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nth Root extraction: Double iteration process and Newton's method[1]

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Abstract

A double iteration process already used to find the nth root of a positive real number is analysed and showed to be equivalent to the Newton's method. These methods are of order two and three. Higher-order methods for finding the nth root are also mentioned. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Finding the nth root of a strictly positive real number is a basic important problem for numerical computation. Several algorithms exist to efficiently solve this problem. In this paper we present a general analysis of two families of algorithms. The first family is based on a double iteration process, while the second family is a consequence of Newton's method. These methods use explicitly only the four elementary arithmetic operations: addition, subtraction, multiplication, and division. Special cases of these two families have already been presented in the literature. These two families are equivalent as indicated below.

The double iteraction process we suggest is the following: let b_0 and z_0 be two given strictly positive real numbers such that $b_0 z_0^n = r$, and

$$b_{k+1} = b_k \left(\frac{(n-\beta)b_k + \beta}{(n+1-\beta)b_k + \beta - 1} \right)^n , \tag{1.1a}$$

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$$z_{k+1} = z_k \left(\frac{(n+1-\beta)b_k + \beta - 1}{(n-\beta)b_k + \beta} \right) \tag{1.1b}$$

$k = 0, 1, 2, \ldots$, where $\beta \in \mathbb{R}$ is fixed. We remark that $b_k z_k^n = r$, and it will be shown that $\{b_k\}_{k=0}^{+\infty}$ converges to 1, and hence $\{z_k\}_{k=0}^{+\infty}$ converges to $r^{1/n}$. Gower [8] has analysed a double iteration process that corresponds to $\beta = 0$ for $r \in (0,1)$. Also Knill [10] has related the case $\beta = 0$ and $n = 2$ for $r \in (1, +\infty)$ to Newton's method for square-root extraction. A general analysis of this family will be done in Section 2.

In Section 3 we analyse Newton's method applied to the equation $F(x) = x^{\beta-n}(x^n - r) = x^\beta(1 - r/x^n)$ to find $r^{1/n}$ where $\beta \in \mathbb{R}$ is fixed. This scheme generates a sequence $\{x_k\}_{k=0}^{+\infty}$, where

$$x_0 \text{ is given (sufficiently close to } r^{1/n}\text{), and} \tag{1.2a}$$

$$x_{k+1} = x_k - \frac{F(x_k)}{F^{(1)}(x_k)} = x_k \left[\frac{(n+1-\beta)r + (\beta - 1)x_k^n}{(n-\beta)r + \beta x_k^n} \right] \quad \text{for } k = 0, 1, 2, \ldots . \tag{1.2b}$$

The two families of methods are equivalent. To see this, let $b_k = r/x_k^n$ in (1.2b), then we obtain

$$x_{k+1} = x_k \left(\frac{(n+1-\beta)b_k + \beta - 1}{(n-\beta)b_k + \beta} \right),$$

and

$$b_{k+1} = \frac{r}{x_{k+1}^n} = \frac{r}{x_k^n} \left(\frac{(n-\beta)b_k + \beta}{(n+1-\beta)b_k + \beta - 1} \right)^n = b_k \left(\frac{(n-\beta)b_k + \beta}{(n+1-\beta)b_k + \beta - 1} \right)^n,$$

which corresponds to (1.1). The two methods produce the same sequences, i.e. for given r and n, if $z_0 = x_0$ then $z_k = x_k$ for all k.

In Sections 2 and 3 we obtain second-order methods for $\beta \neq \frac{1}{2}(n+1)$ and third-order methods for $\beta = \frac{1}{2}(n+1)$. Finally, in Section 4, we indicate how it is possible to obtain methods of order p, for arbitrary p, for obtaining the square root [9, 13] and the nth root [5].

2. Double iteration process

The double iteration process we consider here is based on the following fact:

Lemma 2.1. *Let r be a strictly positive real number. If $\{b_k\}_{k=0}^{+\infty}$ and $\{z_k\}_{k=0}^{+\infty}$ are two sequences of strictly positive real numbers such that (a) $b_k z_k^n = r$ for $k = 0, 1, 2, \ldots$, and (b) $\lim_{k\to+\infty} b_k = 1$, then $\lim_{k\to+\infty} z_k = r^{1/n}$.*

Proof. If b_k and z_k are strictly positive real numbers and $b_k z_k^n = r$ then $z_k = (r/b_k)^{1/n}$. If $b_k \xrightarrow[k\to+\infty]{} 1$, then $z_k \xrightarrow[k\to+\infty]{} r^{1/n}$. □

Scheme (1.1) we suggest here is a generalisation and/or an extension of Gower's [8] and Knill's [10] schemes. The convergence analysis is based on the next three lemmas in which we use the following functions:

$$\varphi(x)=\frac{x+\beta}{x+\beta-n} \quad \text{and} \quad \psi(x)=\left(\frac{x}{x+1}\right)^n \varphi(x).$$

Lemma 2.2. *Let n be a strictly positive integer and $\beta\in\mathbb{R}$. Then $\varphi(x)$ is a strictly decreasing function. Moreover,* $\lim_{k\to+\infty}\varphi(x)=1=\lim_{k\to+\infty}\varphi(x)$, *and* $\lim_{x\to(n-\beta)^+}\varphi(x)=+\infty=-\lim_{x\to(n-\beta)^-}\varphi(x)$.

Lemma 2.3. *Let n be a strictly positive integer and $\beta\in\mathbb{R}$. For*
(a) *$n=1$ and $\beta<1$ or $n\geqslant 2$ and $\beta\leqslant\frac{1}{2}(n+1)$, $\psi(x)$ is a strictly decreasing function on $(n-\beta,+\infty)$ such that* $\lim_{x\to(n-\beta)^+}\psi(x)=+\infty, \lim_{x\to+\infty}\psi(x)=1$, *and* $1<\psi(x)<\varphi(x)$,
(b) *$n=1$ and $\beta>1$ or $n\geqslant 2$ and $\beta\geqslant\frac{1}{2}(n+1)$, $\psi(x)$ is a strictly decreasing function on $(-\infty,-\beta]$ such that* $\lim_{x\to-\infty}\psi(x)=1, \psi(-\beta)=0$, *and* $0<\varphi(x)<\psi(x)<1$.

Proof. The result follows from the analysis of the sign of

$$\psi^{(1)}(x)=\frac{nx^{n-1}}{(x+1)^{n+1}(x+\beta-n)^2}\left[x\left[2\beta-(n+1)\right]+\beta(\beta-n)\right] \quad \square$$

Lemma 2.4. *Let n be a strictly positive integer and $\beta\in\mathbb{R}$.*
(a) *Let $n=1$ and $\beta<1$ or $n\geqslant 2$ and $\beta\leqslant\frac{1}{2}(n+1)$, and $b_0>1$. Let the sequences $\{a_k\}_{k=0}^{+\infty}$ and $\{b_k\}_{k=0}^{+\infty}$ be generated such that $\varphi(a_k)=b_k$ and $b_{k+1}=\psi(a_k)$ for $k=0,1,2,\ldots$. Then $\{a_k\}_{k=0}^{+\infty}$ is a strictly increasing sequence, $a_0>n-\beta>0$ and $\lim_{k\to+\infty}a_k=+\infty$, $\{b_k\}_{k=0}^{+\infty}$ is a strictly decreasing sequence and $\lim_{k\to+\infty}b_k=1$.*
(b) *Let $n=1$ and $\beta>1$ or $n\geqslant 2$ and $\beta\geqslant\frac{1}{2}(n+1)$, and $0<b_0<1$. Let the sequences $\{a_k\}_{k=0}^{+\infty}$ and $\{b_k\}_{k=0}^{+\infty}$ be generated such that $\varphi(a_k)=b_k$ and $b_{k+1}=\psi(a_k)$ for $k=0,1,2,\ldots$. Then $\{a_k\}_{k=0}^{+\infty}$ is a strictly decreasing sequence, $a_0<-\beta<-1$ and $\lim_{k\to+\infty}a_k=-\infty$, $\{b_k\}_{k=0}^{+\infty}$ is a strictly increasing sequence and $\lim_{k\to+\infty}b_k=1$.*

Proof. (a) If $b_k>1$, $\varphi(a_k)=b_k$ implies

$$a_k=(n-\beta)+\frac{n}{b_{k-1}}>n-\beta.$$

Then $1<b_{k+1}=\psi(a_k)<\varphi(a_k)=b_k$. Hence $\{a_k\}_{k=0}^{+\infty}$ is a strictly increasing sequence, and $\{b_k\}_{k=0}^{+\infty}$ is a strictly decreasing sequence lower bounded by 1. The result follows from the properties of $\psi(\bullet)$ and $\varphi(\bullet)$ of Lemma 2.3(a). (b) If $0<b_k<1$, $\varphi(a_k)=b_k$ implies

$$a_k=-\beta-\frac{nb_k}{1-b_k}<-\beta.$$

Then $1>b_{k+1}=\psi(a_k)>\varphi(a_k)=b_k$. Hence, $\{a_k\}_{k=0}^{+\infty}$ is a strictly decreasing sequence, and $\{b_k\}_{k=0}^{+\infty}$ is a strictly increasing sequence upper bounded by 1. The result follows from the properties of $\psi(\bullet)$ and $\varphi(\bullet)$ of Lemma 2.3(b). $\square$

Let us observe that

$$b_{k+1} = \psi(a_k) = \varphi(a_k)\left(\frac{a_k}{a_k+1}\right)^n = b_k\left(\frac{(n-\beta)b_k+\beta}{(n+1-\beta)b_k+\beta-1}\right)^n,$$

and

$$z_{k+1} = z_k\left(\frac{(n+1-\beta)b_k+\beta-1}{(n-\beta)b_k+\beta}\right) = z_k\left(\frac{a_k+1}{a_k}\right) = z_k\left(1+\frac{1}{a_k}\right).$$

We are now ready to prove the convergence of scheme (1.1).

Theorem 2.5. *Let n be a strictly positive integer and $\beta \in \mathbb{R}$.*
(a) *Let $n=1$ and $\beta < 1$ or $n \geqslant 2$ and $\beta \leqslant \frac{1}{2}(n+1)$, and let $\rho > 0$ be given such that $b_0 = r/\rho^n > 1$ and $z_0 = \rho$. Let $\{a_k\}_{k=0}^{+\infty}$ and $\{b_k\}_{k=0}^{+\infty}$ be generated as in Lemma 2.4 (a), and let $\{z_k\}_{k=0}^{+\infty}$ be defined by $z_{k+1} = z_k(1+1/a_k)$ for $k=0,1,2,\ldots$. Then $\{z_k\}_{k=0}^{+\infty}$ is a strictly increasing sequence and $\lim_{k\to+\infty} z_k = r^{1/n}$.*
(b) *Let $n=1$ and $\beta > 1$ or $n \geqslant 2$ and $\beta \geqslant \frac{1}{2}(n+1)$, and let $\rho > 0$ be given such that $0 < b_0 = r/\rho^n < 1$ and $z_0 = \rho$. Let $\{a_k\}_{k=0}^{+\infty}$ and $\{b_k\}_{k=0}^{+\infty}$ be generated as in Lemma 2.4 (b), and let $\{z_k\}_{k=0}^{+\infty}$ be defined by $z_{k+1} = z_k(1+1/a_k)$ for $k=0,1,2,\ldots$. Then $\{z_k\}_{k=0}^{+\infty}$ is a strictly decreasing sequence and $\lim_{k\to+\infty} z_k = r^{1/n}$.*
Moreover, the convergence is quadratic for $\beta \neq \frac{1}{2}(n+1)$ and cubic for $\beta = \frac{1}{2}(n+1)$ and $n \geqslant 2$.

Proof. From Lemma 2.4a, $\{z_k\}_{k=0}^{+\infty}$ is a strictly increasing sequence of strictly positive real numbers because $z_0 > 0$ and $(1+1/a_k) > 1$. Also from Lemma 2.4b, $\{z_k\}_{k=0}^{+\infty}$ is a strictly decreasing sequence of strictly positive real numbers because $z_0 > 0$ and $0 < (1+1/a_k) < 1$. Moreover, the sequences $\{b_k\}_{k=0}^{+\infty}$ and $\{z_k\}_{k=0}^{+\infty}$ satisfy Lemma 2.1 because $b_0 z_0^n = r$ follows from the definition of b_0 and z_0, and $b_k z_k^n = r$ for $k=0,1,2,\ldots$, follows by induction since

$$b_{k+1}z_{k+1}^n = \psi(a_k)z_n^k\left(1+\frac{1}{a_k}\right)^n = \varphi(a_k)\left(\frac{a_k}{a_k+1}\right)^n z_k^n\left(\frac{a_k+1}{a_k}\right)^n = b_k z_k^n.$$

To obtain the rate of convergence, we observe that

$$r - z_k^n = \left(r^{1/n} - z_k\right)\sum_{i=0}^{n-1} r^{(n-1-i)/n} z_k^i$$

and

$$\lim_{k\to+\infty} \frac{r - z_k^n}{r^{1/n} - z_k} = nr^{(n-1)/n}.$$

Also, from $b_k z_k^n = r$ we have

$$\frac{r - z_{k+1}^n}{(r - z_k^n)^m} = \frac{r(1-1/b_{k+1})}{r^m(1-1/b_k)^m} = \frac{b_k^m}{r^{m-1}b_{k+1}}\frac{(b_{k+1}-1)}{(b_k-1)^m}.$$

But

$$b_{k+1} = b_k\left(1 - \frac{b_k-1}{(n-\beta+1)b_k+(\beta-1)}\right)^n = b_k\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(\frac{b_k-1}{(n-\beta+1)b_k+(\beta-1)}\right)^i,$$

then

$$b_{k+1}-1=\left[\frac{n\left(\frac{1}{2}(n+1)-\beta\right)+\left[n\left(\frac{1}{2}(n+1)-\beta\right)+(\beta-1)^2\right](b_k-1)}{[(n-\beta+1)b_k+(\beta-1)]^2}\right](b_k-1)^2$$
$$+b_k\sum_{i=3}^{n}\binom{n}{i}(-1)^i\left(\frac{b_k-1}{(n-\beta+1)b_k+(\beta-1)}\right)^i.$$

It follows that

$$\lim_{k\to+\infty}\frac{b_{k+1}-1}{(b_k-1)^2}=\frac{n+1-2\beta}{2n},$$

$$\lim_{k\to+\infty}\frac{r^{1/n}-z_{k+1}}{(r^{1/n}-z_k)^2}=\frac{n+1-2\beta}{2r^{1/n}},$$

and the convergence is quadratic, but for $\beta=\frac{1}{2}(n+1)$ we have

$$\lim_{k\to+\infty}\frac{b_{k+1}-1}{(b_k-1)^3}=\frac{(n-1)(n+1)}{12n^2},$$

$$\lim_{k\to+\infty}\frac{r^{1/n}-z_{k+1}}{(r^{1/n}-z_k)^3}=\frac{(n-1)(n+1)}{12r^{2/n}},$$

and the convergence is cubic. □

Remark 2.6. The convergence is not uniform in r (see also [10] for $\beta=0$ and $n=2$). Indeed, for $\beta\leqslant\frac{1}{2}(n+1)$, let $r\to+\infty$, then $b_0=r/\rho^n\to+\infty$, and for any fixed k we have $b_k\to+\infty, a_k\to n-\beta$, and $z_k\to z_0\left((n-\beta+1)/(n-\beta)\right)^k$. For $\beta\geqslant\frac{1}{2}(n+1)$, let $r\to 0^+$, then $b_0=r/\rho^n\to 0$, and for any fixed k we have $b_k\to 0, a_k\to-\beta$, and $z_k\to z_0\left((\beta-1)/\beta\right)^k$.

Example 2.7. For $\beta=\frac{1}{2}(n+1)$, let $\rho>0$, set $b_0=r/\rho^n$, $z_0=\rho$, we have methods of order 3 given by

$$b_{k+1}=b_k\left(\frac{(n-1)b_k+(n+1)}{(n+1)b_k+(n-1)}\right)^n \quad\text{and}\quad z_{k+1}=z_k\left(\frac{(n+1)b_k+(n-1)}{(n-1)b_k+(n+1)}\right).$$

Remark 2.8. The double iterative procedure introduced by Gower [8] corresponds to $\beta=0$ since for $b_k=1/(1+c_k)$ we have $(1+c_{k+1})=(1+c_k)(1-c_k/n)^n$ and $z_{k+1}=z_k(1-c_k/n)$. For $n=1$, the process can be used to find r^{-1} for $r\in(0,1)$ using only additions, subtractions, and multiplications. If we start with $b_0z_0=r^{-1}, z_0=1$ and $c_0=r-1$, then

$$c_{k+1}=c_k^2 \quad\text{and}\quad z_{k+1}=z_k(1-c_k).$$

This is the process described in [8].

3. Newton's method

Let us recall the following result about Newton's method:

Theorem 3.1 (Dennis and Schnabel [3], Ford amd Pennline [6], Gerlach [7], Traub [12]). *Let p be an integer $\geqslant 1$, and let $F(x)$ be a regular function such that $F(\alpha)=0$, $F^{(1)}(\alpha)\neq 0$, $F^{(j)}(\alpha)=0$ for $j=2,\dots,p$, and $F^{(p+1)}(\alpha)\neq 0$. Then Newton's method applied to the equation $F(x)=0$ generates a sequence $\{x_k\}_{k=0}^{+\infty}$, where*

$$x_{k+1}=x_k-\frac{F(x_k)}{F^{(1)}(x_k)}\quad (k=0,1,2,\dots)$$

converges to α for x_0 given sufficiently close to α. Moreover, the convergence is of order $p+1$, and the asymptotic constant is

$$K_{p+1}(\alpha)=\lim_{k\to+\infty}\frac{\alpha-x_{k+1}}{(\alpha-x_k)^{p+1}}=(-1)^p\frac{p}{(p+1)!}\frac{F^{(p+1)}(\alpha)}{F^{(1)}(\alpha)}.$$

In order to find the nth root of a positive real number r, we consider Newton's method applied to the equation

$$F(x)=x^{\beta-n}(x^n-r)=x^\beta\left(1-\frac{r}{x^n}\right), \tag{3.1}$$

where $\beta\in\mathbb{R}$. Direct computation gives

$$\begin{aligned}F^{(1)}(x)&=[\beta x^n+(n-\beta)r]/x^{n+1-\beta},\\ F^{(2)}(x)&=[\beta(\beta-1)x^n-(n+1-\beta)(n-\beta)r]/x^{n+2-\beta},\\ F^{(3)}(x)&=[\beta(\beta-1)(\beta-2)x^n+(n+2-\beta)(n+1-\beta)(n-\beta)r]/x^{n+3-\beta}.\end{aligned}$$

Hence $F^{(1)}\left(r^{1/n}\right)=nr^{(\beta-1)/n}\neq 0$ and $F^{(2)}\left(r^{1/n}\right)=-n(n+1-\beta)r^{(\beta-2)/n}$. Moreover $F^{(2)}\left(r^{1/n}\right)=0$ if and only if $\beta=\frac{1}{2}(n+1)$, and then

$$F^{(3)}\left(r^{1/n}\right)=n\frac{(n-1)(n+1)}{4}r^{(n-5)/2n}.$$

We now have the following algorithm:

Let x_0 given (sufficiently close to $r^{1/n}$), and

$$x_{k+1}=x_k-\frac{F(x_k)}{F^{(1)}(x_k)}=x_k\left[\frac{(n+1-\beta)r+(\beta-1)x_k^n}{(n-\beta)r+\beta x_k^n}\right]\quad\text{for } k=0,1,2,\dots,$$

it generates a sequence $\{x_k\}_{k=0}^{+\infty}$ which converges to $r^{1/n}$. The convergence is of order 2 for $\beta\neq\frac{1}{2}(n+1)$ with $K_2\left(r^{1/n}\right)=(n+1-2\beta)/2r^{1/n}$, and of order 3 for $\beta=\frac{1}{2}(n+1)$ with $K_3\left(r^{1/n}\right)=((n-1)(n+1))/12r^{2/n}$. Moreover, from the variation of the sign of $F^{(1)}(x)$ and $F^{(2)}(x)$ on $(0,+\infty)$ with respect to the value of β, we have:

(a) for $n=1$ and $\beta<1$ or $n\geqslant 2$ and $\beta\leqslant\frac{1}{2}(n+1)$, $F(x)$ is strictly increasing and concave on $(0,r^{1/n}]$, and for any $x_0\in(0,r^{1/n})$ the sequence $\{x_k\}_{k=0}^{+\infty}$ increases and converges to $r^{1/n}$;

(b) for $n = 1$ and $\beta > 1$ or $n \geqslant 2$ and $\beta \geqslant \frac{1}{2}(n+1), F(x)$ is strictly increasing and convex on $[r^{1/n}, +\infty)$, and for any $x_0 \in (r^{1/n}, +\infty)$ the sequence $\{x_k\}_{k=0}^{+\infty}$ decreases and converges to $r^{1/n}$.

Example 3.2. Square root extraction: $n = 2$,

$$x_{k+1} = x_k \left[\frac{(3-\beta)r + (\beta-1)x_k^2}{(2-\beta)r + \beta x_k^2} \right],$$

and for $\beta = 3$ it is one of the methods obtained by Heron of Alexandria (about 100 B.C.) [2]

Example 3.3. $\beta = \frac{1}{2}(n+1)$, method of order 3 for nth root extraction

$$x_{k+1} = x_k \left[\frac{(n+1)r + (n-1)x_k^n}{(n-1)r + (n+1)x_k^n} \right].$$

These schemes have a long history [1]. They have also been considered by Kogbetliantz [11] for n odd. For $n = 2$ and $n = 3$, these methods have already been obtained by Heron of Alexandria [2, 4].

4. Higher order methods

High-order methods for approximating square roots have been presented recently. In [9, 13], the scheme generates the sequence $\{x_k\}_{k=0}^{+\infty}$, where

$x_0 > 0$ is given, and

$$x_{k+1} = \frac{\sum_{i \text{ even}} \binom{p}{i} x_k^{p-i} r^{i/2}}{\sum_{i \text{ odd}} \binom{p}{i} x_k^{p-i} r^{(i-1)/2}} \quad \text{for } k = 0, 1, 2, \ldots,$$

which converges to $r^{1/2}$. The convergence is of order p, and the asymptotic constant is

$$K_p\left(r^{1/2}\right) = \lim_{k \to +\infty} \frac{r^{1/2} - x_{k+1}}{(r^{1/2} - x_k)^p} = \frac{1}{(-2r^{1/2})^{p-1}}.$$

In [13] the scheme is related to continued fraction expansions of $r^{1/2}$.

To get higher-order method for finding $r^{1/n}$, we can try to find a function $g(x)$ such that

$$f(x) = g(x)(x^n - r)$$

satisfies the assumptions of Theorem 3.1 [6, 7]. Let us observe that $g(x) = x^{\beta - n}$ in (3.1). One such function $g(x)$ is

$$g(x) = \sum_{i=1}^{p} a_i (x^n - r)^{i-1} = \sum_{i=1}^{p} \binom{1/n}{i} \frac{(x^n - r)^{i-1}}{r^i}.$$

It can be shown [5] that

$$F(x) = g(x)(x^n - r) = \sum_{i=1}^{p} \binom{1/n}{i} \left(\frac{x^n - r}{r}\right)^i$$

verifies $F\left(r^{1/n}\right) = 0$, $F^{(1)}\left(r^{1/n}\right) = r^{-1/n}$, $F^{(j)}\left(r^{1/n}\right) = 0$ for $j = 2, \ldots, p$, and

$$F^{(p+1)}\left(r^{1/n}\right) = -\binom{1/n}{p}(p+1)!n^{p+1}r^{-(p+1)/n}.$$

The sequence $\{x_k\}_{k=0}^{+\infty}$, where

x_0 given, sufficiently close to $r^{1/n}$, and

$$x_{k+1} = x_k - \frac{F(x_k)}{F^{(1)}(x_k)} = x_k - \frac{(x_k^n - r)\sum_{i=1}^{p} \binom{1/n}{i} ((x_k^n - r)/r)^{i-1}}{nx_k^{n-1} \sum_{i=1}^{p} i \binom{1/n}{i} ((x_k^n - r)/r)^{i-1}} \quad \text{for } k = 0, 1, 2, \ldots,$$

converges to $r^{1/n}$. Moreover, the convergence is of order $p + 1$, and the asymptotic constant is

$$K_{p+1}\left(r^{1/n}\right) = \lim_{k \to +\infty} \frac{r^{1/n} - x_{k+1}}{\left(r^{1/n} - x_k\right)^{p+1}} = -pn^{p+1}\binom{1/n}{p} r^{-p/n}.$$

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