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On Vector-Valued Functional Representations of Topological Algebras

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1. INTRODUCTION

Representations of (commutative) m-barreled topological algebras as suitable algebras of complex-valued functions on their spectra have already been considered (cf., [4] and the related references). The representations in question are based on topological properties of the spectra, yielded by those of the algebras. On the other hand, analogous properties of the generalized spectra for suitable topological algebras also have been derived (cf. [9, Sect. 3]). Moreover, certain results of [6] (cf. Section 3 below), referred to the barreledness of the locally convex space $\mathscr{C}(X, F)$ of continuous locally convex F-valued maps on a completely regular space X, equipped with the topology of compact convergence in X, which results constitute extended forms of the Nachbin–Shirota theorem (cf. [5, 8]), in connection with the techniques of [4, Sect. 3], motivate the present work. The results obtained therein have a special bearing on some respective results of [4] and [9].

In the present paper we are interested in (functional) representations of (not necessarily commutative) m-barreled locally convex algebras as suitable algebras of (locally convex) algebra-valued maps on the respective generalized spectra. Thus, let E be an m-barreled (locally convex) algebra and let F be a locally m-convex semi-Montel one. Then, the equicontinuous subsets, the relatively (simply) compact subsets and the (simply) bounded subsets of the corresponding generalized spectrum $\mathscr{M}(E, F)$ are the same (cf. Theorem 2.1 below). In particular, $\mathscr{M}(E, F)$ is a Nachbin–Shirota space (cf. Corollary 3.5 below), so that if, in addition, F is a Fréchet (or a nuclear complete barreled) algebra, $\mathscr{C}(\mathscr{M}(E, F), F)$ is a barreled locally convex space (Theorem 3.3). Finally, into the context of the preceding considerations with the additional assumption that E is a Pták locally convex space, it is proved that the algebraic exactness of the sequence

$$0 \to E \xrightarrow{g} \mathscr{C}(\mathscr{M}(E, F), F) \to 0,$$

where g is the respective generalized Gel'fand map, implies the topological exactness, so that the topology of the algebra E is that of the closed equicontinuous convergence in $\mathscr{M}(E,F)$ (generalized Michael topology; cf. Theorem 4.1 and its Corollaries 4.2 and 4.3 below).

2. Preliminaries: Generalized Spectra of Locally Convex Algebras

All vector spaces and (associative linear) algebras considered in the following are over the field $\mathbb{C}$ of complex numbers. The topological spaces involved are assumed to be Hausdorff unless otherwise indicated.

By an *m-barreled algebra* we mean a (locally convex) algebra E such that every m-barrel (i.e., absorbing balanced convex closed and idempotent subset of E) is a 0-neighborhood in E [3]. For a detailed discussion of m-barreled topological algebras we also refer to [3] and [4]. On the other hand, by a *semi-Montel algebra* we mean a (locally convex) algebra E such that the underlying locally convex space E is a semi-Montel space [1]. Finally, a locally convex algebra E is said to be a *Pták algebra* if the locally convex space E is a Pták space (cf. [1, p. 299, Definition 2]).

Now, let E and F be locally convex algebras. Then, by the *generalized spectrum of E (for F given)* [4a] we mean the set $\mathscr{M}(E,F)$ of nonzero continuous (algebra) homomorphisms of E into F, topologized as a subset of $\mathscr{L}_s(E,F)$ (i.e., the space $\mathscr{L}(E,F)$ of continuous linear maps between the topological vector spaces E and F, equipped with the topology of simple convergence in E). Thus, the spectrum $\mathscr{M}(E,F)$ is, of course, a completely regular (Hausdorff) topological space. Moreover, if the algebras E and F have identity elements, the elements of the spectrum $\mathscr{M}(E,F)$ are assumed to be *identity preserving*. In this case, $\mathscr{M}(E,F)$ is clearly a closed subset of $\mathscr{L}_s(E,F)$. On the other hand, the set $\mathscr{M}(E,F)^+ := \mathscr{M}(E,F) \cup \{0\} \subseteq \mathscr{L}(E,F)$, topologized as a subset of $\mathscr{L}_s(E,F)$, is called the *extended generalized spectrum* of E (for F given). It is clear that $\mathscr{M}(E,F)^+$ is a closed subset of $\mathscr{L}_s(E,F)$ whenever F has continuous multiplication.

Now, let E be an m-barreled (locally convex) algebra, F a locally convex one, $\mathscr{M}(E,F)$ the respective generalized spectrum of E (for F given) and let H be a (nonempty) subset of $\mathscr{M}(E,F)$. First, if H is relatively (simply) compact, then it is clearly (simply) bounded. Moreover, if F is a locally m-convex algebra and H (simply) bounded, then H is equicontinuous (cf. [9, p. 176, Proposition 3.1]). On the other hand, in case, F is a semi-Montel algebra and H is equicontinuous, then H is also relatively (simply) compact (cf. [9, p. 175, Theorem 3.1]), so that we get the following result, which extends it [4, p. 470, Theorem 2.1, Corollary 2.1]. That is, we have

Theorem 2.1. *Let E be an m-barreled (locally convex) algebra, F be a semi-Montel locally m-convex algebra, $\mathscr{M}(E,F)$ be the corresponding generalized* *spectum of E (for F given), and let H be a (nonempty) subset of $\mathscr{M}(E,F)$. Then, the following assertions are equivalent:*

(1) *H is equicontinuous.*

(2) *H is relatively (simply) compact.*

(3) *H is (simply) bounded.*

Now, let E and F be locally convex algebras and let $\mathscr{M}(E,F)$ be the generalized spectrum of E (for F given). Then, the *generalized Gel'fand transform* of an element x of E is the (continuous) map

$$\hat{x}\colon \mathscr{M}(E,F) \mapsto F\colon h \mapsto \hat{x}(h) := h(x).$$

On the other hand, the *generalized Gel'fand map* is the (algebra) homomorphism $g\colon E \mapsto \mathscr{C}(\mathscr{M}(E,F),F)\colon x \mapsto g(x) := \hat{x}$, where $\mathscr{C}(\mathscr{M}(E,F),F)$ is the algebra of continuous F-valued maps on $\mathscr{M}(E,F)$. The algebra E is called *F-bounded*, if for every $x \in E$ the set $\hat{x}(\mathscr{M}(E,F))$ is a bounded subset of F. As is easily verified, E is F-bounded if, and only if, $\mathscr{M}(E,F)$ is a (simply) bounded subset of $\mathscr{L}_s(E,F)$. Moreover, one obviously has

$$\mathscr{M}(E,F)^+ \subseteq \overline{\mathscr{M}(E,F)} \cup \{0\},$$

where $\overline{\mathscr{M}(E,F)}$ is the closed hull of $\mathscr{M}(E,F)$ in $\mathscr{L}_s(E,F)$. Thus, $\mathscr{M}(E,F)$ is clearly relatively (simply) compact if, and only if, $\mathscr{M}(E,F)^+$ is relatively (simply) compact, so that by the foregoing and Theorem 2.1 above we now have the following result, which has a special bearing on [4, p. 471, Corollaries 2.2, 2.3].

COROLLARY 2.2. *Let E be an m-barreled (locally convex) algebra, F be a semi-Montel locally m-convex algebra, and let $\mathscr{M}(E,F)$ and $\mathscr{M}(E,F)^+$ be the generalized and extended generalized spectrum of E (for F given), respectively. Then, the following assertions are equivalent:*

(1) *E is F-bounded.*

(2) *$\mathscr{M}(E,F)$ is relatively (simply) compact.*

(3) *$\mathscr{M}(E,F)^+$ is (simply) compact.*

3. On Nachbin–Shirota Spaces

Let X be a completely regular (Hausdorff) topological space, F a locally convex space, whose topology is defined by a saturated family Γ of seminorms, and let $\mathscr{C}(X,F)$ be the locally convex space of continuous F-valued maps on

X, equipped with the topology of compact convergence in X [2]. Moreover, for $f \in \mathscr{C}(X,F)$, $p \in \Gamma$, $\epsilon > 0$ and a (non empty) subset A of X we put: $p_A(f) := \sup\{p(f(x)): x \in A\}$ and $V(A, p, \epsilon) := \{h \in \mathscr{C}(X,F): p_A(h) < \epsilon\}$. It is clear that $V(A, p, \epsilon)$ is a 0-neighborhood in $\mathscr{C}(X,F)$ if and only if, A is relatively compact. On the other hand, if $F = \mathbb{C}$, we will write $\mathscr{C}(X)$ for $\mathscr{C}(X, \mathbb{C})$.

In this section, we are mainly interested in spaces X and F such that $\mathscr{C}(X,F)$ be a barreled locally convex space. First, the well-known Nachbin–Shirota theorem asserts that $\mathscr{C}(X)$ is a barreled locally convex space if and only if, for every closed noncompact subset B of X there exists a continuous real-valued function f on X, which is unbounded on B (cf. [5, p. 471, Theorem 1] and [8, p. 294, Theorem 1]). Thus, a completely regular (Hausdorff) topological space X is called a *Nachbin–Shirota space* if for every closed noncompact subset B of it there exists a continuous real-valued function f on X, which is unbounded on B.

PROPOSITION 3.1. *Let X be a completely regular (Hausdorff) topological space. Then, the following assertions are equivalent:*

(1) *X is a Nachbin–Shirota space.*

(2) *For some (in fact, clearly for every) (Hausdorff) locally convex space F and for every closed noncompact subset B of X there exists an element h of $\mathscr{C}(X,F)$, which is unbounded on B.*

Proof. (1) implies (2). Let F be a (Hausdorff) locally convex space and let B be a closed noncompact subset of X. Moreover, let $y \in F$ with $y \neq 0$ and $p \in \Gamma$ with $p(y) = 1$. Then, for the map $h := f \otimes y \in \mathscr{C}(X,F)$ one obviously has $p(h(B)) = \{|f(b)| : b \in B\}$, and hence, by hypothesis for f, h is unbounded on B.

(2) implies (1). Let B be a closed noncompact subset of X. Then, by hypothesis, $h(B)$ is an unbounded subset of F and hence there exists a seminorm $p \in \Gamma$ with $p(h(B))$ unbounded, so that the function $f := p \circ h$ is unbounded on B and the proof is finished.

On the other hand, by using the respective arguments of [5, p. 471, Theorem 2], we get the following.

PROPOSITION 3.2. *Let X be a completely regular (Hausdorff) topological space and let F be a locally convex space such that $\mathscr{C}(X,F)$ be a barreled locally convex space. Then, X is a Nachbin–Shirota space.*

Proof. Let B be a closed noncompact subset of X, $p \in \Gamma$ and $\epsilon > 0$. Then, $V(B, p, \epsilon)$ is obviously an absolutely convex and closed subset of $\mathscr{C}(X,F)$, which, by the foregoing, fails to be a 0-neighborhood in $\mathscr{C}(X,F)$. Thus, by hypothesis for $\mathscr{C}(X,F)$, $V(B,p,\epsilon)$ fails to be absorbing, that is, there exists an element h of $\mathscr{C}(X,F)$, with $h \notin \lambda V(B,p,\epsilon)$ for every $\lambda > 0$, and hence, h is unbounded on B, which, by Proposition 3.1 above, proves the assertion.

Furthermore, it is proved in [6] that if X is a Nachbin–Shirota space and F a Fréchet (or, a complete nuclear barreled) locally convex space then $\mathscr{C}(X,F)$ is barreled, so that by Proposition 3.2 above, we are now in a position to state the following result, which will be used in the next section. That is, we have

THEOREM 3.3 (Nachbin–Shirota). *Let X be a completely regular (Hausdorff) topological space and let F be a Fréchet (or a complete nuclear barreled) locally convex space. Then, the following assertions are equivalent:*

(1) *X is a Nachbin–Shirota space.*

(2) *$\mathscr{C}(X,F)$ is a barreled locally convex space.*

PROPOSITION 3.4. *Let E and F be locally convex algebras and let $\mathscr{M}(E,F)$ be the generalized spectrum of E (for F given) such that every (simply) bounded subset of $\mathscr{M}(E,F)$ be relatively (simply) compact. Then, the spectrum $\mathscr{M}(E,F)$ is a Nachbin–Shirota space.*

Proof. Let H be a (simply) closed noncompact subset of $\mathscr{M}(E,F)$. Then, by hypothesis, H is (simply) unbounded, and hence, there exists $x \in E$ such that the set $\{h(x): h \in H\}$ is an unbounded subset of F. Thus, the respective generalized Gel'fand transform $\hat{x} \in \mathscr{C}(\mathscr{M}(E,F),F)$ is unbounded on H, so that the assertion is now obtained by Proposition 3.1 above, and the proof is finished.

Finally, by Proposition 3.4 and Theorem 2.1 in the foregoing, we clearly get the next result, which is needed for what follows.

COROLLARY 3.5. *Let E be an m-barreled (locally convex) algebra and let F be a locally m-convex semi-Montel algebra. Then, the corresponding generalized spectrum $\mathscr{M}(E,F)$ of E is a Nachbin–Shirota space.*

4. FUNCTIONAL REPRESENTATIONS

Let E and F be locally convex algebras, $\mathscr{M}(E,F)$ the generalized spectrum of E (for F given) and let $g: E \mapsto \mathscr{C}(\mathscr{M}(E,F),F)$ be the corresponding generalized Gel'fand map. Then, the algebra E is called *(functionally) semisimple (with respect to F)* (or, briefly *F-semisimple*), if g is injective. On the other hand, E is called *full (with respect to F)* (or briefly *F-full*), if g is bijective.

Now, let E and F be locally convex algebras and let $\mathscr{C}_k(\mathscr{M}(E,F),F)$ be the locally convex space of continuous F-valued maps on $\mathscr{M}(E,F)$, equipped with the topology τ_k of closed equicontinuous convergence in $\mathscr{M}(E,F)$ [7]. Then the inverse image (initial) topology $m(E,F)$ on E, defined by the corresponding generalized Gel'fand map g (i.e., $m(E,F) = g^{-1}(\tau_k)$), is called the *generalized Michael topology of E (with respect to F)*. If F is a locally m-convex algebra, then clearly $m(E,F)$ is a locally m-convex topology. Now, a locally convex algebra $E[\tau]$ is called a *generalized Michael algebra* (*with respect to a given locally convex algebra F*) (or, briefly *F-generalized Michael algebra*), if $\tau = m(E,F)$.

In this respect, we now have the following

THEOREM 4.1. *Let E be an m-barreled locally convex Pták algebra, let F be a locally m-convex semi-Montel algebra and let $\mathscr{M}(E,F)$ be the corresponding generalized spectrum of E (for F given) such that E be F-full and the space $\mathscr{C}(\mathscr{M}(E,F),F)$ of continuous F-valued maps on $\mathscr{M}(E,F)$, equipped with the topology of compact convergence in $\mathscr{M}(E,F)$* [2], *be barreled. Then the respective generalized Gel'fand map $g\colon E \mapsto \mathscr{C}(\mathscr{M}(E,F),F)$ is an homeomorphism. Moreover, the topology of E coincides with $m(E,F)$ (i.e., E is an F-generalized Michael algebra).*

Proof. First, by hypothesis and [9, p. 177, Corollary 4.1], the map g is continuous. Moreover, the continuity of the inverse map g^{-1} follows clearly by hypothesis and [1, p. 296, Theorem 2]. On the other hand, the final assertion is now obviously obtained by hypothesis, Theorem 2.1 in the foregoing and the definition of the topology $m(E,F)$, and the proof is completed.

COROLLARY 4.2. *Let E be an m-barreled locally convex Pták algebra and let F be a Fréchet (locally m-convex) semi-Montel algebra such that E be F-full. Then, the corresponding generalized Gel'fand map $g\colon E \mapsto \mathscr{C}(\mathscr{M}(E,F),F)$ is an homeomorphism. Moreover, the topology of E coincides with $m(E,F)$ (i.e., E is an F-generalized Michael (locally m-convex) algebra).*

Proof. First, by hypothesis and Corollary 3.5 above, the spectrum $\mathscr{M}(E,F)$ is a Nachbin–Shirota space. Moreover, by hypothesis for E and Theorem 3.3 in the foregoing, $\mathscr{C}(\mathscr{M}(E,F),F)$ is a barreled space, so that the assertion is reduced to that of Theorem 4.1 above and the proof is finished.

On the other hand, by Theorem 4.1 above and the same arguments of the preceding corollary we finally get the following.

COROLLARY 4.3. *Let E be an m-barreled locally convex Pták algebra and let F be a complete nuclear barreled locally m-convex (hence, semi-Montel) algebra* *such that E is F-full. Then, the respective generalized Gel'fand map g is a homeomorphism. Moreover, E is an F-generalized Michael (locally m-convex) algebra.*

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