x_n) > 0$ *for* $x \geqslant 0$, $x_1 > 0$;

(ii) *For all* $\alpha > 0$ *and* $A_i > 0$, $i = 0, 1, 2, \ldots, n-2$,

$$\int_0^{\infty} f(t, A_{n-2}(t-\alpha)^{n-2}, \ldots, A_1(t-\alpha), A_0, 0)\, dt = \infty. \tag{2.19}$$

Then for all $\beta \geqslant 0$ *the principal solution* $y(x) \equiv y(x, \beta)$ *of* (1.1) *has at least one zero in* (β, ∞), *i.e.,* (1.1) *is oscillatory.*

Proof. If the theorem were false there should exist a $\beta \geqslant 0$ such that $y(x, \beta) > 0$, $x > \beta$. It follows from (i) that

$$y^{(n)}(x) = -f(x, y(x), \ldots, y^{(n-1)}(x)) < 0,$$

$x > \beta$, and so $y^{(n-1)}(x)$ is strictly decreasing in (β, ∞). Also condition (2.1) is true and by Lemma 2.1 for $k = n$ it follows that $y^{(n-1)}(\infty) \geqslant 0$. Therefore $y^{(n-1)}(x) > 0$ for $x > \beta$. So $y^{(n-2)}(x)$ is strictly increasing and since $y^{(n-2)}(\beta) = 0$ it follows that $y^{(n-2)}(x) > 0$, $x > \beta$. With the same reasoning and an induction we prove that

$$y^{(k)}(x) > 0, \qquad x > \beta, \qquad k = 0, 1, 2, \ldots, n-1. \tag{2.20}$$

Applying Taylor's theorem to $y^{(k)}(x)$ around $\alpha > \beta$ we obtain

$$y^{(k)}(x) = y^{(k)}(\alpha) + y^{(k+1)}(\alpha)(x-\alpha) + \cdots + y^{(n-2)}(\alpha)\frac{(x-\alpha)^{n-2-k}}{(n-2-k)!} + y^{(n-1)}(\xi_k)\frac{(x-\alpha)^{n-1-k}}{(n-1-k)!},$$

where $\alpha \leqslant \xi_k \leqslant x$, $k = 0, 1, 2, \ldots, n-2$.
Defining

$$A_k = \frac{y^{(n-2)}(\alpha)}{k!}, \qquad k = 0, 1, 2, \ldots, n-2,$$

it follows that

$$y^{(k)}(x) > A_{n-2-k}(x - \alpha)^{n-2-k}, \qquad k = 0, 1, 2, \ldots, n-2, \tag{2.21}$$

and $y^{(n-1)}(x) > 0$ for $x > \alpha$.

Integrating (1.1) over $[\alpha, x]$ and using (i) and (2.21) it follows that

$$\begin{aligned} 1 \geqslant y^{(n-1)}(\alpha) - y^{(n-1)}(x) &= \int_\alpha^x f(t, y(t), y'(t), \ldots, y^{(n-1)}(t))\, dt \\ &\geqslant \int_\alpha^x f(t, A_{n-2}(t-\alpha)^{n-2}, \ldots, A_1(t-\alpha), A_0, 0)\, dt. \end{aligned}$$

Since the left side is independent of x while the right side, by (2.19), goes to ∞ as $x \to \infty$, the above relation is a contradiction and Theorem (2.4) is proved.

References

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